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Invariant-EKF Design for a Unicycle Robot under Linear Disturbances

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***Abstract*— We consider a nonlinear estimation problem where a unicycle vehicle moves with unknown disturbances generated from linear time-invariant systems. The vehicle measures its position to estimate its state and disturbance information simultaneously. We show that this system is invariant under the action of a Lie group and design an Invariant Extended Kalman Filter (IEKF). We propose a first-order approximation of the noise covariance in the invariant frame. Through Monte-Carlo simulations, we demonstrate that the first-order approximation improves the performance of the IEKF and that the IEKF yields superior transient performance over the standard EKF.**

I. Introduction

Estimation and filtering of nonlinear systems is an important problem in research and in industry. There are several filters capable of dealing with nonlinear systems, such as the extended Kalman filter (EKF) [1], unscented Kalman filter (UKF) [2] and particle filters [3]. When applied to robotic applications, these filters provide simple, 'off-the-shelf' solutions. However, they do not take advantage of properties present in robotic dynamics, such as symmetries. There has been much interest lately in designing observers that can leverage symmetries of certain nonlinear dynamics to improve estimation performance. These are known more generally as symmetry preserving observers.

The theory behind symmetry preserving observers is given in [4]. When the EKF equations are used to compute the gain matrix of a symmetry preserving observer, it is referred to as an invariant EKF (IEKF) [5]. More recently, the IEKF has gained attention as a tool well suited for applications in localization of mobile robots [6][7] and sensor fusion for navigation of unmanned aerial vehicles (UAVs) [5][8]. Reference [9] uses the IEKF in a visual inertial navigation system. In [10] the authors show that an IEKF based SLAM (simultaneous localization and mapping) algorithm has better consistency and convergence properties over other EKF based SLAM techniques. More recently, the authors in [11] propose a matrix Lie group framework for the IEKF and show that it possesses guaranteed local stability properties.

In this paper, we consider an estimation problem for a planar vehicle modeled as a unicycle moving in a dynamic flow field. We model the flow impact on the vehicle as disturbances to its motion and assume that the disturbances are the output of a linear time-invariant system. This assumption holds, e.g., for small unmanned aerial vehicles moving in unknown uniform wind fields and for underwater vehicles moving subject to sinusoidal wave disturbances. The vehicle employs its location measurements to estimate its state (position and heading) and disturbance information.

While a standard EKF is applicable to our estimation problem, we construct an IEKF by identifying an invariance property of the system. The invariance property allows us to transform the conventional estimation error into an invariant coordinate and derive the IEKF under this coordinate. Since the transformation also changes noise characteristics, we propose to carefully characterize the covariance of the transformed noise so that the IEKF can operate at its full potential. In particular, we improve the result in [11] and derive a first order approximation for the covariance of the transformed noise to better represent its statistics. Using Monte Carlo simulations, we show that the introduction of the first order approximation indeed improves the performance of the IEKF, particularly for non-isotropic sensor noise, and that the designed IEKF yields superior transient performance over the EKF.

The contribution of this paper is three-fold. First, we prove that unicycle kinematics subject to linear disturbances are invariant with respect to a Lie group action. Second, we design an IEKF that leverages this invariance property. Third, we propose a transformation of the filter covariance matrices that more accurately represents uncertainties for the IEKF. Simulation results demonstrate the advantages of the proposed IEKF.

The rest of the paper is organized as follows. In Section II we formulate the problem. In Section III we prove that the system is invariant and present the steps for designing the IEKF. In particular, in Section III-B we derive the rotated noise terms and summarize the IEKF algorithm. In Section IV we provide simulation results and analysis. Section V contains conclusions and future work.

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II. PROBLEM FORMULATION

Consider a unicycle robot subject to velocity disturbances (d_x, d_y). The kinematic model of the robot is given by

$$\begin{aligned}\dot{x} &= v\cos\theta + d_x \\ \dot{y} &= v\sin\theta + d_y \\ \dot{\theta} &= \omega,\end{aligned} \tag{1}$$

where (x, y) is the position of the robot, θ is the heading, v is the linear velocity and ω is the turning rate. We assume that (d_x, d_y) are outputs from a linear system given by

$$\dot{d} = Ad \tag{2}$$

$$\begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix} d, \tag{3}$$

where $d \in \mathbb{R}^{m\times 1}$, $A \in \mathbb{R}^{m\times m}$ and $C, D \in \mathbb{R}^{1\times m}$. The matrices A, C, and D are assumed known and constant. For example, d_x and d_y can represent constant and sinusoidal disturbances with known frequencies.

The robot is equipped with a positioning device, such as a GPS or a suite of range and bearing sensors, measuring its position (x, y). The position measurement can be in a global frame or with respect to a known landmark. In the latter case, without loss of generality, we assume that the landmark is at the origin. Then (x, y) represents the relative position between the robot and the landmark. The measurement equation of the system is

$$Y = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{4}$$

The system (1)–(4) models the kinematics of a differential drive mobile vehicle, underwater vehicle motion [12], and the simplified kinematics of a fixed wing aerial vehicle in planar flight [13]. Furthermore, some applications include estimating the states of these types of vehicles for the purpose of localization [14], trajectory tracking [15], or flow field reconstruction [16], [17].

The state in (1)–(4) can be estimated with a standard EKF. However, the EKF does not take into account symmetry properties present in the dynamics. Our objective is to design a novel nonlinear filter that leverages symmetries inherent in (1) to estimate the vehicle state (x, y, θ) and the disturbances (d) using the measurements in (4).

III. IEKF DESIGN

We augment (1) with the disturbance dynamics (2)–(3) and obtain

$$\begin{aligned}\dot{x} &= v\cos\theta + Cd \\ \dot{y} &= v\sin\theta + Dd \\ \dot{\theta} &= \omega \\ \dot{d} &= Ad.\end{aligned} \tag{5}$$

Motivated by [4], we first show that the system in (5) is invariant with respect to a particular Lie group action. We then derive an IEKF that takes advantage of this invariance property to achieve better estimation performance than the standard EKF.

We now establish the invariance property of (5) with respect to the group of translations and rotations in two dimensions, special Euclidean or $SE(2)$. Let $G \in SE(2)$. Any element of G can be represented by (x_g, y_g, θ_g). Let $X = [x, y, \theta, d^\top]^\top$ and define a transformation as

$$\varphi_g(X) = \begin{pmatrix} x\cos\theta_g - y\sin\theta_g + x_g \\ x\sin\theta_g + y\cos\theta_g + y_g \\ \theta + \theta_g \\ d \end{pmatrix}. \tag{6}$$

Let $\mathcal{U} = (v, \omega, C, D, A)$ and define a transformation of $\mathcal{U}$ as

$$\psi_g(\mathcal{U}) = \begin{pmatrix} v \\ \omega \\ C\cos\theta_g - D\sin\theta_g \\ C\sin\theta_g + D\cos\theta_g \\ A \end{pmatrix}. \tag{7}$$

Proposition 1. The system in (5) is invariant with respect to G, under the transformations in (6) and (7). □

Proof. To show that (5) is invariant with respect to G, we use the invariance condition [18, Definition 2]

$$f(\varphi_g(X), \psi_g(\mathcal{U})) = \begin{bmatrix} v\cos(\theta+\theta_g) + [C\cos\theta_g - D\sin\theta_g]d \\ v\sin(\theta+\theta_g) + [C\sin\theta_g + D\cos\theta_g]d \\ \omega \\ Ad \end{bmatrix},$$

which equals

$$\frac{\partial}{\partial X}\varphi_g(X)\cdot f(X,\mathcal{U}) = \begin{bmatrix} \cos\theta_g & -\sin\theta_g & 0 & 0 \\ \sin\theta_g & \cos\theta_g & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} v\cos\theta + Cd \\ v\sin\theta + Dd \\ \omega \\ Ad \end{bmatrix} = \begin{bmatrix} v\cos(\theta+\theta_g) + [C\cos\theta_g - D\sin\theta_g]d \\ v\sin(\theta+\theta_g) + [C\sin\theta_g + D\cos\theta_g]d \\ \omega \\ Ad \end{bmatrix},$$

where I_m is the identity matrix of dimension m. Thus (5) is invariant under the transformations in (6) and (7).

Similarly, according to [18, Definition 3], the measurement equation (4) is G-equivariant with the transformation ϱ_g defined as

$$\varrho_g(x, y) = \begin{pmatrix} x\cos\theta_g - y\sin\theta_g + x_g \\ x\sin\theta_g + y\cos\theta_g + y_g \end{pmatrix}. \tag{8}$$

■

A. Deriving the Error Dynamics of the IEKF

Following the methods outlined in [4], $\varphi_g(X)$ can be split into $\varphi_g^a(X)$ and $\varphi_g^b(X)$ such that $\varphi_g^a(X)$ is invertible with respect to g. Setting $\varphi_g^a(X) = 0$ gives the normalization equation

$$\begin{pmatrix} x_g \\ y_g \\ \theta_g \end{pmatrix} = \gamma \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} -x\cos\theta - y\sin\theta \\ x\sin\theta - y\cos\theta \\ -\theta \end{pmatrix}. \tag{9}$$

The invariants are

$$I(\hat{X},\mathcal{U}) = \left(\varphi^b_{\gamma(\hat{X})}\left(\hat{X}\right), \psi_{\gamma(\hat{X})}(\mathcal{U})\right) = \left(\hat{d}, v, \omega, C\cos\hat{\theta} + D\sin\hat{\theta}, -C\sin\hat{\theta} + D\cos\hat{\theta}, A\right), \tag{10}$$

where $\hat{X}$ is the estimate of X. The invariant output error is given by

$$\begin{aligned} E &= \varrho_g(\hat{x},\hat{y}) - \varrho_g(x,y) \\ &= \begin{pmatrix} \hat{x}\cos\theta_g - \hat{y}\sin\theta_g - x\cos\theta_g + y\sin\theta_g \\ \hat{x}\sin\theta_g + \hat{y}\cos\theta_g - x\sin\theta_g - y\cos\theta_g \end{pmatrix} \\ &= T(\hat{\theta}) \begin{bmatrix} \hat{x} - x \\ \hat{y} - y \end{bmatrix}, \end{aligned} \tag{11}$$

where

$$T(\hat{\theta}) = \begin{bmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix}. \tag{12}$$

The invariant frame is given by

$$W(\hat{\theta}) = \begin{bmatrix} T(\hat{\theta})^\top & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_m \end{bmatrix}. \tag{13}$$

Thus, the observer equation has the following form

$$\dot{\hat{X}} = f(\hat{X}) + W(\hat{\theta}) \cdot L \cdot T(\hat{\theta}) \left(Y - \hat{Y}\right), \tag{14}$$

where L is a gain matrix to be designed. For notation convenience, we let

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \\ L_{31} & L_{32} \\ L_{d1} & L_{d2} \end{bmatrix}, \tag{15}$$

where L_{ij} are scalars for $i = 1, 2, 3$, $j = 1, 2$, and $L_{d1}, L_{d2} \in \mathbb{R}^{m\times 1}$. The invariant state error is given by

$$\begin{aligned} \sigma(\hat{X}, X) &= \varphi_{\gamma(\hat{X})}(X) - \varphi_{\gamma(\hat{X})}(\hat{X}) \\ &= \begin{pmatrix} x\cos\hat{\theta} + y\sin\hat{\theta} - \hat{x}\cos\hat{\theta} - \hat{y}\sin\hat{\theta} \\ -x\sin\hat{\theta} + y\cos\hat{\theta} + \hat{x}\sin\hat{\theta} - \hat{y}\cos\hat{\theta} \\ \theta - \hat{\theta} \\ d - \hat{d} \end{pmatrix} \\ &= W(\hat{\theta})^\top \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \\ \theta - \hat{\theta} \\ d - \hat{d} \end{bmatrix}. \end{aligned} \tag{16}$$

To find the invariant error dynamics, we differentiate (16) with respect to time and obtain

$$\begin{aligned} \dot{\sigma} = W(\hat{\theta})^\top &\left[\begin{pmatrix} v\cos\theta + Cd - v\cos\hat{\theta} - C\hat{d} \\ v\sin\theta + Dd - v\sin\hat{\theta} - D\hat{d} \\ 0 \\ Ad - A\hat{d} \end{pmatrix} \right. \\ &\left. - W(\hat{\theta})L \begin{bmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix} \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix} \right] + \begin{bmatrix} \dot{\hat{\theta}}\sigma_y \\ -\dot{\hat{\theta}}\sigma_x \\ 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{17}$$

which yields

$$\begin{aligned} \dot{\sigma}_x &= v\left(\cos\sigma_\theta - 1\right) + \omega\sigma_y + \left(C\cos\hat{\theta} + D\sin\hat{\theta}\right)\sigma_d \\ &\quad + L_{11}\sigma_x + L_{12}\sigma_y + L_{31}\sigma_x\sigma_y + L_{32}\sigma_y^2 \\ \dot{\sigma}_y &= v\sin\sigma_\theta - \omega\sigma_x + \left(-C\sin\hat{\theta} + D\cos\hat{\theta}\right)\sigma_d \\ &\quad + L_{21}\sigma_x + L_{22}\sigma_y - L_{31}\sigma_x^2 - L_{32}\sigma_x\sigma_y \\ \dot{\sigma}_\theta &= L_{31}\sigma_x + L_{32}\sigma_y \\ \dot{\sigma}_d &= A\sigma_d + L_{d1}\sigma_x + L_{d2}\sigma_y. \end{aligned} \tag{18}$$

Note that the invariant error dynamics (18) depend only on σ and the invariants $I(\hat{X},\mathcal{U})$ in (10).

Linearizing (18) around $\sigma = 0$ yields the state matrix needed for implementing the IEKF at time step k:

$$A_k = \begin{bmatrix} 0 & \omega_k & 0 & C\cos\hat{\theta}_k + D\sin\hat{\theta}_k \\ -\omega_k & 0 & v_k & -C\sin\hat{\theta}_k + D\cos\hat{\theta}_k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A \end{bmatrix}. \tag{19}$$

B. The IEKF Algorithm

To derive the IEKF algorithm, we note that the invariant state error (16) rotates the conventional estimation error to another frame. Thus, the initial state covariance, process, and measurement noise matrices, P, Q and R, respectively, can no longer accurately represent the uncertainty in the transformed system. We propose that these matrices be transformed to ensure the IEKF operates at its full potential for different cases of sensor noise and initial error. We next discuss how to rotate the covariance to the invariant error frame.

For the system in (5), the invariant state error and invariant output error are defined as:

$$\sigma = W(\hat{\theta})^\top (X - \hat{X}), \quad E = T(\hat{\theta})(\hat{Y} - Y), \tag{20}$$

respectively, where $W(\hat{\theta})^\top$ and $T(\hat{\theta})$ are given in (13) and (12), respectively. We now propose a transformation rule for the measurement noise matrix R. We use the notation $\mathcal{N}(\mu, \Sigma)$ to denote the Gaussian distribution with mean μ and covariance Σ.

Proposition 2. Let $\epsilon = \hat{Y} - Y \sim \mathcal{N}(0, R)$. Let $\hat{\theta}$ be the estimate of θ such that $\hat{\theta} - \theta \sim \mathcal{N}(0, q_\theta)$. Then

$$\mathrm{cov}(T(\hat{\theta})\epsilon) \approx T(\theta) R T(\theta)^\top + q_\theta \frac{\partial T}{\partial \theta} R \frac{\partial T}{\partial \theta}^\top \tag{21}$$

for a sufficiently small q_θ. □

Proposition 2 is proved by expanding $T(\hat{\theta})\epsilon$ up to its first order term and then computing the covariance. The proof of Proposition 2 is omitted due to space.

Similarly, let the initial state error at time $t = 0$ and the process noise be defined by $\eta_0 = \hat{X}_0 - X_0 \sim \mathcal{N}(0, P_0)$ and $\nu \sim \mathcal{N}(0, Q)$, respectively. Then the corresponding transformations for P_0 and Q are given by

$$\mathrm{cov}(W(\hat{\theta})^\top \eta_0) \approx W(\theta_0)^\top P_0 W(\theta_0) + q_\theta \frac{\partial W}{\partial \theta}^\top P_0 \frac{\partial W}{\partial \theta} \tag{22}$$

$$\mathrm{cov}(W(\hat{\theta})^\top \nu) \approx W(\theta)^\top Q W(\theta) + q_\theta \frac{\partial W}{\partial \theta}^\top Q \frac{\partial W}{\partial \theta}. \tag{23}$$

In implementation, we replace θ with its estimate $\hat{\theta}$, assuming that they are close. We set $q_\theta = P_k^\theta$ because it corresponds to the θ component of the state covariance matrix, P, for each time step k.

Note that [11] uses only the first term in (21) in their examples (Section IV-B-3), which corresponds to the zeroth-order approximation of the covariance. Through simulations in Section IV, we demonstrate the significant improvement due to the second term when the measurement noise is non-isotropic.

Having found the rotated covariances, we present the IEKF algorithm in Algorithm 1. Algorithm 1 follows the standard steps of an EKF except lines 2, 7, and 9 where the covariance matrices are modified, line 5 where the linearized A_k (19) is computed based on the invariant error dynamics (18), and line 11 where the update equation is modified with transformations of the innovation.

Algorithm 1 The IEKF

1: Initialize X_0, P_0 in the original coordinates.
2: $P = W(\theta_0)^\top P_0 W(\theta_0) + P_0^\theta \frac{\partial W}{\partial \theta}^\top P_0 \frac{\partial W}{\partial \theta}$
3: **for** $k = 1$ **to** n **do**
4: $\quad \hat{X}_k^- = f(\hat{X}_{k-1}^+, \mathcal{U})$
5: $\quad$ Compute A_k from (19)
6: $\quad$ Compute H_k from (4)
7: $\quad Q_{rot} = W(\hat{\theta})^\top Q W(\hat{\theta}) + P_{k-1}^\theta \frac{\partial W}{\partial \hat{\theta}}^\top Q \frac{\partial W}{\partial \hat{\theta}}$
8: $\quad P_k^- = A_k P_{k-1}^+ A_k^\top + Q_{rot}$
9: $\quad R_{rot} = T(\hat{\theta}) R T(\hat{\theta})^\top + P_k^\theta \frac{\partial T}{\partial \hat{\theta}} R \frac{\partial T}{\partial \hat{\theta}}^\top$
10: $\quad L_k = P_k^- H_k^\top \left(H_k P_k^- H_k^\top + R_{rot}\right)^{-1}$
11: $\quad \hat{X}_k^+ = \hat{X}_k^- + W(\hat{\theta}) L_k T(\hat{\theta})(Y - h(\hat{X}_k^-, \mathcal{U}))$
12: $\quad P_k^+ = (I - L_k H_k) P_k^-$
13: **end for**

IV. SIMULATIONS

In this section we compare the performance between the EKF and the IEKF. We conduct Monte Carlo simulations of 100 trials. Each simulation represents a robot maneuvering for 6 minutes, with measurements collected at a rate of 10 Hz. We have found that the designed IEKF is most effective when the disturbances are given by a system of the following form:

$$A = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix} \tag{24}$$

$$C = \begin{bmatrix} \bar{C} & 0 \end{bmatrix} \tag{25}$$

$$D = \begin{bmatrix} 0 & \bar{C} \end{bmatrix}. \tag{26}$$

Note that (24)–(26) decouples the disturbances d_x and d_y and assumes that d_x and d_y share the same generating model. In the simulations, we choose

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{27}$$

$$\bar{C} = \begin{bmatrix} 1 & 2 \end{bmatrix}. \tag{28}$$

The robot subject to the disturbance in (24)–(28) experiences sustained oscillations at a frequency of 1 rad/s. The following parameters are used in all the simulations:

$$v = 13 \text{ m/s}$$
$$\omega = 4 \text{ deg/s}$$
$$\mu_0 = \mathbf{0}_{n\times 1}$$
$$P_0 = \mathrm{diag}(10^2, 10^2, (\pi/2)^2, 2^2, 2^2, 2^2, 2^2)$$
$$X_0 \sim \mathcal{N}(\mu_0, P_0)$$
$$\hat{X}_0 = \mu_0.$$

The noise corrupting the measurements satisfies a zero mean, Gaussian distribution with covariance

$$R = \begin{bmatrix} 3^2 & 8 \\ 8 & 3^2 \end{bmatrix}. \tag{29}$$

The covariance of the process noise ν in the model is

$$Q = \begin{bmatrix} 10^{-6} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 10^{-6} \end{bmatrix}. \tag{30}$$

Our metric of performance is the root mean square error (RMSE), calculated at every time step. Let $x_i(t)$ be the ith element of the state X at time t. Then the RMSE of the ith state at each time t is given by

$$RMSE_i(t) = \sqrt{\frac{\sum_{j=1}^{n} (x_i(t) - \hat{x}_i(t))^2}{n}} \tag{31}$$

where n is the number of trials. Since all the states follow similar trends from scenario to scenario, for the sake of brevity we will only include RMSE graphs for x, θ, and d_1.

A. *Effect of Transformed Noises*

We demonstrate the effect of different rotated noise terms on the performance of the IEKF. In the simulation the IEKF is run using 3 different approaches to handling the covariance matrices. The first approach, denoted as '0 term' in Figure 1, does not transform the P, Q, and R matrices, i.e., in Algorithm 1, $P = P_0$, $Q_{rot} = Q$, and $R_{rot} = R$. The second approach, referred to as '1 term', includes only the first term on the right side of (21)–(23), excluding the first derivative terms. This '1 term' approach corresponds to noise covariance used in [11, Section IV-B-3]. Lastly, 'both terms' refers to the case when the transformations in (21)–(23) are used. From Figure 1 we see that using both terms from equations (21)–(23) results in the best transient performance and fastest convergence rate for estimating x, θ, and d_1. The rest of the states all have similar trends.

Figure 2 shows the same comparison for the case when the measurement noise has an isotropic distribution, i.e., zero mean with the off diagonal terms of the covariance in (29) set to zero. From Figure 2, we see no contribution due to including the first term of (21)–(23) and little contribution when the second terms are included. This is because $T(\hat{\theta})$ (12) used in (21) is an element of the special orthogonal group $SO(2)$ and transformations by this group leave matrices of the form described unchanged. However, the rotated covariance terms do not degrade the performance in the isotropic case, either. Since using both rotation terms results in the best performance, this approach ('both terms') will be used next, when comparing the performance of the IEKF with that of the EKF.

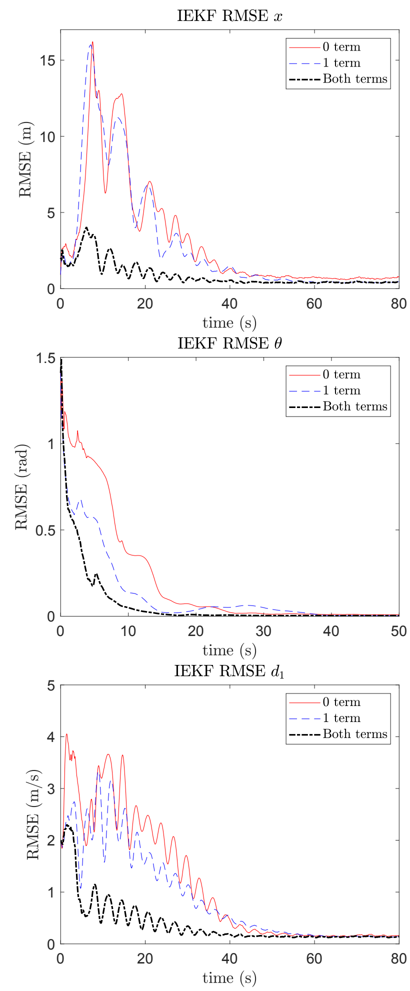

Fig. 1: Effects of different noise covariance rotation terms for the IEKF subjected to the non-isotropic noise whose covariance is given in (29). '0 term' refers to no transformation of the P, Q, or R matrices. '1 term' refers to using just the first terms on the right side of (21)–(23). 'Both terms' refers to using (21)–(23). *Top:* RMSE of x estimate. *Middle:* RMSE of θ estimate. *Bottom:* RMSE of d_1 estimate.

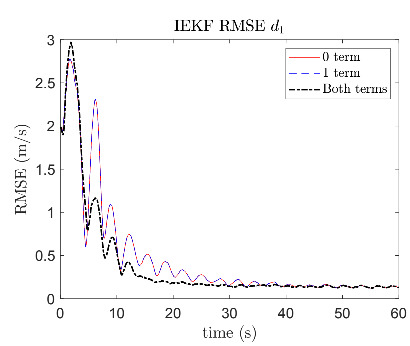

Fig. 2: Effects of different noise covariance rotation terms for the IEKF. The measurement noise is made isotropic by setting the off diagonal terms of the covariance in (29) to zero.

B. EKF/IEKF Comparison

We now compare the performance of x, θ, and d_1 estimation between the EKF and the IEKF. Figure 3 shows that the IEKF yields significantly better estimation accuracy than the EKF during the transient stage. Unlike the EKF, which has large transient oscillations and does not converge until around 150 seconds of simulation time, the error for the IEKF decreases rapidly and converges to the lower bound within 40 seconds.

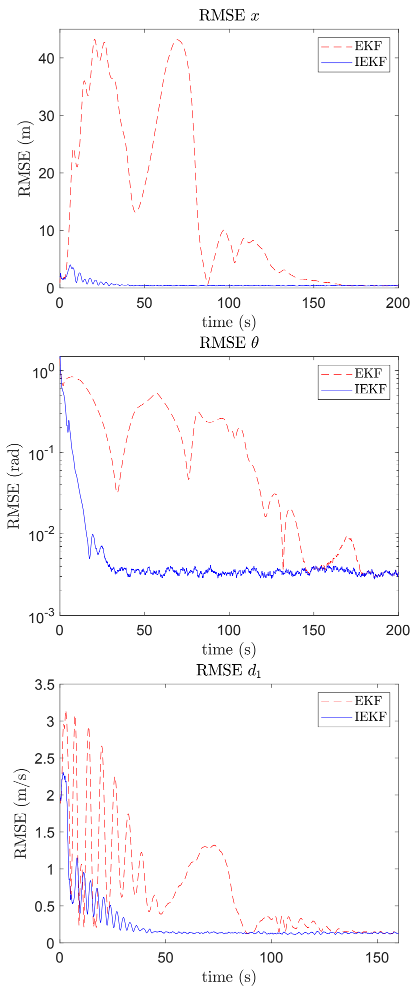

Fig. 3: Comparisons of RMSE of estimates versus truth for the EKF and IEKF. *Top:* RMSE comparison of x estimate. *Middle:* RMSE comparison of θ estimate. *Bottom:* RMSE comparison of d_1 estimate.

V. Conclusion

In this paper we prove that the classic unicycle kinematics, when augmented with disturbances from a linear system, are invariant with respect to $SE(2)$. We take advantage of this invariance property to design an Invariant-EKF that yields better estimation performance than the standard EKF. We propose the first order approximation of the filtering covariance matrices and show that this approximation improves the performance of the IEKF in the presence of non-isotropic measurement noise. Future work includes experimental validation of the designed filter for a small UAV in windy environments.

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