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Nash Equilibrium in Discontinuous Games

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Abstract

We review the discontinuous games literature, with a sharp focus on conditions that ensure the existence of pure and mixed strategy Nash equilibria in strategic form games and of Bayes-Nash equilibria in Bayesian games.

1. INTRODUCTION

The purpose of this article is to provide a review of some of the main results in the literature on the existence of Nash equilibria in discontinuous games. Our objective is to give practitioners and nonexperts in the field a useful compilation of results that can be consulted for use in their work, and to give some sense of why the results are true. A few results here are new. These new results either unify different assumptions into a single, more permissive assumption or illuminate how known assumptions can be adapted to provide new existence results in familiar contexts.[1] We do not provide here a comprehensive survey of the literature, nor do we always give the most general versions of results that are available. One example of an important topic that could not be covered here is the work by Simon & Zame (1990) and of Jackson et al. (2002) on sharing-rule equilibria. Section 6 contains some brief remarks on this topic and also includes suggested readings for other topics not covered here.[2]

Discontinuous games are commonplace in economics. A Bertrand price competitor experiences a discontinuous reduction in profits when their price rises above the lowest price among their competitors. With fixed costs of production, a Cournot oligopolist's losses are close to their fixed costs when production is close to zero but discontinuously jump to zero when production is exactly zero. A bidder in an auction experiences a discontinuous jump in their utility when their bid on some unit increases to the point where it is no longer a losing bid. A politician's utility jumps when their policy position shifts just enough to bring their total vote count from just below a winning threshold to just above it. In all of these cases, the discontinuities in payoffs preclude the use of Nash's (1950) theorem or even of Glicksberg's (1952) theorem to guarantee the existence of a pure or mixed strategy Nash equilibrium. Nevertheless, by making use of theorems developed over the last 35 years, the existence of a Nash equilibrium can be guaranteed under quite general conditions.

The basic method of proof underlying the existence theorems in this literature (whether for pure or for mixed strategy Nash equilibria) is simple. Suppose that N is a finite set of players and that for each $i \in N$, X_i is player i's strategy space. Let $X := \times_{i\in N} X_i$ be the space of strategy profiles. Call $b : X \twoheadrightarrow X$ a better-reply correspondence iff, for every $x \in X$, we have $\emptyset \neq b(x) = (b_i(x_{-i}))_{i\in N} \subseteq X$ and there is a player i for whom every $z_i \in b_i(x_{-i}) \in X_i$ is a strictly better reply than x_i against $x_{-i} = (x_j)_{j\neq i}$. The proofs of virtually all of the existence theorems in the recent discontinuous games literature proceed as follows. They begin by assuming, by way of contradiction, that the game does not possess a Nash equilibrium. Consequently, for every $x \in X$, some player i has a strictly better reply than x_i against x_{-i}. From all of these better replies, a better-reply correspondence, b, is constructed. Finally, it is shown that the constructed better-reply correspondence has a fixed point $x^* \in b(x^*)$. The existence of such a fixed point provides the desired contradiction because, by the definition of a better-reply correspondence, there is a player i for whom every $z_i \in b_i(x^*_{-i})$ is a strictly better reply than x^*_i against x^*_{-i}. However, by the fixed point property we have $x^*_i \in b_i(x^*_{-i})$, and so x^*_i is a strictly better reply than itself against x^*_{-i}! Thus, the assumption that the game has no Nash equilibrium leads to a contradiction, and so we can conclude that the game has a Nash equilibrium after all.

As simple as this method of proof may seem, producing a good and useful existence result can be tricky. Ideally, the hypotheses that are given should be easy to verify and widely applicable. Ultimately however, there is a trade-off between these two desiderata, and so we will give here a variety of results that lie along the easy-to-verify/widely applicable frontier.

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[1] Examples of the former are Theorems 3(j, k), 6, 7, and 12. Examples of the latter are Theorems 13, 14, and 15.
[2] Carmona (2013) provides an excellent treatment of the overall theory of discontinuous games.

The remainder of this review is organized as follows. Section 2 lays out the basic notation that is used throughout the article. Section 3 covers results on the existence of pure strategy equilibria when payoff functions are own-strategy quasiconcave, including results for quasi-symmetric games. Theorem 3 puts all of the various existence results together in one place for easy reference, and **Figure 1** in that section indicates how those results are related to one another. Section 4 covers games in which payoffs need not be own-strategy quasiconcave. Because mixed strategy equilibria can be thought of as pure strategy equilibria of a game's mixed extension, all of the results on pure strategy equilibria can be applied to a game's mixed extension to yield results on the existence of a mixed strategy Nash equilibrium. This is pointed out in Section 5. However, the multilinearity of the players' payoffs in all of their mixed strategies yields additional structure that permits powerful mixed strategy equilibrium existence results that go beyond the application of the pure strategy results. The main contribution of Section 5 is to present some of these mixed strategy results, including several that apply specifically to Bayesian games. Section 6 provides a few remarks on some related literature that is not covered here.

2. PRELIMINARIES

There is a nonempty finite set of players, $N = \{1, 2, \ldots, |N|\}$, with typical element i. Each player $i \in N$ has a nonempty set of (pure) strategies X_i that is a subset of a Hausdorff linear topological space. Each X_i is endowed with its relative topology, and the product of any number of topological spaces is endowed with the product topology.[3] The set of strategy profiles is $X := \times_{i \in N} X_i$. Each player i has a payoff function $u_i : X \to \mathbb{R}$. Altogether, these items define a strategic form game $G := (X_i, u_i)_{i \in N}$. The symbol $-i$ indicates all players but i. In particular, we have $X_{-i} := \times_{j \neq i} X_j$, and x_{-i} denotes a typical element of X_{-i}. Unless stated otherwise, the game G refers to the strategic form game $(X_i, u_i)_{i \in N}$.

Say that G is (*a*) compact iff each X_i is compact, (*b*) convex iff each X_i is convex and locally convex, (*c*) bounded iff each $u_i : X \to \mathbb{R}$ is bounded, and (*d*) quasiconcave iff each $u_i(x_i, x_{-i})$ is quasiconcave in $x_i \in X_i$ for each $x_{-i} \in X_{-i}$. Define $u : X \to \mathbb{R}^{|N|}$, the vector payoff function of the game G by $u(x) := (u_i(x))_{i \in N}$ for every $x \in X$. The graph of the vector payoff function u is $\mathrm{Gr}u := \{(x, \alpha) \in X \times \mathbb{R}^{|N|} : u(x) = \alpha\}$. The closure of $\mathrm{Gr}u$ is denoted by $\overline{\mathrm{Gr}u}$. A neighborhood of any point in any topological space is any open set that contains that point.

Let A be any subset of X, and let $b : A \twoheadrightarrow X$ be any correspondence mapping A into subsets (including the empty set) of X. The graph of b is $\mathrm{Gr}b := \{(x, y) \in A \times X : y \in b(x)\}$, and we say that $b : A \twoheadrightarrow X$ is closed iff its graph is closed in $A \times X$, i.e., iff for every closed $C \subseteq A$, $(\mathrm{Gr}b) \cap (C \times X)$ is a closed subset of $X \times X$. For any subset A of any convex set, $\mathrm{co}A$ denotes the convex hull of A.

3. PURE STRATEGY EQUILIBRIA IN QUASICONCAVE GAMES

A strategy profile $x^* \in X$ is a pure strategy Nash equilibrium of G iff $u_i(x_i, x^*_{-i}) \leq u_i(x^*)$ for every player $i \in N$ and every $x_i \in X_i$.

In an important early paper, Sion (1958) showed that, for two-person zero-sum compact and convex games, if player 1's payoff function is quasiconcave and upper semicontinuous in the player's own strategy (for any fixed strategy of player 2) and quasiconvex and lower

[3]Therefore, a subset U of X_i is open iff $U = X_i \cap V$ for some open subset V of the linear topological space that contains X_i. For example, if the set of real numbers is given its usual topology and $X_i = [0, 1]$ is given its relative topology, then $[0, 1/2)$ is an open subset of X_i, since it is the intersection of X_i with the open set of all real numbers less than $1/2$.

semicontinuous in player 2's strategy (for any fixed strategy of player 1), then the game has a pure strategy Nash equilibrium.[4]

Sion (1958) gives a simple example that shows that these semicontinuity conditions are needed. The example is a two-person zero-sum game on the unit square in which player 1 gets a payoff of 1 if either they choose $x_1 = 0$ and player 2 chooses $x_2 \leq 1/2$, or they choose $x_1 = 1$ and player 2 chooses $x_2 > 1/2$. Otherwise, player 1 gets a payoff of 0. This game is compact, convex, bounded, and quasiconcave, but it has no pure strategy Nash equilibrium.

There is no equilibrium because, for any strategy of player 2, player 1 can get a payoff of 1, and for any strategy of player 1, player 2 can get a payoff of 0 (so we have that $\min\max u_1 = 1 > 0 = \max\min u_1$). The only assumption that is violated is that player 1's payoff is not lower semicontinuous in player 2's strategy. Even then, this failure of lower semicontinuity occurs at just one point, namely at the point $(x_1, x_2) = (0, 1/2)$ where player 1 gets a payoff of 1. Lower semicontinuity fails at this point because when $x_1 = 0$ and when player 2's strategy converges to $1/2$ from above, player 1's payoff jumps up from 0 to 1 at the limit $x_2 = 1/2$.

In their seminal work on the existence of pure and mixed strategy Nash equilibria in discontinuous games, Dasgupta & Maskin (1986a) provide a pure strategy equilibrium existence result for multi-person games that, like Sion's result, makes use of various semicontinuity properties.

For any player $i \in N$, define i's value function $v_i : X_{-i} \to \mathbb{R}$ by $v_i(x_{-i}) := \sup_{x_i \in X_i} u_i(x_i, x_{-i})$ for every $x_{-i} \in X_{-i}$. We have the following result [see Tian & Zhou (1992, 1995) for related results for abstract games/economies].

Theorem 1 (Dasgupta & Maskin 1986a). Suppose that G is compact, convex, and quasiconcave, where each X_i is a subset of Euclidean space with its usual topology. If each u_i is upper semicontinuous on X, and if each v_i is lower semicontinuous on X_{-i}, then G has a pure strategy Nash equilibrium.

To prove Theorem 1, Dasgupta & Maskin (1986a) argue that, under the stated hypotheses, each player has a best-reply correspondence that is closed and that has nonempty and convex values. Hence, Kakutani's (1949) theorem can be applied.

Remark 1. If u_i is upper semicontinuous, then i's best-reply correspondence $\hat{b}_i$ is nonempty-valued and, by quasiconcavity, convex-valued. Because the upper semicontinuity of u_i implies that of v_i, the hypotheses of Theorem 1 actually imply that v_i is continuous. Consequently, if x is not a Nash equilibrium, then there is a player i, an $\varepsilon > 0$, and a neighborhood U of x such that $v_i(x'_{-i}) > u_i(x) + \varepsilon \geq u_i(y)$ holds for all $x', y \in U$, where the weak inequality follows because u_i is upper semicontinuous, in particular, at x. Hence, $u_i(z_i, x'_{-i}) = v_i(x'_{-i}) > u_i(x) + \varepsilon \geq u_i(y)$ holds for all x', y, and z_i with $x', y \in U$ and $z_i \in \hat{b}_i(x'_{-i})$. This last fact will be helpful when comparing Theorem 1 with some of the results below.

Returning to Sion's (1958) result, notice that by virtue of the zero-sum property, we can state the quasiconcave-convex and upper-lower-semicontinuity hypotheses of Sion's theorem another way. We could instead just say that the game is convex and quasiconcave, and that for each player, if we fix any of their strategies, then their payoff function is lower semicontinuous in the other player's strategy. As we shall see, lower semicontinuity of payoffs in the others' strategies, and conditions with a similar flavor, play an important role in the theory of discontinuous games.

In their work on the existence of mixed strategy Nash equilibria, Dasgupta & Maskin (1986a) introduce the condition that the sum of the players' payoff functions is upper semicontinuous, a

[4]Kneser (1952), Fan (1953), and Berge (1954) previously made the same semicontinuity assumptions but required player 1's payoff to be concave(-like) in his own strategy and convex(-like) in player 2's strategy.

condition that has proven to be extremely useful in economic contexts. By combining Dasgupta & Maskin's (1986a) upper-semicontinuous-sum condition with the individual-payoff semicontinuity conditions of Sion (1958), we can extend Sion's result from two-person zero-sum games to multi-person games.

Theorem 2 (Reny 1999). Suppose that G is compact, convex, bounded, and quasiconcave, and that the sum of the players' payoff functions $\sum_{i \in N} u_i(x)$ is upper semicontinuous in $x \in X$. If for each player i and for any $x_i \in X_i$ we obtain that $u_i(x_i, x_{-i})$ is lower semicontinuous in $x_{-i} \in X_{-i}$, then G has a pure strategy Nash equilibrium.

While the hypotheses of Theorems 1 and 2 are certainly easy to state and to understand, these theorems tend not to be very useful in economic environments. While payoff sums are often upper semicontinuous in economic games (because discontinuities tend to result when one player captures resources from another player, so that one player's payoff jumps up and the other's jumps down, in many games by the same amount), individual payoffs are not generally upper semicontinuous on X and are not generally lower semicontinuous in others' strategies.

For example, in a Bertrand pricing game, if a competitor's price converges from below to a firm's price that is above their common marginal cost, then the firm's profits will jump up. Hence, the firm's payoff is not lower semicontinuous in the competitor's price.[5] On the other hand, if the competitor's price converges instead from above the firm's price, then the firm's profits will jump down, and payoffs are not upper semicontinuous either. So, in a Bertrand duopoly game, neither of the two theorems above applies. Since many economic games feature discontinuities like these that arise in Bertrand duopoly, we need to develop more useful conditions than those stated in Theorems 1 and 2.

Simon (1987) observed that Dasgupta & Maskin's (1986a) upper semicontinuous payoff-sum hypothesis is needed only to ensure that whenever some player's payoff jumps down, some other player's payoff jumps up. Therefore, in an economic context, if discontinuities arise only when resources (e.g., customers or voters) suddenly shift from one player to another, it is only necessary that if the player who loses resources experiences a payoff loss, then the player who gains resources experiences a payoff gain. There is no need for the gain to weakly exceed the loss, as is needed for the payoff sum to be upper semicontinuous.

Simon called this property "complementary discontinuities." Simon's important idea is now called "reciprocal upper semicontinuity," because, when specialized to a single-player game, it reduces to the assumption that the player's payoff function is upper semicontinuous. Thus, it is a generalization of upper semicontinuity to vector-valued functions. The formal definition is as follows.

Following Simon (1987), say that the vector payoff function $u : X \to \mathbb{R}^{|N|}$ is reciprocally upper semicontinuous iff for any $(x, \alpha) \in X \times \mathbb{R}^{|N|}$ that is in the closure of the graph of u, if $u_i(x) > \alpha_i$ for some $i \in N$, then there exists $j \in N$ such that $u_j(x) < \alpha_j$.

If X were a metric space, then this definition would be equivalent to saying that for any $x \in X$ and for any sequence x^n in X that converges to x, if $u(x^n)$ converges to some point in $\mathbb{R}^{|N|}$ and $\lim_n u_i(x^n) > u_i(x)$ for some player i, then there is a player j such that $\lim_n u_j(x^n) < u_j(x)$ (i.e., if some i's payoff jumps down, then some j's payoff jumps up).

Clearly, if $\sum_i u_i$ is upper semicontinuous, then the vector payoff function u is reciprocally upper semicontinuous. But the reciprocal upper semicontinuity of u does not imply that $\sum_i u_i$ is upper

[5]Notice, however, that the payoff sum does not jump because the competitor's revenues will jump down by precisely the amount that the firm's revenues jump up, since any customers who switch from the competitor to the firm do so at the common limit price.

semicontinuous (and so reciprocally upper semicontinuity is a strictly more permissive condition). For example, consider the two-player game on the unit square in which both players get payoffs of 0 unless they both choose 1, in which case player 1 gets $u_1(1, 1) = -2$ and player 2 gets $u_2(1, 1) = 1$.

In addition to introducing the idea of reciprocal upper semicontinuity, Simon (1987), like Sion (1958) and others, recognized the importance of discontinuities in a player's payoff that result from changes in the others' strategies. Relatedly, while studying variational inequalities and binary relations, Tian (1992a) introduced a generalized concept of lower semicontinuity (called γ-transfer lower semicontinuity) of a function in one of its two variables. These important semicontinuity ideas are all closely related to the notion of a payoff-secure game.

Following Reny (1999), say that player i can secure the payoff $\alpha_i \in \mathbb{R}$ at $x \in X$ iff for every $\varepsilon > 0$, there is $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x'_{-i}) \geq \alpha_i$ for all x'_{-i} in some neighborhood of x_{-i}. Therefore, for any strategy profile and for any payoff number, a player can secure that payoff number at that profile if they have a strategy that gives them at least that payoff even if the other players deviate slightly from their strategies.

Following Reny (1999), say that the game G is payoff secure iff for every $x \in X$ and for every $\varepsilon > 0$, each player i can secure the payoff $u_i(x) - \varepsilon$ at x.[6] Notice that if $u_i(x_i, x_{-i})$ is lower semicontinuous in $x_{-i} \in X_{-i}$ for each $x_i \in X_i$, then, for any $\varepsilon > 0$, player i can secure the payoff $u_i(x) - \varepsilon$ at x simply because, for $\hat{x}_i = x_i$, lower semicontinuity in x_{-i} implies that $u_i(\hat{x}_i, x'_{-i}) \geq u_i(x) - \varepsilon$ for all x'_{-i} in some neighborhood of x_{-i}. Hence, requiring G to be payoff secure is a relaxation of the requirement that each player's payoff function should be lower semicontinuous in the others' strategies.

As we have already seen, a Bertrand duopolist's profits are not lower semicontinuous in the price chosen by the other firm. However, for any price pair (p_1, p_2) and for any $\varepsilon > 0$, firm i can secure the payoff $u_i(p_1, p_2) - \varepsilon$ either by pricing at marginal cost if $u_i(p_1, p_2) \leq 0$, or, if $u_i(p_1, p_2) > 0$, by reducing its price slightly. Indeed, if $u_i(p_1, p_2) > 0$, then there is $\delta > 0$ small enough [this δ will depend on the price-pair (p_1, p_2) and ε] that $u_i(p_i - \delta, p'_j) > u_i(p_i, p_j) - \varepsilon$ for all $p'_j > p_i - \delta$. So the Bertrand duopoly game is payoff secure.

Payoff security and reciprocal upper semicontinuity are relatively easy conditions to check in practice, and in compact, convex, and quasiconcave games, they suffice for the existence of a pure strategy Nash equilibrium [see Theorem 3(*c*) below].[7]

While many discontinuous games are payoff secure and reciprocally upper semicontinuous, there are important exceptions. Consider, for example, a first-price single-unit auction between two risk-neutral bidders with private values that are drawn independently and uniformly from [1, 3]. Each bidder can submit any number in [0, 4] as a sealed bid. The higher bid wins, with ties broken by tossing a fair coin. This game is compact, convex, bounded, and quasiconcave when we define the pure strategy sets X_i to be the sets of (mixed) behavioral strategies that map a bidder's value into a probability distribution over bids.[8] However, this very standard auction game is not reciprocally upper semicontinuous, as the following example shows.

For any positive integer n, consider the pure strategy profile in which each bidder bids $1 - 1/n$ when their value is in [1, 2] and bids 1 when their value is in (2,3]. Under this strategy profile, each

[6]Prokopovych (2011) observed that Reny's (1999) concept of a payoff-secure game can be equivalently stated by saying that each player's payoff function is γ-transfer lower semicontinuous in the others' strategies for every $\gamma \in \mathbb{R}$.
[7]This result remains true with a weaker concept of reciprocal upper semicontinuity (see Bagh & Jofre 2006).
[8]We can use the weak* on the space of distributional strategies (as done by Milgrom & Weber 1985). We will have more to say about Bayesian games in Section 5.1.

bidder's ex-ante expected payoff is given by

$$\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}-\left(1-\frac{1}{n}\right)\right)\right)+\frac{1}{2}0\right]+\frac{1}{2}\left[\frac{1}{2}\left(\frac{5}{2}-1\right)+\frac{1}{2}\left(\frac{1}{2}\left(\frac{5}{2}-1\right)\right)\right].$$

To see why this formula is correct, note that the first term in the sum arises because a bidder's value is in $[1, 2]$ with probability $1/2$; in this case, they bid $1 - 1/n$ and so they can win only when the other bidder's value is also in $[1, 2]$, which occurs with probability $1/2$. But even then, the bids will be the same, and so the first bidder wins with probability $1/2$, and in this event their expected surplus is $3/2 - (1 - 1/n)$. The second term in the sum arises because a bidder's value is in $(2, 3]$ with probability $1/2$, and in this case they bid 1. So they win the auction and receive an expected surplus of $5/2 - 1$ with probability 1 when the other bidder's value is in $[1, 2]$, which occurs with probability $1/2$; they win and receive that same expected surplus, but now only with probability $1/2$, when the other bidder ties their bid of 1 because the other bidder's value is in $(2,3]$, which occurs with probability $1/2$. So, for each n, each bidder's expected payoff under this strategy profile is given by

$$\frac{5}{8}+\frac{1}{8n}.$$

The limit strategy profile (after sending $n \to \infty$) has both bidders bidding 1 no matter what their value is. At this limit strategy profile, each bidder's expected payoff is again the sum of two terms and is

$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}-1\right)\right)+\frac{1}{2}\left(\frac{1}{2}\left(\frac{5}{2}-1\right)\right)=\frac{1}{2}<\frac{5}{8}=\lim_{n}\left(\frac{5}{8}+\frac{1}{8n}\right).$$

Consequently, both bidders' payoffs jump down at the limit, and so reciprocal upper semicontinuity fails.

Therefore, in order to encompass important games like first-price auctions in the independent private-values framework, we need to generalize our conditions further. We will do so in the next three sections. We will then put all of these conditions together in a single existence theorem that covers a large set of results in the literature. All of the conditions that we will consider amount to placing restrictions on a player's set of better replies, i.e., strategies that increase the player's payoff starting from some initial strategy profile. We begin by presenting a number of conditions that build on the payoff-security idea.

3.1. Payoff-Securing Better Replies

The basic idea behind the following concepts of payoff-securing better replies is that whenever some strategy profile is not a Nash equilibrium, there should be some player who not only has a better reply than their current strategy against the strategies of the others, but also has a better reply that improves upon their original payoff even if the others deviate slightly from their strategies.

That said, the various security conditions below differ in how many strategies the players are permitted to use in order to secure their better replies as the non-Nash-equilibrium strategy profile in question varies in any small open set. Better-reply security (Reny 1999) allows the players only a single strategy, multiple security (McLennan et al. 2011) allows finitely many strategies, and continuous security (Barelli & Meneghel 2013) allows the players any number of strategies.

3.1.1. Better-reply security. Following Reny (1999), say that the game G is better-reply secure iff for any $(x, \alpha) \in \overline{\text{Gr}u}$, if x is not a Nash equilibrium, then some player i can secure a payoff at x that is strictly greater than α_i.

To get a sense of what this condition is saying, suppose that x is any strategy profile that is not a Nash equilibrium. For simplicity, suppose that X is a metric space, and let x^n be any sequence of strategy profiles converging to x such that $u(x^n)$ converges to, for example, $\alpha \in \mathbb{R}^{|N|}$. Notice that α need not be equal to $u(x)$ because u need not be continuous.

Better-reply security requires that there is $\varepsilon > 0$ such that some player i has a strategy $\hat{x}_i$ that secures the payoff $\alpha_i + \varepsilon$. Combined with $u_i(x^n) \to \alpha_i$, we can conclude that there is n_0 large enough so that for every $n \geq n_0$, we obtain

$$u_i(\hat{x}_i, x'_{-i}) \geq \alpha_i + \varepsilon > u_i(x^n) \text{ for every } x'_{-i} \text{ in some neighborhood } U_{-i} \text{ of } x_{-i}. \qquad 1.$$

In particular, since for n large enough x^n_{-i} is in U_{-i}, we must have $u_i(\hat{x}_i, x^n_{-i}) > u_i(x^n)$—and so x^n is not a Nash equilibrium—for all n that are sufficiently large.[9] Since the sequence x^n was arbitrary, we see that better-reply security implies that for any strategy profile that is not a Nash equilibrium, there is a neighborhood of profiles that are also not Nash equilibria. Furthermore, for each profile in the neighborhood, some player can profitably deviate and can even secure a payoff that is above their payoff before deviating.

Notice that for different vectors $\alpha \in \mathbb{R}^{|N|}$ satisfying $(x, \alpha) \in \overline{\text{Gr}u}$, there can be a different player i who secures a payoff above α_i. This flexibility is useful because in many economic games, it will not always be the case that any one player has a single strategy that is a better reply to every strategy profile in a neighborhood of a nonequilibrium strategy. For example, in a Bertrand duopoly game with zero costs, if both firms price at 1 and earn positive profits, then we can specify for each firm a small price reduction of $\varepsilon > 0$. For any price vector sufficiently nearby the original price vector (1, 1), at least one firm's deviation to the price $1 - \varepsilon$ will be profitable. However, the other firm's deviation to $1 - \varepsilon$ need not be profitable. For example, consider any price vector $(1 - \delta, 1)$ where $\delta \in (0, \varepsilon/2)$. Against any such price vector, deviating to the price $1 - \varepsilon$ is profitable only for firm 2.

Returning to the first-price auction example from the previous section, let us note that despite the fact that both bidders' payoffs jump down at the limit strategy in which they both always bid 1, if player 1 were to always bid 9/8 no matter what their value is, then for any strategy of player 2 that is close enough to that player's limit strategy of always bidding 1 (i.e., the ex-ante probability that bidder 2 bids less than 9/8 is at least 6/7), bidder 1's payoff from always bidding 9/8 will give player 1 an expected surplus of at least $6/7(2 - 9/8) = 3/4$, which is strictly greater than the player's limit payoff of 5/8 under the sequence of strategy profiles that was used in that example. So Equation 1 holds for that particular sequence of strategy profiles, even though reciprocal upper semicontinuity fails for that sequence. This is no coincidence. Reny (1999) shows that a large class of first-price auction games are better-reply secure, which allows for a general theorem on the existence of mixed strategy equilibria in first-price auctions.

3.1.2. Multiply secure games. A significant advance in the theory was made by McLennan et al. (2011). They realized that Reny's (1999) better-reply security condition could be made significantly more permissive by allowing players to use more than one strategy to secure the requisite payoff. They give the following definitions.

For any $\alpha_i \in \mathbb{R}$ and for any $x \in X$, say that player i can multiply secure the payoff α_i at x iff there are finitely many strategies $x_i^1, \ldots, x_i^K \in X_i$ and there are finitely many closed subsets $F_{-i}^1, \ldots, F_{-i}^K$

[9]This last inequality defines the concept of point security (see Reny 2016b).

of X_{-i} whose union contains a neighborhood of x_{-i} such that, for every $k \in \{1, \ldots, K\}$, we obtain

$$u_i(x_i^k, x'_{-i}) \geq \alpha_i \text{ for every } x'_{-i} \in F_{-i}^k.$$

Notice that if player i can secure the payoff α_i at x, then it is possible to multiply secure that payoff by taking $k = 1$ in the above definition. Hence, the requirement that player i can secure a payoff at x is more restrictive than the requirement that player i can multiply secure that payoff at x.

Say that the game G is multiply secure iff, for any $x \in X$ that is not a Nash equilibrium, there is a vector $\alpha \in \mathbb{R}^{|N|}$ such that each player i can multiply secure α_i at x, and, for every y in some neighborhood of x, there is a player i such that $\alpha_i > u_i(y)$.[10] Notice that the condition that for every y in some neighborhood of x there is a player i such that $\alpha_i > u_i(y)$, when combined with $u_i(x_i^k, x'_{-i}) \geq \alpha_i$ for every $x'_{-i} \in F_{-i}^k$, implies that $u_i(x_i^k, x'_{-i}) \geq \alpha_i > u_i(y)$ for every $x'_{-i} \in F_{-i}^k$. But this last string of inequalities is implied by Equation 1. Hence, if G is better-reply secure, then G is multiply secure.

3.1.3. Continuously secure games.

The next significant advance was made by Barelli & Meneghel (2013). They took the insight of McLennan et al. (2011) to its logical limit and allowed players to use infinitely many strategies to secure their payoffs. They give the following definitions.

For any $\alpha_i \in \mathbb{R}$ and for any $x \in X$, say that player i can continuously secure the payoff α_i at x iff there is a nonempty-valued, convex-valued,[11] closed correspondence $b_i : U_{-i} \twoheadrightarrow X_i$ defined on a neighborhood U_{-i} of x_{-i} such that we have

$$u_i(z_i, x'_{-i}) \geq \alpha_i \text{ for every } x'_{-i} \in U_{-i} \text{ and for every } z_i \in b_i(x'_{-i}). \qquad 2.$$

Notice that if player i can secure the payoff α_i at x, it is possible to continuously secure that payoff by defining $b_i(x'_{-i})$ to be a singleton on a neighborhood of x_{-i}. And if player i can multiply secure α_i at x with the finitely many strategies $x_i^1, \ldots, x_i^K$ and the finitely many closed sets $F_i^1, \ldots, F_i^K$, then for any x'_{-i} in the neighborhood of x_{-i} that is covered by the union of the F_i^k sets, one can define $b_i(x'_{-i})$ to be the convex hull of the x_i^k for those k such that $x'_{-i} \in F_i^k$. Then the multiply secure condition and the fact that u_i is quasiconcave in i's strategy ensure that the continuous security condition is satisfied. So continuous security is more permissive than both multiple security and payoff security.

Say that the game G is continuously secure iff for any $x \in X$ that is not a Nash equilibrium, there is a vector $\alpha \in \mathbb{R}^{|N|}$ such that each player i can continuously secure α_i at x, and, for every y in some neighborhood of x, there is a player i such that $\alpha_i > u_i(y)$.[12] Notice that if each u_i is upper semicontinuous and each player's value function v_i is continuous, as is assumed in Theorem 1, then by Remark 1, the game G is continuously secure.

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[10]The definition given here specializes the definition by McLennan et al. (2011) in two ways: by restricting attention to quasiconcave games and by focusing on what the authors call the universal restriction operator (see Section 4 below for nonquasiconcave games).

[11]Barelli & Meneghel (2013) do not assume convex values. However, their proof does not go through without some additional hypotheses. The difficulty arises when they state that the correspondence Φ is compact-valued (Barelli & Meneghel 2013, p. 823). This statement is not generally true, but it is true with the additional convex-values assumption made here.

[12]We have given here the definition that applies to quasiconcave games. Barelli & Meneghel's (2013) definition applies also to nonquasiconcave games and is presented in Section 4 below.

One might wonder whether the security aspect of the better replies that is required in these conditions is necessary. The answer is yes, at least to some extent. For example, to ensure the existence of a pure strategy Nash equilibrium, it is not sufficient for a compact, convex, bounded, and quasiconcave game to satisfy the following single better-reply property: For every $x \in X$ that is not a Nash equilibrium there is a strategy profile $\hat{x} \in X$ such that for every x' in a neighborhood of x, there is a player i such that $u_i(\hat{x}_i, x'_{-i}) > u_i(x')$. Reny (2009, 2016b) and Prokopovych (2013) offer a counterexample; Prokopovych (2013) also offers a proof that this single better-reply property does suffice when there are just two players and $X_1 = X_2 = [0, 1]$; Kukushkin (2018) extends such results to games with strategic complements and potential games.[13] We will explore in Section 5 a related property (the finite better-reply property) in the context of mixed strategy equilibria, where we will find that it can be very fruitful.

Next, we describe two conditions that do not incorporate the idea of payoff security. Rather, these conditions require that for any strategy profile x that is not a Nash equilibrium, there is a single player who can profitably deviate from every strategy profile in some neighborhood of x. The two conditions differ in whether the deviating player must use only one strategy or can use many strategies to profitably deviate throughout the neighborhood. The first of these conditions was introduced by Nessah & Tian (2008) and is connected with the diagonal transfer continuity condition introduced by Baye et al. (1993) in a paper that will play a significant role in Sections 3.5 and 5 below (see Remark 9). The second of these conditions is due to Prokopovych (2016).

3.1.4. Secure better replies. As the reader can readily verify, the conditions defined so far in this section are not ordinal. That is, they might hold in one game, but they fail to hold after applying increasing transformations to the players' utility functions. This is not ideal. After all, the set of pure strategy equilibria depends only on the ordinal properties of utility, so properties guaranteeing the existence of an equilibrium should depend only on ordinal properties as well. Furthermore, when the binary relations defining the players' preferences over X are not continuous, which is of course the whole domain of study here, there is no guarantee that a utility representation even exists. With this in mind, Reny (2016b) introduced the following ordinal generalization of the better-reply security condition.

Say that any $\hat{x} \in X$ is a secure better-reply profile at $x \in X$ iff, for every y in some neighborhood U of x, there is a player $i \in N$ such that

$$u_i(\hat{x}_i, x'_{-i}) > u_i(y) \text{ for every } x' \in U.$$

The game G has the secure better-reply property iff, for any $x \in X$ that is not a Nash equilibrium, there is a secure better-reply profile at x.[14]

It is easy to see that this secure better-reply property is ordinal. Less obvious, but nonetheless true, is that it strictly generalizes Reny's (1999) better-reply security condition (see the discussion above leading up to Equation 1; for even more general conditions along these lines, see Reny 2016b). We now move on to provide two more ordinal conditions, local better replies and robust better replies, the latter of which unifies all of the conditions introduced in this section on pure strategy equilibria.

[13]For related concepts and results, the reader is referred to Nessah & Tian (2016) and Scalzo (2019a, 2020).
[14]This is equivalent to Reny's (2016b, definition 3.1) notion of point secure game.

3.2. Local Better Replies

We next define the concept of a local better reply. This idea is due to Nessah & Tian (2008), who instead use Tian's (1992a) terminology of transfer continuity. The local better-reply terminology used here emphasizes the connection to the unifying concept of better replies.

For any player $i \in N$, and for any $\hat{x}_i \in X_i$, say that $\hat{x}_i$ is a local better reply for i at $x \in X$ iff

$$u_i(\hat{x}_i, x'_{-i}) > u_i(x') \text{ for every } x' \text{ in a neighborhood of } x.$$

The game G has the local better-reply property iff, for any $x \in X$ that is not a Nash equilibrium, some player has a local better reply at x.

We have already seen that Bertrand duopoly games do not have the local better-reply property because both firms may be needed to upset every strategy profile in an open set of non-Nash equilibria when the players are each constrained to use only one strategy for their deviation. Like Barelli & Meneghel's (2013) generalization of McLennan et al. (2011) and of Reny (1999), we can relax this local better-reply condition by allowing more flexibility in how the better reply is chosen. The next definition (but with different terminology) is due to Prokopovych (2016).[15]

For any player $i \in N$, say that a correspondence $b_i : U \twoheadrightarrow X_i$ is a local better-reply correspondence for i at $x \in X$ iff U is a neighborhood of x such that

$$u_i(z_i, x'_{-i}) > u_i(x') \text{ for every } x' \in U \text{ and for every } z_i \in b_i(x').$$

Notice that this definition reduces to a local better reply when b_i is a constant singleton.

The game G has the local better-reply-correspondence property iff, for any $x \in X$ that is not a Nash equilibrium, some player has a local better-reply correspondence at x, and this correspondence is nonempty-valued, convex-valued, and closed.

Notice that if each u_i is upper semicontinuous and each player's value function v_i is continuous, as is assumed in Theorem 1, then by Remark 1, the game G has the local better-reply-correspondence property.

3.3. Robust Better Replies

The conditions in this section are new and are an attempt to unify the payoff-securing better-reply conditions in Section 3.1 with the local better-reply conditions in Section 3.2.

Say that any $\hat{x} \in X$ is a robust better reply at $x \in X$ iff, for every y in some neighborhood U of x, there is a player $i \in N$ such that

$$u_i(\hat{x}_i, x'_{-i}) > \min(u_i(x'), u_i(y)) \text{ for every } x' \in U. \qquad 3.$$

The game G has the robust better-reply property iff, for any $x \in X$ that is not a Nash equilibrium, there is a robust better reply at x.

Notice that if $\hat{x}_i \in X_i$ is a local better reply for some player i at x, then we have $u_i(\hat{x}_i, x'_{-i}) > u_i(x')$ for every $x' \in U$, and so Equation 3 will be satisfied. Also, if $\hat{x} \in X$ is a secure better-reply profile at x, then for every $y \in U$, there is a player i such that $u_i(\hat{x}_i, x'_{-i}) > u_i(y)$ for every $x' \in U$, and so again Equation 3 will be satisfied. Consequently, if G has either the local better-reply property or the secure better-reply property, then G has the robust better-reply property.

[15]Nessah (2011) provides a very similar condition that is just slightly more restrictive.

We have already seen that Bertrand duopoly games do not have the local better-reply property because both firms may be needed to upset every strategy profile in an open set of non-Nash equilibria when the players are each constrained to use only one strategy for their deviation. However, since zero-cost Bertrand games are better-reply secure, and hence have the secure better-reply property, they necessarily have the robust better-reply property.

We can relax the robust better-reply property in the now familiar way by allowing more flexibility in how the better reply is chosen.

Say that a correspondence $b : U \twoheadrightarrow X$, where $b(w) = \times_{i \in N} b_i(w) \subseteq \times_{i \in N} X_i$ for every $w \in U$, is a robust better-reply correspondence at $x \in X$ iff $U \subseteq X$ is a neighborhood of x, and for every $y \in U$ there is a player i such that

$$u_i(z_i, x'_{-i}) > \min(u_i(x'), u_i(y)) \text{ for every } x' \in U \text{ and for every } z_i \in b_i(x'). \qquad 4.$$

Notice that this definition reduces to a robust better reply when b is a constant singleton on U.

The game G has the robust better-reply-correspondence property iff, for any $x \in X$ that is not a Nash equilibrium, there is a robust better-reply correspondence at x, and this correspondence is nonempty-valued, convex-valued, and closed.

Notice that if G is continuously secure, then for any x that is not a Nash equilibrium, there is a vector $\alpha \in \mathbb{R}^{|N|}$ and there is a single neighborhood U of x such that Equation 2 holds and such that for every $y \in U$ there is a player i such that $\alpha_i > u_i(y)$.[16] Then, combining $\alpha_i > u_i(y)$ with Equation 2 implies that $u_i(z_i, x'_{-i}) \geq \alpha_i > u_i(y)$ for every $x' \in U$ and for every $z_i \in b_i(x'_{-i})$, which is strictly more restrictive than Equation 4. Consequently, if G is continuously secure, then G has the robust better-reply property.

Remark 2. If, for $x \in X$, we let $v_i(x_{-i}) := \sup_{z_i \in X_i} u_i(z_i, x_{-i})$ denote i's value function, then observe that the inequality in Equation 4 implies that

$$v_i(x'_{-i}) > \min(u_i(x'), u_i(y)). \qquad 5.$$

Consequently, an even more permissive condition on G would be to ask merely that whenever x is not a Nash equilibrium, there should be a neighborhood U of x such that for every $y \in U$ there is a player i such that Equation 5 holds for every $x' \in U$. However, this condition is not strong enough to give an existence result, because it is satisfied in the game without an equilibrium described by Sion (1958) that was discussed at the beginning of Section 3.

3.4. A Collection of Sufficient Conditions

To prepare for the theorem below, say that G is upper semicontinuous iff each u_i is upper semicontinuous on X, and G is opponent lower semicontinuous iff for each player i and for each $x_i \in X_i$, $u_i(x_i, \cdot)$ is lower semicontinuous on X_{-i}.

We can now collect all of the various conditions that we have considered and put them into a single existence theorem. Because results (j) and (k) below are new, a proof is provided in Section 7. It should be noted that condition (k) is the most permissive of all of the sufficient conditions given here (see Remark 3 and **Figure 1**).

Theorem 3. Suppose that G is a compact, convex, bounded, and quasiconcave game; then, G has a pure strategy Nash equilibrium if any one of the following conditions holds.

[16]For example, for each player j, let U^j be the neighborhood of x on which j can secure α_j, and let V be the neighborhood of x such that for any y in V there is a player i such that $u_i(y) < \alpha_i$. Then we may take $U = V \cap (\cap_{j \in N} U^j)$.

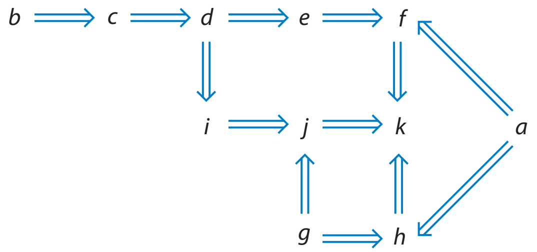

Figure 1

The arrows in the figure indicate the logical relationships among conditions (*a*)–(*k*) in Theorem 3.

(*a*) G is upper semicontinuous and each player's value function is continuous (Dasgupta & Maskin 1986a).
(*b*) G is opponent lower semicontinuous and reciprocally upper semicontinuous (Reny 1999).
(*c*) G is payoff secure and reciprocally upper semicontinuous (Reny 1999).
(*d*) G is better-reply secure (Reny 1999).
(*e*) G is multiply secure (McLennan et al. 2011).
(*f*) G is continuously secure (Barelli & Meneghel 2013).
(*g*) G has the local better-reply property (Nessah & Tian 2008).
(*h*) G has the local better-reply-correspondence property (Prokopovych 2016).
(*i*) G has the secure better-reply property (Reny 2016b).
(*j*) G has the robust better-reply property.
(*k*) G has the robust better-reply-correspondence property.

Remark 3. Conditions (*a*)–(*k*) in Theorem 3 are related as shown in **Figure 1**. (For (*b*) $\Rightarrow$ (*c*) $\Rightarrow$ (*d*), see Reny 1999; for (*d*) $\Rightarrow$ (*e*), see McLennan et al. 2011; for (*e*) $\Rightarrow$ (*f*), see Barelli & Meneghel 2013.) The implications (*g*) $\Rightarrow$ (*h*) and (*j*) $\Rightarrow$ (*k*) follow trivially from the definitions. The remaining implications, (*a*) $\Rightarrow$ (*f*), (*a*) $\Rightarrow$ (*h*), (*d*) $\Rightarrow$ (*i*) $\Rightarrow$ (*j*), (*g*) $\Rightarrow$ (*j*), (*f*) $\Rightarrow$ (*k*), and (*h*) $\Rightarrow$ (*k*), have all been discussed in the text above.

Remark 4. The boundedness condition in Theorem 3, which in any event is only needed for results (*b*)–(*f*), can be dropped if, by the closure of the graph of u, we mean the closure within the topological space $X \times [-\infty, +\infty]^{|N|}$, where each $[-\infty, +\infty]$ has the topology of the extended reals (and so is compact) and, as we always assume, product spaces are endowed with their product topologies. Equivalently, if any u_i is unbounded on X, replace u_i with the bounded payoff function $e^{u_i(x)}/(1 + e^{u_i(x)})$ and apply Theorem 3 as stated.

Remark 5. The proof in Section 7 shows that the assumption that each X_i is locally convex can be dropped for results (*b*)–(*e*), (*g*), (*i*), and (*j*).

Remark 6. The following result on the existence of ε-Nash equilibria is proposed by Reny (1996) and (in independent work) by Prokopovych (2011). If G is compact, convex, bounded, quasiconcave, and payoff secure, then G has an ε-Nash equilibrium for every $\varepsilon > 0$ if, in addition, each player i's value function, $v_i(x_{-i}) = \sup_{x_i \in X_i} u_i(x_i, x_{-i})$, is continuous in x_{-i}. Carmona (2005) offers an example showing that the condition on the value function cannot be removed, and Bich & Laraki (2017) provide additional results on ε-Nash equilibria.

Remark 7. Let $\phi_i : X \to \mathbb{R}$ be any bounded function such that $\phi_i(x_i, x_{-i})$ is quasiconcave in x_i on X_i for each $x_{-i} \in X_{-i}$. Then Theorem 3 would remain true if in any of the conditions (*d*)–(*k*), any player *i*'s payoff function u_i was replaced with ϕ_i (but Nash equilibria were still defined by the players' payoff functions u_i). So, for example, since $\underline{u}_i(x_i, x_{-i}) := \liminf_{x'_{-i} \to x_{-i}} u_i(x_i, x'_{-i})$ inherits quasiconcavity in x_i and boundedness from u_i, we can replace any of the u_i in any of the conditions (*d*)–(*k*) with $\underline{u}_i$ and the result remains true (see Nessah & Tian 2016 for results along these lines).

Remark 8. The convex-valuedness restriction on the correspondences in the various definitions above (i.e., continuous security, the local better-reply-correspondence property, and the robust better-reply-correspondence property) can be replaced with the weaker restriction of contractible valuedness when the X_i's are metric spaces. In the proof of Theorem 3, one then uses the fixed point theorem due to Eilenberg & Montgomery (1946) instead of Glicksberg's (1952) theorem.

Remark 9. The pure strategy equilibrium existence result by Baye et al. (1993, theorem 1) is absent from the present section on pure strategies because their diagonal transfer quasiconcavity (DTQ) hypothesis is often difficult to check in practice. For example, even if each player's payoff function is own-strategy quasiconcave, DTQ may fail. In contrast, DTQ can be naturally satisfied in symmetric games and is trivially satisfied in the mixed extension of a game. Consequently, theorem 1 by Baye et al. (1993) is a powerful result for symmetric games and for mixed strategy equilibria, and it is discussed in Sections 3.5 and 5.[17]

3.5. Symmetric Games

Let us briefly touch on symmetric games, where it is possible to improve upon the conditions above by taking advantage of the symmetry. Among the very best results for pure strategy equilibria in symmetric games are a result due to Baye et al. (1993) and its generalization by Bich & Laraki (2012).

Say that the game $G = (X_i, u_i)_{i \in N}$ is quasi-symmetric iff $X_1 = \cdots = X_{|N|}$, and $u_1(w, z, \ldots, z) = u_2(z, w, z, \ldots, z) = \cdots = u_{|N|}(z, \ldots, z, w)$ for all $w, z \in X_1$. When $|N| = 3$, note that we do not require $u_1(v, w, z) = u_2(z, v, w) = u_3(w, z, v)$ for all $v, w, z \in X_1$ as would typically be required in a symmetric game.

Let G be quasi-symmetric. We need several definitions. First, say that $x^* \in X$ is a symmetric Nash equilibrium of G iff $x^* = (z^*, \ldots, z^*) \in X$ and $u_1(z, z^*, \ldots, z^*) \leq u_1(z^*, \ldots, z^*)$ hold for every $z \in X_1$. Next, say that G is diagonally quasiconcave iff, for any finite subset F of X_1 and for any $\bar{z} \in \mathrm{co}F$, we have $u_1(\bar{z}, \ldots, \bar{z}) \geq \min_{z \in F} u_1(z, \bar{z}, \ldots, \bar{z})$.[18] In particular, if $u_1(w, z, \ldots, z)$ is quasiconcave in $w \in X_1$ for each $z \in X_1$, then G is diagonally quasiconcave. Finally, say that G has the local better-reply property on the diagonal iff, for every $x = (z, \ldots, z)$ that is not a symmetric Nash equilibrium of G, there is $\hat{z} \in X_1$ such that $u_1(\hat{z}, z', \ldots, z') > u(z', \ldots, z')$ holds for every z' in some neighborhood of z in X_1.

The next result is obtained by following the proof of theorem 1 by Baye et al. (1993), but with their function $\mathcal{U}(\cdot, \cdot)$ defined instead by $\mathcal{U}(w, z) := u_1(w, z, \ldots, z)$ for every $w, z \in X_1$.[19,20]

[17] Nessah & Tian (2008, 2016) and Scalzo (2019b) offer other strengthenings of own-strategy quasiconcavity.
[18] Therefore, G is diagonally quasiconcave iff $\phi(z, w) := u_1(z, w, \ldots, w)$ is diagonally quasiconcave in z in the sense of Zhou & Chen (1988).
[19] Theorem 1 in Baye et al. (1993) assumes Euclidean strategy spaces, but their proof goes through without change under our more general assumptions here.
[20] Remark 7 applies here as well (see Nessah & Tian 2016 for related results).

Theorem 4 (Baye et al. 1993). Suppose that G is compact, convex, quasi-symmetric, and diagonally quasiconcave. If G has the local better-reply property on the diagonal, then G has a symmetric pure strategy Nash equilibrium.

Bich & Laraki (2012, theorems 46, 47) generalized Theorem 4 by introducing correspondences.

Say that a quasi-symmetric game G has the local better-reply-correspondence property on the diagonal iff, for every $x = (z, \ldots, z)$ that is not a symmetric Nash equilibrium of G, there is a nonempty-valued, convex-valued closed correspondence $b : U \twoheadrightarrow X_1$ such that $U \subseteq X_1$ is a neighborhood of z and $u_1(w', z', \ldots, z') > u(z', \ldots, z')$ holds for every w' and z', with $z' \in U$ and $w' \in b(z')$.

We can state the following generalization of Theorem 4.

Theorem 5 (Bich & Laraki 2012). Suppose that G is compact, convex, quasi-symmetric, and diagonally quasiconcave. If G has the diagonal local better-reply-correspondence property, then G has a symmetric pure strategy Nash equilibrium.

We give Bich & Laraki's (2012) elegant proof here because it nicely illustrates the proof technique described in Section 1, namely that, under the given assumptions, if an equilibrium were to fail to exist, there would be a better-reply correspondence with a fixed point, which is absurd. The simplicity of this particular proof stems from the fact that, in quasi-symmetric games, there is effectively just one utility function, and so tying together the better-reply correspondences of different players is much simpler than it is when the players are not quasi-symmetric.

Proof. Suppose, by way of contradiction, that there is no symmetric Nash equilibrium. Then, for every $z \in X_1$, there is a neighborhood U^z of z and there is a nonempty-valued, convex-valued closed correspondence $b^z : U^z \twoheadrightarrow X_1$ such that $u_1(w', z', \ldots, z') > u(z', \ldots, z')$ holds for every w' and z', with $z' \in U^z$ and $w' \in b^z(z')$. Since $\{U^z\}_{z \in X_1}$ is an open cover of X_1, and since X_1 is compact, there is a finite subcover, $\{U^z\}_{z \in F}$, where F is a finite subset of X_1. Let $\{\beta_z\}_{z \in F}$ be a partition of unity subordinated to the finite cover $\{U^z\}_{z \in F}$ (see, e.g., Munkres 1975). Hence, each $\beta_z : X_1 \to [0, 1]$ is continuous, $\beta_z(w) > 0$ implies $w \in U^z$, and $\sum_{z \in F} \beta_z(w) = 1$ for every $w \in X_1$. For any $w \in X_1$, define $b(w) := \sum_{z \in F : \beta_z(w) > 0} \beta_z(w) b^z(w)$.[21] Then, $b : X_1 \twoheadrightarrow X_1$ is a nonempty-valued, convex-valued, closed correspondence. By Glicksberg's (1952) theorem, there is a fixed point $w^* \in b(w^*)$. Hence, we obtain $w^* = \sum_{z \in F : \beta_z(w^*) > 0} \beta_z(w^*) w^z$, where each $w^z \in b^z(w^*)$. Consequently, we have that $u_1(w^z, w^*, \ldots, w^*) > u_1(w^*, w^*, \ldots, w^*)$ for every z with $\beta_z(w^*) > 0$ [since $\beta_z(w^*) > 0$ implies that $w^* \in U^z$]. But then diagonal quasiconcavity implies that $u_1(w^*, w^*, \ldots, w^*) > u_1(w^*, w^*, \ldots, w^*)$, which is a contradiction. □

Remark 10. If we strengthen the diagonal quasiconcavity hypothesis by assuming that $u_1(w, z, \ldots, z)$ is quasiconcave in w for each z, then by the same logic as in the last three sentences of the proof of Theorem 5 given above, the correspondence $b(\cdot)$ defined there is seen to be a better-reply correspondence along the diagonal; that is, for every $z \in X_1$ and for every $w \in b(z)$, we have $u_1(w, z, \ldots, z) > u_1(z, \ldots, z)$.

4. PURE STRATEGY EQUILIBRIA IN NONQUASICONCAVE GAMES

Nishimura & Friedman (1981) provide a pure strategy equilibrium existence result for compact games whose strategy sets are convex subsets of Euclidean space and whose payoff functions are

[21] Each $b^z(w)$ in this sum is well defined, because $\beta_z(w) > 0$ implies that $w \in U^z$.

continuous but not necessarily quasiconcave. As they show, their results cover Cournot competition as well as price competition models with differentiated products. A key assumption in Nishimura & Friedman's (1981) paper is that for any $x \in X$ that is not a Nash equilibrium, there should be a player i and a coordinate k such that $(\hat{x}^1_{ik} - x_{ik})(\hat{x}^2_{ik} - x_{ik}) > 0$ for any two best replies $\hat{x}^1_i$ and $\hat{x}^2_i$ for i against x_{-i}. We will see below that this condition is a special case of a more general condition that suffices for the existence of a pure strategy Nash equilibrium in nonquasiconcave games.

McLennan et al. (2011) were the first to bring nonquasiconcave games into the scope of analysis of discontinuous games. Their insight is to make use of the convex hull of an appropriate utility upper-contour set, an idea that can be traced back to Sonnenschein (1971) and Shafer & Sonnenschein (1975). Barelli & Meneghel (2013) followed suit in their analysis of quasiconcave and nonquasiconcave games.

For the sake of brevity, rather than extending each of the various properties in the above section on quasiconcave games to nonquasiconcave games, we will extend only the most permissive property there, namely the robust better-reply-correspondence property. The basic idea, once again, is due to Sonnenschein (1971), Shafer & Sonnenschein (1975), and McLennan et al. (2011).

Following Reny (2016b), say that a correspondence $b : U \twoheadrightarrow X$ is coclosed iff the correspondence whose value is co$b(x)$ for each $x \in U$ is closed.[22] Requiring b to be coclosed does not require it to be either convex-valued or closed.[23]

A correspondence $b : U \twoheadrightarrow X$, where $b(w) = \times_{i \in N} b_i(w) \subseteq \times_{i \in N} X_i$ for every $w \in U$, is a robust better-reply correspondence at $x \in X$ iff $U \subseteq X$ is a neighborhood of x, and for every $y \in U$ there is a player i such that, for each (z_i, x') with $x' \in U$ and $z_i \in b_i(x')$, we obtain either

$$y_i \notin \text{co}\{w_i : u_i(w_i, y_{-i}) \geq u_i(z_i, x'_{-i})\} \qquad 6.$$

or

$$x'_i \notin \text{co}\{w_i : u_i(w_i, x'_{-i}) \geq u_i(z_i, x'_{-i})\}.^{24} \qquad 7.$$

Notice that when $u_i(w_i, x'_{-i})$ is quasiconcave in w_i, Equation 6 reduces to the inequality $u_i(z_i, x'_{-i}) > u_i(y)$, and Equation 7 reduces to the inequality $u_i(z_i, x'_{-i}) > u_i(x')$. Therefore, since for each (z_i, x'_{-i}) only one of these two inequalities needs to be satisfied, the whole condition reduces to $u_i(z_i, x'_{-i}) > \min(u_i(x'), u_i(y))$, which is exactly the condition given in Equation 4 in the quasiconcave case. Consequently, this broader definition of a robust better-reply correspondence coincides with our previous definition for quasiconcave games. We can now extend the definition of the robust better-reply property to nonquasiconcave games.

A (quasiconcave or nonquasiconcave) game G has the robust better-reply-correspondence property iff whenever $x \in X$ is not a Nash equilibrium, there is a robust better-reply correspondence at x, and this correspondence is nonempty-valued and coclosed.

Remark 11. When utility functions are not quasiconcave, it is important not to require the robust better-reply correspondence b to be convex-valued, since the utility value at a convex combination of two points might be much lower than the value at each of the two points. On the other hand, the existence proof requires the convex hull of b to be closed. Therefore, coclosed correspondences are the right objects here.

[22] For example, a closed correspondence $b : U \twoheadrightarrow X$ is coclosed if the values of b are all contained in a fixed finite dimensional subspace of the ambient topological vector space, and so in particular when X itself is finite dimensional.

[23] Consider, for example, the correspondence mapping each point in [0, 1] into the set of all rational numbers with the usual topology.

[24] Therefore, for some (z_i, x') Equation 6 may hold, while for other (z_i, x') Equation 7 may hold.

Remark 12. When the players' utility functions are quasiconcave, the conditions in Equations 6 and 7 reduce to $u_i(z_i, x'_{-i}) > \min(u_i(x'), u_i(y))$, which, if satisfied for all $z_i \in b_i(x'_{-i})$, is also satisfied for all $z_i \in \mathrm{co}b_i(x'_{-i})$. So when the game is quasiconcave, there is no loss of generality in requiring b to be convex-valued and closed.

We can now state the following result, which generalizes Theorem 3. A proof is in Section 7.

Theorem 6. Suppose that the game G is compact and convex but is not necessarily quasiconcave. If G has the robust better-reply-correspondence property, then G has a pure strategy Nash equilibrium.

Theorem 6 generalizes theorem 2.2 by Barelli & Meneghel (2013),[25] and, because closed correspondences mapping into subsets of a fixed finite subset of a convex space are coclosed, it also generalizes theorem 3.4 by McLennan et al. (2011) with their universal restriction operator.

Finally, following an analogous argument by McLennan et al. (2011), let us show that the assumptions by Nishimura & Friedman (1981) imply that the game has the robust better-reply-correspondence property. So suppose that $x \in X$ is not a Nash equilibrium. The Nishimura-Friedman condition implies that there is a player i and a coordinate k such that $(\hat{x}^1_{ik} - x_{ik})(\hat{x}^2_{ik} - x_{ik}) > 0$ for any two of the player's best replies, $\hat{x}^1_i$ and $\hat{x}^2_i$, against x_{-i}. So either we have $\hat{x}_{ik} > x_{ik}$ for every $\hat{x}_i$ that is a best reply against x_{-i}, or we have $\hat{x}_{ik} < x_{ik}$ for every $\hat{x}_i$ that is a best reply against x_{-i}. Either way, we may conclude that x_i is not in the convex hull of i's set of best replies against x_{-i}.

Let $\hat{x}_i$ be any best reply against x_{-i}. Then, for a small-enough open ball U around x, we claim that there can be no $x' \in U$ such that x'_i is in the convex hull of the set $\{w_i \in X_i : u_i(w_i, x'_{-i}) \geq u_i(\hat{x}_i, x'_{-i})\}$. Otherwise there would be a sequence of such x' converging to x and, by continuity and compactness (assumed by Nishimura and Friedman), x_i would be in the convex hull of the set $\{w_i \in X_i : u_i(w_i, x_{-i}) \geq u_i(\hat{x}_i, x_{-i})\}$. But since this latter set is just the set of best replies against x_{-i} (recall that $\hat{x}_i$ is a best reply), this would yield a contradiction. Hence, for every $x' \in U$, we have $x'_i \notin \mathrm{co}\{w_i : u_i(w_i, x'_{-i}) \geq u_i(\hat{x}_i, x'_{-i})\}$, i.e., Equation 7 holds when we define the correspondence b so that for all $x' \in U$, $b_i(x'_{-i}) = \{\hat{x}_i\}$ and for all other players, $j \neq i$, $b_j(x'_{-j})$ is any constant singleton. Consequently, this game has the robust better-reply-correspondence property.

5. MIXED STRATEGY EQUILIBRIA

The game G is measurable iff each $u_i : X \to \mathbb{R}$ is measurable, where we use the Borel sigma algebra on each X_i and the product sigma algebra on X. In order to calculate expected payoffs, we shall assume throughout this section that G is bounded and measurable.

Because the X_i's are compact subsets of a Hausdorff linear topological space, if M_i denotes the set of (regular, countably additive) probability measures on the Borel subsets of X_i, then M_i is compact and convex and is locally convex in the weak* topology, which is the topology that is to be understood throughout this section.[26] Extend each u_i to $M = \times_{i=1}^N M_i$ by defining $u_i(m) = \int_X u_i(x)dm$ for all $m \in M$. The mixed extension of G is the game $\bar{G} = (M_i, u_i)_{i \in N}$.

A strategy profile $m \in M$ is a mixed strategy Nash equilibrium of G iff m is a pure strategy Nash equilibrium of the mixed extension $\bar{G}$ of G.

[25] See footnote 11. Note, however, that instead of adding the assumption that the ϕ_x correspondences are convex-valued, it would suffice according to Barelli & Meneghel (2013, theorem 2.2) to replace the assumption that the ϕ_x correspondences are closed with the assumption that they are coclosed.

[26] Compactness follows from the Riesz representation theorem and Alaoglu's theorem (see, for example, Dunford & Schwartz 1988).

Dasgupta & Maskin (1986a) provide a pathbreaking result on the existence of mixed strategy Nash equilibria in discontinuous games. Because their result applies to such a large class of important economic games, the field immediately became important to economists, and research in the area blossomed. Their result is based on the idea of approximating the infinite game with a sequence of finite games. For any finite discretization of the players' pure strategy sets, the resulting finite game has a mixed strategy Nash equilibrium. Dasgupta and Maskin provide conditions on G that ensure that any weak* limit of any sequence of mixed strategy equilibria of any finite strategy approximations of G, as the discretizations become finer and finer, is a mixed strategy Nash equilibrium of G.

Dasgupta & Maskin (1986a) limit their attention to pure strategy sets that are nonempty, compact, convex subsets of Euclidean space, and, apart from boundedness and measurability, they make three assumptions about payoffs, which we describe only informally here.

First, they assume that each player's payoff function is weakly lower semicontinuous in the player's own pure strategy. This assumption has the effect that for any $x \in X$ and for any $\varepsilon > 0$, if λ_i is a uniform distribution over a small-enough open ball around x_i, then $u_i(\lambda_i, x_{-i}) \geq u_i(x) - \varepsilon$ holds for every $x_{-i} \in X_{-i}$. It is here that the unusual assumption that pure strategy sets are convex plays a role (convexity of pure strategy sets is not usually required when searching for mixed strategy Nash equilibria, and convexity of X_i is not required for any of the results below).

Second, they assume that any payoff discontinuities occur along finitely many diagonal sets, as for example occurs in Bertrand duopoly, where all of the discontinuities lie along the main diagonal on which the players choose the same price. The main effect of this diagonal discontinuities assumption is to ensure that if any player employs an atomless mixed strategy, then the player's payoff is continuous in the others' strategies.

Notice that, together, these first two assumptions, weak lower semicontinuity and diagonal discontinuities, imply that the game's mixed extension is payoff secure. Indeed, for any mixed strategy m and for any $\varepsilon > 0$, player i can adjust their strategy m_i as follows. First, the player employs m_i to choose a provisional pure strategy x_i', and then they choose their actual pure strategy x_i using a uniform distribution on a small ball around x_i'. For a small-enough ball, and by weak lower semicontinuity, the player loses less than $\varepsilon/2$ in utility from this uniform randomization, no matter what pure strategies the other players would choose. So if we let $\hat{m}_i$ denote this adjusted strategy for player i, we have $u_i(\hat{m}_i, m_{-i}) \geq u_i(m) - \varepsilon/2$. Next, notice that the constructed $\hat{m}_i$ is atomless. Consequently, by the diagonal discontinuities assumption, $u_i(\hat{m}_i, x_{-i})$ is continuous in $x_{-i} \in X_{-i}$; but then $u_i(\hat{m}_i, \cdot)$ is continuous on M_{-i} in the weak* topology. Hence, there is a weak* neighborhood U_{-i} of m_{-i} such that $u_i(\hat{m}_i, m'_{-i}) \geq u_i(m) - \varepsilon$ holds for every $m'_{-i} \in U_{-i}$. This means that player i can secure the payoff $u_i(m) - \varepsilon$. Since m, ε, and i were arbitrary, this shows that the game's mixed extension is payoff secure.

Dasgupta & Maskin's (1986a) third assumption is that the sum of the players' payoffs $\sum u_i(x)$ is upper semicontinuous in $x \in X$, which implies that $\sum_i u_i(m)$ is upper semicontinuous in $m \in M$ (see, e.g., Reny 1999, proposition 5.1).

Taken together, these three assumptions on payoffs imply that the mixed extension of G is payoff secure and its payoff sum is upper semicontinuous on M. Since the mixed extension is compact, convex, bounded, and quasiconcave, Theorem 3(*c*) can be applied to conclude that the mixed extension has a pure strategy equilibrium, which is a mixed strategy equilibrium of G.

Dasgupta & Maskin's (1986a) proof establishes more than just the existence of a mixed strategy Nash equilibrium. Their proof shows that, under their hypotheses, the original infinite discontinuous game can be well approximated by finite discretizations of the players' pure strategy sets in the following sense: Any limit of mixed strategy Nash equilibria of games restricted to finite subsets of the players' pure strategies, as the discretizations become finer and finer, is a mixed strategy Nash equilibrium of the original infinite game.

The existence result by Dasgupta & Maskin (1986a) inspired, and did much to prepare the way for, a large body of work on the existence of mixed strategy equilibria in discontinuous games. We will now present the fruits of some of that work.

Obviously, one obtains theorems on the existence of mixed strategy equilibria for G by applying any of the results in the previous sections to the game's mixed extension, where each $m \in M$ is considered a pure strategy in the mixed extension $\bar{G}$.[27,28]

Theorem 7. Suppose that G is compact, bounded, and measurable. Then G has a mixed strategy Nash equilibrium if its mixed extension $\bar{G}$ satisfies any one of the conditions in Theorem 3(*a–k*), where all mixed strategy sets are endowed with the weak* topology.

Another very general result is due to Baye et al. (1993). Their theorem 1, together with their proposition 1a, yields the following result.

Theorem 8 (Baye et al. 1993). Suppose that G is compact, bounded, and measurable. Then G has a mixed strategy Nash equilibrium if, for any $m \in M$ that is not a mixed strategy Nash equilibrium of G, there exists $\hat{m} \in M$ such that $\sum_{i \in N} u_i(\hat{m}_i, m'_{-i}) > \sum_{i \in N} u_i(m')$ holds for every m' in some weak* neighborhood of m.

Remark 13. Theorem 8 can be obtained from Theorem 4 by considering the two-person game in which, when player 1 chooses $m \in M$ and player 2 chooses $m' \in M$, player 1's payoff is $\sum_{i \in N} u_i(m_i, m'_{-i})$ and player 2's payoff is $\sum_{i \in N} u_i(m'_i, m_{-i})$. Under the hypotheses of Theorem 8, this two-person game is compact, quasi-symmetric, diagonally quasiconcave, and has the local better-reply property along the diagonal. Hence, it has a symmetric Nash equilibrium (m^*, m^*). But then m^* is a pure strategy Nash equilibrium of $\bar{G}$, and so it is a mixed strategy Nash equilibrium of G.

Let us take a closer look at the inequality condition in Theorem 8. This condition requires that, for any $m \in M$ that is not a mixed strategy Nash equilibrium of G, there is a weak* neighborhood U of m and there is $\hat{m} \in M$ such that

$$\sum_{i \in N} u_i(\hat{m}_i, m'_{-i}) > \sum_{i \in N} u_i(m') \text{ for every } m' \in U. \qquad 8.$$

Clearly, if $\sum_{i \in N} u_i(\hat{m}_i, m'_{-i}) > \sum_{i \in N} u_i(m')$, then there must be some player i such that $u_i(\hat{m}_i, m'_{-i}) > u_i(m')$.

Consequently, Equation 8 implies that, for any $m \in M$ that is not a Nash equilibrium, there exists $\hat{m} \in M$ and a weak* neighborhood U of m such that, for every $m' \in U$, there is a player i such that $u_i(\hat{m}_i, m'_{-i}) > u_i(m')$.[29] We now weaken this latter condition even further.

Following Reny (2009, 2016b), say that a product subset of mixed strategy profiles $\times_{i \in N} F_i$ is a finite better-reply set at $m \in M$ iff each F_i is a finite subset of M_i and, for every m' in some weak* neighborhood of m, there is a player i and there is $\hat{m}_i \in F_i$ such that

$$u_i(\hat{m}_i, m'_{-i}) > u_i(m').$$

[27]For a proof, simply note that, if G is compact, bounded, and measurable, then with the weak* topology, $\bar{G}$ is compact, convex, bounded, and quasiconcave. Now apply Theorem 3.

[28]To apply the symmetric pure strategy equilibrium results above, one must ensure that the game's mixed extension is quasi-symmetric, which need not be the case if G is quasi-symmetric. But if the payoff functions in G are symmetric, then $\bar{G}$ is also symmetric and hence quasi-symmetric.

[29]Nessah & Tian (2008) call this the weak transfer quasi-continuity property, and (independently) Reny (2009) calls this the single deviation property.

The game G has the finite better-reply property on M iff, whenever $m \in M$ is not a mixed strategy Nash equilibrium of G, there is a subset of mixed strategy profiles that is a finite better-reply set at m.[30]

As we have just argued, the finite better-reply property is more permissive than Equation 8. Indeed, requiring G to have the finite better-reply property on M is more permissive than requiring $\bar{G}$ to be either multiply secure or to have the robust better-reply property. So the finite better-reply property on M is a rather permissive condition. We can state the following result.

Theorem 9 (Reny 2009, 2016b). If G is compact, bounded, and measurable, and G has the finite better-reply property on M, then G has a mixed strategy Nash equilibrium.

Despite its generality, the proof of this result is refreshingly short and simple, and so we will present it here.

Proof. Suppose, by way of contradiction, that no Nash equilibrium exists. Then, for every $m \in M$, each player has finitely many mixed strategies such that, for every m' in a weak* neighborhood of m, one of these mixed strategies is a profitable deviation from m for some player. These neighborhoods form an open cover of M that has a finite subcover, by compactness. So in fact each player has finitely many mixed strategies—call them better-reply strategies—such that for every $m \in M$, some better-reply strategy is a profitable deviation from m for some player. However, by Nash's theorem, the finite game whose set of pure strategy profiles is the product of the players' finite sets of better-reply strategies has a mixed strategy Nash equilibrium, whose mixture yields an element of M that no player can profitably deviate from using any of their better-reply strategies. This contradiction completes the proof. □

Remark 14. Given the discussion preceding Theorem 9, and based on Remark 3, we see that if the mixed extension of G satisfies any one of the conditions (*b*)–(*e*), (*g*), (*i*), or (*j*) of Theorem 3, or if the hypotheses of Theorem 8 hold, then G has the finite better-reply property. Hence, Theorem 9 is more permissive than each of these others.

Theorems 7, 8, and 9 all provide quite general mixed strategy equilibrium existence results whose hypotheses are satisfied in many economic games. However, it can sometimes be difficult to check these conditions because they all require the consideration of a neighborhood of mixed strategies, which means that one must deal with the topology of weak convergence of measures. While this is not always difficult, there is no doubt that it would be simpler if there were sufficient conditions that depended only on the players' payoffs on the set of pure strategy profiles.

For example, as we have already mentioned, if $\sum_i u_i(x)$ is upper semicontinuous in x on the space of pure strategy profiles X, then $\sum_i u_i(m)$ will be upper semicontinuous in m on the space of mixed strategy profiles M. So the simple condition that $\sum_i u_i(x)$ is upper semicontinuous in x on X suffices for the mixed extension $\bar{G}$ to be reciprocally upper semicontinuous. Unfortunately, things are not as simple with payoff security. Even if G is payoff secure, it need not be the case that its mixed extension $\bar{G}$ is payoff secure (see, e.g., Carmona 2005). This is because, in $\bar{G}$, payoff security requires the players to secure their payoffs not only at all pure strategy profiles, but also at all of the nondegenerate mixed strategy profiles.[31] So the literature has sought to find security conditions that, while less general, are easier to verify. Monteiro & Page (2007) led the way here with the following definition (see also Allison & Lepore 2014).

[30]Reny (2009, 2016b) calls this the finite deviation property (see also Prokopovych 2013).
[31]The reverse implication also fails. That is, $\bar{G}$ can be payoff secure even when G is not. This is because, in $\bar{G}$, players have more than just their pure strategies available for the purposes of securing payoffs.

The game G is uniformly payoff secure iff, for every $\varepsilon > 0$, for every player i, and for every $x \in X$, there is $\hat{x}_i \in X_i$ such that, for every $y_{-i} \in X_{-i}$, $u_i(\hat{x}_i, x'_{-i}) > u_i(x_i, y_{-i}) - \varepsilon$ holds for every x'_{-i} in some neighborhood of y_{-i} in X_{-i}.

Monteiro & Page (2007) show that if G is uniformly payoff secure, then $\bar{G}$ is payoff secure. Consequently, we have the following result, whose conditions on payoffs depend only on the values of payoffs on the set of pure strategy profiles X.[32]

Theorem 10 (Monteiro & Page 2007). Suppose that G is compact and bounded, and that $\sum_{i \in n} u_i(x)$ is upper semicontinuous in x on X. If G is uniformly payoff secure, then G has a mixed strategy Nash equilibrium.

Another useful result whose conditions on payoffs depend only on the values that payoffs take on X is the following, due to Prokopovych & Yannelis (2014).

The game G is uniformly diagonally secure iff, for every $\varepsilon > 0$ and for every $x \in X$, there is $\hat{x} \in X$ such that for every $y \in X$, $\sum_{i \in N} u_i(\hat{x}_i, x'_{-i}) - \sum_{i \in N} u_i(x') > \sum_{i \in N} u_i(x_i, y_{-i}) - \sum_{i \in N} u_i(y) - \varepsilon$ holds for every x' in some neighborhood of y.

Prokopovych & Yannelis (2014) show that if G is uniformly diagonally secure, then $\bar{G}$ satisfies the hypotheses of Theorem 8. Hence, we have the following result.

Theorem 11 (Prokopovych & Yannelis 2014). Suppose that G is compact, bounded, and measurable. If G is uniformly diagonally secure, then G has a mixed strategy Nash equilibrium.

5.1. Bayesian Games

For each $i \in N$, let A_i be player i's nonempty metric space of actions and let T_i be player i's nonempty, separable metric space of types.[33] All of these spaces are endowed with their Borel sigma algebras, and all product spaces are endowed with their product topologies and product sigma algebras. Let $A = \times_{i \in N} A_i$, and let $T = \times_{i \in N} T_i$. Let $v_i : A \times T \to \mathbb{R}$ denote i's (ex-post) bounded and measurable payoff function, and let $v = (v_i)_{i \in N}$ be the profile of (ex-post) payoff functions.

For any measurable space Z, let $\Delta(Z)$ denote the set of countably additive probability measures on the measurable subsets of Z. Let $f : T \to [0, \infty)$ be a nonnegative measurable function, and, for each $i \in N$, let $p_i \in \Delta(T_i)$ be a probability measure on the Borel subsets of T_i. The prior on the set of types T is $p \in \Delta(T)$, where, for any measurable subset C of T, we have

$$p(C) := \int_C f(t_1, \ldots, t_{|N|}) p_1(dt_1) \ldots p_{|N|}(dt_{|N|}). \quad 9.$$

Altogether, $BG := (A, T, v, f, p_1, \ldots, p_{|N|})$ constitutes an absolutely continuous Bayesian game.[34] BG is compact iff each A_i is compact. A behavioral strategy for player i is any transition probability $s_i : T_i \to \Delta(A_i)$.[35] Let S_i denote player i's set of behavioral strategies, and let $S = \times_{i \in N} S_i$. For any

[32] For a proof, note that $\bar{G}$ is compact, convex, quasiconcave, payoff secure (since G is uniformly payoff secure), and reciprocally upper semicontinuous [since the upper semicontinuity of $\sum u_i(x)$ on X implies the upper semicontinuity of $\sum u_i(m)$ on M]. Now apply Theorem 3(*a*).

[33] He & Yannelis (2015) provide interesting results for the special case of countable type spaces.

[34] The formulation of an absolutely continuous Bayesian game here automatically builds in the important assumption, due to Milgrom & Weber (1985), that the prior on the type space is absolutely continuous with respect to the product of its marginals. Without this absolute continuity condition, Simon (2003) has shown that there need be no equilibrium even if the players' action sets A_i are all finite.

[35] To say that $s_i : T_i \to \Delta(A_i)$ is a transition probability means that $s_i(\cdot\ |t_i) \in \Delta(A_i)$ for every $t_i \in T_i$, and, for every Borel subset C of A_i, $s_i(C|t_i)$ is a measurable function of t_i on T_i.

$s=(s_i)_{i\in N}\in S$ and for any $t=(t_i)_{i\in N}\in T$, define $s(\cdot\,|t)$ to be the product measure $\times_{i\in N}s_i(\cdot\,|t_i)$, and define $V_i(s):=\int_T(\int_A v_i(a,t)\,s(da|t))\,p(dt)$. Say that $s^*\in S$ is a Bayes-Nash equilibrium of BG (in behavioral strategies) iff $V_i(s_i,s^*_{-i})\le V_i(s^*)$ for every $s_i\in S_i$ and for every $i\in N$.

We next construct a surrogate game that will be helpful in determining whether BG has a Bayes-Nash equilibrium.

For any $i\in N$, let $X_i=\{x_i\in\Delta(A_i\times T_i):x_i(A_i\times C)=p_i(C)$ for every Borel subset C of $T_i\}$. Therefore, each $x_i\in X_i$ is a probability measure on $A_i\times T_i$ whose marginal on T_i is p_i. Such strategies are called distributional strategies by Milgrom & Weber (1985).

Let $X=\times_{i\in N}X_i$ and, for each $i\in N$, define $u_i:X\to\mathbb{R}$ as follows. For each $x=(x_i)_{i\in N}\in X$, define

$$u_i(x):=\int v_i(a,t)f(t)[x_1\times\cdots\times x_{|N|}](d(a,t)),$$

where $x_1\times\cdots\times x_{|N|}$ is the product measure on $\times_{i\in N}(A_i\times T_i)$ whose marginal on $A_i\times T_i$ is x_i.

Call the strategic form game $DG:=(X_i,u_i)_{i\in N}$ the distributional strategic form of BG.

The significance of the strategic form game DG is that any pure strategy Nash equilibrium of DG induces a Bayes-Nash equilibrium of BG. To see this, let us first define, for any player $i\in N$ and for any $s_i\in S_i$, the probability measure $s_i*p_i\in\Delta(A_i\times T_i)$ so that for any Borel subsets C of A_i and D of T_i, we have

$$[s_i*p_i](C\times D):=\int_D s_i(C|t_i)p_i(dt_i). \qquad 10.$$

Notice that the marginal of s_i*p_i on T_i is p_i (because $[s_i*p_i](A_i\times D)=\int_D s_i(A_i|t_i)p_i(dt_i)=p_i(D)$ for any Borel subset D of T_i). Consequently, $s_i*p_i\in X_i$ for every $s_i\in S_i$. So, if $s\in S$ and we let $x_i=s_i*p_i$ for every $i\in N$, then we have $x=(x_i)_{i\in N}\in X$ and

$$\begin{aligned}u_i(x)&=\int v_i(a,t)s_1(da_1|t_i)\ldots s_{|N|}(da_{|N|}|t_{|N|})f(t)p_1(da_1)\ldots p_{|N|}(da_{|N|})\\&=\int v_i(a,t)s(da|t)p(dt)\\&=V_i(s).\end{aligned} \qquad 11.$$

Now suppose that x^* is a pure strategy Nash equilibrium of DG. For each player $i\in N$, because the marginal of x_i^* on T_i is p_i, there exists a transition probability (i.e., a behavioral strategy) $s_i^*:T_i\to\Delta(A_i)$ such that $x_i^*=s_i^*\ast p_i$ (see, e.g., Bertsekas & Shreve 1978, proposition 7.27; Milgrom & Weber 1985). Let $s^*=(s_i^*)_{i\in N}$. Then, for any player $i\in N$ and for any $s_i\in S_i$, letting $x_i=s_i*p_i$, we have

$$V_i(s_i,s^*_{-i})=u_i(x_i,x^*_{-i})\le u_i(x^*)=V_i(s^*),$$

where the two equalities follow from Equation 11 and the inequality follows because $x_i\in X_i$ and x^* is a Nash equilibrium of DG. Hence, s^* is a Bayes-Nash equilibrium of BG.

Therefore, to ensure that BG has a Bayes-Nash equilibrium, it suffices to find conditions on BG that ensure that its distributional strategic form DG has a pure strategy Nash equilibrium. Some observations are helpful in this direction. First, if for each player $i\in N$ the action space

A_i is compact, then the strategy space X_i is compact and locally convex in the weak* topology.[36] Second, each $u_i(x_i, x_{-i})$ is concave (linear), and therefore quasiconcave, in x_i on X_i for each $x_{-i} \in X_{-i}$. Hence, DG is a (weak*) compact, convex, bounded, and quasiconcave game. Consequently, if DG satisfies any of the conditions (*a*)–(*j*) of Theorem 3, then DG has a pure strategy Nash equilibrium and so BG has a Bayes-Nash equilibrium. This immediately gives us the following result.

Theorem 12. Suppose that the absolutely continuous Bayesian game BG is compact. If DG satisfies any one of the conditions (*a*)–(*k*) of Theorem 3 when each X_i is given the weak* topology, then BG has a Bayes-Nash equilibrium in behavioral strategies.

Since the behavioral strategy spaces S_i are convex and the payoff functions $V_i(s_1, \ldots, s_{|N|})$ are not only linear in s_i for each s_{-i}, but are also linear in s_j for each s_{-j} for any $j \in N$, an appropriate extension of the finite better-reply property to Bayesian games can yield an existence result.

The Bayesian game BG has the finite better-reply property iff, whenever $s \in S$ is not a Bayes-Nash equilibrium, there is a weak* neighborhood $U \subseteq X$ of $(s_i * p_i)_{i \in N} \in X$ and there are finite sets of behavioral strategies $F_1 \subseteq S_1, \ldots, F_{|N|} \subseteq S_{|N|}$ such that, for every $s' \in S$ with $(s'_i * p_i)_{i \in N} \in U$, there is a player i and $\hat{s}_i \in F_i$ such that $V_i(\hat{s}_i, s'_{-i}) > V_i(s')$.

With this definition, we can state the following result, which is a corollary of Theorem 15 below.

Theorem 13. Suppose that the absolutely continuous Bayesian game BG is compact. If BG has the finite better-reply property, then BG has a Bayes-Nash equilibrium in behavioral strategies.

Remark 15. Analogous to Remark 14, Theorem 13 generalizes the existence results stated in Theorem 12 when DG there satisfies conditions (*b*)–(*e*), (*g*), (*i*), and (*j*) of Theorem 3.

A condition that uses actions or even distributions over actions instead of distributional strategies when looking for better replies can be simpler to employ. We next give a definition that goes in this direction.

Say that BG has the finite action-distribution better-reply property iff, whenever $s \in S$ is not a Bayes-Nash equilibrium, there are finite sets of (possibly degenerate) distributions over actions $F_1 \subseteq \Delta(A_1), \ldots, F_{|N|} \subseteq \Delta(A_{|N|})$, and there is a weak* neighborhood $U \subseteq X$ of $(s_i * p_i)_{i \in N} \in X$ such that, for every $s' \in S$ with $(s'_i * p_i)_{i \in N} \in U$, there is a player i and there is $\hat{s}_i \in S_i$ such that $\hat{s}_i(\cdot|t_i) \in F_i$ for every $t_i \in T_i$, and $V_i(\hat{s}_i, s'_{-i}) > V_i(s')$.

Under the finite action-distribution better-reply property, the set of available deviations for player i is the potentially infinite set of all behavioral strategies that can be constructed piecewise from the finite set of action-distributions in F_i.

The finite action-distribution better-reply property for BG is unrelated to the finite better-reply property for BG. Indeed, on the one hand, the finite action-distribution better-reply property is more restrictive because the deviating players are restricted to behavioral strategies that map into a finite set of action-distributions. On the other hand, the set of deviation strategies that are available in the finite action-distribution case is the set of all behavioral strategies that have support contained in F_i for every t_i, which is typically an infinite set, while the collection of deviation strategies under the finite better-reply condition is finite.

[36]Because T_i is a separable metric space, when the metric space A_i is compact (and hence separable), X_i is metrizable following Bertsekas & Shreve (1978, theorem 7.20). Then, weak* compactness of X_i follows from the sequential compactness result of Balder [1994, theorems 3.15 and 3.19($a \Leftrightarrow c$)], and by noting that the inequality in Balder's theorem 3.19*c* holds for both g and $-g$ (and so is an equality) when g is bounded and continuous.

The following result is a corollary of Theorem 15 below.

Theorem 14. Suppose that the absolutely continuous Bayesian game BG is compact. If BG has the finite action-distribution better-reply property, then BG has a Bayes-Nash equilibrium in behavioral strategies.

We can generalize both of these last two results by replacing the actions in the previous theorem with what we may call strategic actions.[37]

Say that BG has the finite strategic-action better-reply property iff, whenever $s \in S$ is not a Bayes-Nash equilibrium of BG, there are finite sets of behavioral strategies $F_1 \subseteq S_1, \ldots, F_{|N|} \subseteq S_{|N|}$, and there is a weak* neighborhood $U \subseteq X$ of $(s_i * p_i)_{i \in N}$ such that, for every $s' \in S$ with $(s'_i * p_i)_{i \in N} \in U$, there is a player i and there is $\hat{s}_i \in S_i$ such that $\hat{s}_i(\cdot|t_i) \in \{\bar{s}_i(\cdot|t_i) : \bar{s}_i \in F_i\}$ for every $t_i \in T_i$, and $V_i(\hat{s}_i, s'_{-i}) > V_i(s')$.

Under this most permissive of the three finite better-reply properties for BG, the set of available deviations for player i is the potentially infinite set of all behavioral strategies that can be constructed piecewise from the behavioral strategies in F_i.

The following result generalizes Theorems 13 and 14. A proof is in Section 7.

Theorem 15. Suppose that the absolutely continuous Bayesian game BG is compact. If BG has the finite strategic-action better-reply property, then BG has a Bayes-Nash equilibrium in behavioral strategies.

Theorems 12–15 can be powerful and useful tools. However, working with open sets in the space of distributional strategies X can sometimes be challenging, and so it is beneficial to have sufficient conditions that are easier to check.

We give two such conditions here, both due to Carbonell-Nicolau & McLean (2018). The first of their conditions extends the uniform payoff-security condition of Monteiro & Page (2007) to Bayesian games as follows.

For any $i \in N$, say that $s_i \in S_i$ is pure iff, for every $t_i \in T_i$, there is $a_i(t_i) \in A_i$ such that $s_i(\{a_i(t_i)\}|t_i) = 1$. Say that $s = (s_1, \ldots, s_{|N|}) \in S$ is pure iff each s_i is pure.

Say that BG is uniformly payoff secure iff, for every $\varepsilon > 0$, for every $i \in N$, and for every pure $s_i \in S_i$, there exists a pure $\hat{s}_i \in S_i$ such that for every $(t, a_{-i}) \in T \times A_{-i}$, $u_i(\hat{s}_i(t_i), a'_{-i}, t) > u_i(s_i(t_i), a_{-i}, t) - \varepsilon$ holds for every a'_{-i} in some neighborhood in A_{-i} of a_{-i}.

Carbonell-Nicolau & McLean (2018) show that if BG is uniformly payoff secure, then the strategic form game DG is uniformly payoff secure in the sense of Monteiro & Page (2007). Hence, Monteiro & Page's (2007) result would apply if the sum of the players' payoffs was upper semicontinuous. So we have the following result.

Theorem 16 (Carbonell-Nicolau & McLean 2018). Suppose that the absolutely continuous Bayesian game BG is compact. If for each $t \in T$, $\sum_{i \in N} v_i(a, t)$ is upper semicontinuous in $a \in A$, and if BG is uniformly payoff secure, then BG has a Bayes-Nash equilibrium in behavioral strategies.

The second result of Carbonell-Nicolau & McLean (2018) extends Prokopovych & Yannelis's (2018) uniform diagonal security condition to Bayesian games as follows.

Say that BG is uniformly diagonally secure iff, for every $\varepsilon > 0$, and for every pure $s \in S$, there exists a pure $\hat{s} \in S$ such that for every $(t, a) \in T \times A$, $\sum_{i \in N} u_i(\hat{s}_i(t_i), a'_{-i}, t) - \sum_{i \in N} u_i(a', t) > \sum_{i \in N} u_i(s_i(t_i), a_{-i}, t) - \sum_{i \in N} u_i(a, t) - \varepsilon$ holds for every a' in some neighborhood in A of a.

[37]The term "strategic action" stems from discussions with Roger Myerson in our joint work on sequential equilibria in infinite games, though we ultimately did not use the idea.

Carbonell-Nicolau & McLean (2018) show that if BG is uniformly diagonally secure, then DG satisfies the hypotheses of Baye et al. (1993, theorem 1). So we can state the following result.

Theorem 17 (Carbonell-Nicolau & McLean 2018). Suppose that the absolutely continuous Bayesian game BG is compact. If BG is uniformly diagonally secure, then BG has a Bayes-Nash equilibrium in behavioral strategies.

6. FURTHER REMARKS

6.1. Sharing-Rule Equilibria

An important topic not covered here is the sharing-rule approach pioneered by Simon & Zame (1990). This approach is motivated by the fact that in many strategic settings, discontinuities in payoffs arise because those payoffs are a reduced form of a dynamic game in which the bang-bang optimal choices of later players reacting to small changes in the choices of earlier players lead to large changes in the earlier players' payoffs.

A good example is Bertrand duopoly. Discontinuities there arise only when the two firms' prices are the same. In that case, consumers are indifferent between the two firms. While it is customary to assume that half of the consumers purchase from one firm and the other half purchase from the other firm, there is no game-theoretic reason to compel this equal-split sharing rule. So it is entirely reasonable to allow the actual split to be endogenous.

Considerations such as these led Simon & Zame (1990) to consider games in which payoffs are not fully determined. Instead, payoffs are partially specified by a correspondence $Q : X \twoheadrightarrow \mathbb{R}^{|N|}$ where, following our usual notation, $X = \times_{i \in N} X_i$. They call (X, Q) a game with an endogenous sharing rule and establish the following result. If each X_i is a nonempty, compact metric space, and the correspondence $Q : X \twoheadrightarrow \mathbb{R}^{|N|}$ has a compact graph and has nonempty and convex values, then there is a measurable function $u : X \to \mathbb{R}^{|N|}$ satisfying $u(x) = (u_i(x))_{i \in N} \in Q(x)$ for every $x \in X$ and there is $m^* \in M$ such that m^* is a mixed strategy Nash equilibrium of the game $G = (X_i, u_i)_{i \in N}$.

There are many economic games in which discontinuities arise as the result of a particular sharing rule, such as Bertrand competition, auctions, or voting. In each of these cases, Simon & Zame's (1990) endogenous sharing-rule approach can be a useful technique to employ.

A major advance in the sharing-rule approach was made by Jackson et al. (2002), who extended Simon & Zame's (1990) existence result to Bayesian games. One might think that this should be a trivial exercise, consisting in simply applying Simon & Zame's (1990) result to the distributional strategic form of the given Bayesian game. However, this would not work, because the natural payoff indeterminacies are indeterminacies in the underlying ex-post payoffs, while the strategic form approach only gives payoffs as a function of behavioral strategy profiles, with no guarantee that those payoffs can actually be obtained as expected utilities from a feasible selection of ex-post utilities.

A second major issue in the Bayesian context is the players' private information. Typically, the endogenous sharing rule will be type dependent. For example, deciding which bidder in an auction wins when bids are tied will typically depend on the tied bidders' values. A good rule of thumb is to break ties in favor of the bidder with the highest value (because this makes payoff sums upper semicontinuous!). But because values are private information, we cannot necessarily rely on the bidders to truthfully report their values when winning or losing is at stake. So it is not at all obvious that there are implementable sharing rules that admit equilibria. Jackson et al. (2002) beautifully solve this problem by showing that there always exists a sharing rule that, if type dependent, is incentive compatible.

There are interesting and subtle connections between the discontinuous games literature and the sharing-rule approach (see, e.g., Balder 2011; de Castro 2011; Bich & Laraki 2017; Carmona & Podczeck 2018a,b). Carbonell-Nicolau & Ok (2007) propose an interesting application.

6.2. More Discontinuous Game Topics

Some additional topics not touched upon here are listed below, with a short list of references.

1. **Abstract economies, ordinal games.** The following authors' results apply to settings in which the players' preferences are given by binary relations instead of utility functions: Tian (1992b,c), Tian & Zhou (1992, 1995), Carmona & Podczeck (2016), He & Yannelis (2016), Prokopovych (2016), Reny (2016a,b).
2. **Applications.** The following authors show how the various results can be applied in practice: Dasgupta & Maskin (1986b), Jackson & Swinkels (2005), Carbonell-Nicolau & Ok (2007), Duggan (2007), Monteiro & Page (2008), Jackson (2009), Barelli et al. (2013), Olszewski & Siegel (2016, 2019, 2020), Scalzo (2019b).
3. **Approximating games and equilibria.** The following authors include results in which equilibrium existence can be obtained by approximating the discontinuous game with a sequence of finite games: Dasgupta & Maskin (1986a), de Castro (2010), Balder (2011), Carmona (2011, 2013), Prokopovych (2011), Reny (2011), Bich & Laraki (2017).
4. **Refinements.** Carbonell-Nicolau (2011), Scalzo (2013), and Bich (2019) consider various refinements of Nash equilibrium in discontinuous games.
5. **Strategic complements, potential games.** Kukushkin (2018) provides various Nash equilibrium existence results for games with strategic complements and for potential games.

7. PROOFS

For any player $i \in N$, and for any subset S of $X \times X$, let $D^i(S) := \{y \in X : u_i(z) > \min(u_i(x), u_i(y))$ for every $(x, z) \in S\}$.

Lemma 1. For any player $i \in N$, and for any subsets S^0 and S^1 of $X \times X$, either $D^i(S^0) \subseteq D^i(S^1)$ or $D^i(S^1) \subseteq D^i(S^0)$.

Proof. If the assertion is false, then there exist $y^0, y^1 \in X$ such that (a) $y^0 \in D^i(S^0) \backslash D^i(S^1)$ and (b) $y^1 \in D^i(S^1) \backslash D^i(S^0)$. By (a), we have: (a_1) $u_i(z) > \min(u_i(x), u_i(y^0))$ for every $(x, z) \in S^0$, and (a_2) $u_i(z^1) \leq \min(u_i(x^1), u_i(y^0))$ for some $(x^1, z^1) \in S^1$. By (b), we have: (b_1) $u_i(z) > \min(u_i(x), u_i(y^1))$ for every $(x, z) \in S^1$, and (b_2) $u_i(z^0) \leq \min(u_i(x^0), u_i(y^1))$ for some $(x^0, z^0) \in S^0$.
Putting $(x, z) = (x^0, z^0)$ in (a_1) and combining the result with (b_2) gives

$$\min(u_i(x^0), u_i(y^0)) < u_i(z^0) \leq \min(u_i(x^0), u_i(y^1)),$$

and so $\min(u_i(x^0), u_i(y^0)) < \min(u_i(x^0), u_i(y^1))$, from which we conclude that $u_i(y^0) < u_i(y^1)$.
Putting $(x, z) = (x^1, z^1)$ in (b_1) and combining the result with (b_2) gives

$$\min(u_i(x^1), u_i(y^1)) < u_i(z^1) \leq \min(u_i(x^1), u_i(y^0)),$$

and so $\min(u_i(x^1), u_i(y^1)) < \min(u_i(x^1), u_i(y^0))$, from which we conclude that $u_i(y^1) < u_i(y^0)$.
This contradiction establishes the result. □

Proof of Theorem 3. Since, by Remark 3, each of the conditions (*a*)–(*j*) implies condition (*k*), it suffices to prove that a Nash equilibrium exists when condition (*k*) holds. So suppose that (*k*) holds, i.e., suppose that G has the robust better-reply-correspondence property. Also, suppose by way of contradiction that there is no Nash equilibrium. Then, by the robust better-reply-correspondence property, for every $x \in X$ there is a neighborhood U^x of x and a nonempty-valued, convex-valued, closed correspondence $b^x : U \twoheadrightarrow X$ [with $b^x(x') = \times_{i \in N} b_i^x(x')$ for all $x' \in U^x$] such that, for every $y \in U^x$, there is a player $i \in N$ for whom

$$u_i(z_i, x'_{-i}) > \min(u_i(x'), u_i(y)) \text{ for every } x' \in U^x \text{ and for every } z_i \in b_i^x(x'). \qquad 12.$$

Since X is a compact Hausdorff space, for each $x \in X$ there is an open set that contains x and whose closure, C^x, is contained in U^x. So for every $x \in X$, C^x is a closed subset of U^x and the interior of C^x, denoted $\text{int}C^x$, contains x. So $\{\text{int}C^x\}_{x \in X}$ is an open cover of X. Since X is compact, there is a finite subcover, $\{\text{int}C^k\}_{k \in K}$, where K is some finite subset of X.

For any player $i \in N$, and for any $k \in K$, we need several definitions. First, define $D^{i,k} := \{y \in X : u_i(z_i, x'_{-i}) > \min(u_i(x'), u_i(y))$ for every $x' \in U^k$ and for every $z_i \in b_i^k(x')\}$. So $D^{i,k}$ is the set of $y \in X$ such that Equation 12 holds for $x = k$ there. Next, define $Q^{i,k} := C^k \cap (\cap_j (X \setminus \text{int}C^j))$, where the intersection in parentheses is over those $j \in K$ such that $D^{i,k}$ does not contain $D^{i,j}$. Then $Q^{i,k}$ is a closed subset of X, being the finite intersection of closed sets. Finally, define $F_i^k(y) := b_i^k(y)$ if $y \in Q^{i,k}$, and define $F_i^k(y) := \emptyset$ otherwise. Then the correspondence $F_i^k : X \twoheadrightarrow X_i$ has a closed graph because $Q^{i,k}$ is closed and because b_i^k has a closed graph. Also, F_i^k is convex-valued because each b_i^k is convex-valued and because the empty set is trivially convex.[38]

For any player $i \in N$ and for any $y \in X$, define $F_i(y) := \{x_i \in X_i$. There exists some nonnegative vector $(\lambda_k)_{k \in K}$ with $\sum_{\kappa \in K} \lambda_k = 1$ such that $\lambda_k > 0$ implies $F_i^k(y) \neq \emptyset$ and such that $x_i \in \sum_{k \in K} \lambda_k F_i^k(y)\}$. Then, $F_i : X \twoheadrightarrow X_i$ has a closed graph because K is finite and because each F_i^k has a closed graph with values that are subsets of the compact set X_i. Also, F_i is convex-valued because each F_i^k is convex-valued. We next show that F_i is nonempty-valued.

Consider any $i \in N$ and any $y \in X$. We must show that $F_i(y)$ is nonempty. By Lemma 1, for every $k, j \in \{1, \ldots, K\}$, we have either $D^{i,k} \subseteq D^{i,j}$ or $D^{i,j} \subseteq D^{i,k}$.[39] Consequently, the sets $D^{i,k}$ for $k \in K$ are totally ordered by set inclusion. Therefore, because there is at least one j with $y \in \text{int}C^j$ ($\{\text{int}C^k\}_{k \in K}$ covers X), there is $k^* \in K$ such that D^{i,k^*} is the largest set among all of the sets $D^{i,j}$ such that $y \in \text{int}C^j$. Hence, we have $y \in Q^{i,k^*}$, since by the definition of k^*, $y \in \text{int}C^{k^*} \subseteq C^{k^*}$ and $D^{i,k^*} \supseteq D^{i,j}$ for every $j \in K$ such that $y \in \text{int}C^j$. Consequently, we obtain $F_i^{k^*}(y) = b_i^{k^*}(y)$, and therefore, because $F_i(y) \supseteq F_i^{k^*}(y) = b_i^{k^*}(y)$ and because $b_i^{k^*}(y)$ is nonempty, we may conclude that $F_i(y)$ is nonempty.

So for every player $i \in N$, F_i is nonempty-valued, convex-valued, and has a closed graph. Therefore, by Glicksberg's (1952) theorem, the correspondence $\times_{i \in N} F_i : X \twoheadrightarrow X$ has a fixed point, $y^* \in \times_{i \in N} F_i(y^*)$.[40] Since $\{\text{int}C^k\}_{k \in K}$ covers X, there is $k_0 \in K$ such that $y^* \in \text{int}C^{k_0} \subseteq U^{k_0}$. Hence, there is a player $i_0 \in N$ such that Equation 12 holds for player i_0 when the x there is set equal to k_0 and the y there is set equal to y^*. Consequently, we have $y^* \in D^{i_0,k_0}$.

[38]As an aside, let us remark here that, under any one of the conditions (*a*)–(*d*), (*f*), or (*h*), there is a finite subset X_i^k of X_i such that $F_i^k(y) \subseteq \text{co}X_i^k$ for every $y \in X$, a condition that we can describe by saying that F_i^k is polyhedral.

[39]To apply Lemma 1, for each $k \in K$, let $S^{i,k} := \{(x', z) \in U^k \times X : x' \in U^k, z_{-i} = x'_{-i}$ and $z_i \in b_i^k(x')\}$. Then observe that the set $D^i(S^{i,k})$ defined just before Lemma 1 is equal to $D^{i,k}$ here.

[40]Continuing with the aside in footnote 38, when each F_i^k is polyhedral we do not actually need to assume that the X_i are locally convex, since in that case we can use Kakutani's (1941) theorem instead of Glicksberg's (1952) theorem. The proofs of the main results found by Reny (1999) and McLennan et al. (2011) are instances of this technique.

Since $y^*_{i_0} \in F_{i_0}(y^*)$, the definition of $F_{i_0}(y^*)$ implies that $y^*_{i_0}$ can be written as a convex combination, $\sum \lambda_k x^k_{i_0}$, such that for every k with $\lambda_k > 0$, $x^k_{i^0} \in F^k_{i^0}(y^*) = b^k_{i^0}(y^*)$ and therefore $y^* \in Q^{i,k}$, which implies that $y^* \in C^k$ and $D^{i_0,k} \supseteq D^{i_0,k_0}$ (the latter since $y^* \in \text{int}C^{k_0}$). In particular, since $y^* \in D^{i_0,k_0}$, we have $y^* \in D^{i_0,k}$ for every k such that $\lambda_k > 0$. Hence, if $\lambda_k > 0$, then we have $y^* \in C^k$ and, because $y^* \in D^{i_0,k}$,

$$u_{i_0}(z_{i_0}, x'_{-i_0}) > \min(u_{i_0}(x'), u_{i_0}(y^*)) \text{ for every } x' \in U^k \text{ and every } z_{i^0} \in b^k_{i^0}(x'). \qquad 13.$$

In particular Equation 13 is satisfied when $x' = y^*$ and $z_{i^0} = x^k_{i^0}$, since $y^* \in C^k \subseteq U^k$ and since $x^k_{i^0} \in F^k_{i^0}(y^*) = b^k_{i^0}(y^*)$. But this means that $u_{i_0}(x^k_{i_0}, y^*_{-i_0}) > u_{i_0}(y^*)$ for every k such that $\lambda_k > 0$. The quasiconcavity of $u_{i_0}(\cdot, y^*_{-i_0})$, together with $y^*_{i_0} = \sum \lambda_k x^k_{i_0}$, yields the contradiction $u_{i_0}(y^*) > u_{i_0}(y^*)$. □

Proof of Theorem 6. It suffices to follow the proof of Theorem 3, but with the following adjustments. First, notice that Lemma 1 holds when $D^i(S)$ is redefined as $D^i(S) := \{y \in X$. For each $(x, z) \in S$, we then have either $x_i \notin \text{co}\{w_i : u_i(w_i, x_{-i}) \geq u_i(z)\}$ or $y_i \notin \text{co}\{w_i : u_i(w_i, y_{-i}) \geq u_i(z)\}\}$.[41] Second, replace the occurrence of any inequality of the form $u_i(a) > \min(u_i(b), u_i(c))$—as occurs in Equations 12 and 13 and in the definition of $D^{i,k}$—with $b_i \notin \text{co}\{w_i : u_i(w_i, b_{-i}) \geq u_i(a)\}$ or $c_i \notin \text{co}\{w_i : u_i(w_i, c_{-i}) \geq u_i(a)\}$. Third, replace the definition $F^k_i(y) := b^k_i(y)$ with $F^k_i(y) := \text{co}b^k_i(y)$, and notice that because the b^k_i correspondences can be chosen to be coclosed, each correspondence F^k_i is closed and has convex values. Finally, replace the last two sentences of the proof with the following two sentences: "But this means that $y^*_{i_0} \notin \text{co}\{w_{i_0} : u_{i_0}(w_{i_0}, y^*_{-i_0}) \geq u_{i_0}(x^k_{i_0}, y^*_{-i_0})\}$ for every k such that $\lambda_k > 0$. In particular, choosing $\bar{k}$ to solve $\min_{k:\lambda_k>0} u_{i_0}(x^k_{i_0}, y^*_{-i_0})$, we have $x^k_{i_0} \in \{w_{i_0} : u_{i_0}(w_{i_0}, y^*_{-i_0}) \geq u_{i_0}(x^{\bar{k}}_{i_0}, y^*_{-i_0})\}$ for every k with $\lambda_k > 0$, and we have $y^*_{i_0} \notin \text{co}\{w_{i_0} : u_{i_0}(w_{i_0}, y^*_{-i_0}) \geq u_{i_0}(x^{\bar{k}}_{i_0}, y^*_{-i_0})\}$, which, taken together, contradict the fact that $y^*_{i_0} = \sum \lambda_k x^k_{i_0}$." □

Proof of Theorem 15. Suppose, by way of contradiction, that BG has no Bayes-Nash equilibrium. Then, for every $s \in S$, because s is not a Bayes-Nash equilibrium and because G has the finite strategic-action better-reply property, there are finite sets of behavioral strategies $F^s_1 \subseteq S_1, \ldots, F^s_{|N|} \subseteq S_{|N|}$, and there is a weak* neighborhood $U^s \subseteq X$ of $(s_i * p_i)_{i \in N} \in X$ such that for every $s' \in S$ with $(s'_i * p_i)_{i \in N} \in U^s$, there is a player i and there is $\hat{s}_i \in S_i$ such that $\hat{s}_i(\cdot|t_i) \in \{\bar{s}_i(\cdot|t_i) : \bar{s}_i \in F_i\}$ for every $t_i \in T_i$, and $V_i(\hat{s}_i, s'_{-i}) > V_i(s')$.

Since every $x \in X$ is of the form $x = (s_i * p_i)_{i \in N}$ for some $s \in S$, the collection of weak* open sets $\{U^s\}_{s \in S}$ covers X. Since X is weak* compact there is a finite subcover. Thus, there is a finite subset K of S such that $\{U^k\}_{k \in K}$ covers X.

For each player i, define $F_i := \cup_{k \in K} F^k_i$. Hence, each F_i is a finite subset of S_i. Consider a surrogate Bayesian game BG^* that is identical to BG, except that each player i's action set is F_i instead of A_i, and each player i's payoff function is $v^*_i : (\times_{i \in N} F_i) \times T \to \mathbb{R}$ instead of $v_i : A \times T \to \mathbb{R}$, where $v^*_i(s, t) := v_i(s(t), t)$ for each $s \in \times_{i \in N} F_i$ and for each $t \in T$. Consequently, BG^* is an absolutely continuous Bayesian game in which each player's action set is finite. Following Balder (1988, theorem 3.1), BG^* has a Bayes-Nash equilibrium, σ^*, where $\sigma^* = (\sigma^*_i)_{i \in N}$, and each $\sigma^*_i : T_i \to \Delta(F_i)$ is a transition probability.

[41]The proof of Lemma 1 proceeds as before. Then, the analogues of (a_1) and (b_2) imply that $y^0_i \notin \text{co}\{w_i : u_i(w_i, y^0_{-i}) \geq u_i(z^0)\}$ and $y^1_i \in \text{co}\{w_i : u_i(w_i, y^1_{-i}) \geq u_i(z^0)\}$, and the analogues of (a_2) and (b_1) imply that $y^0_i \in \text{co}\{w_i : u_i(w_i, y^0_{-i}) \geq u_i(z^1)\}$ and $y^1_i \notin \text{co}\{w_i : u_i(w_i, y^1_{-i}) \geq u_i(z^1)\}$. The first and third of these imply that $u_i(z^0) > u_i(z^1)$, and the second and fourth imply that $u_i(z^1) > u_i(z^0)$. This contradiction proves the lemma.

For each player $i \in N$, define $s_i' \in S_i$ so that for every $t_i \in T_i$ and for every Borel subset C of A_i, we have

$$s_i'(C|t_i) := \sum_{s_i \in F_i} \sigma_i^*(\{s_i\}|t_i) s_i(C|t_i).$$

Since $s' \in S$, we have $(s_i' * p_i)_{i \in N} \in X$. Therefore, since $\{U^k\}_{k \in K}$ covers X, there is $k \in K$ such that $(s_i' * p_i)_{i \in N} \in U^k$. Then, by the properties of the sets $F_1^k, \ldots, F_{|N|}^k$, there is a player i and there is $\hat{s}_i \in S_i$ such that, for every $t_i \in T_i$, we obtain

$$\hat{s}_i(\cdot|t_i) \in \{\bar{s}_i(\cdot|t_i) : \bar{s}_i \in F_i^k\} \tag{14.}$$

and

$$V_i(\hat{s}_i, s_{-i}') > V_i(s'). \tag{15.}$$

By Equation 14, we may let $[\hat{s}_i]$ denote the feasible behavioral strategy for player i in BG^* that, for each $t_i \in T_i$, gives probability 1 to the element $\bar{s}_i \in F_i^k$ that satisfies $\hat{s}_i(\cdot|t_i) = \bar{s}_i(\cdot|t_i)$.[42] Then, we have

$$V_i^*([\hat{s}_i], \sigma_{-i}^*) = V_i(\hat{s}_i, s_{-i}') > V_i(s') = V_i^*(\sigma^*),$$

where the two equalities follow from the definitions of $[\hat{s}_i]$ and s', and the inequality follows from Equation 15. But then $V_i^*([\hat{s}_i], \sigma_{-i}^*) > V_i^*(\sigma^*)$ contradicts the fact that σ^* is a Bayes-Nash equilibrium of BG^*. □

DISCLOSURE STATEMENT

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[42]So defined, $[s_i] : T_i \to \Delta(A_i^*)$ is a transition probability because, for any $\bar{s}_i \in F_i$, $\{t_i \in T_i : \hat{s}_i(\cdot|t_i) = \bar{s}_i(\cdot|t_i)\}$ is measurable since it is equal to $\cap_C \{t_i \in T_i : \hat{s}_i(C|t_i) = \bar{s}_i(C|t_i)\}$, where the intersection is over any countable basis of open subsets C of A_i (a compact metric space).

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