

A dynamic contagion risk model with recovery features ^{*}

Hamed Amini[†]

Andreea Minca[‡]

Agnès Sulem[§]

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Abstract

We introduce threshold growth in the classical threshold contagion model, or equivalently a network of Cramér-Lundberg processes in which nodes have downward jumps when there is a failure of a neighboring node. Choosing the configuration model as underlying graph, we prove fluid limits for the baseline model, as well as extensions to the directed case, state-dependent inter-arrival times and the case of growth driven by upward jumps. We obtain explicit ruin probabilities for the nodes according to their characteristics: initial threshold and in- (and out-) degree. We then allow nodes to choose their connectivity by trading off link benefits and contagion risk. We define a rational equilibrium concept in which nodes choose their connectivity according to an expected failure probability of any given link, and then impose condition that the expected failure probability coincides with the actual failure probability under the optimal connectivity. We show existence of an asymptotic equilibrium as well as convergence of the sequence of equilibria on the finite networks. In particular, our results show that systems with higher overall growth may have higher failure probability in equilibrium.

Keywords: Collective risk theory, systemic risk, default contagion, random graphs, interbank network, insurance-reinsurance networks, financial stability.

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[†]Georgia State University, Department of Risk Management and Insurance, Atlanta, GA 30303, USA, email: hamini@gsu.edu.

[‡]Cornell University, School of Operations Research and Information Engineering, Ithaca, NY, 14850, USA, email: acm299@cornell.edu.

[§]INRIA Paris, 2 rue Simone Iff, CS 42112, 75589 Paris Cedex 12, France, and Université Paris-Est, email: agnes.sulem@inria.fr.

1 Introduction

The random graph approach is a tool for systemic risk modeling when uncertainty stems from missing information on linkages. Such is the case for financial networks, see e.g. [12, 36, 40]. Instead of who is connected to whom, only aggregated information at the level of each node (agent) is available. One can think of these as node characteristics, and examples include capital, asset or liability size, degree of connectivity. Under some degree (distribution) regularity conditions, the random graph approach allows one to establish a limit (when the size of the network is large) of the fraction of agents that fail after an exogenous shock. The assumption is that one can categorize nodes according to some of their characteristics, and within each category, nodes are exchangeable. Along this direction, [6] assume that connectivity of each node is known and that the underlying graph is the configuration model, chosen uniformly over all graphs with the prescribed degree sequence. Their exchangeability assumptions on the linkage weights ensure that a limit exists for the fraction of nodes with an initial threshold to contagion. The final fraction of affected nodes is given in closed form for all values of degrees and initial thresholds.

Using random graphs models requires some caution in the real world application. For instance, networks must be sufficiently large in order for the asymptotic analysis to be relevant. In financial and risk networks, this is true at the level of a large economic zone. Real-world networks may have small cycles, while most existing literature on sparse random graphs (nongeometric models with finite average degree distribution) feature a locally tree-like property when the size of network goes to infinity, e.g., Erdős-Rényi random graph, configuration model, inhomogeneous random graphs, preferential attachment model [56]. However, recent literature shows that the basic configuration model can be extended to incorporate clustering; see e.g., [31, 53, 54, 57]. In light of its tractability and interpretability, as well as its potential to be enriched with clustering, in this paper we use the configuration model as our base model. Moreover, in the case of financial networks, closed form interpretable results based on random graphs can serve as a mandate for regulators to collect data on that specific network and assess contagion via more intensive computational methods. Without such mandate, data collection is only at aggregate level and existing methods for filling in the gaps are entropy methods, likely to understate contagion, or bayesian methods, which are computationally intensive even to generate one network compatible with the aggregated values.

Our main contribution in this paper is to extend the independent threshold contagion on the configuration model to the case when nodes' thresholds receive growth from the linkages. Threshold models of cascades in social network have been used primarily to study the spread of influence, see e.g., [44, 47, 50, 59] and references therein. Because losses from the linkages and growth are intertwined, we call this *the recovery feature* of the threshold. We are motivated by the application to financial and insurance-reinsurance networks. Indeed, in financial networks thresholds represent –depending on the context – either capital or liquidity. An initial set of nodes fail exogenously and affect the nodes connected to them as they default on financial obligations. If those nodes' capital or liquidity is insufficient to absorb the losses, they will fail in turn. In other terms, if the number of failed neighbors reaches a node's threshold, then this node will fail as well, and so on. Since contagion takes time, there is the potential for the capital to recover before the next failure. It is therefore important to introduce a notion of growth.

Networks of insurers and reinsurers are the closest example of what we model here, for several reasons. First, as the recent literature [21, 46, 48] emphasizes, network effects are critical in insurance-reinsurance networks, where reinsurers insure the primary insurers and other reinsurers. Contagion proceeds as failed reinsurers cannot honor contracts to other institutions, and such failures can propagate via chains of reinsurance contracts. Such networks contain several thousand nodes with infrequent reporting, and consequently uncertainty about the reinsurance links, [52].

Therefore asymptotic results on the scope of contagion on a random graph model are appealing. Second, the growth model is particularly adequate, as both primary insurers and reinsurers receive premiums rather deterministically between instances of large losses.

The model we consider in this paper can be seen as a set of Cramér-Lundberg processes living on the nodes of a graph and which interact through the graph links. The capital grows linearly over time. In contrast to the Cramér-Lundberg process, losses do not arrive according to an exogenous Poisson process. Nodes have downward jumps when there is a failure of a neighboring node. When a node's capital or liquidity reaches zero, the node fails and it leads to downward jumps to its own neighbors. The notion of time is also important. Calendar time governs the growth of capital. On the other hand, jumps are governed by the interaction between nodes (specifically between a failed one and one of its neighbors, chosen according to a probability law dictated by the random graph model). There is a natural notion of interaction time and the link revealing filtration. Consequently, jump arrival times have to be translated from interaction time to calendar time. We assume that inter-arrival times are exponentials with mean inversely proportional to the size of the network.

We assume that in each time unit, nodes' growth is proportional to nodes' number of linkages. The linear growth as in the Cramér-Lundberg is also consistent with models in the wider network literature that attribute a fixed reward (respectively cost in some models) to each link as a tradeoff to more contagion risk (respectively network rewards), see [22] and references therein.

Our model allows us to study a connectivity game in which agents choose their degrees. More connectivity leads to more benefits from the links, but also to more exposure to contagion risk. Although stylized, this tradeoff is highly relevant to the last two global crises. In the financial crisis, it was the banks and other financial institutions that became highly interconnected and posed what is now well known as systemic risk, a risk that the financial system collapses and imposes extreme losses in the overall economy. In the current Covid-19 crisis, it is individuals' choices to connect (or rather socially disconnect) to their peers that drives the path of the epidemic.

We provide several extensions and applications of the baseline model.

- First, we propose how to apply this model to an epidemic, whose course is determined by the overall social distancing decision of the agents.
- Second, we allow nodes' in- and out-degrees to differ. Further, we allow the inter-arrival times of interactions to depend on the state of the system, and in particular on the number of failed links (out-going links of failed nodes). This essentially provides a self-exciting process for the interactions on the network: the more failed linkages in the system, the higher the intensity with which counterparties are affected.
- Third, we consider the case when the intensity of the growth process can be prescribed and we provide fluid limits for the number of surviving agents.
- The relation to the Cramér-Lundberg networked risk processes is discussed in Section 5.4. In the classical model, a firm's (insurer's) capital is modeled as

$$C(t) = x + \alpha t - \sum_{k=1}^{\mathcal{N}_{\beta}(t)} L_k, \quad (1)$$

where x denotes the initial capital, L_k are i.i.d. non-negative random variable and $\mathcal{N}_{\beta}(t)$ is a Poisson process with intensity β independent of L_k . The ruin time for the insurer is defined by $\tau_x := \inf\{t \mid C(t) \leq 0\}$, (with the convention that $\inf \emptyset = \infty$) and the central question is to

find the ruin probability $\psi(x) = \mathbb{P}(\tau_x < \infty)$. In our threshold model with linear growth, there is capital growth α on every link. Most importantly, losses are not driven by an independent Poisson process, but by the ruin times of the neighbours. When those losses are constant but their arrival is driven by the neighbours' ruin, our model can be seen as the networked version of (1).

Relation to prior literature. For the Cramér-Lundberg process, many extensions have been proposed and there is an extensive machinery for a variety of first passage problems. However the multi-dimensional case is considerably harder. In [28], authors provide Pollaczek-Khinchine type formula for the transform of ruin probabilities in the two-dimensional case. A risk network with a central branch has been proposed recently in [14]. A particular dependence that allows for tractability is when one of the processes models a central branch and another one a subsidiary, and the jumps from the central branch are driven by bailouts of the subsidiary. This particular two-dimensional case allows for approximations via a reduction to the one-dimensional case. Other explicit computations for the two dimensional case are provided in [15–17]. Recent work in insurance mathematics [20] finds ruin probabilities for risk processes living on bipartite networks, on one side the firms and on the other side the insured objects. Losses are associated to the objects and are divided among firms according to the bipartite network. In contrast to our paper, there is no contagion among firms. In the case of two-dimensional coupled Levy queues, two-dimensional Laplace transform for the "or" ruin probability is given in [26]. Our coupling of the individual risk processes is by the random graph given by the configuration model. This allows us to give explicit results which are asymptotic and thus quite different in nature.

Limiting systemic risk requires new analytical and computational tools. Several previous works point to the link between network structure and financial stability, see e.g., [1–3, 9, 11, 19, 30, 34, 35, 38, 39, 43]. In particular, [6] study financial contagion on configuration model and derive a criterion for the resilience of a financial network to insolvency contagion, based on connectivity and the structure of contagious links (i.e., those exposures of a bank larger than its capital). The model extends the results on bootstrap percolation model (when the thresholds are fixed) in configuration model studied in [4, 5, 18]. Further, [7, 8] studied bootstrap percolation in inhomogeneous random graphs and show that when the degree distribution has infinite variance, then a sublinear number of initially infected vertices is enough to spread the infection over a linear fraction of the vertices of the random graph, with high probability. [32] apply and extend the techniques developed in [7, 8] to determine minimal capital requirements for financial institutions sufficient to make the network resilient to small shocks in the context of inhomogeneous random graphs.

Parallel to the development of network models for systemic risk, a recent series of works [27, 37, 41, 42, 42] introduced a reduced form approach to systemic risk analysis, based on mean field interaction models. Two works on interacting diffusions stand out as highly relevant to our work [49, 51]. The first one, [51] is closest in spirit. They study diffusions that live on the nodes of an infinite (directed, weighted) and complete graph, with weights that depend on the end nodes' types. There is an underlying stochastic kernel on the type set, and which encodes the networks structure of the model. Nodes impact each other via upward and downward jumps prescribed by the kernel when the processes hit certain barriers. Their network, while deterministic, is connected to the inhomogeneous random graph in [24]. Consistently with past results on contagion in inhomogeneous random graphs [7–9], the interacting particle system exhibits large macroscopic jumps (that we can interpret as cascades) that are linked to the largest eigenvalue of a matrix related to the type kernel. The type kernel can be controlled by the agents via a mean-field game. While close in spirit, the interactions in our model live on a random graph that is a finite configuration model. We study

the scaling limit, when the size of this random graph tends to infinity. The process that lives on the graph nodes is not a diffusion, but a Cramér-Lundberg process, with downward jumps at the failure times. Note that our model also features types, namely the initial thresholds, and can be extended to have more categorization, e.g. by having different in- and out-degrees. Agents control here the connectivity in a rational expectations equilibrium for which we prove convergence as the number of agents tends to infinity.

In [49], the authors study the scaling limit of interacting diffusions, where infractions do occur according to a random graph. While remarkably comprehensive, their model is not intended for cascades and there is no mechanism by which "small" jumps can add to macroscopic effects and our results on the size of a dynamic process on the network could not be obtained using their methodology. Indeed, we could not study the (scaled) limit of the stopping time of the contagion by analyzing the graph within a finite number of hops from a root. Indeed, as the size of the graph grows, so does the stopping time of the cascade.

Outline. The paper is organized as follows. In section 2 we give the main results on the scaled limit of failures when the model allows for recovery. We introduce the problem of choosing connectivity optimally in Section 3 and we give the equilibrium solution in the limit, the convergence of the equilibrium in the finite network as well as a numerical analysis of the equilibrium. Proofs of the main result are provided in Section 4. Finally, in Section 5, we give further results on different extensions.

Notations. We let $\mathbb{N}$ be the set of non-negative integers. For non-negative sequences x_n and y_n , we write $x_n = O(y_n)$ if there exist $N \in \mathbb{N}$ and $C > 0$ such that $x_n \leq Cy_n$ for all $n \geq N$, and $x_n = o(y_n)$, if $x_n/y_n \rightarrow 0$, as $n \rightarrow \infty$.

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued random variables on a probability space $(\Omega, \mathbb{P})$. If $c \in \mathbb{R}$ is a constant, we write $X_n \xrightarrow{p} c$ to denote that X_n converges in probability to c . That is, for any $\epsilon > 0$, we have $\mathbb{P}(|X_n - c| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$.

Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers that tends to infinity as $n \rightarrow \infty$. We write $X_n = o_p(a_n)$, if $|X_n|/a_n$ converges to 0 in probability. Additionally, we write $X_n = O_p(a_n)$, to denote that for any positive sequence $\omega(n) \rightarrow \infty$, we have $\mathbb{P}(|X_n|/a_n \geq \omega(n)) = o(1)$. If $\mathcal{E}_n$ is a measurable subset of Ω , for any $n \in \mathbb{N}$, we say that the sequence $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$ occurs with high probability (w.h.p.) if $\mathbb{P}(\mathcal{E}^{(n)}) = 1 - o(1)$, as $n \rightarrow \infty$.

$\text{Bin}(k, p)$ denotes a binomial distribution corresponding to the number of successes of a sequence of k independent Bernoulli trials each having probability of success p .

2 Model and results

2.1 A dynamic threshold model of cascades

We consider a system of n nodes (agents) $[n] = \{1, 2, \dots, n\}$ endowed with a sequence of initial thresholds $(\theta_i)_{i \in [n]}$. To simplify the analysis, we assume that in-degree equals out-degree for each node $i \in [n]$, i.e., $\lambda_i^+ = \lambda_i^- = \lambda_i$. In Section 5.2, we state our main theorem for the case when in- and out-degree are allowed to differ. Given a connectivity λ_i , $i \in [n]$, nodes form links according to the random matching from the configuration model. In the (directed) configuration model with given degree sequence λ_i , $i \in [n]$, each node i is assigned λ_i in-coming half edges and λ_i outgoing half edges. The (multi)graph results from uniform matching of the in-coming half edges and the

outgoing half edges [23, 55]. Although self-loops may occur, these become rare as $n \rightarrow \infty$ (under the assumption that the maximum degree is $o(n)$, see e.g. [55]).

This graph is denoted by $\mathcal{G}^{(n)}$ and we write $(i, j) \in \mathcal{G}^{(n)}$ for the event that there's a link between i and j . We let $\mu_{\lambda, \theta}^{(n)}$ be the fraction of nodes with degree λ and threshold θ ,

$$\mu_{\lambda, \theta}^{(n)} := \frac{\#\{i \in [n] \mid \theta_i = \theta, \lambda_i = \lambda\}}{n}. \quad (2)$$

We assume the following regularity conditions on the degree sequence and thresholds:

(C₁) For every $\lambda, \theta \in \mathbb{N}$ and for some distribution $\mu : \mathbb{N}^2 \rightarrow [0, 1]$, as $n \rightarrow \infty$,

$$\mu_{\lambda, \theta}^{(n)} \rightarrow \mu_{\lambda, \theta}. \quad (3)$$

(C₂) We also assume that the average connectivity converges to a finite limit, as $n \rightarrow \infty$

$$\bar{\lambda}^{(n)} := \sum_{\lambda, \theta} \lambda \mu_{\lambda, \theta}^{(n)} \rightarrow \sum_{\lambda, \theta} \lambda \mu_{\lambda, \theta} =: \bar{\lambda} \in (0, \infty). \quad (4)$$

We let the fraction of nodes with threshold θ be defined as

$$\mu_{\theta}^{(n)} := \sum_{\lambda} \mu_{\lambda, \theta}^{(n)}.$$

Remark 1. We state our results for the random multigraph $\mathcal{G}^{(n)}$ constructed by configuration model. However, they could be transferred by conditioning on the multigraph being a simple graph (without loops and multiple edges). The resulting random graph, denoted by $\mathcal{G}_*^{(n)}$, will be uniformly distributed among all directed graphs with the same degree sequence. In order to transfer the results, we would need to assume that the degree distribution has a finite second moment and (as $n \rightarrow \infty$)

$$\sum_{\lambda=0}^{\infty} \sum_{\theta=0}^{\infty} \lambda^2 \mu_{\lambda, \theta}^{(n)} \rightarrow \sum_{\lambda=0}^{\infty} \sum_{\theta=0}^{\infty} \lambda^2 \mu_{\lambda, \theta} \in (0, \infty),$$

which from [29, 45] implies that the probability of $\mathcal{G}^{(n)}$ being simple is bounded away from zero as $n \rightarrow \infty$. This means that, under the above second moment condition, any property that holds with high probability on the configuration model, also holds with high probability conditional on this random graph being simple (for the random graph $\mathcal{G}_*^{(n)}$). Indeed, under the above assumption, [29, 45] show that $\liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{G}^{(n)} \text{ simple}) > 0$ for the undirected and respectively directed configuration model. Consequently, for any sequence of properties (events) $\{\mathcal{E}^{(n)}\}$ such that $\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{G}^{(n)} \in \mathcal{E}^{(n)}) = 1$, then also $\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{G}_*^{(n)} \in \mathcal{E}^{(n)}) = 1$ since

$$\mathbb{P}(\mathcal{G}_*^{(n)} \notin \mathcal{E}^{(n)}) = \mathbb{P}(\mathcal{G}^{(n)} \notin \mathcal{E}^{(n)} \mid \mathcal{G}^{(n)} \text{ simple}) \leq \frac{\mathbb{P}(\mathcal{G}^{(n)} \notin \mathcal{E}^{(n)})}{\mathbb{P}(\mathcal{G}^{(n)} \text{ simple})} \rightarrow 0 \quad (\text{as } n \rightarrow \infty).$$

However, we conjecture that all results hold for the uniformly chosen random graph $\mathcal{G}_*^{(n)}$ even without the second moment assumption. Indeed, [25] have recently shown results for the size of the giant component from the multigraph case without using the second moment assumption; they prove that even with the (exponentially) small probability that the multigraph is simple, the error probabilities are even smaller.

This network is subject to contagion risk. After the network is formed, a shock occurs. We let $\mathcal{D}_0$ represent an exogenous set of failed nodes. This initial set of failed nodes triggers a failure cascade, as failed nodes affect the nodes connected to them. Whenever a node's threshold is smaller than the number of failed (in-coming) links, i.e., linkages starting from failed nodes, then it fails due to contagion.

Recovery feature. During the cascade processes, there is a growth feature in the whole system. This feature is captured by introducing the global growth rate (per unit time) $\alpha \cdot n$ for the system with n nodes, where $0 \leq \alpha < 1$. We think of $\alpha \cdot n$ is the rate of growth per unit time of the entire capital. We assume this global growth is distributed proportionally to the node's number of links. That is, the threshold of nodes with connectivity λ will grow with rate (per unit time)

$$\alpha \cdot n \cdot \frac{\lambda}{\sum_{\lambda, \theta} \lambda \mu_{\lambda, \theta}^{(n)} n} = \frac{\alpha \lambda}{\bar{\lambda}^{(n)}},$$

where $\sum_{\lambda, \theta} \lambda \mu_{\lambda, \theta}^{(n)} n = n \bar{\lambda}_n$ gives the total number of links in the system.

A very important feature is that only surviving nodes can grow.

We now introduce the dynamic model of contagion. At time 0 nodes in set the $\mathcal{D}_0$ fail exogenously. Each failed link, defined as a link between any node and a failed node, represents an interaction and the number of interactions is always lower than the total number of linkages in the network $\mathcal{G}^{(n)}$. In the dynamic model, we introduce the calendar time and relate it to interaction time. We will study the scaling limit of contagion size and we assume that the total (calendar) time for all interactions is independent of n . Since the number of links scales linearly with n (see (4)) then the average time between interactions must scale with $\frac{1}{n}$. For a system with n nodes, we define $T_k^{(n)}$ the calendar time of the k^{th} interaction and we refer to k as the interaction time.

We assume that the duration in calendar time between the two successive interactions is given by a random variable $\Delta_k^{(n)}$ follows an exponential distribution of parameter n , i.e.,

$$\Delta_k^{(n)} = T_k^{(n)} - T_{k-1}^{(n)} \sim \text{Exp}(n).$$

This reward mechanism in the dynamic case allows the threshold to grow $\frac{\alpha \lambda}{\bar{\lambda}^{(n)}} \Delta_k^{(n)}$ between the two interactions. Further extensions will be discussed in Section 5.

The dynamics of interactions is as follows: links that belong to failed nodes are revealed one by one (initially all such links are unrevealed). At each interaction time, a link belonging to a failed node is revealed¹ and the survival condition of the counterparty node is checked according to its current threshold. If the number of failed links of the counterparty exceeds its current threshold, the node fails and its links become unrevealed failed links. The cascade progresses until there are no more unrevealed failed links. Therefore it stops at most after $n \bar{\lambda}^{(n)}$ interactions.

Formally, we let $S_{\lambda, \theta, \ell}^{(n)}(k)$ be the number of surviving nodes with initial threshold θ , λ outgoing links and ℓ failed (incoming) links at time $T_k^{(n)}$. We have that $\mathbf{S}^{(n)}(k) = \left(S_{\lambda, \theta, \ell}^{(n)}(k) \right)_{\lambda, 0 \leq \ell \leq \lambda, 0 \leq \theta \leq \lambda}$ is a Markov chain and its transitions are given in Section 4.1.

Remark 2 (Threshold at k -th interaction). *It is easy to see that any (surviving) node with λ outgoing links and initial threshold θ will have a threshold*

$$\theta + \alpha \frac{\lambda}{\bar{\lambda}^{(n)}} T_k^{(n)}$$

at the k -th interaction.

¹The choice is uniform among all unrevealed links belonging to failed nodes.

The number of failed nodes among those with connectivity λ and initial threshold θ is then

$$D_{\lambda,\theta}^{(n)}(k) = n\mu_{\lambda,\theta}^{(n)} - \sum_{0 \leq \ell < \lceil \theta + \alpha \frac{\lambda}{\bar{\lambda}^{(n)}} T_k^{(n)} \rceil} S_{\lambda,\theta,\ell}^{(n)}(k).$$

Contagion stops at a time when all failed links have been revealed

$$k_{\text{stop}}^{(n)} = \min\{k = 0, 1, \dots, n\bar{\lambda}^{(n)} : \sum_{\lambda,\theta} \lambda D_{\lambda,\theta}^{(n)}(k) = k\}.$$

We let $\mathcal{D}_f^{(n)}$ the set of failed nodes at the end of the contagion process. The number of failed nodes at the end of the contagion process is thus given by

$$|\mathcal{D}_f^{(n)}| = \sum_{\lambda,\theta} D_{\lambda,\theta}^{(n)}(k_{\text{stop}}^{(n)}).$$

2.2 Limit theorem in the case without growth

In the case without recovery feature, i.e. $\alpha = 0$, the asymptotic fraction of failed nodes is characterized by the following theorem:

Theorem 3 ([6]). *Consider the above threshold contagion process without growth ($\alpha = 0$) on random graph $\mathcal{G}^{(n)}$ satisfying Conditions $(C_1) - (C_2)$. Let π^* be the smallest fixed point of the map J in $[0, 1]$, where*

$$J(\pi) := \sum_{\lambda,\theta} \frac{\lambda \mu_{\lambda,\theta}}{\bar{\lambda}} \cdot B_{\lambda,\theta}(\pi),$$

and,

$$B_{\lambda,\theta}(\pi) := \mathbb{P}(\text{Bin}(\lambda, \pi) \geq \theta) = \sum_{\ell=\theta}^{\lambda} \binom{\lambda}{\ell} \pi^\ell (1-\pi)^{\lambda-\ell}.$$

- (i) *If $\pi^* = 1$, i.e., if $J(\pi) > \pi$ for all $\pi \in [0, 1)$, then asymptotically (as $n \rightarrow \infty$) almost all nodes fail during the cascade.*
- (ii) *If $\pi^* < 1$ and π^* is a stable fixed point of J , i.e., $J'(\pi^*) < 1$, then the final fraction of failed nodes converges in probability to*

$$\frac{|\mathcal{D}_f^{(n)}|}{n} \xrightarrow{p} \sum_{\lambda,\theta} \mu_{\lambda,\theta} B_{\lambda,\theta}(\pi^*). \quad (5)$$

Furthermore, the asymptotic fraction of surviving nodes with degree λ and initial threshold θ converges in probability (as $n \rightarrow \infty$)

$$\frac{S_{\lambda,\theta}^{(n)}(k_{\text{stop}}^{(n)})}{n} \xrightarrow{p} \mu_{\lambda,\theta} (1 - B_{\lambda,\theta}(\pi^*)).$$

Theorem 3 states that in the asymptotic limit the fraction of failed nodes can be described by means of the binomial distribution (which governs the solution on the infinite Galton-Watson tree): given the probability π^* that a link failed, then the probability of failure of a node with connectivity λ and threshold θ is approximated by $B_{\lambda,\theta}(\pi^*)$, the probability of failure *as if* the links failure events were independent. For this reason we will refer to π^* as the global failure probability, which is the probability that a link chosen at random leads to a failed node at the end of the cascade process, in the limit when $n \rightarrow \infty$. The fraction of failed nodes is given by (49). Note that a fixed point as in theorem always exists as we check that $J(0) \geq 0$ and $J(1) \leq 1$ and the function J is continuous.

2.3 Scaled limit of contagion with recovery features

We can now extend Theorem 3 to the case where the global growth rate is $\alpha > 0$. We will prove that a similar convergence result also holds in this case. The proof is more involved because the threshold at any point in time is no longer constant and equal to θ , but grows at a rate α . Nodes that fail during the contagion process will not benefit from the recovery feature, only surviving nodes can grow at any given time. This presents challenges in the description and the analysis of the system.

In the case without growth, it was sufficient at any time to keep track of the number of failed linkages since failure happens when the number of failed linkages reaches the initial threshold. In contrast, here nodes fail at the first time when the number of failed links reaches the initial threshold plus the growth up to that time, so it is insufficient to keep track only of the current number of failed links. We need to keep track of cumulative failed links process (which is an increasing jump process with jump size one). If this process has ever crossed the threshold (with linear growth), then the node has failed, so the failure of a node is a first passage problem.

Remarkably, one can give a heuristic to compute the probability that a node fails based on a notion of average growth and the notion of global failure probability. The rigorous proof is given below as proof to Theorem 5.

Heuristic of node failure probability computation. We introduce $B_{\lambda,\theta}^\alpha(\pi^*)$ as the failure probability of a node with degree λ , threshold θ when the growth rate is α and the global failure probability is π^* . We also introduce $\beta_{\lambda,\theta,\ell}^\alpha(\pi^*)$ as the probability that the nodes with connectivity λ , initial threshold θ and under the growth threshold rate α survive when the global failure probability is π^* and have ℓ failed links at the end of cascade. The quantity $B_{\lambda,\theta}^\alpha(\pi^*)$ can be calculated by:

$$B_{\lambda,\theta}^\alpha(\pi^*) := 1 - \sum_{\ell=0}^{\lambda} \beta_{\lambda,\theta,\ell}^\alpha(\pi^*).$$

By Remark 2 the threshold at the end of the cascade is (approximately)

$$\theta + \alpha \frac{\lambda}{\bar{\lambda}(n)} \frac{k_{\text{stop}}^{(n)}}{n}.$$

As we will see, side result of the proof to Theorem 5 is that

$$\frac{k_{\text{stop}}^{(n)}}{\bar{\lambda}(n)n} \xrightarrow{p} \pi^*.$$

The global failure probability π^* can also be thought of as the duration of the contagion in calendar time: the higher the global failure probability, the longer the contagion lasts. In turn, if the cascade lasts for longer then nodes that survive have also recovered for longer. Heuristically, the final threshold $\theta + \alpha\lambda\pi^*$ is the initial threshold plus the growth. A (necessary) condition for the node to survive is that the number of failed links ℓ does not exceed the final threshold. Hence,

$$\beta_{\lambda,\theta,\ell}^\alpha(\pi^*) = 0 \quad \text{for } \ell \geq \lceil \theta + \alpha\lambda\pi^* \rceil.$$

Note that $B_{\lambda,0}^\alpha(\pi^*) = 1$. This follows by definition, since $\beta_{\lambda,0,\ell}^\alpha(\pi^*) = 0$. Moreover, we can check that $B_{\lambda,\theta}^\alpha(0) = 0$ for $\theta > 0$ (all not initially failed nodes survive). This is due to the following:

- (i) If $\ell = 0$, then $\beta_{\lambda,\theta,\ell}^\alpha(0) = \binom{\lambda}{0} 0^0 (1-0)^\lambda = 1$;

(ii) If $0 < \ell \leq \theta$, then $\beta_{\lambda,\theta,\ell}^\alpha(0) = \binom{\lambda}{\ell} 0^\ell (1-0)^{\lambda-\ell} = 0$;

(iii) If $\ell > \theta$, then $\beta_{\lambda,\theta,\ell}^\alpha(0) = 0$.

In general, when $0 \leq \pi \leq \frac{\ell-\theta}{\alpha\lambda}$, we have $\beta_{\lambda,\theta,\ell}^\alpha(\pi) = 0$ (see Theorem 5 (ii)).

We now proceed to give the heuristic for the computation of β . For the case when the number of failed links ℓ is smaller or equal than the initial threshold θ , then the survival probability $\beta_{\lambda,\theta,\ell}^\alpha(\pi^*)$ is simply the binomial distribution (the probability to have ℓ failed links)

$$\beta_{\lambda,\theta,\ell}^\alpha(\pi^*) = \binom{\lambda}{\ell} (\pi^*)^\ell (1-\pi^*)^{\lambda-\ell} \quad \text{for } \ell \leq \theta,$$

since each link is exposed to failure probability π^* . For the case when ℓ is larger than the initial threshold θ , the calculation of $\beta_{\lambda,\theta,\ell}^\alpha(\pi^*)$ is more involved.

If the number of failed links is $\ell > \theta$, then the node is definitely failed if the growth cannot cover $\ell - \theta$, i.e., if $\alpha\pi^*\lambda \leq \ell - \theta$, which gives

$$\beta_{\lambda,\theta,\ell}^\alpha(\pi) = 0 \quad \text{for } 0 \leq \pi \leq \frac{\ell - \theta}{\alpha\lambda}.$$

In contrast, if $\pi > \frac{\ell-\theta}{\alpha\lambda}$ then we need to make sure that the node survives. Namely, it needs to satisfy the survival condition at the end of contagion and also at each time before. This makes the computation more involved. Remarkably, the solution has a combinatorial representation, that we show in Figure 1. The key point is to identify the critical times when the threshold process could be crossed by the cumulative failed links process. Let t be the current time in the spread of contagion. As we will show, the cascade will end at a time (step) $k_{\text{stop}}^{(n)}$ when all failed linkages have been explored and is related to the global link failure probability π^* by a scaling constant: $t_{\text{stop}} \approx \bar{\lambda}\pi^*$. The longer the contagion lasts the larger the global link failure probability. We note that the threshold process for a node with initial threshold θ and connectivity λ is $\theta + \alpha\frac{\lambda}{\lambda}t$, $t \in [0, t_{\text{stop}}]$. Recall that we are computing the survival probability of such a node, *given* that its final number of failed linkages is ℓ . For every $u \in [\theta + 1, \ell]$ we let

$$t_u = t_{\lambda,\theta,u} := \frac{u - \theta}{\alpha\lambda},$$

the scaled time (real time divided by average degree $\bar{\lambda}$) when the node's threshold process is equal to u and thus the node can withstand up to u failed links at this time. In order to ensure that the node survives, we need to check that the number of failed links at time t_ℓ is lower than ℓ . The first case is when this number is $r \leq \theta$. In this case, we are sure that the cumulative failed links process has not crossed the threshold process at any time $s \in [0, t_{\lambda,\theta,\ell}]$. Therefore the survival probability in this case is simply the probability that there are r failed linkages between 0 and t_ℓ times the probability that there are $\ell - r$ failed linkages between the time t_ℓ and t_{stop} . The proof of Theorem 5 suggests that the failure probability of a linkage in any time interval is proportional to the length of that interval (with the scaling $\frac{1}{\lambda}$).

The second case is when the number of failed links at time t_ℓ is $\theta + u_m$, for some $u_m \in [1, \ell - \theta - 1]$. Then we have a backward recursion by which we determine previous times when we need to check that the crossing has not happened. The previous time when such crossing could have happened is $t_{\theta+u_m}$, since between $t_{\theta+u_m}$ and t_ℓ , the threshold processes is definitely above $\theta + u_m$. Thus we only need to check that the number of failed links at time $t_{\theta+u_m}$ is given by $\theta + u_{m-1}$ for $u_{m-1} < u_m$. By the same reasoning, we need to check at time $t_{\theta+u_{m-1}}$ the number of failed links is given by

$\theta + u_{m-2}$ for a $u_{m-2} < u_{m-1}$ and so on, until at time $t_{\theta+u_1}$ we need to check that the number of failed links is $r \leq \theta$. Then the survival probability is the product of the probabilities that there are $r, \theta + u_1 - r, \dots, u_{m-1} - u_{m-2}, u_m - u_{m-1}$ and $\ell - \theta - u_m$ in the respective time intervals $[0, t_{\theta+u_1}], [t_{\theta+u_1}, t_{\theta+u_2}] \dots, [t_{\theta+u_{m-1}}, t_{\theta+u_m}], [t_{\theta+u_m}, t_\ell]$ and finally $[t_\ell, t_{\text{stop}}]$. It is understood that m is a discrete variable which takes values in $[1, \ell - 1 - \theta]$ (this is the number of times it would be possible to cross the threshold process).

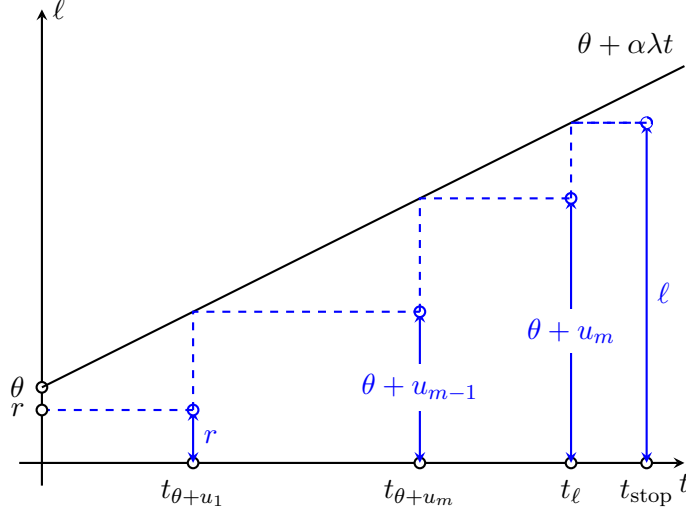


Figure 1: Heuristic of node failure probability computation

It turns out that

$$\begin{aligned} \frac{\beta_{\lambda, \theta, \ell}^\alpha(\pi)}{\binom{\lambda}{\ell} \pi^\ell (1-\pi)^{\lambda-\ell}} &= \sum_{r=0}^{\theta} \sum_{m=1}^{\ell-\theta-1} \sum_{0 < u_1 < \dots < u_m < \ell-\theta} \frac{\ell!}{r! (\theta + u_1 - r)! (u_2 - u_1)! \dots (u_m - u_{m-1})! (\ell - \theta - u_m)!} \\ &\quad \left(\frac{u_1}{\alpha \lambda \pi} \right)^r \left(\frac{u_2 - u_1}{\alpha \lambda \pi} \right)^{\theta + u_1 - r} \left(\frac{u_3 - u_2}{\alpha \lambda \pi} \right)^{u_2 - u_1} \dots \left(\frac{u_m - u_{m-1}}{\alpha \lambda \pi} \right)^{u_{m-1} - u_{m-2}} \\ &\quad \left(\frac{\ell - \theta - u_m}{\alpha \lambda \pi} \right)^{u_m - u_{m-1}} \left(1 - \frac{\ell - \theta}{\alpha \lambda \pi} \right)^{\ell - \theta - u_m}. \end{aligned}$$

Let $U_1^\pi, U_2^\pi, \dots, U_\ell^\pi$ be i.i.d. uniform distribution on $[0, \pi]$ and the order statistics be

$$U_{(1)}^\pi \leq U_{(2)}^\pi \leq \dots \leq U_{(\ell)}^\pi.$$

As the above heuristic arguments suggest, we expect to have

- for $\ell = 0, 1, \dots, \theta$:

$$\beta_{\lambda, \theta, \ell}^\alpha(\pi) = \binom{\lambda}{\ell} \pi^\ell (1-\pi)^{\lambda-\ell}.$$

- for $\ell = \theta + 1, \dots, \lambda$:

$$\beta_{\lambda, \theta, \ell}^\alpha(\pi) = \binom{\lambda}{\ell} \pi^\ell (1-\pi)^{\lambda-\ell} \mathbb{P} \left(U_{(\theta+1)}^\pi > t_{\theta+1}, U_{(\theta+2)}^\pi > t_{\theta+2}, \dots, U_{(\ell)}^\pi > t_\ell \right),$$

where $t_u = t_{\lambda, \theta, u} = \frac{u-\theta}{\alpha\lambda}$ for all $u = \theta + 1, \dots, \lambda$.

Let us denote by

$$P_{\lambda,\theta,\ell}(\pi) := \mathbb{P} \left(U_{(\theta+1)}^\pi > t_{\theta+1}, U_{(\theta+2)}^\pi > t_{\theta+2}, \dots, U_{(\ell)}^\pi > t_\ell \right), \quad (6)$$

for $\ell = \theta + 1, \dots, \lambda$ and $P_{\lambda,\theta,\ell}(\pi) = 1$ for $\ell = 0, 1, \dots, \theta$.

Remark 4. The density of $U_{(\theta+1)}^\pi, U_{(\theta+2)}^\pi, \dots, U_{(\ell)}^\pi$ is given by

$$f(x_{\theta+1}, x_{\theta+2}, \dots, x_\ell) = \frac{\ell! (x_{\theta+1})^\theta}{\theta! \pi^\ell} \mathbb{1}_{\{0 < x_{\theta+1} < x_{\theta+2} < \dots < x_\ell < \pi\}}.$$

With the intuition behind the survival probability computation, we can now turn to finding the global link failure probability π^* . Our theorem below shows that this quantity is the solution to the fixed point equation $\pi = J^\alpha(\pi)$, where the function J^α makes use of the survival probability B^α . To see that this fixed point represents the global link failure probability, let us multiply both sides of the fixed point equation with $n\bar{\lambda}$, which represents the total number of links in the network. Then on the lefthand side we have $\pi^* n\bar{\lambda}$ which represents the expected total number of failed links present in the system at the end of the cascade. The number of failed links at the end of the cascade can be also accounted as the sum of expected failed links across different nodes. Indeed, $\lambda n \mu_{\lambda,\theta} B_{\lambda,\theta}^\alpha(\pi^*)$ gives the expected number of failed links from nodes with threshold θ and connectivities λ , since $B_{\lambda,\theta}^\alpha(\pi^*)$ represents the failure probability of node with threshold θ and connectivities λ while $\lambda n \mu_{\lambda,\theta}$ counts the total number of links belongs to such nodes. Summing up over θ and λ we have that $\sum_{\lambda,\theta} \lambda n \mu_{\lambda,\theta} B_{\lambda,\theta}^\alpha(\pi^*)$ also gives the total number of failed links. The fixed point equation $\pi^* = J^\alpha(\pi^*)$ states that the second way to account for failed links reaches the same value as the first one.

Theorem 5. Consider the above threshold contagion process with growth ($\alpha > 0$) on random graph $\mathcal{G}^{(n)}$ satisfying Conditions $(C_1) - (C_2)$. Let π^* be the relaxed fixed point of the map J^α defined as

$$\pi^* := \min\{\pi \in [0, 1] \mid J^\alpha(\pi) \leq \pi\},$$

where

$$J^\alpha(\pi) := \sum_{\lambda,\theta} \frac{\lambda \mu_{\lambda,\theta}}{\bar{\lambda}} B_{\lambda,\theta}^\alpha(\pi),$$

and for fixed $\alpha, \lambda, \theta, \pi$,

$$B_{\lambda,\theta}^\alpha(\pi) := 1 - \sum_{\ell=0}^{\min\{\lceil \theta + \alpha \lambda \pi \rceil - 1, \lambda\}} \binom{\lambda}{\ell} \pi^\ell (1 - \pi)^{\lambda - \ell} P_{\lambda,\theta,\ell}(\pi).$$

We have for all $\epsilon > 0$, w.h.p.,

$$\left| \frac{\mathcal{D}_f^{(n)}}{n} \right| \geq \sum_{\lambda,\theta} \mu_{\lambda,\theta} B_{\lambda,\theta}^\alpha(\pi^*) - \epsilon.$$

Moreover,

- (i) if $\pi^* = 1$, i.e., if $J^\alpha(\pi) > \pi$ for all $\pi \in [0, 1)$, then asymptotically (as $n \rightarrow \infty$) almost all nodes fail during the cascade.

(ii) if $\pi^* < 1$ and π^* is a stable fixed point of J^α , then the final fraction of failed nodes converges in probability to

$$\frac{|\mathcal{D}_f^{(n)}|}{n} \xrightarrow{p} \sum_{\lambda, \theta} \mu_{\lambda, \theta} B_{\lambda, \theta}^\alpha(\pi^*). \quad (7)$$

Furthermore, the asymptotic fraction of surviving nodes with degree λ and initial threshold θ converges in probability (as $n \rightarrow \infty$)

$$\frac{S_{\lambda, \theta}^{(n)}(k_{\text{stop}}^{(n)})}{n} \xrightarrow{p} \mu_{\lambda, \theta} (1 - B_{\lambda, \theta}^\alpha(\pi^*)).$$

Note that when $\alpha = 0$, we have $B_{\lambda, \theta}^0(\pi) = B_{\lambda, \theta}(\pi)$ and we recover the result of Theorem 3.

Remark 6 (Existence and stability of a fixed point). *The relaxed fixed point always exists, even if J^α is discontinuous, since $J^\alpha(0) \geq 0$ and $J^\alpha(1) \leq 1$. When the function J^α is continuous on $[0, 1]$, the relaxed fixed point is the same as a standard fixed point. We request that π^* is a stable fixed point, i.e., it is not a local minimum of the function $J^\alpha(\pi) - \pi$. The stability condition of the fixed point solution therefore implies that $J^\alpha(\pi) - \pi$ becomes negative after π^* . As we will see, $J^\alpha(\pi) - \pi$ is (asymptotically) the fraction of remaining failed links (for which interaction has not yet taken place). Intuitively, contagion stops when there are no more failed linkages, so when π^* is a stable fixed point, w.h.p contagion stops at time $\bar{\lambda}\pi^* + o(1)$. On the other hand, if π^* is not a stable fixed point, it cannot be guaranteed that there are no remaining failed links and more precise error bounds around π^* would be needed. This case remains an open question.*

The main theorem is established by describing the contagion process using a Markov chain of lower dimension than the initial system, in which we aggregate nodes according to their connectivity, threshold, and number of failed counterparties. From the point of view of the evolution of the cascade, the nodes in the same class are exchangeable. We use the standard machinery of Wormald' differential equation method to prove that the rescaled Markov chain converges in probability to a limit described by a system of ordinary differential equations. However, the application is quite tricky, especially due to complications like the state space increasing with time. This readily gives us the asymptotic fraction of surviving nodes (in each class of connectivity and threshold) at each time of the cascade spread. The stopping time of the cascade $k_{\text{stop}}^{(n)}$ is the first time when there are no more unexplored failed linkages. We can relate the stopping time of the cascade to the global failure risk captured by π^* and in the sequel we will use this quantity to define the nodes' performance criteria and define their connectivity optimization problem.

3 Agents' optimal connectivity choice in equilibrium

In this section we introduce the problem of optimal connectivity choice. In choosing their connectivity, agents (nodes) face the following tradeoff: as they add more connectivity, they increase the risk of contagion. At the same time they derive more rewards from their linkages. In the finance example, one can lend more and earn more interest, but with more connections comes more counterparty default risk. In an epidemic example, that we push further in Section 5.1, more connectivity can mean more social contacts or economic activity, and at the same time more local and global infection risk.

Here we study this trade-off in the simplest form. Surviving agents with connectivity λ receive λ (units of a numéraire) at the end of the cascade. They receive no reward (and later we will allow for potential loss) if they fail. We define the agents' reward as follows.

Definition 7 (Nodes' asymptotic reward). *We define the asymptotic reward for a node with degree λ and threshold θ as the expected benefit of linkages, and it is given in the asymptotic limit by*

$$\lambda(1 - B_{\lambda,\theta}^\alpha(\pi^*)),$$

with π^* given in Theorem 5.

The timeline is as follows. At time 0, each node is given a certain positive threshold (its initial capacity to absorb losses from potential counterparties). The distribution of thresholds is common knowledge. Based on the reward function and its threshold each node chooses its connectivity. The network is then formed according to the configuration model and is subject to an exogenous initial shock. The contagion then ensues starting from the set of exogenous initial failures (chosen with fixed probability among all nodes, for simplicity).

3.1 Asymptotic Nash equilibrium analysis

The global link failure probability π^* depends on the connectivity choice of all agents. Therefore, the optimal connectivity is an outcome of an equilibrium. In a large network of size n , nodes with degree λ and threshold θ compute a reward

$$\lambda \left(1 - \frac{D_{\lambda,\theta}^{(n)}(k_{\text{stop}}^{(n)})}{n} \right) = \lambda(1 - B_{\lambda,\theta}^\alpha(\pi^*) + o_p(1)). \quad (8)$$

We now proceed in two steps to determine this equilibrium with the asymptotic criterion. In the first step, we let nodes choose their connectivity according to an expected failure probability π of a link, i.e., a node with threshold θ chooses a connectivity $\lambda_\theta^*(\pi)$ which satisfies

$$\lambda_\theta^*(\pi) \in \arg \max_{\lambda} \lambda(1 - B_{\lambda,\theta}^\alpha(\pi)). \quad (9)$$

In the second step, we will impose an equilibrium condition that the expected failure probability of a link coincides with the actual failure probability of a link, under the optimal connectivity.

It is understood that all other nodes' connectivities and nodes optimize the same asymptotic criterion in all networks of size n . A finite optimizer exists, and as such it can be used to obtain an equilibrium.

Proposition 8 (Existence). *The optimization problem (43) admits a finite optimizer $\lambda_\theta^*(\pi)$.*

Proof. We define $V(\lambda) := \lambda(1 - B_{\lambda,\theta}^\alpha(\pi))$ which is (upper) bounded by the following quantity:

$$U(\lambda) := \lambda \left(\sum_{\ell \leq \theta + \alpha \lambda \pi} \binom{\lambda}{\ell} (1 - \pi)^{\lambda - \ell} \pi^\ell \right).$$

We recognize here that $1 - B_{\lambda,\theta}^\alpha(\pi)$ is bounded by an Incomplete beta function $I_{1-\pi}(\lambda - \theta + 1, \theta)$. It follows that

$$U(\lambda) = \lambda I_{1-\pi}(\lambda - \theta - \alpha \lambda \pi, \theta + \alpha \lambda \pi + 1).$$

We next recall the following estimates: if $k \leq n\pi$, then

$$I_{1-\pi}(n - k, k + 1) \leq \exp \left(-\frac{(n\pi - k)^2}{2\pi n} \right).$$

Since $\alpha < 1$, there exist λ_0 such that when $\lambda > \lambda_0$, we have $\theta + \alpha\lambda\pi \leq \lambda\pi$. This gives

$$V(\lambda) \leq U(\lambda) \leq \lambda \exp\left(-\frac{(\lambda\pi - \theta - \alpha\lambda\pi)^2}{2\pi\lambda}\right) = \lambda \exp\left(-\frac{(1-\alpha)^2}{2}\lambda\pi + \frac{(1-\alpha)}{2}\theta - \frac{\theta^2}{2\lambda\pi}\right)$$

The righthand side tends to 0 when $\lambda \rightarrow +\infty$. Thus the maximizer exists and is finite. $\square$

Remark 9. *The quantity U gives the value function for the same optimization problem for a modified system in which we allow failed nodes to receive rewards from their linkages and even recover, so it is intuitive that $V(\lambda) \leq U(\lambda)$. We formally check that $V(\lambda) \leq U(\lambda)$ by using the inequality $\beta_{\lambda,\theta,\ell}^\alpha(\pi) \leq \binom{\lambda}{\ell}(1-\pi)^{\lambda-\ell}\pi^\ell$.*

We now impose the equilibrium condition that the anticipated global failure probability coincides with the global failure probability given by Theorem 5 when nodes are at their optimal connectivity.

Definition 10 (Equilibrium). *We call $(\pi^*, (\lambda_\theta^*)_{\theta \geq 0})$ a rational expectations equilibrium if*

- given π^* ,

$$\lambda_\theta^* \in \arg \max_{\lambda} \lambda(1 - B_{\lambda,\theta}^\alpha(\pi^*)), \quad \text{for each } \theta; \quad (10)$$

- π^* is the smallest solution of the fixed point equation :

$$\pi^* = \sum_{\theta} \frac{\lambda_\theta^* \mu_\theta}{\sum_{\theta} \lambda_\theta^* \mu_\theta} B_{\lambda_\theta^*, \theta}^\alpha(\pi^*). \quad (11)$$

The fixed point equation in the equilibrium definition is derived from the fixed point equation in Theorem 5, where the connectivity is set to λ_θ^* and we let $\mu_\theta = \mu_{\lambda_\theta^*, \theta}$. Combining the two conditions above, we see that in equilibrium the failure probability of a link π^* is the smallest fixed point of the map

$$J^\alpha(\pi) := \sum_{\theta} \frac{\lambda_\theta^*(\pi) \mu_\theta}{\sum_{\theta} \lambda_\theta^*(\pi) \mu_\theta} \cdot B_{\lambda_\theta^*(\pi), \theta}^\alpha(\pi), \quad (12)$$

where $\lambda_\theta^*(\pi)$ is defined in (43).

Now we study the existence of the equilibrium.

Proposition 11 (Existence of equilibrium). *When $\lambda_\theta^*(\pi)$ is continuous in $\pi \in [0, 1]$, the function J^α admits at least one fixed point π^* . When $\lambda_\theta^*(\pi)$ is not continuous in π , the map J^α admits a relaxed fixed point π^* defined as*

$$\pi^* := \min\{\pi \in [0, 1] \mid J^\alpha(\pi) \leq \pi\}.$$

For both cases, we let $\lambda_\theta^* = \lambda_\theta^*(\pi^*)$.

Proof. The proof is immediate: when $\lambda_\theta^*(\pi)$ is continuous in $\pi \in [0, 1]$ then the function J^α is continuous on $[0, 1]$. It thus admits at least one fixed point since

$$\lim_{\pi \rightarrow 1} J^\alpha(\pi) \leq \sum_{\theta} \mu_\theta \frac{\lambda_\theta^*(\pi)}{\sum_{\theta} \lambda_\theta^*(\pi) \mu_\theta} = 1$$

and $\lim_{\pi \rightarrow 0} J^\alpha(\pi) \geq 0$. When $\lambda_\theta^*(\pi)$ is not continuous, the relaxed fixed point always exists since $J^\alpha(0) \geq 0$ and $J^\alpha(1) \leq 1$. $\square$

Proposition 12. *The continuity of the map $\pi \rightarrow \lambda_\theta^*(\pi)$ holds under uniqueness of the optimal connectivity $\lambda_\theta^*(\pi)$.*

Proof. Suppose uniqueness holds. For a fixed θ , if $\pi_i \rightarrow \pi$, let

$$\lambda_i = \arg \max_{\lambda} \lambda(1 - B_{\lambda, \theta}^\alpha(\pi_i)).$$

We may assume $\lambda_i \leq K$ for some constant K . Now suppose λ^* is an accumulation point of λ_i , that is, $\lambda_{(i)} \rightarrow \lambda$ over some subsequence. Since $\lambda(1 - B_{\lambda, \theta}^\alpha(\pi_i))$ is continuous in λ and π_i , we have

$$\lambda^* = \arg \max_{\lambda} \lambda(1 - B_{\lambda, \theta}^\alpha(\pi)).$$

In particular, any accumulation point of the sequence π_i is an optimizer. Since the optimizer is unique, the only accumulation point of the sequence λ_i is the optimizer λ^* . Thus we have proved that any subsequence of λ_i has a further subsequence that converges to λ^* , which indicates $\lambda_i \rightarrow \lambda^*$. This gives $\lambda(\pi_i) \rightarrow \lambda(\pi)$, so the map $\pi \rightarrow \lambda_\theta^*(\pi)$ is continuous. $\square$

Remark 13 (Asymptotic Nash equilibrium). *We can relate the equilibrium of Definition 10 to a Nash equilibrium of the following game. Any player with threshold θ_0 , given the connectivity $\{\lambda_\theta\}_{\theta \neq \theta_0}$ of the other players, computes their optimal connectivity*

$$\lambda_{\theta_0}^*(\pi) \in \arg \max_{\lambda} \lambda(1 - B_{\lambda, \theta_0}^\alpha(\pi)),$$

under the constraint that

$$\sum_{\theta \neq \theta_0} \mu_\theta \frac{\lambda_\theta}{\sum_{\theta \neq \theta_0} \mu_\theta \lambda_\theta + \mu_{\theta_0} \lambda} B_{\lambda_\theta, \theta}^\alpha(\pi) + \mu_{\theta_0} \frac{\lambda}{\sum_{\theta \neq \theta_0} \mu_\theta \lambda_\theta + \mu_{\theta_0} \lambda} B_{\lambda, \theta_0}^\alpha(\pi) = \pi.$$

We can rewrite the above constraint as $\pi = J(\{\lambda_\theta^\}_{\theta \neq \theta_0}, \lambda)$ for some function J . Thus, the optimization criterion can be rewritten as :*

$$\lambda_{\theta_0}^* \in \arg \max_{\lambda} \lambda(1 - B_{\lambda, \theta_0}^\alpha(J(\{\lambda_\theta^*\}_{\theta \neq \theta_0}, \lambda))).$$

In this sense, $(\lambda_\theta^, \theta \geq 0)$ is a Nash equilibrium.*

The following theorem establishes that the equilibrium in the network of size n converges to the asymptotic equilibrium.

Theorem 14. *For the network of size n , define a rational expectations equilibrium $(T^{*(n)}, (\lambda_\theta^{*(n)}, \theta \geq 0))$ as follows:*

- Given $T^{*(n)}$

$$\lambda_\theta^{*(n)} \in \arg \max_{\lambda} \lambda \left(1 - \frac{D_{\lambda, \theta}^{(n)}(T^{*(n)})}{n} \right), \quad \text{for each } \theta; \quad (13)$$

- $T^{*(n)}$ is the smallest solution of the fixed point equation :

$$T^{*(n)} = \sum_{\theta} \lambda_\theta^{*(n)} \mu_\theta^{(n)} \cdot D_{\lambda_\theta^{*(n)}, \theta}^{(n)}(T^{*(n)}).$$

Suppose $\lambda_\theta^{*(n)}$ is unique in (13). If we define $\pi^{*(n)} := \frac{T^{*(n)}}{\sum_\theta \lambda_\theta^{*(n)} \mu_{\lambda_\theta^{*(n)}}^{(n)} n}$, then we have (as $n \rightarrow \infty$)

$$\pi^{*(n)} \longrightarrow \pi^* \quad \text{and} \quad \lambda_\theta^{*(n)} \longrightarrow \lambda_\theta^*. \quad (14)$$

Proof. Firstly we notice that if $(T^{*(n)}, (\lambda_\theta^{*(n)}, \theta \geq 0))$ is the standard equilibrium of (13), then $\pi^{*(n)}$ is the fixed point of the map

$$J^{(n)}(\pi) := \sum_\theta \frac{\lambda_\theta^{*(n)}(\pi) \mu_\theta^{(n)}}{\sum_\theta \lambda_\theta^{*(n)}(\pi) \mu_{\lambda_\theta^{*(n)}}^{(n)}} \cdot \frac{D_{\lambda_\theta^{*(n)}, \theta}^{(n)}(\lambda_\theta^{*(n)}(\pi) \mu_\theta^{(n)} n \pi)}{n}, \quad \text{for } \pi \in [0, 1]. \quad (15)$$

By the continuity of $\arg \max$, we have $\lambda_\theta^{*(n)}(\pi) \longrightarrow \lambda_\theta^*(\pi)$ as $n \rightarrow \infty$. By the uniform convergence results (32), we obtain

$$\frac{D_{\lambda_\theta^{*(n)}, \theta}^{(n)}(\lambda_\theta^{*(n)}(\pi) \mu_\theta^{(n)} n \pi)}{n} \xrightarrow{p} B_{\lambda_\theta^*, \theta}^\alpha(\pi).$$

These give $J^{(n)}(\pi) \xrightarrow{p} J(\pi)$ as $n \rightarrow \infty$ for $\pi \in [0, 1]$. Obviously we have $|J^{(n)}(\pi)| \leq 1$ for each n . If we furthermore assume the sequence $\{J^{(n)}\}_{n \in \mathbb{N}}$ is equicontinuous, i.e. for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|J^{(n)}(\pi_1) - J^{(n)}(\pi_2)| < \epsilon \quad (16)$$

whenever $|\pi_1 - \pi_2| < \delta$ for all functions $J^{(n)}$ in the sequence. Then by Arzela-Ascoli theorem, the sequence $\{J^{(n)}\}_{n \in \mathbb{N}}$ contains a uniformly convergent subsequence $\{J^{(n_k)}\}_{k \in \mathbb{N}}$. Now suppose $\{\pi^{(n_k)}\}_{k \in \mathbb{N}}$ are the fixed points of the maps $\{J^{(n_k)}\}_{k \in \mathbb{N}}$. i.e. $J^{(n_k)}(\pi^{(n_k)}) = 0$. Since $\{\pi^{(n_k)}\}_{k \in \mathbb{N}}$ are bounded in $[0, 1]$, π^* is an accumulation point of $\pi^{(n_k)}$, that is, $\pi_{(i)}^{(n_k)} \rightarrow \pi^*$ over some subsequence.

Taking limit of both sides on subsequence, since $J^{(n)}$ is continuous in π , we obtain $J(\pi^*) = 0$. In particular, any accumulation point of the sequence $\pi_{(i)}^{(n_k)}$ is a zero point. Since the zero point is unique, the only accumulation point of the sequence $\pi^{(n_k)}$ is the optimizer π^* . Thus we have proved that any subsequence of $\pi^{(n_k)}$ has a further subsequence that converge to π^* , which implies $\pi^{(n_k)} \rightarrow \pi^*$. $\square$

3.2 Numerical analysis of equilibrium

While the equilibrium connectivity cannot be given in closed form, it can be efficiently investigated numerically as it benefits of the simple closed form equations for the probability of survival and the fixed point equation for the global link failure probability.

We first investigate for regular homogeneous networks how agents choose their connectivity when they all have the same initial threshold θ . The initial failure probability is $p(0) = 0.05$. Figure 2 shows that equilibrium connectivity is non decreasing with the initial threshold and the growth parameter α . However, the monotonicity of the failure probability does not hold neither with θ nor with α . This is a case of moral hazard: the individual agent chooses a higher connectivity because it leads to a higher increase in utility. Then, the overall failure risk increases in the system.

We now assume that nodes' initial thresholds θ are randomly distributed over a given range $[0, \theta_{\max}]$. We keep the mean constant and we change the variance of this distribution. We assume a conditional Gaussian distribution with mean 5 and standard deviation $\sigma \in [1, 5]$ (We then condition on the interval $[0, 30]$ and take the integer part). As we vary σ from 1 to 6 we have more heterogeneity in the initial threshold.

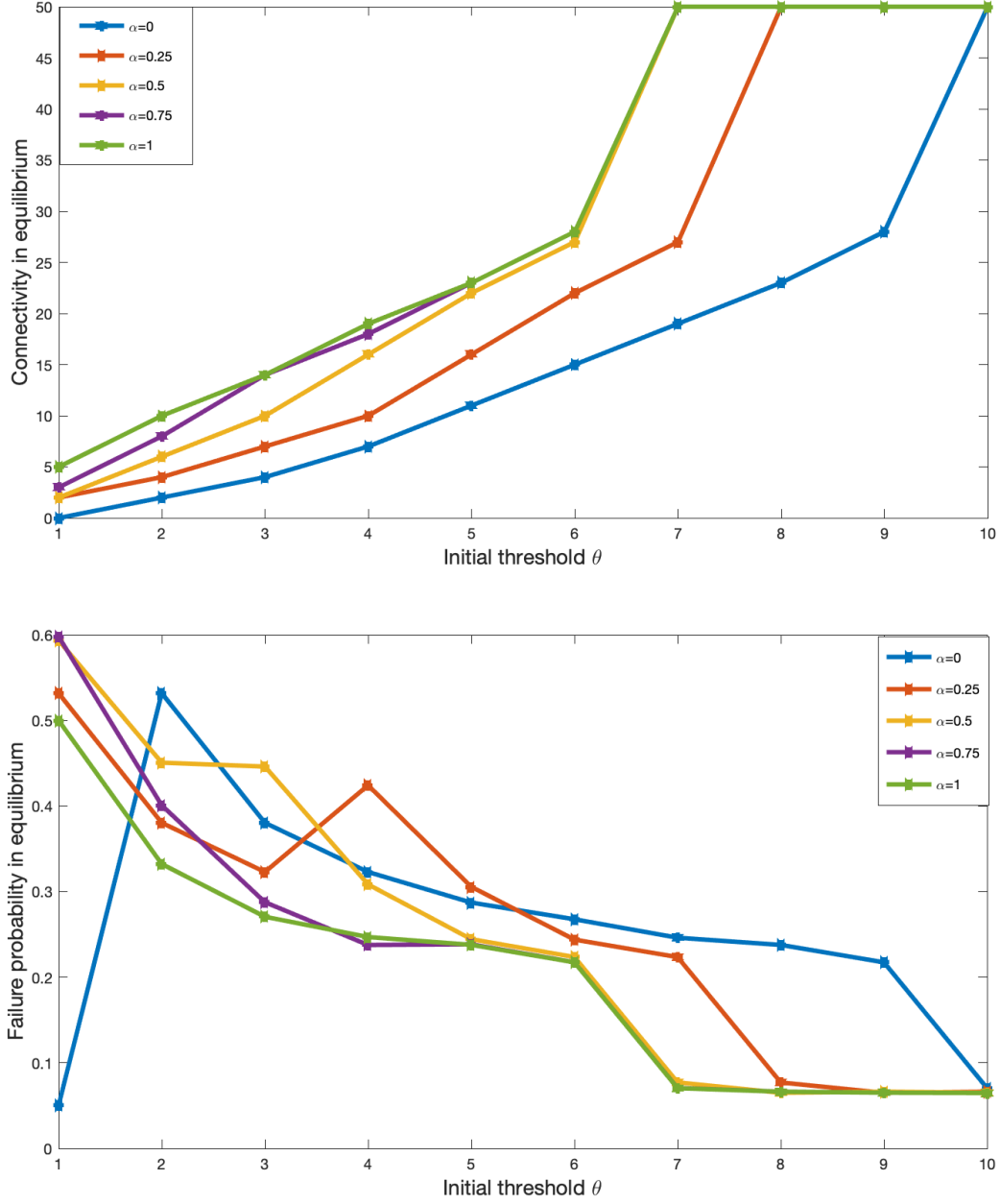


Figure 2: Failure probability and average connectivity in equilibrium with different initial threshold θ for regular networks with initial failure probability $p(0) = 0.05$.

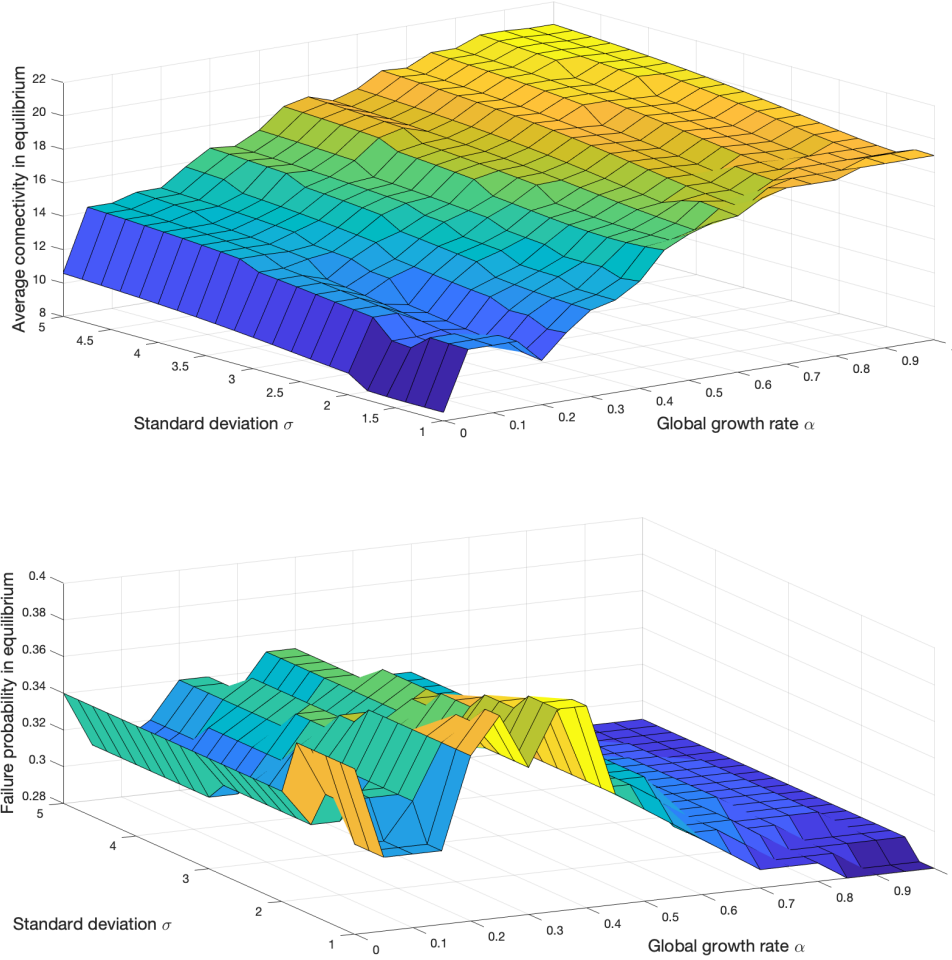


Figure 3: Failure probability and final fraction of failed nodes in equilibrium with different standard deviations and global growth rates. Here the thresholds are normally distributed with mean 5, conditioned to be between 0 and 20. The initial failure probability is $p(0) = 0.05$.

Figure 3 shows that while most of the time, the average connectivity increases with the global growth rate and with the standard deviation, the failure probability is a more irregular and complex function of these variables.

We find that for both cases with medium growth ($\alpha \approx 0.4$) and small growth ($\alpha \approx 0$) the link failure probability in equilibrium decreases as initial thresholds become more heterogeneous (as modeled by larger standard deviation for the initial threshold distribution). As agents become more dissimilar in terms of their thresholds, the system is more diversified and turns out more robust. This is true even as larger standard deviation in the initial distribution of θ gives raise to larger average connectivities in equilibrium.

As we vary the growth rate $\alpha \in [0, 0.4]$, we find that the link failure probability and final fraction of failed nodes in equilibrium both increase as the growth rate increases. This unintuitive result can be explained as follows. When growth increases, agents may engage in over-lending (the equilibrium connectivity λ_θ^* increases). Nodes with higher thresholds can act as stabilizers in the system when they choose a lower connectivity, but this is no longer true if they increase their connectivity as it happens when α increases. The growth effect on the thresholds can be outweighed (for α in small and medium range) by the increase in connectivity throughout the system and especially for the most connected nodes. In this case more instability can ensue as the growth rate increases. As expected, when α is large enough ($\alpha \in [0.4, 1]$), then the growth effect outweighs the effect of larger connectivity.

4 Proofs and asymptotic results

In this section, we present the proof of Theorem 5. In [6], the authors extend the differential equation method of [60] to show that as the network size increases, the rescaled Markov chains that describe the contagion converge in probability to a limit described by a system of ordinary differential equations. The case solved there corresponds to the zero growth case $\alpha = 0$ of this paper. Here, with non-zero growth, the size of the state space also increases with time as the threshold increases. We show similar convergence results to a limit described by a more involved system of ordinary differential equations for the case $\alpha > 0$ and we obtain an analytical result on the final fraction of failed nodes in the network. The convergence result is shown in section 4.3.

4.1 Markov chain transitions

The set of survived nodes in different states $\mathbf{S}_n(k) = \left(S_{\lambda, \theta, \ell}^{(n)}(k) \right)_{\lambda, 0 \leq \theta \leq \lambda, 0 \leq \ell \leq \lambda}$ represents a Markov chain whose transition probabilities are as in [6]. The key difference is that the number of possible solvent states changes with time as nodes' threshold grows, in particular it is possible to have surviving nodes with ℓ failed links for $0 < \ell < \theta + \frac{\alpha}{\lambda^{(n)}} \cdot \lambda \cdot T_k^{(n)}$.

Formally, we let $\Delta_k^{(n)}$ be the difference operator: $\Delta_k^{(n)} S := S^{(n)}(k+1) - S^{(n)}(k)$. We let $\mathcal{F}_k^{(n)}$ be the natural filtration of the Markov chain. We obtain the following equations for the expectation

of $\mathbf{S}_n(k+1)$, conditional on $\mathcal{F}_k^{(n)}$, by averaging over the possible transitions:

$$\begin{aligned}\mathbb{E}[\Delta_k^{(n)} S_{\lambda,\theta,0}^{(n)} | \mathcal{F}_k^{(n)}] &= - \left(\frac{\lambda}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda,\theta,0}^{(n)}(k), \\ \mathbb{E}[\Delta_k^{(n)} S_{\lambda,\theta,\ell}^{(n)}(k) | \mathcal{F}_k^{(n)}] &= \left(\frac{\lambda - \ell + 1}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda,\theta,\ell-1}^{(n)}(k) - \left(\frac{\lambda - \ell}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda,\theta,\ell}^{(n)}(k), \quad 0 < \ell < \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \cdot \lambda \cdot T_k^{(n)}, \\ \mathbb{E}[\Delta_k^{(n)} S_{\lambda,\theta,\ell}^{(n)}(k) | \mathcal{F}_k^{(n)}] &= \left(\frac{\lambda - \ell + 1}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda,\theta,\ell-1}^{(n)}(k), \quad \ell = \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \cdot \lambda \cdot T_k^{(n)}, \\ S_{\lambda,\theta,\ell}^{(n)} &= 0, \quad \ell \geq \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \cdot \lambda \cdot T_k^{(n)}.\end{aligned}$$

The initial condition is

$$S_{\lambda,\theta,\ell}^{(n)}(0) = \mu_{\lambda,\theta}^{(n)} n \mathbf{1}(\ell = 0) \mathbf{1}(0 < \theta \leq \lambda).$$

The last two equations capture the feature that failed nodes will not benefit from the threshold growth. Recall that the number of failed nodes among those with connectivity λ and initial threshold θ is

$$D_{\lambda,\theta}^{(n)}(k) = n \mu_{\lambda,\theta}^{(n)} - \sum_{0 \leq \ell < \lceil \theta + \alpha \frac{\lambda}{\bar{\lambda}^{(n)}} T_k^{(n)} \rceil} S_{\lambda,\theta,\ell}^{(n)}(k),$$

and contagion stops at a time when all failed links have been revealed

$$k_{\text{stop}}^{(n)} = \inf \left\{ k = 0, 1, \dots, n \bar{\lambda}^{(n)} : \sum_{\lambda,\theta} \lambda D_{\lambda,\theta}^{(n)}(k) = k \right\}. \quad (17)$$

Indeed, the number of (alive) outgoing half-edges belonging to failed nodes at time k is

$$D_{\text{out}}^{(n)} = \sum_{\lambda,\theta} \lambda D_{\lambda,\theta}^{(n)}(k) - k$$

and $k_{\text{stop}}^{(n)}$ is the first time that $D_{\text{out}}^{(n)} = 0$. The number of failed nodes at the end of the contagion process is thus given by

$$|\mathcal{D}_f^{(n)}| = \sum_{\lambda,\theta} D_{\lambda,\theta}^{(n)}(k_{\text{stop}}^{(n)}).$$

We introduce (for fixed $\lambda, \theta, \ell \leq \lambda$)

$$\hat{k}_{\lambda,\theta,\ell}^{(n)} := \inf \left\{ k \mid \ell \leq \left(\theta + \frac{\alpha \lambda}{\bar{\lambda}_n} \cdot T_k^{(n)} \right) \right\} \wedge \lambda \quad (18)$$

as the first interaction time when a node starting with threshold θ has accumulated enough growth to withstand $\ell \leq \lambda$ failed links. Clearly, for $\ell \leq \theta$, the node can withstand the failed links using only the initial threshold, so we have $\hat{k}_{\lambda,\theta,\ell}^{(n)} = 0$ for $\ell \leq \theta$.

The above transition probabilities equations can be rewritten as following

$$\begin{cases} \mathbb{E}[\Delta_k^{(n)} S_{\lambda,\theta,0}^{(n)} | \mathcal{F}_k^{(n)}] = - \left(\frac{\lambda}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda,\theta,0}^{(n)}(k), \\ \mathbb{E}[\Delta_k^{(n)} S_{\lambda,\theta,\ell}^{(n)} | \mathcal{F}_k^{(n)}] = \left(\frac{\lambda - \ell + 1}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda,\theta,\ell-1}^{(n)}(k) - \left(\frac{\lambda - \ell}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda,\theta,\ell}^{(n)}(k) & \text{for } k \geq \hat{k}_{\lambda,\theta,\ell}^{(n)}, \\ S_{\lambda,\theta,\ell}^{(n)}(k) = 0 & \text{for } k < \hat{k}_{\lambda,\theta,\ell}^{(n)}, \end{cases} \quad (\text{SDE})$$

with initial condition

$$S_{\lambda,\theta,\ell}^{(n)}(0) = \mu_{\lambda,\theta}^{(n)} n \mathbf{1}(\ell = 0) \mathbf{1}(0 < \theta \leq \lambda).$$

4.2 Fluid limit

We show that the Markov chain admits a fluid limit, namely that rescaled by the size of the network n it is close to the solution of the ordinary differential equations stated below.

We define the following set of differential equations:

$$\begin{cases} \frac{ds_{\lambda,\theta,0}}{dt}(t) = -(\frac{\lambda}{\bar{\lambda}-t})s_{\lambda,\theta,0}(t), \\ \frac{ds_{\lambda,\theta,\ell}}{dt}(t) = (\frac{\lambda-\ell+1}{\bar{\lambda}-t})s_{\lambda,\theta,\ell-1}(t) - (\frac{\lambda-\ell}{\bar{\lambda}-t})s_{\lambda,\theta,\ell}(t) & \text{for } t \geq \hat{t}_{\lambda,\theta,\ell}, \\ s_{\lambda,\theta,\ell}(t) = 0 & \text{for } t < \hat{t}_{\lambda,\theta,\ell}, \end{cases} \quad (\text{DE})$$

with initial conditions

$$s_{\lambda,\theta,\ell}(0) = \mu_{\lambda,\theta} \mathbf{1}(\ell = 0) \mathbf{1}(0 < \theta \leq \lambda),$$

where for $\theta \leq \ell \leq \lambda$,

$$\hat{t}_{\lambda,\theta,\ell} = \frac{(\ell - \theta)\bar{\lambda}}{\alpha\lambda}, \quad (19)$$

and $\hat{t}_{\lambda,\theta,\ell} = 0$ for $0 \leq \ell \leq \theta$.

Lemma 15. *The system of ordinary differential equations (DE) admits the unique solution*

$$\mathbf{s}(t) = (s_{\lambda,\theta,\ell}(t))_{\lambda, 0 \leq \theta \leq \lambda, 0 \leq \ell \leq \lambda}$$

in the interval $0 \leq t < \bar{\lambda}$, with

- for $\ell = 0, 1, \dots, \theta$:

$$s_{\lambda,\theta,\ell}(t) = \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^\ell \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell}. \quad (20)$$

- for $\ell = \theta + 1, \dots, \lambda$:

$$s_{\lambda,\theta,\ell}(t) = \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^\ell \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \mathbb{P}\left(U_{(\theta+1)}^t > \hat{t}_{\lambda,\theta,\theta+1}, U_{(\theta+2)}^t > \hat{t}_{\lambda,\theta,\theta+2}, \dots, U_{(\ell)}^t > \hat{t}_{\lambda,\theta,\ell}\right),$$

where $U_{(1)}^t \leq U_{(2)}^t \leq \dots \leq U_{(\ell)}^t$ denotes the order statistics of ℓ i.i.d. uniformly distributed random variables on $[0, t]$.

The solution of the corresponding DE without growth is given by [6, Lemma 5.8]. Here the solution is significantly more involved, and we have to proceed piecewise and setting the initial condition at successive times $\hat{t}_{\lambda,\theta,\theta+k}$ (the first time when the threshold has grown from θ to $\theta + k$).

Proof. Let $\tau = \tau(t) = -\ln(\bar{\lambda} - t)$. Then $\tau(0) = -\ln(\bar{\lambda})$, τ is strictly increasing and so is the inverse function $t = t(\tau)$. We write the system of differential equations (DE) with respect to τ :

$$\begin{aligned} s'_{\lambda,\theta,0}(\tau) &= -\lambda s_{\lambda,\theta,0}(\tau), \\ s'_{\lambda,\theta,\ell}(\tau) &= (\lambda - \ell + 1)s_{\lambda,\theta,\ell-1}(\tau) - (\lambda - \ell)s_{\lambda,\theta,\ell}(\tau) && \text{for } \tau \geq \hat{\tau}_{\lambda,\theta,\ell}, \\ s_{\lambda,\theta,\ell}(\tau) &= 0 && \text{for } \tau < \hat{\tau}_{\lambda,\theta,\ell}, \end{aligned}$$

where for $\theta + 1 \leq \ell \leq \lambda$,

$$\hat{\tau}_{\lambda,\theta,\ell} = -\ln\left(\bar{\lambda} - \frac{(\ell - \theta)\bar{\lambda}}{\alpha\lambda}\right),$$

and $\hat{\tau}_{\lambda,\theta,\ell} = \tau(0) = -\ln(\bar{\lambda})$ for $0 \leq \ell \leq \theta$. Then we have (for $\ell \geq 1$)

$$\frac{d}{d\tau} \left(e^{(\lambda-\ell)(\tau-\tau(0))} s_{\lambda,\theta,\ell}(\tau) \right) = (\lambda - \ell) e^{(\lambda-\ell)(\tau-\tau(0))} s_{\lambda,\theta,\ell-1}(\tau) \mathbb{1}(\tau \geq \hat{\tau}_{\lambda,\theta,\ell}). \quad (21)$$

(i) For the case $0 \leq \ell \leq \theta$, similar to [6, Lemma 5.8], by induction we find

$$s_{\lambda,\theta,\ell}(\tau) = e^{-(\lambda-\ell)(\tau-\tau(0))} \sum_{r=0}^{\ell-1} \binom{\lambda-r}{\ell-r} \left(1 - e^{-(\tau-\tau(0))}\right)^{\ell-r} s_{\lambda,\theta,r}(\tau(0)). \quad (22)$$

By going back to t , we have

$$s_{\lambda,\theta,\ell}(t) = \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \sum_{r=0}^{\ell-1} s_{\lambda,\theta,r}(0) \binom{\lambda-r}{\ell-r} \left(\frac{t}{\bar{\lambda}}\right)^{\ell-r}.$$

Then, by using the initial conditions, we find (for $\theta > 0$)

$$s_{\lambda,\theta,\ell}(t) = \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^{\ell} \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell}.$$

(ii) Consider the case $\ell = \theta + k$ for $k \geq 1$. We proceed the proof by induction on k . For $k = 1$, when $\ell = \theta + 1$, by Equation 21, we have

$$\frac{d}{d\tau} \left(e^{(\lambda-\theta-1)(\tau-\tau(0))} s_{\lambda,\theta,\theta+1}(\tau) \right) = (\lambda - \theta - 1) e^{(\lambda-\theta-1)(\tau-\tau(0))} s_{\lambda,\theta,\theta}(\tau) \mathbb{1}(\tau \geq \hat{\tau}_{\lambda,\theta,\theta+1}).$$

For $\tau \geq \hat{\tau}_{\lambda,\theta,\theta+1}$, by setting $\hat{\tau}_{\lambda,\theta,\theta+1}$ as the initial time, we have (similar to Equation 22)

$$s_{\lambda,\theta,\theta+1}(\tau) = e^{-(\lambda-\theta-1)(\tau-\hat{\tau}_{\lambda,\theta,\theta+1})} \sum_{r=0}^{\theta} \binom{\lambda-r}{\ell-r} \left(1 - e^{-(\tau-\hat{\tau}_{\lambda,\theta,\ell})}\right)^{\ell-r} s_{\lambda,\theta,r}(\hat{\tau}_{\lambda,\theta,\theta+1}).$$

By going back to t , we find for $\ell = \theta + 1$ and $\hat{t}_{\lambda,\theta,\ell} \leq t < \bar{\lambda}$,

$$\begin{aligned} s_{\lambda,\theta,\ell}(t) &= \left(\frac{\bar{\lambda} - t}{\bar{\lambda} - \hat{t}_{\lambda,\theta,\ell}}\right)^{\lambda-\ell} \sum_{r=0}^{\theta} \binom{\lambda-r}{\ell-r} \left(\frac{t - \hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda} - t}\right)^{\ell-r} s_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}) \\ &= \left(\frac{\bar{\lambda} - t}{\bar{\lambda} - \hat{t}_{\lambda,\theta,\ell}}\right)^{\lambda-\ell} \sum_{r=0}^{\theta} \binom{\lambda-r}{\ell-r} \left(\frac{t - \hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda} - \hat{t}_{\lambda,\theta,\ell}}\right)^{\ell-r} \mu_{\lambda,\theta} \binom{\lambda}{r} \left(\frac{\hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda}}\right)^r \left(\frac{\bar{\lambda} - \hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda}}\right)^{\lambda-r} \\ &= \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^{\ell} \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \frac{1}{t^{\ell}} \sum_{r=0}^{\theta} \binom{\ell}{r} (t - \hat{t}_{\lambda,\theta,\ell})^{\ell-r} \hat{t}_{\lambda,\theta,\ell}^r \\ &= \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^{\ell} \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \frac{t^{\ell} - \hat{t}_{\lambda,\theta,\ell}^{\ell}}{t^{\ell}} \\ &= \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}}\right)^{\ell} \left(1 - \frac{t}{\bar{\lambda}}\right)^{\lambda-\ell} \mathbb{P}\left(U_{(\theta+1)}^t > \hat{t}_{\lambda,\theta,\theta+1}\right), \end{aligned}$$

since (for $\ell = \theta + 1$)

$$\mathbb{P}\left(U_{(\ell)}^t > \hat{t}\right) = 1 - \mathbb{P}\left(U_1^t \leq \hat{t}, U_2^t \leq \hat{t}, \dots, U_{\ell}^t \leq \hat{t}\right) = 1 - \left(\frac{\hat{t}}{t}\right)^{\ell} = \frac{t^{\ell} - \hat{t}^{\ell}}{t^{\ell}}.$$

Suppose now that the statement is true up to $\ell = \theta + k$. We now proceed with the proof for $\ell = \theta + k + 1$. Let us denote by

$$P_{\lambda,\theta,\ell}(t) := \mathbb{P} \left(U_{(\theta+1)}^t > \hat{t}_{\lambda,\theta,\theta+1}, U_{(\theta+2)}^t > \hat{t}_{\lambda,\theta,\theta+2}, \dots, U_{(\ell)}^t > \hat{t}_{\lambda,\theta,\ell} \right), \quad (23)$$

for $\ell = \theta + 1, \dots, \lambda$ and $P_{\lambda,\theta,\ell}(t) = 1$ for $\ell = 0, 1, \dots, \theta$.

Lemma 16. *For all $\ell \geq \theta + 1$, the following recursive identity holds:*

$$P_{\lambda,\theta,\ell}(t) = \frac{1}{t^\ell} \sum_{r=0}^{\ell-1} \binom{\ell}{r} (t - \hat{t}_{\lambda,\theta,\ell})^{\ell-r} \hat{t}_{\lambda,\theta,\ell}^r P_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}).$$

Proof. Let $U_1^t, U_2^t, \dots, U_\ell^t$ be i.i.d. uniformly distributed random variables on $[0, t]$. Then $\mathbb{P}(U_i^t \leq \hat{t}_{\lambda,\theta,\ell}) = \frac{\hat{t}_{\lambda,\theta,\ell}}{t}$. We write $P_{\lambda,\theta,\ell}(t)$ by conditioning on the number of points in $[0, \hat{t}_{\lambda,\theta,\ell}]$. Let $\mathcal{A}_r$ be the event that there are exactly r points in $[0, \hat{t}_{\lambda,\theta,\ell}]$. Then we have (for $r = 0, \dots, \ell - 1$)

$$\mathbb{P}(\mathcal{A}_r) = \binom{\ell}{r} \left(\frac{t - \hat{t}_{\lambda,\theta,\ell}}{t} \right)^{\ell-r} \left(\frac{\hat{t}_{\lambda,\theta,\ell}}{t} \right)^r.$$

Moreover, since $\hat{t}_{\lambda,\theta,\ell}$ is increasing in ℓ and $(U_i^t \mid U_i^t \leq \hat{t}_{\lambda,\theta,\ell})$ is uniformly distributed on $[0, \hat{t}_{\lambda,\theta,\ell}]$, given there are r points in $[0, \hat{t}_{\lambda,\theta,\ell}]$ we have

$$\mathbb{P} \left(U_{(\theta+1)}^t > \hat{t}_{\lambda,\theta,\theta+1}, U_{(\theta+2)}^t > \hat{t}_{\lambda,\theta,\theta+2}, \dots, U_{(\ell)}^t > \hat{t}_{\lambda,\theta,\ell} \mid \mathcal{A}_r \right) = P_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}).$$

Thus we obtain the recursive equation

$$P_{\lambda,\theta,\ell}(t) = \sum_{r=0}^{\ell-1} \binom{\ell}{r} \left(\frac{t - \hat{t}_{\lambda,\theta,\ell}}{t} \right)^{\ell-r} \left(\frac{\hat{t}_{\lambda,\theta,\ell}}{t} \right)^r P_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}),$$

as desired. $\square$

Using the above lemma and by following the same steps as the case $\ell = \theta + 1$, we obtain for $\ell = \theta + k + 1$ and $\hat{t}_{\lambda,\theta,\ell} \leq t < \bar{\lambda}$,

$$\begin{aligned} s_{\lambda,\theta,\ell}(t) &= \left(\frac{\bar{\lambda} - t}{\bar{\lambda} - \hat{t}_{\lambda,\theta,\ell}} \right)^{\lambda-\ell} \sum_{r=0}^{\theta+k} \binom{\lambda-r}{\ell-r} \left(\frac{t - \hat{t}_{\lambda,\theta,\ell}}{\bar{\lambda} - t} \right)^{\ell-r} s_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}) \\ &= \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}} \right)^\ell \left(1 - \frac{t}{\bar{\lambda}} \right)^{\lambda-\ell} \frac{1}{t^\ell} \sum_{r=0}^{\theta+k} \binom{\ell}{r} (t - \hat{t}_{\lambda,\theta,\ell})^{\ell-r} \hat{t}_{\lambda,\theta,\ell}^r P_{\lambda,\theta,r}(\hat{t}_{\lambda,\theta,\ell}) \\ &= \mu_{\lambda,\theta} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}} \right)^\ell \left(1 - \frac{t}{\bar{\lambda}} \right)^{\lambda-\ell} P_{\lambda,\theta,\ell}(t), \end{aligned}$$

which completes the proof. $\square$

A key idea to prove Theorem 5 is to approximate, following [60], the Markov chain by the solution of a system of differential equations in the large network limit. We summarize here the main result of [60].

For a set of variables $Y_1, \dots, Y_b$ and for $\mathcal{D} \subseteq \mathbb{R}^{b+1}$, define the stopping time

$$T_{\mathcal{D}} = T_{\mathcal{D}}(Y_1, \dots, Y_b) = \inf\{t \geq 1, (t/n; Y_1(t)/n, \dots, Y_b(t)/n) \notin \mathcal{D}\}.$$

Lemma 17 ([58, 60]). *Given integers $b, n \geq 1$, a bounded domain $\mathcal{D} \subseteq \mathbb{R}^{b+1}$, functions $(F_L)_{1 \leq \ell \leq b}$ with $f_\ell : \mathcal{D} \rightarrow \mathbb{R}$, and σ -fields $\mathcal{F}_{n,0} \subseteq \mathcal{F}_{n,1} \subseteq \dots$, suppose that the random variables $(Y_\ell^{(n)}(t))_{1 \leq \ell \leq b}$ are $\mathcal{F}_{n,t}$ -measurable for $t \geq 0$. Furthermore, assume that, for all $0 \leq t < T_{\mathcal{D}}$ and $1 \leq \ell \leq b$, the following conditions hold*

- (i) (Boundedness). $\max_{1 \leq \ell \leq b} |Y_\ell^{(n)}(t+1) - Y_\ell^{(n)}(t)| \leq \beta$,
- (ii) (Trend-Lipschitz). $|\mathbb{E}[Y_\ell^{(n)}(t+1) - Y_\ell^{(n)}(t) | \mathcal{F}_{n,t}] - f_\ell(t/n, Y_1^{(n)}(t)/n, \dots, Y_\ell^{(n)}(t)/n)| \leq \delta$, where the function (f_ℓ) is L -Lipschitz-continuous on $\mathcal{D}$,

and that the following condition holds initially:

- (iii) (Initial condition). $\max_{1 \leq \ell \leq b} |Y_\ell^{(n)}(0) - \hat{y}_\ell n| \leq \alpha n$, for some $(0, \hat{y}_1, \dots, \hat{y}_b) \in \mathcal{D}$.

Then there are $R = R(\mathcal{D}, L) \in [1, \infty)$ and $C = C(\mathcal{D}) \in (0, \infty)$ such that, whenever $\alpha \geq \delta \min\{C, L^{-1}\} + R/n$, with probability at least $1 - 2be^{-n\alpha^2/(8C\beta^2)}$ we have

$$\max_{0 \leq t \leq \sigma n} \max_{1 \leq \ell \leq b} |Y_\ell^{(n)}(t) - y_\ell(t/n)n| < 3e^{CL}\alpha n,$$

where $(y_\ell(t))_{1 \leq \ell \leq b}$ is the unique solution to the system of differential equations

$$\frac{dy_\ell(t)}{dt} = f_\ell(t, y_1, \dots, y_b) \quad \text{with} \quad y_\ell(0) = \hat{y}_\ell, \quad \text{for } \ell = 1, \dots, b,$$

and $\sigma = \sigma(\hat{y}_1, \dots, \hat{y}_b) \in [0, C]$ is any choice of $\sigma \geq 0$ with the property that $(t, y_1(t), \dots, y_b(t))$ has ℓ^∞ -distance at least $3e^{LC}\alpha$ from the boundary of $\mathcal{D}$ for all $t \in [0, \sigma)$.

In the next section, we apply Lemma 17 to the contagion model described in Section 4.1. Let us define, for $0 \leq t \leq \bar{\lambda}$

$$\begin{aligned} \delta_{\lambda, \theta}(t) &:= \mu_{\lambda, \theta} - \sum_{\ell} s_{\lambda, \theta, \ell}(t), \\ \delta^-(t) &:= \sum_{\lambda, \theta} \lambda \delta_{\lambda, \theta}(t) - t, \quad \text{and} \\ \delta(t) &:= \sum_{\lambda, \theta} \delta_{\lambda, \theta}(t), \end{aligned}$$

with $s_{\lambda, \theta, \ell}$ given in Lemma 15. Hence, we have

$$\delta_{\lambda, \theta}(t) = \mu_{\lambda, \theta} \left(1 - \sum_{\ell} \binom{\lambda}{\ell} \left(\frac{t}{\bar{\lambda}} \right)^\ell \left(1 - \frac{t}{\bar{\lambda}} \right)^{\lambda - \ell} P_{\lambda, \theta, \ell}(t) \right) = \mu_{\lambda, \theta} \left(1 - B_{\lambda, \theta}^\alpha \left(\frac{t}{\bar{\lambda}} \right) \right), \quad (24)$$

$$\delta^-(t) = \sum_{\lambda, \theta} \lambda \delta_{\lambda, \theta}(t) - t = \bar{\lambda} \left(J^\alpha \left(\frac{t}{\bar{\lambda}} \right) - \frac{t}{\bar{\lambda}} \right), \quad \text{and}, \quad (25)$$

$$\delta(\tau) = \sum_{\lambda, \theta} \mu_{\lambda, \theta} B_{\lambda, \theta}^\alpha \left(\frac{t}{\bar{\lambda}} \right). \quad (26)$$

4.3 Proof of Theorem 5

We now proceed to the proof of Theorem 5 whose aim is to approximate the value $\frac{D^{(n)}(T_{\text{stop}}^{(n)})}{n}$ as $n \rightarrow \infty$. We build on the techniques used in [6, Theorem 3.8]. In contrast to [6], the number of states of the Markov chain grows with time. We first prove the convergence for the rescaled number of nodes that have $\ell < \theta$ failed neighbors, i.e. which are guaranteed to survive. This part of the proof follows from [6, Theorem 3.8]. Next we consider $\ell \geq \theta$ and partition the time interval according to the possibility that growth is sufficient for survival. For every $\ell \geq \theta$ there is a minimal time $\hat{t}_{\lambda, \theta, \ell}$ such that a node with initial threshold θ can survive after $\hat{t}_{\lambda, \theta, \ell}$. As the induction initial step we have convergence on the entire time interval $[0, 1]$ of the rescaled vector $S_{\lambda, \theta, \ell}^{(n)}$ for $\ell < \theta$. As induction step, we show that convergence on the interval $[\hat{t}_{\lambda, \theta, \tilde{\ell}}, 1]$ of $S_{\lambda, \theta, \ell}^{(n)}$ for $\ell < \tilde{\ell}$ implies convergence on the interval $[\hat{t}_{\lambda, \theta, \tilde{\ell}+1}, 1]$ of $S_{\lambda, \theta, \ell}^{(n)}$ for $\ell < \tilde{\ell} + 1$.

Let $k = \gamma \bar{\lambda}^{(n)} n + o(n)$ for some $\gamma \in [0, 1]$. We first show that $T_k^{(n)} \xrightarrow{p} \gamma \bar{\lambda}$. Since

$$\mathbb{E}(T_k^{(n)}) = \sum_{i=1}^k \mathbb{E}(T_i^{(n)} - T_{i-1}^{(n)}) = \sum_{i=1}^k \frac{1}{n} = \frac{k}{n}, \quad (27)$$

and

$$\text{Var}(T_k^{(n)}) = \sum_{i=1}^k \text{Var}(T_i^{(n)} - T_{i-1}^{(n)}) = \sum_{i=1}^k \frac{1}{n^2} = \frac{k}{n^2} \sim O\left(\frac{1}{n}\right), \quad (28)$$

we have by Chebysev's inequality, as $n \rightarrow \infty$, in probability

$$T_k^{(n)} \xrightarrow{p} \gamma \bar{\lambda}.$$

This gives us that (for fixed $\lambda, \theta, \ell \leq \lambda$)

$$\frac{\hat{k}_{\lambda, \theta, \ell}^{(n)}}{n} = \hat{t}_{\lambda, \theta, \ell} + o_p(1). \quad (29)$$

We also need to bound the contribution of higher order terms in the infinite sums (25) and (26). Fix $\epsilon > 0$. By regularity conditions on the degree sequence, we know

$$\bar{\lambda} = \sum_{\lambda} \sum_{\theta=0}^{\lambda} \lambda \mu_{\lambda, \theta} \in (0, \infty).$$

Then, there exists an integer K_ϵ , such that

$$\sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda \mu_{\lambda, \theta} < \epsilon.$$

It follows that for all $0 \leq t \leq \bar{\lambda}$,

$$\sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda \delta_{\lambda, \theta}(t) = \sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda \mu_{\lambda, \theta} \left(1 - B_{\lambda, \theta}^\alpha\left(\frac{t}{\lambda}\right)\right) < \epsilon. \quad (30)$$

The number of vertices with degree λ and initial threshold θ is $n \mu_{\lambda, \theta}^{(n)}$. Again, by regularity conditions,

$$\sum_{\lambda, \theta} \lambda \mu_{\lambda, \theta}^{(n)} \rightarrow \bar{\lambda} \in (0, \infty).$$

Therefore, for n large enough,

$$\sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda \mu_{\lambda, \theta}^{(n)} < \epsilon,$$

which implies for all $0 \leq k \leq n\bar{\lambda}^{(n)}$,

$$\sum_{\lambda \geq K_\epsilon} \sum_{\theta} \lambda D_{\lambda, \theta}^{(n)}(k)/n < \epsilon. \quad (31)$$

For $K \geq 1$, we denote by

$$\begin{aligned} \mathbf{s}^K(t) &:= (s_{\lambda, \theta, \ell}(t))_{\lambda \leq K, 0 \leq \ell, \theta \leq \lambda} \quad \text{and} \\ \mathbf{S}_n^K(k) &:= \left(S_{\lambda, \theta, \ell}^{(n)}(k) \right)_{\lambda \leq K, 0 \leq \ell, \theta \leq \lambda}, \end{aligned}$$

both of dimension $b(K)$.

We now show by induction that

$$\sup_{0 \leq t \leq \sigma n} \left| \mathbf{S}_n^K(k)/n - \mathbf{s}^K(k/n) \right| \leq C\epsilon + o_p(1), \quad (32)$$

where $\sigma = \bar{\lambda} - \epsilon$. As the induction first step, we consider the subvector with $\ell < \theta$.

For $K \geq 1$, we denote this subvector where $\ell < \theta$ by

$$\begin{aligned} \mathbf{s}^{K, \theta}(t) &:= (s_{\lambda, \theta, \ell}(t))_{\lambda \leq K, 0 \leq \ell < \theta \leq \lambda} \quad \text{and} \\ \mathbf{S}_n^{K, \theta}(k) &:= \left(S_{\lambda, \theta, \ell}^{(n)}(k) \right)_{\lambda \leq K, 0 \leq \ell < \theta \leq \lambda}, \end{aligned}$$

where the superscript θ marks the upper bound on ℓ . For an arbitrary constant $\epsilon > 0$, we define the domain $\mathcal{D}_\epsilon^\theta$ as

$$\mathcal{D}_\epsilon^\theta = \{(\tau, s^K) \in \mathbb{R}^{K\theta+1} : -\epsilon < \tau < \bar{\lambda} - \epsilon, -\epsilon < s_{\lambda, \theta, \ell} < 1\}$$

and we verify the conditions of Lemma 17 which shows that the fluid limit holds on the domain $\mathcal{D}_\epsilon^\theta$. We obtain that for a sufficiently large constant C

$$\sup_{0 \leq k \leq \sigma n} \left| \mathbf{S}_n^{K, \theta}(k)/n - \mathbf{s}^{K, \theta}(k/n) \right| \leq C\epsilon + o_p(1)$$

with $\sigma = \bar{\lambda} - \epsilon$.

For $\tilde{\ell} \geq \theta$ we denote the subvector $(s^{K, \tilde{\ell}}, S_n^{K, \tilde{\ell}})_{\lambda \leq K, \theta \leq \lambda, 0 \leq \ell < \tilde{\ell}}$ with index $\ell < \tilde{\ell}$ by

$$\begin{aligned} \mathbf{s}^{K, \tilde{\ell}}(t) &:= (s_{\lambda, \theta, \ell}(t))_{\lambda \leq K, \theta \leq \lambda, 0 \leq \ell < \tilde{\ell}} \quad \text{and} \\ \mathbf{S}_n^{K, \tilde{\ell}}(k) &:= \left(S_{\lambda, \theta, \ell}^{(n)}(k) \right)_{\lambda \leq K, \theta \leq \lambda, 0 \leq \ell < \tilde{\ell}}. \end{aligned}$$

Now consider on the domain

$$\mathcal{D}_\epsilon^{\tilde{\ell}} = \{(t, s^K) \in \mathbb{R}^{K\tilde{\ell}+1} : -\epsilon < t < \bar{\lambda} - \epsilon, -\epsilon < s_{\lambda, \theta, \ell} < 1\}.$$

We prove by induction on $\tilde{\ell}$ that the fluid limit holds on the domain $\mathcal{D}_\epsilon^{\tilde{\ell}}$

$$\sup_{0 \leq k \leq \sigma n} \left| \mathbf{S}_n^{K, \tilde{\ell}}(k)/n - \mathbf{s}^{K, \tilde{\ell}}(k/n) \right| \leq C\tilde{\ell}\epsilon + o_p(1). \quad (33)$$

We have seen that the fluid limit holds for $\tilde{\ell} \leq \theta - 1$, so we use this as the initial induction step. We now suppose that it holds for an $\tilde{\ell} \geq \theta - 1$ and we need to show that it holds for $\tilde{\ell} + 1$. We consider the ODEs solution $\tilde{\mathbf{s}}^{K,\theta} := (\tilde{s}_{\lambda,\theta,\ell}(t))_{\lambda < K, \theta \leq \lambda, 0 \leq \ell < \tilde{\ell}+1}$ on the domain

$$\mathcal{D}_\epsilon^{\tilde{\ell}+1} = \{(t, s^K) \in \mathbb{R}^{K(\tilde{\ell}+1)+1} : -\epsilon \leq t < \bar{\lambda} - \epsilon, -\epsilon < s_{\lambda,\theta,\ell} < 1\},$$

with initial condition $\tilde{\mathbf{s}}_n^{K,\tilde{\ell}+1}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) = (\mathbf{S}_n^{K,\tilde{\ell}}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n), 0)$; namely we start the same system from the time $\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n$. At this time it is guaranteed that $S_n^{K,\tilde{\ell}+1} = 0$.

Then from Lemma 17, we have

$$\sup_{\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)} \leq k \leq \sigma n} \left| \mathbf{S}_n^{K,\tilde{\ell}+1}(k)/n - \tilde{\mathbf{s}}^{K,\tilde{\ell}+1}(k/n) \right| \leq C\epsilon + o_p(1). \quad (34)$$

By the induction hypothesis, namely (33), we have

$$\left| \mathbf{S}_n^{K,\tilde{\ell}}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) - \mathbf{s}^{K,\tilde{\ell}}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) \right| \leq C^{\tilde{\ell}}\epsilon + o_p(1),$$

and using that gives

$$\left| \tilde{\mathbf{s}}^{K,\tilde{\ell}}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) - \mathbf{s}^{K,\tilde{\ell}}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) \right| \leq C^{\tilde{\ell}}\epsilon + o_p(1). \quad (35)$$

By definition

$$s_{\lambda,\theta,\tilde{\ell}+1}(t) = 0 \text{ for } t \leq \hat{t}_{\lambda,\theta,\tilde{\ell}+1}$$

Thus by continuity property of ODEs,

$$\left| \tilde{s}_{\lambda,\theta,\tilde{\ell}+1}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) - s_{\lambda,\theta,\tilde{\ell}+1}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) \right| \leq C\epsilon + o_p(1).$$

Combining with (35) we obtain

$$\left| \tilde{\mathbf{s}}^{K,\tilde{\ell}+1}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) - \mathbf{s}^{K,\tilde{\ell}+1}(\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}/n) \right| \leq C_1\epsilon + o_p(1).$$

Thus by the stability results of ODEs we have

$$\sup_{\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)} \leq k \leq \sigma n} \left| \mathbf{S}_n^{K,\tilde{\ell}+1}(k/n) - \tilde{\mathbf{s}}^{K,\tilde{\ell}+1}(k/n) \right| \leq C_2\epsilon + o_p(1)$$

Combined with (34) this gives

$$\sup_{\hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)} \leq k \leq \sigma n} \left| \mathbf{S}_n^{K,\tilde{\ell}+1}(k)/n - \mathbf{s}^{K,\tilde{\ell}+1}(k/n) \right| \leq (C_2 + C)\epsilon + o_p(1) \quad (36)$$

By definition

$$S_{\lambda,\theta,\tilde{\ell}+1}^{(n)}(k) = 0 \text{ for } k \leq \hat{k}_{\lambda,\theta,\tilde{\ell}+1}^{(n)}$$

and

$$s_{\lambda,\theta,\tilde{\ell}+1}(t) = 0 \text{ for } t \leq \hat{t}_{\lambda,\theta,\tilde{\ell}+1}.$$

Combining with (33), it gives

$$\sup_{0 \leq k \leq \min\{\hat{t}_{\lambda, \theta, \tilde{\ell}+1} n, \hat{k}_{\lambda, \theta, \tilde{\ell}+1}^{(n)}\}} \left| \mathbf{S}_n^{K, \tilde{\ell}+1}(k)/n - \mathbf{s}^{K, \tilde{\ell}+1}(k/n) \right| = 0. \quad (37)$$

Thus by the continuity of solution $(s_{\lambda, \theta, \tilde{\ell}+1}(t))_{t \geq 0}$ between $\min\{\hat{t}_{\lambda, \theta, \tilde{\ell}+1} n, \hat{k}_{\lambda, \theta, \tilde{\ell}+1}^{(n)}\}$ and $\hat{k}_{\lambda, \theta, \tilde{\ell}+1}^{(n)}$, combining with (33), (36) and (37), we obtain

$$\sup_{0 \leq k \leq \sigma n} \left| \mathbf{S}_n^{K, \tilde{\ell}+1}(k)/n - \mathbf{s}^{K, \tilde{\ell}+1}(k/n) \right| \leq (C_2 + C)\epsilon + o_p(1).$$

Thus by mathematical induction, (32) holds.

When the solution reaches the boundary of $\mathcal{D}_\epsilon$, it violates the first constraint, determined by $\sigma = \bar{\lambda} - \epsilon$. By convergence of $\bar{\lambda}^{(n)}$ to λ , there is a value n_0 such that $\forall n \geq n_0$, $\bar{\lambda}^{(n)} > \lambda - \epsilon$, which ensures that $\sigma n \leq \bar{\lambda}^{(n)} n$. Using (30) and (31), we have, for $0 \leq k \leq \sigma n$ and $n \geq n_0$:

$$\begin{aligned} \left| D_{\text{out}}^{(n)}(k)/n - \delta^-(k/n) \right| &= \left| \sum_{\lambda} \sum_{\theta \leq \lambda} \lambda (D_{\lambda, \theta}^{(n)}(t)/n - \delta_{\lambda, \theta}(t/n)) \right| \\ &\leq \sum_{\lambda} \sum_{\theta \leq \lambda} \lambda \left| D_{\lambda, \theta}^{(n)}(t)/n - \delta_{\lambda, \theta}(t/n) \right| \\ &\leq \sum_{\lambda \leq K_\epsilon} \sum_{\theta \leq \lambda} \lambda \left| D_{\lambda, \theta}^{(n)}(t)/n - \delta_{\lambda, \theta}(t/n) \right| + 2\epsilon, \end{aligned} \quad (38)$$

and

$$\left| D^{(n)}(k)/n - \delta(k/n) \right| \leq \sum_{\lambda \leq K_\epsilon} \sum_{\theta \leq \lambda} \left| D_{\lambda, \theta}^{(n)}(k)/n - \delta_{\lambda, \theta}(k/n) \right| + 2\epsilon. \quad (39)$$

We then obtain by (32) that

$$\sup_{0 \leq k \leq \sigma n} \left| D_{\text{out}}^{(n)}(k)/n - \delta^-(k/n) \right| \leq 2C\epsilon + o_p(1), \text{ and} \quad (40)$$

$$\sup_{0 \leq k \leq \sigma n} \left| D^{(n)}(k)/n - \delta(k/n) \right| \leq 2C\epsilon + o_p(1). \quad (41)$$

We now study the stopping time $k_{\text{stop}}^{(n)}$ defined in (50) and the size of the default cascade $D^{(n)}(k_{\text{stop}}^{(n)})$. First assume $J^\alpha(\pi) > \pi$ for all $\pi \in [0, 1)$, i.e., $\pi^* = 1$. Then we have for all $t < \sigma$,

$$\delta^-(t) = \sum_{\lambda, \theta} \lambda \delta_{\lambda, \theta}(t) - t > 0.$$

We have then that $k_{\text{stop}}^{(n)}/n = \sigma + O(\epsilon) + o_p(1)$ and from convergence (41), since $\delta(\sigma) = 1 - O(\epsilon)$, we obtain by tending ϵ to 0 that $|D^{(n)}(k_{\text{stop}}^{(n)})| = n - o_p(n)$. This proves the first part of the theorem.

Now consider the case $\pi^* < 1$, and furthermore π^* is a stable fixed point of $J^\alpha(\pi)$. Then by definition of π^* and by using the fact that $J^\alpha(1) \leq 1$, we have $J^\alpha(\pi) < \pi$ for some interval $(\pi^*, \pi^* + \tilde{\pi})$. Then $\delta^-(t)$ is negative in an interval $(t^*, t^* + \tau)$, with $t^* = \bar{\lambda}\pi^*$.

Let ϵ such that $2\epsilon < -\inf_{t \in (t^*, t^* + \tau)} \delta^-(t)$ and denote $\hat{\sigma}$ the first iteration at which it reaches the minimum. Since $\delta^-(\hat{\sigma}) < -2\epsilon$ it follows that with high probability $D_{\text{out}}^{(n)}(\hat{\sigma}n)/n < 0$, so $k_{\text{stop}}^{(n)}/n = t^* + O(\epsilon) + o_p(1)$. The conclusion follows by taking the limit $\epsilon \rightarrow 0$.

5 Applications, extensions and open questions

In this section we propose several extensions to our baseline model and define some open questions.

5.1 Agents optimal connectivity choice during a pandemic

With the current Covid-19 health crisis underway, it becomes apparent that the control of the epidemic hinges on the decisions of all individuals. The basic trade-off underlying the agents' connectivity decision in Section 3 can be reinterpreted: more connectivity brings the benefits of social interaction, and at the same time increase the infection risk. In recent work [10], we model the benefit of social interaction as a fraction of the Value of Statistical Live Year (VSLY)². If the individual is infected, then they will face a loss. In the case of Covid-19 or a deadly infectious disease, that loss can be modeled as the Value of Statistical Life (VSL). Of course, these values are heterogeneous across the population, for example they clearly depend on age. In [10], we consider a susceptible-infected-recovered (SIR) epidemic and the individual chooses a social distancing level. The connectivity is fixed there and the social distancing level modifies the infection probability on each link. In the following, we suggest how to extend our model of Section 3 in order to investigate the optimal connectivity decision of agents during an epidemic.

Assume that the agents have heterogeneous loss in case they are infected. Let ℓ_i denote the loss for agent i in case he is infected by the end of the contagion process. For example, ℓ_i can incorporate the death (or severe illness) rate given infection and VSL. The distribution of agents' losses is denoted by F_ℓ . We assume that the threshold distribution for an agent choosing degree λ follows $p(\theta; \lambda)$ for $\theta = 0, 1, \dots, \lambda$. In the SIR epidemic (or contact process) model, if we denote by γ the infection probability over each link and γ_0 as the initial infection probability, we can easily express

$$p(0; \lambda) = \gamma_0, \quad p(\theta; \lambda) = (1 - \gamma_0)\gamma(1 - \gamma)^{\theta-1} \quad \text{for } \theta = 1, 2, \dots, \lambda.$$

In this case, we can simply set the growth rate $\alpha = 0$. We will nonetheless give the mathematical formulation for all α .

We restrict our analysis by assuming that the threshold distribution is (stochastically) decreasing in the connectivity λ , i.e., higher connectivity gives higher payoff but comes along with increased the risk of infection (failure). Equivalently, $p(0; \lambda) + p(1; \lambda) + \dots + p(\theta; \lambda)$ is an increasing function of λ for all $\theta \in \mathbb{N}$. Note that for the contact process this assumption is always satisfied because

$$p(0; \lambda) + p(1; \lambda) + \dots + p(\theta; \lambda) = \gamma_0 + (1 - \gamma_0)(1 - (1 - \gamma)^\theta),$$

is increasing in γ for all $\theta = 0, 1, \dots, \lambda$.

The infection probability for an individual choosing connectivity λ (when neighbors are infected independently with probability π) is given by $B_\lambda^\alpha(\pi) := \sum_{\theta=0}^{\infty} p(\theta; \lambda) B_{\lambda, \theta}^\alpha(\pi)$ for all $\pi \in [0, 1]$ and $\lambda \in \mathbb{N}_0$. We write the asymptotic reward for agent i with degree λ as the expected benefit of linkages (normalized to one for each link) minus the expected loss (in case the agent fails) by

$$\lambda(1 - B_\lambda^\alpha(\pi^*)) - \ell_i B_\lambda^\alpha(\pi^*), \tag{42}$$

with π^* is given in Theorem 5.

Note that the global failure probability of a link π^* depends on the connectivity choice of all nodes. Therefore, the optimal connectivity is an outcome of the equilibrium, similarly defined as

²Value of Statistical Life (VSL) and Value of Statistical Life Year (VSLY) are estimated on the basis of how much an individual would be willing to pay in order to decrease death risk by a small percentage point.

in Section 3. We proceed again in two steps to determine this equilibrium with the asymptotic criterion. Let π_e denote the expected infection probability via a randomly chosen link. So agent i with loss ℓ_i computes a reward

$$\lambda(1 - B_\lambda^\alpha(\pi_e)) - \ell_i B_\lambda^\alpha(\pi_e).$$

In the first step, nodes choose their connectivity according to an expected infection probability π of a link, i.e., a node with threshold θ and facing loss ℓ (in case infected) chooses a connectivity

$$\lambda^*(\pi; \ell) \in \arg \max_{\lambda \in \mathbb{N}_0} \{ \lambda(1 - B_\lambda^\alpha(\pi)) - \ell B_\lambda^\alpha(\pi) \}. \quad (43)$$

In the second step, we impose an equilibrium condition that the expected infection probability via a link coincides with the actual infection probability via a link, under the optimal connectivity.

A representative agent with loss ℓ prefers connectivity λ_0 over λ_1 with $\lambda_0 < \lambda_1$ for a given infection probability π if and only if

$$\ell > \frac{(\lambda_1 - \lambda_0) - (\lambda_1 B_{\lambda_1}^\alpha(\pi) - \lambda_0 B_{\lambda_0}^\alpha(\pi))}{B_{\lambda_1}^\alpha(\pi) - B_{\lambda_0}^\alpha(\pi)}.$$

For a given $\pi \in [0, 1]$ and $\lambda \in [0, \lambda_{\max}]$, let us denote by $\mathcal{A}_\lambda(\pi)$ the set of all loss values $\ell \in \mathbb{R}^+$ such that $\lambda \in [0, \lambda_{\max}]$ is the optimal connectivity for agent with loss ℓ and global infection probability π . Considering the contact process with infection probability γ and $\alpha = 0$, Theorem 5 implies that

$$B_\lambda^0(\pi) = 1 - (1 - \gamma_0)(1 - \gamma\pi)^\lambda \quad (44)$$

and π_* is the smallest solution in $[0, 1]$ to the equation

$$\pi = 1 - (1 - \gamma_0) \sum_{\lambda} \mu_\lambda(\pi) (1 - \gamma\pi)^\lambda. \quad (45)$$

Here the fraction of individuals who choose connectivity λ (under infection probability π) is precisely the fraction of individuals whose loss is in the set $\mathcal{A}_\lambda(\pi)$ and has therefore the very simple form

$$\mu_\lambda(\pi) = F_\ell(\mathcal{A}_\lambda(\pi)). \quad (46)$$

5.2 Different in/out-degree distribution

So far we have considered the baseline case where nodes have the same in- and out-degree (that we call connectivity) and where the relation between the interaction time and the calendar time is not dependent on the state of the system. In this section we show that both these assumptions can be relaxed to yield more realism to the model.

In the following, we consider the case when in- and out-degrees are allowed to differ. The in-degree is a channel by which one node is impacted by its neighbors, whereas the out-degree is the channel by which the node impacts others in the case of distress. Growth depends mainly on the in-degree: the higher the in-degree, the more a bank is exposed to others and in return it receives higher interest or fees. In particular, for a reinsurer its capital grows with the number of firms that it provides reinsurance to, as they pay premiums. However, growth can also be a function of the out-degree: banks' leverage and growth depends on how much debt they issue; reinsurers' business growth depends on how many other firms reinsure them.

We provide the asymptotic limit of the survival probability (under growth benefits). We leave it for future research the analysis of an equilibrium in which both in- and out-degrees are bank choices. Such equilibrium can be defined as in the baseline case, but with the additional constraint that average in-degree matches average out-degree.

Let $\mu_{\lambda_+, \lambda_-, \theta}^{(n)}$ be the fraction of nodes with in-degree λ_+ , out-degree λ_- and threshold θ . Assume the following regularity conditions $\mu_{\lambda_+, \lambda_-, \theta}^{(n)} \rightarrow \mu_{\lambda_+, \lambda_-, \theta}$, as $n \rightarrow \infty$, for some distribution $\mu : \mathbb{N}^3 \rightarrow [0, 1]$. We also assume again that the average connectivity converges to a finite limit

$$\bar{\lambda}^{(n)} := \sum_{\lambda_+, \lambda_-, \theta} \lambda_+ \mu_{\lambda_+, \lambda_-, \theta}^{(n)} = \sum_{\lambda_+, \lambda_-, \theta} \lambda_- \mu_{\lambda_+, \lambda_-, \theta}^{(n)} \rightarrow \sum_{\lambda_+, \lambda_-, \theta} \lambda_+ \mu_{\lambda_+, \lambda_-, \theta} =: \bar{\lambda} \in (0, \infty). \quad (47)$$

Suppose that growth benefits arrive uniformly over time according to the “growth parameter” α and both the in- and out-degrees. Given a growth function g , $g(\alpha, \lambda_+, \lambda_-)$, one can define similarly to (19) the minimal time when the node could survive ℓ failed neighbors

$$\hat{t}_{\lambda_+, \lambda_-, \theta, \ell} = \frac{(\ell - \theta) \bar{\lambda}}{g(\alpha, \lambda_+, \lambda_-)}. \quad (48)$$

We let $P_{\lambda_+, \lambda_-, \theta, \ell}(\pi)$ as in (23) but with the new definition of $\hat{t}$.

Theorem 18. *Let π^* be the relaxed fixed point of the map J^α defined as*

$$\pi^* := \min\{\pi \in [0, 1] \mid J^\alpha(\pi) \leq \pi\},$$

where

$$J^\alpha(\pi) := \sum_{\lambda_+, \lambda_-, \theta} \frac{\lambda_- \mu_{\lambda_+, \lambda_-, \theta}}{\bar{\lambda}} \cdot B_{\lambda_+, \lambda_-, \theta}^\alpha(\pi),$$

and

$$B_{\lambda_+, \lambda_-, \theta}^\alpha(\pi) := 1 - \sum_{\ell=0}^{\min\{\lceil \theta + g(\alpha, \lambda_+, \lambda_-) \pi \rceil - 1, \lambda_+\}} \binom{\lambda_+}{\ell} \pi^\ell (1 - \pi)^{\lambda_+ - \ell} P_{\lambda_+, \lambda_-, \theta, \ell}(\pi).$$

We have:

- (i) If $\pi^* = 1$, i.e., if $J^\alpha(\pi) > \pi$ for all $\pi \in [0, 1)$, then asymptotically (as $n \rightarrow \infty$) almost all nodes fail during the cascade.
- (ii) If $\pi^* < 1$ and π^* is a stable fixed point of J^α , i.e., $J^{\alpha'}(\pi^*) < 1$, then the final fraction of failed nodes converges in probability to

$$\frac{|\mathcal{D}_f^{(n)}|}{n} \xrightarrow{p} \sum_{\lambda_+, \lambda_-, \theta} \mu_{\lambda_+, \lambda_-, \theta} B_{\lambda_+, \lambda_-, \theta}^\alpha(\pi^*). \quad (49)$$

Furthermore, the asymptotic fraction of surviving nodes with degree λ and initial threshold θ converges in probability (as $n \rightarrow \infty$)

$$\frac{S_{\lambda_+, \lambda_-, \theta}(k_{\text{stop}}^{(n)})}{n} \xrightarrow{p} \mu_{\lambda_+, \lambda_-, \theta} \left(1 - B_{\lambda_+, \lambda_-, \theta}^\alpha(\pi^*)\right).$$

Proof. We discuss the changes needed to adapt the proof of Theorem 5 for the case of different in/out-degree distribution. In this case, the set of surviving nodes in different states is given by $\mathbf{S}_n(k) = \left(S_{\lambda_+, \lambda_-, \theta, \ell}^{(n)}(k) \right)_{\lambda_+, \lambda_-, 0 \leq \theta, \ell \leq \lambda_+}$, which represents a Markov chain whose transition probabilities are

$$\begin{aligned}\mathbb{E}[\Delta_k^{(n)} S_{\lambda_+, \lambda_-, \theta, 0}^{(n)} | \mathcal{F}_k^{(n)}] &= - \left(\frac{\lambda_+}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda_+, \lambda_-, \theta, 0}^{(n)}(k), \\ \mathbb{E}[\Delta_k^{(n)} S_{\lambda_+, \lambda_-, \theta, \ell}^{(n)}(k) | \mathcal{F}_k^{(n)}] &= \left(\frac{\lambda_+ - \ell + 1}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda_+, \lambda_-, \theta, \ell-1}^{(n)}(k) - \left(\frac{\lambda_+ - \ell}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda_+, \lambda_-, \theta, \ell}^{(n)}(k),\end{aligned}$$

for $0 < \ell < \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \lambda_+ T_k^{(n)}$ and

$$\begin{aligned}\mathbb{E}[\Delta_k^{(n)} S_{\lambda_+, \lambda_-, \theta, \ell}^{(n)}(k) | \mathcal{F}_k^{(n)}] &= \left(\frac{\lambda_+ - \ell + 1}{\bar{\lambda}^{(n)} n - k} \right) S_{\lambda_+, \lambda_-, \theta, \ell-1}^{(n)}(k), \quad \ell = \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \lambda_+ T_k^{(n)}, \\ S_{\lambda_+, \lambda_-, \theta, \ell}^{(n)} &= 0, \quad \ell > \theta + \frac{\alpha}{\bar{\lambda}^{(n)}} \lambda_+ T_k^{(n)}.\end{aligned}$$

The initial condition changes to

$$S_{\lambda_+, \lambda_-, \theta, \ell}^{(n)}(0) = \mu_{\lambda_+, \lambda_-, \theta}^{(n)} n \mathbf{1}(\ell = 0) \mathbf{1}(0 < \theta \leq \lambda_+).$$

The number of failed nodes among those with connectivity (λ_+, λ_-) and initial threshold θ is

$$D_{\lambda_+, \lambda_-, \theta}^{(n)}(k) = n \mu_{\lambda_+, \lambda_-, \theta}^{(n)} - \sum_{0 \leq \ell < \lceil \theta + \alpha \frac{\lambda_+}{\bar{\lambda}^{(n)}} T_k^{(n)} \rceil} S_{\lambda_+, \lambda_-, \theta, \ell}^{(n)}(k),$$

and contagion stops at a time when all failed links have been revealed

$$k_{\text{stop}}^{(n)} = \inf \{ k = 0, 1, \dots, n \bar{\lambda}^{(n)} : \sum_{\lambda_+, \lambda_-, \theta} \lambda_- D_{\lambda_+, \lambda_-, \theta}^{(n)}(k) = k \}. \quad (50)$$

The number of outgoing half-edges belonging to failed nodes at time k is

$$D_{\text{out}}^{(n)} = \sum_{\lambda_+, \lambda_-, \theta} \lambda_- D_{\lambda_+, \lambda_-, \theta}^{(n)}(k) - k$$

and $k_{\text{stop}}^{(n)}$ is the first time that $D_{\text{out}}^{(n)} = 0$. The number of failed nodes at the end of the contagion process is thus given by

$$|\mathcal{D}_f^{(n)}| = \sum_{\lambda_+, \lambda_-, \theta} D_{\lambda_+, \lambda_-, \theta}^{(n)}(k_{\text{stop}}^{(n)}).$$

The rest of the proof follows exactly as in the proof of Theorem 5. $\square$

Note that in this case, in order to transfer the results to the uniformly chosen simple random graph $G_*^{(n)}$, one needs to assume the second moment condition, which becomes now the convergence of second moment for both in- and out-degrees. In addition, one needs to assume the convergence of the covariance of the in and out degree (see [29, Theorem 3.6] and Remark 1).

5.3 Calendar time between interactions

So far, we have analyzed systemic risk under the assumption that the duration in calendar time between the two successive interactions follows an exponential distribution with parameter n .

In this subsection we consider the case when the calendar time between two interactions is an exponential whose parameter depends on the state of the system. In particular, the exponential time between interactions can depend on the current duration of the contagion process or on the number of unrevealed outgoing links from the failed nodes. The larger the number of unrevealed links, the higher the intensity with which we learn of affected counterparties. Growth is unlikely to stabilize the system if it happens uniformly over time. On the other hand, if the number of initial failures is small, then so is the intensity with which we learn the counterparties of failed linkages. This will slow contagion as there is a threshold growth in the meanwhile for the counterparties, and the system is likely to stabilize itself even without intervention.

In order to allow the growth intensity to vary over time, we can change the distribution of inter-arrival times to an exponential of mean given by a general function of k and n . The higher the mean, the more time nodes have to grow their thresholds before failed links are revealed. This gives us the flexibility to control the different growth rates depending on the stage of the cascade. For k/n smaller we are in the early stages of the cascade and for $k/n \sim \bar{\lambda}^{(n)}$ we are in the later stages of the cascade.

In particular, our results extend to the case when

$$\Delta_k^{(n)} = T_k^{(n)} - T_{k-1}^{(n)} \sim \text{Exp}(F_k^{(n)}),$$

where $F_k^{(n)}$ satisfies

$$F_k^{(n)} = nf\left(\frac{k}{n}\right) + o(1)$$

for some (squared) integrable function $1/f$ on $[0, \bar{\lambda} - \epsilon]$. The proof of Theorem 5 follows through as in Section 4.3. Indeed, we note that only two changes are necessary. Let $k = \gamma \bar{\lambda}^{(n)} n + o(n)$ for some $\gamma \in [0, 1]$. First, we show that in this case $T_k^{(n)} \xrightarrow{p} \int_0^{\gamma \bar{\lambda}} \frac{1}{f(s)} ds$. Indeed, (27)-(28) become

$$\mathbb{E}(T_k^{(n)}) = \sum_{i=1}^k \frac{1}{F_i^{(n)}} = \sum_{i=1}^k \left(\frac{i}{n} - \frac{i-1}{n}\right) \frac{1}{\frac{F_i^{(n)}}{n}} \approx \int_0^{\gamma \bar{\lambda}} \frac{1}{f(s)} ds + o(1),$$

and,

$$\text{Var}(T_k^{(n)}) = \sum_{i=1}^k \frac{1}{\left(F_i^{(n)}\right)^2} = \frac{1}{n} \sum_{i=1}^k \left(\frac{i}{n} - \frac{i-1}{n}\right) \frac{1}{\left(\frac{F_i^{(n)}}{n}\right)^2} \approx \frac{1}{n} \int_0^{\gamma \bar{\lambda}} \frac{1}{f^2(s)} ds + o\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right).$$

Thus (again) by using Chebysev's inequality, we have as $n \rightarrow \infty$, in probability

$$T_k^{(n)} \xrightarrow{p} \int_0^{\gamma \bar{\lambda}} \frac{1}{f(s)} ds.$$

Second, let $V_1^\pi, V_2^\pi, \dots, V_\ell^\pi$ be i.i.d. random variables on $[0, \pi \bar{\lambda}]$ with density

$$g(s) = \frac{\frac{1}{f(s)}}{\int_0^{\pi \bar{\lambda}} \frac{1}{f(s)} ds} \quad \text{for } s \in [0, \pi \bar{\lambda}]$$

and the order statistics be

$$V_{(1)}^\pi \leq V_{(2)}^\pi \leq \dots \leq V_{(\ell)}^\pi.$$

In this case, we need to set

- for $\ell = 0, 1, \dots, \theta$:

$$\beta_{\lambda, \theta, \ell}^\alpha(\pi) = \binom{\lambda}{\ell} \pi^\ell (1 - \pi)^{\lambda - \ell}.$$

- for $\ell = \theta + 1, \dots, \lambda$:

$$\beta_{\lambda, \theta, \ell}^\alpha(\pi) = \binom{\lambda}{\ell} \pi^\ell (1 - \pi)^{\lambda - \ell} \mathbb{P} \left(V_{(\theta+1)}^\pi > t_{\theta+1}, V_{(\theta+2)}^\pi > t_{\theta+2}, \dots, V_{(\ell)}^\pi > t_{\lambda, \theta, \ell} \right),$$

where

$$\hat{t}_u := \frac{u - \theta}{\alpha \lambda} \bar{\lambda}$$

for all $u = \theta + 1, \dots, \lambda$.

Using these changes, Theorem 5 will be still valid.

Remark 19 (Lender of last resort). *Here we considered a fixed schedule f that drives the inter-arrival times. Of course, it is desirable to have inter-arrival times as large as possible to allow for maximum growth and recovery. The natural case is when events happen faster in the beginning of the cascade, then decrease in intensity towards the end. Decreasing the intensity of inter-arrival times would come at a cost. We can envision an extension of this model where a lender of last resort or government can intervene to decrease inter-arrival time intensity or alternatively increase the growth intensity α .*

We now consider a more concrete example where the intensity between the two successive interactions depends on the number of unrevealed links in the system, i.e.

$$\Delta_k^{(n)} = T_k^{(n)} - T_{k-1}^{(n)} \sim \text{Exp}(D_{\text{out}}^{(n)}(k)),$$

where $D_{\text{out}}^{(n)}(k)$ denotes the number of unrevealed outgoing links belonging to failed agents at the k -th interaction time $T_k^{(n)}$. As shown in Section 4, for $k = \gamma \bar{\lambda}^{(n)} n + o(n)$ and $\gamma < \pi^*$,

$$\frac{D_{\text{out}}^{(n)}(k)}{n} \xrightarrow{p} \bar{\lambda} (J^\alpha(\gamma) - \gamma).$$

In this case we conjecture all our results hold, since

$$\mathbb{E}(T_k^{(n)}) \approx \int_0^\gamma \frac{1}{\bar{\lambda} (J^\alpha(s) - s)} ds + o(1)$$

and the variance is negligible:

$$\text{Var}(T_k^{(n)}) \approx \frac{1}{n} \int_0^\gamma \frac{1}{\bar{\lambda}^2 (J^\alpha(s) - s)^2} ds + o\left(\frac{1}{n}\right) \sim o\left(\frac{1}{n}\right).$$

We leave this to a future work.

5.4 Networked Cramér-Lundberg risk processes

In the Cramér-Lundberg model, or the classical compound Poisson risk process, the aggregate capital of an agent (insurer) who starts with initial capital x is given by

$$C(t) = x + \alpha t - \sum_{k=1}^{\mathcal{N}_\beta(t)} L_k, \quad (51)$$

where L_k are i.i.d. non-negative random variable following distribution F , mean $\bar{L}$ and $\mathcal{N}_\beta(t)$ is a Poisson process with intensity β independent of L_k . The ruin time for the insurer is defined by

$$\tau_x := \inf\{t \mid C(t) \leq 0\},$$

(with the convention that $\inf \emptyset = \infty$) and the central question is to find the ruin probability

$$\psi(x) = \mathbb{P}(\tau_x < \infty).$$

This can be computed exactly using the Pollaczek-Khinchine formula (see e.g. [13, 33]) as

$$\psi(x) = \left(1 - \frac{\beta \bar{L}}{\alpha}\right) \sum_{k=0}^{\infty} \left(\frac{\beta \bar{L}}{\alpha}\right)^k \left(1 - \hat{F}^{*k}(x)\right), \quad (52)$$

where

$$\hat{F}(x) = \frac{1}{\bar{L}} \int_0^x (1 - F(u)) du,$$

and f^{*k} denotes the k -fold convolution of any function f .

We now extend this setup to the case when the agents interact through a network. Namely, when an agent j fails it will impose failure cost or write down on its neighbor i , denoted by L_{ji} . Similar to the classical model, we assume that these losses are i.i.d. random variables with distribution F over all existing links. The aggregate capital for agent i with initial surplus x_i and capital growth α_i at time t is given by

$$C_i(t) = x_i + \alpha_i t - \sum_{j \in \mathcal{D}(t)} L_{ji}, \quad t \leq \tau_i, \quad (53)$$

where $\mathcal{D}(t)$ is the set of failed (ruined) agents at time t and τ_i denotes the ruin time of agent i , i.e.,

$$\tau_i = \inf\{t \mid C_i(t) \leq 0\}. \quad (54)$$

The (initial) threshold distribution for an agent with connectivity λ and initial capital x follows

$$p_x(\theta; \lambda) = \mathbb{P}\left(\sum_{k=1}^{\theta-1} L_k < x \leq \sum_{k=1}^{\theta} L_k\right), \quad \theta = 0, 1, \dots, \lambda, \quad (55)$$

where $\{L_k\}_{k=0}^{\infty}$ are i.i.d. distributed random variables with distribution F . Remark that x incorporates any exogenous macroeconomic shock in the stress scenario, and therefore x might be negative. The initial set of defaults is exogenous and determined by x .

It would be interesting to extend Theorem 5 to this setup, find the analogous of (52) and give the ruin probability for an agent with connectivity λ and initial capital x in the case of networked Cramér-Lundberg risk processes. This seems to be more involved due to the inter-dependence of the inter-arrival times of losses and the loss distribution. For this we need to generalize our approach, allowing for random threshold times $\hat{t}_{\lambda, \theta, \ell}$ depending on loss distribution. While the general case remains open, Theorem 5 can be seen as a first step toward this direction for deterministic loss function $L_{ij} = \bar{L} \mathbf{1}_{\{i \rightarrow j\}}$. These and some other related issues are left to a future work.

Conclusions

We investigated the fluid limit for a network of Cramér-Lundberg processes interacting through a random graph. Nodes receive growth benefits and suffer downward jumps occurring on exponential clocks after the failure of a neighbor. Using the limiting solution for the probability of failure, we define a game in which nodes choose their connectivity under the tradeoff that linkages provide income, and at the same time they bear the risk of contagion. In equilibrium, the risk of contagion depends on the choices of all nodes in the system. Our notion of equilibrium is similar to a mean field game: players take as given a mean-field, namely the conjectured failure probability of a link (which also gives the proportion of failed nodes at the end of a potential contagion process). They then decide on their own connectivity. This leads to an actual failure probability in the network and we check that a fixed point holds: the actual link failure probability is the same as the conjectured link failure probability.

Our results show that a higher heterogeneity in the initial distribution of the threshold (as captured by its standard deviation) implies a lower default probability in equilibrium even as it leads to a larger average connectivity in equilibrium. More importantly, systems with higher growth/recovery rates can have equilibria with higher failure probability as well as higher final fraction of failed agents. The fact that bailouts lead to moral hazard problems is a known fact. Our results point to the fact that even in systems where threshold growth happens over time (as opposed to equity or liquidity infusions) strategic agents will adapt and potentially take more risks in equilibrium as captured by increased connectivity. This result is surprising. In anticipation of future growth agents take higher exposure to systemic risk and therefore the growth effect is hindered by higher exposures. To counteract this effect, the most interconnected agents should have higher thresholds in proportion to their interconnectedness, and this proportion should be even higher in environments with large growth. The effect of threshold growth over time would then allow them to play a role as shock absorbers in the system.

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