

STABILITY PHENOMENA FOR RESONANCE ARRANGEMENTS

NICHOLAS PROUDFOOT AND ERIC RAMOS

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ABSTRACT. We prove that the i^{th} graded pieces of the Orlik–Solomon algebras or Cordovil algebras of resonance arrangements form a finitely generated FS^{op} -module, thus obtaining information about the growth of their dimensions and restrictions on the irreducible representations of symmetric groups that they contain.

1. INTRODUCTION

Let $\mathcal{A}(n)$ be the collection of all hyperplanes in \mathbb{R}^n that are perpendicular to some nonzero vector with entries in the set $\{0, 1\}$. This hyperplane arrangement is called the **resonance arrangement** of rank n . The resonance arrangement has connections to algebraic geometry, representation theory, geometric topology, mathematical physics, and economics; for a survey of these connections, see [4, Section 1]. Of particular interest is the set of chambers of $\mathcal{A}(n)$. Amazingly, despite the simplicity of the definition, no formula for the number of chambers as a function of n is known. A more refined invariant of $\mathcal{A}(n)$ is its characteristic polynomial, whose coefficients (after taking absolute values) have sum equal to the number of chambers. Kühne has made some progress toward understanding the coefficient of t^{n-i} in the characteristic polynomial as a function of n with i fixed. Our purpose is to shed a new light on Kühne’s result, to generalize it to a wider class of arrangements, and to study the action of the symmetric group Σ_n on various algebraic invariants of these arrangements.

Let $S \subset \mathbb{R}$ be any finite set, and let $\mathcal{A}_S(n)$ be the collection of hyperplanes that are perpendicular to a nonzero vector with entries in S . If $S = \{0, 1\}$, $\mathcal{A}_S(n)$ is the resonance arrangement. If $S = \{\pm 1\}$, it is the **threshold arrangement**, which is studied in [3]. For each positive integer d , let $M_S(n, d)$ denote the set of n -tuples of vectors in \mathbb{R}^d such that no nontrivial¹ linear combination of all n vectors with coefficients in S is equal to zero. The cohomology ring of $M_S(n, d)$ is generated in degree $d - 1$ [2, Corollary 5.6]. If d is even, the presentation of this ring in [2] coincides with that of the **Orlik–Solomon algebra** of $\mathcal{A}_S(n)$ (with all degrees multiplied by $d - 1$) [6]. If d is odd and greater than 1, then it coincides with that of the **Cordovil algebra** of $\mathcal{A}_S(n)$ (with all degrees multiplied by $d - 1$) [1]; see

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¹Nontrivial means that, if $0 \in S$, we do not allow all coefficients to be 0.

also [5, Example 5.8].² In particular, for any $n \geq 1$, $d \geq 2$, and $i \geq 0$, the dimension $b_S^i(n) = \dim H^{(d-1)i}(M_S(n, d); \mathbb{Q})$ is equal to $(-1)^i$ times the coefficient of t^{n-i} in the characteristic polynomial of $\mathcal{A}_S(n)$.

These vector spaces carry more information than just their dimension; they also carry actions of the symmetric group Σ_n , which acts by permuting the n vectors. These representations are isomorphic for all even $d \geq 2$ and for all odd $d \geq 3$, but the $d = 2$ and $d = 3$ cases are genuinely different. The total cohomology $H^*(M_S(n, 3); \mathbb{Q})$ with all degrees combined is isomorphic as a representation of Σ_n to $H^0(M_S(n, 1); \mathbb{Q})$, which is the permutation representation with basis indexed by the chambers of $\mathcal{A}_S(n)$ [5, Theorem 1.4(b)].

For fixed $S \subset \mathbb{R}$, $d \geq 2$, and $i \geq 0$, we will define in the next section a contravariant module $B_S^{i,d}$ over the category of finite sets with surjections that takes the set $[n]$ to $H^{(d-1)i}(M_S(n, d); \mathbb{Q})$.

Theorem 1.1. *The module $B_S^{i,d}$ is finitely generated in degrees $\leq |S|^i$.*

Combining Theorem 1.1 with [7, Theorem 4.1], we obtain the following numerical results:³

Corollary 1.2. *Fix a finite set $S \subset \mathbb{R}$ and a pair of integers $d \geq 2$ and $i \geq 0$.*

1. *The generating function*

$$G_S^i(t) := \sum_{n=1}^{\infty} b_S^i(n) t^n$$

is a rational function with poles contained in the set $\{1/j \mid 1 \leq j \leq |S|^i\}$, with at worst a simple pole at $|S|^{-i}$. Equivalently, there exist polynomials $\{c_S^{i,j}(n) \mid 1 \leq j \leq |S|^i\}$ such that, for n sufficiently large,

$$b_S^i(n) = \sum_{j=1}^{|S|^i} c_S^{i,j}(n) j^n,$$

and the last polynomial $c_S^{i,|S|^i}(n)$ is a constant polynomial.

2. *For any partition λ of n , let V_λ denote the irreducible representation of Σ_n indexed by λ . If $\text{Hom}_{\Sigma_n}(V_\lambda, H^{(d-1)i}(M_S(n, d); \mathbb{Q})) \neq 0$, then λ has at most $|S|^i$ rows.*
3. *For any partition λ with $n \geq |\lambda| + \lambda_1$, let $\lambda(n)$ be the **padded partition** of n obtained from λ by adding a row of length $n - |\lambda|$. For any λ , the function*

$$n \mapsto \dim \text{Hom}_{\Sigma_n}(V_{\lambda(n)}, H^{(d-1)i}(M_S(n, d); \mathbb{Q}))$$

is bounded above by a polynomial in n . In particular, if λ is the empty partition, this says that the multiplicity of the trivial representation in $H^{(d-1)i}(M_S(n, d); \mathbb{Q})$ is bounded above by a polynomial in n .

²For d odd, the presentation in [2] incorrectly omits the relations that each of the generators squares to zero.

³The deepest of these statements, namely the fact that the dimension generating function for a finitely generated FS^{op}-module is rational with prescribed poles, is due to Sam and Snowden [8, Corollary 8.1.4].

Remark 1.3. A stronger version of item (1) above for the resonance arrangement appears in [4, Theorem 1.4]. Kühne proves that the polynomials $c_{\{0,1\}}^{i,j}(n)$ are *all* constant (i.e. that all poles of $G_{\{0,1\}}^i(t)$ are simple), obtains bounds on their sizes, and shows that the equality holds for all n , not just sufficiently large n (i.e. that the limit as t goes to ∞ of $G_{\{0,1\}}^i(t)$ is zero). It should be possible to categorify Kühne's theorem by proving that the restriction of $B_{\{0,1\}}^{i,d}$ to the category of **ordered surjections** [8] is isomorphic to a direct sum of shifts of principal projectives, with summands indexed by Kühne's **functional prototypes**. The cost of working with ordered surjections would be that we would lose all information about the action of the symmetric group.

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2. THE PROOF

Let FS denote the category whose objects are nonempty finite sets and whose morphisms are surjective maps. An **FS^{op}-module** over \mathbb{Q} is a contravariant functor from FS to the category of rational vector spaces. For each finite set F , we have the **principal projective** module P_F , which sends a finite set E to the vector space with basis $\text{Hom}_{\text{FS}}(E, F)$, with morphisms defined on basis elements by composition. An FS^{op}-module N is said to be **finitely generated** if it is a quotient of a finite sum $\oplus_i P_{F_i}$ of principal projectives, and it is said to be **finitely generated in degrees $\leq m$** if the sets F_i can all be taken to have cardinality less than or equal to m . This is equivalent to saying that, for all E , the vector space $N(E)$ is finite dimensional and is spanned by the images of the pullbacks along various maps $\varphi : E \rightarrow F$, where F has cardinality less than or equal to m .

Lemma 2.1. *Suppose that N_1 is finitely generated in degrees $\leq m_1$ and N_2 is finitely generated in degrees $\leq m_2$. Then the pointwise tensor product $N_1 \otimes N_2$ is finitely generated in degrees $\leq m_1 m_2$.*

Proof. We immediately reduce to the case where $N_1 = P_{[m_1]}$ and $N_2 = P_{[m_2]}$. For any $\varphi : E \rightarrow [m]$, let e_φ denote the corresponding basis element of $P_{[m]}(E)$. Then $N_1 \otimes N_2$ has basis

$$\{e_{\varphi_1} \otimes e_{\varphi_2} \mid \varphi_1 : E \rightarrow [m_1], \varphi_2 : E \rightarrow [m_2]\}.$$

Given the pair of surjections (φ_1, φ_2) , let $F \subset [m_1] \times [m_2]$ denote the image of $\varphi_1 \times \varphi_2$, let $\varphi = \varphi_1 \times \varphi_2 \in \text{Hom}_{\text{FS}}(E, F)$, and let $\psi_1 : F \rightarrow [m_1]$ and $\psi_2 : F \rightarrow [m_2]$ denote the coordinate projections. It is clear that we have $e_{\varphi_1} \otimes e_{\varphi_2} = \varphi^*(e_{\psi_1} \otimes e_{\psi_2})$. Since the cardinality of F is at most $m_1 m_2$, this completes the proof. \square

Fix a positive integer d and a finite set $S \subset \mathbb{R}$. To any finite set E , we associated the space $M_S(E, d)$ of E -tuples of vectors in \mathbb{R}^d such that any nontrivial linear combination of the vectors with coefficients in S is nonzero. Given a surjection $\varphi : E \rightarrow F$, we obtain a map

$$\varphi_* : M_S(E, d) \rightarrow M_S(F, d)$$

by adding the vectors in each fiber of φ . These maps define a functor from FS to the category of topological spaces. By taking rational cohomology in degree $(d-1)i$,

we obtain an FS^{op} -module $B_S^{i,d}$. We prove the following theorem, which implies the three statements in the introduction.

Proof of Theorem 1.1. As noted above, the cohomology of $M_S(E, d)$ is generated as an algebra in degree $d - 1$, hence $B_S^{i,d}$ is a quotient of $(B_S^{1,d})^{\otimes i}$. By Lemma 2.1, this means that it is sufficient to prove that $B_S^{1,d}$ is finitely generated in degrees $\leq |S|$. For any finite set F , the vector space $B_S^{1,d}(F)$ has a generating set indexed by nonzero elements of S^F [2, Corollary 5.6] (these generators form a basis unless two nonzero elements of S^F are proportional, in which case the corresponding generators are equal). For any nonzero $v \in S^F$, let $x_v \in B_S^{1,d}(F)$ be the corresponding generator. Concretely, if we take $x \in H^{d-1}(\mathbb{R}^d \setminus \{0\}; \mathbb{Q})$ to be the standard generator, then x_v is equal to the pullback of x along the map

$$f_v : M_S(F, d) \rightarrow \mathbb{R}^d \setminus \{0\}$$

that sends an F -tuple of vectors to its linear combination with coefficients determined by v . Given a surjection $\varphi : E \rightarrow F$, we have $f_v \circ \varphi_* = f_{\varphi^*v}$, and therefore

$$\varphi^*(x_v) = \varphi^* \circ f_v^*(x) = f_{\varphi^*v}^*(x) = x_{\varphi^*v} \in B_S^{1,d}(E).$$

Since every element of S^E may be pulled back from a subset of cardinality at most $|S|$, $B_S^{1,d}$ is generated in degrees $\leq |S|$. \square

Remark 2.2. Our construction also works if we replace \mathbb{R} with an arbitrary field k and we take S to be a finite subset of k . We define the arrangement $\mathcal{A}_{k,S}(n)$ in k^n as above, we denote its complement by $M_{k,S}(E, 1)$, and we take $B_{k,S}^{i,1}(E)$ to be the étale cohomology group $H_{\text{ét}}^i(M_{k,S}(E, 1) \otimes_k \bar{k}; \mathbb{Q}_l)$ for some prime l not equal to the characteristic of k , which is isomorphic to the degree i part of the Orlik–Solomon algebra of $\mathcal{A}_{k,S}(n)$. This is an FS^{op} -module over \mathbb{Q}_l , and the same argument shows that it is finitely generated in degrees $\leq |S|^i$.

An interesting special case is where $k = \mathbb{F}_q$ is a finite field and $S = k$, so that our arrangement $\mathcal{A}_{\mathbb{F}_q, \mathbb{F}_q}(n)$ is the collection of all hyperplanes in \mathbb{F}_q^n . This arrangement has characteristic polynomial $(t-1)(t-q) \cdots (t-q^{n-1})$, and therefore the i^{th} Betti number is equal to the evaluation of the i^{th} elementary symmetric polynomial at the values $1, q, \dots, q^{n-1}$. This implies that the Hilbert series of our module is

$$q^{\binom{i}{2}} t^i \prod_{j=0}^{i-1} \frac{1}{1 - q^j t},$$

which has simple poles at q^{-j} for $j = 0, 1, \dots, i$. The projectivization of $M_{\mathbb{F}_q, \mathbb{F}_q}(n, 1) \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ is a Deligne–Lusztig variety for the group $\text{GL}_n(\mathbb{F}_q)$.

REFERENCES

- [1] R. Cordovil, *A commutative algebra for oriented matroids*, Discrete Comput. Geom. **27** (2002), no. 1, 73–84, Geometric combinatorics (San Francisco, CA/Davis, CA, 2000).
- [2] Mark de Longueville and Carsten A. Schultz, *The cohomology rings of complements of subspace arrangements*, Math. Ann. **319** (2001), no. 4, 625–646.
- [3] Samuel C. Gutekunst, Karola Mészáros, and T. Kyle Petersen, *Root cones and the resonance arrangement*, arXiv:1903.06595.
- [4] Lucas Kühne, *The universality of the resonance arrangement and its Betti numbers*, arXiv:2008.10553.
- [5] Daniel Moseley, *Equivariant cohomology and the Varchenko–Gelfand filtration*, J. Algebra **472** (2017), 95–114.

- [6] Peter Orlik and Louis Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), no. 2, 167–189.
- [7] Nicholas Proudfoot and Ben Young, *Configuration spaces, $\mathrm{FS}^{\mathrm{op}}$ -modules, and Kazhdan-Lusztig polynomials of braid matroids*, New York J. Math. **23** (2017), 813–832.
- [8] Steven V Sam and Andrew Snowden, *Gröbner methods for representations of combinatorial categories*, J. Amer. Math. Soc. **30** (2017), no. 1, 159–203.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403
Email address: `njp@uoregon.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403
Email address: `eramos@uoregon.edu`