2021 IEEE International Symposium on Information Theory (ISIT) | 978-1-5386-8209-8/21/$31.00 ©2021 IEEE | DOI: 10.1109/ISIT45174.2021.9518051

# Achievable Error Exponents for Two-Way AWGN Channels

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***Abstract*—We present achievable error exponent regions for the Two-Way AWGN channel under an expected block power constraint and variable-length coding (VLC). We propose an achievability scheme that allows terminals to cooperate via interaction to detect decoding errors and request re-transmissions. Under this scheme, in certain rate-pair regimes both directions are able to simultaneously attain error exponent pairs larger than the feedback-free point-to-point random coding error exponents[1].**

*A full version of this paper is accessible at: [1].*

## I. Introduction

Shannon [2] introduced the two-way channel, consisting of two terminals, $T_i$ for $i \in \{1,2\}$, that exchange messages. For the Two-way AWGN memoryless channel, the capacity region corresponds to a rectangular region [3], [4] determined by the interference-free AWGN capacities at signal-to-noise ratio SNR, $C = \frac{1}{2}\log(1+\text{SNR})$ of each direction (denoted by $C_{12}$ and $C_{21}$, respectively).

The reliability function (or error exponent) $E(R) = \limsup_{N\to\infty} \frac{-\ln \mathsf{P}_e}{N}$ provides a more refined yet still asymptotic characterization of the communication limits, where $\mathsf{P}_e$ is the probability of error of a blocklength-$N$ code. For one-way channels, $E(R)$ has been studied with and without feedback. In memoryless channels, while feedback cannot increase capacity, it may simplify coding schemes and enlarge the error exponent [5].

In the presence of noiseless feedback in one-way AWGN channels, error exponents can be greatly improved as shown in [6]–[10]. When noisy feedback is used, error exponent improvements over non-feedback channels are still possible, in particular when the feedback channel is stronger (less noisy) than the forward channel. A generalization of the Yamamoto-Itoh coding scheme under VLC with perfect feedback [10] to noisy feedback was presented by Sato-Yamamoto [11], and this scheme's reliability tends to Schalkwikj-Barron's [9] as the feedback noise approaches zero.

For two-way parallel memoryless channels (such as the Two-way AWGN channel), terminals send messages and (noisy) feedback over the same channels. This interaction (noisy feedback) does not increase the capacity region of the Two-way AWGN channel. Whether interaction in the Two-way AWGN channel can increase error exponents is addressed in [12] at zero-rate; here we focus on positive rate-pairs.

Apart from the authors' prior work on two-way channels [12], the most related prior work is that for error exponents for one-way channels with noisy feedback in the positive rate regime [11], [13]. In the one-way noisy feedback setting, error exponent gains have mainly been attained when the feedback channel is much stronger than the direct channel, as in [14] where the sphere packing bound is exceeded for a wide rate regime. This works considers an expected block power constraint, as that used in [15] for the zero-rate regime (transmission of two messages). Interestingly, in the two-way setting for positive rate, we are able to attain error exponent gains even when the channels in the two directions are symmetric – one direction need not be much stronger than the other. In fact, the scheme presented here exploits this symmetry, and is hence useful in a wider range of settings, including for example full duplex two-way communications with channel reciprocity. This scheme does rely on the flexibility provided by an expected power constraint.

## II. Problem Statement and Main Result

Consider a two-way AWGN channel as in Figure 1, for the transmission of $|\mathcal{W}_1| = 2^{\{nR_{12}\}}$ and $|\mathcal{W}_2| = 2^{\{nR_{21}\}}$ equally likely messages in the $1 \to 2$ and $1 \leftarrow 2$ directions respectively. The output of this channel at the $i$-th terminal at the $k$-th channel use is modeled as in (1):

$$Y_{i,k} = X_{i,k} + a_{i,k}X_{3-i,k} + N_{i,k}, \quad \text{for } k = 1,2,... \quad (1)$$

where, $a_{i,k}$ is a constant, $X_{i,k} \in \mathbb{R}$ the channel input satisfying a block power constraint, $Y_{i,k} \in \mathbb{R}$ the output, and $N_{i,k} \sim \mathcal{N}(0,\sigma_i^2)$ zero-mean AWGN, each independent and identically distributed across channel uses. Since each terminal may subtract its own input, and setting $a_{i,k} = 1$, (1) simplifies to: $Y_{i,k} = X_{3-i,k} + N_{i,k}$.

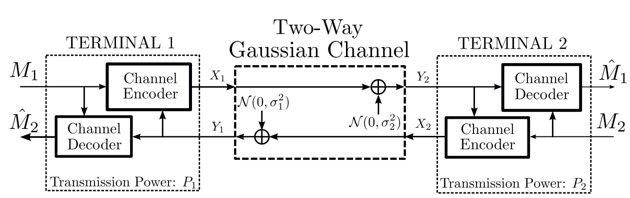


Fig. 1. Two-way AWGN channel.

[1]The work of N. Devroye was partially supported by NSF under award 1815428. The contents of this article are solely the responsibility of the authors and do not necessarily represent the official views of the NSF.

Let $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$ be the set of reals. A *variable-length two-way code* $\mathcal{C}_{\text{vl}}\left(|\mathcal{W}_1|, |\mathcal{W}_2|, P_1, P_2, \sigma_1^2, \sigma_2^2, N\right)$ for the transmission of messages $M_i$ uniformly selected from $\mathcal{W}_i$ in the $i \to (3-i)$ directions for $i = 1, 2$ over a two-way AWGN channel with average transmitter power $P_i$, and noise variances $\sigma_i^2$ respectively, consists of:

1. Two encoding functions: $f_{i,k} : \mathcal{W}_i \times \mathcal{Y}_i^{k-1} \to \mathcal{X}_i$ for $i = 1, 2$ and $k = 1, 2, ...$ leading to channel inputs $X_{i,k} = f_{i,k}\left(M_i, Y_i^{k-1}\right)$ satisfying an expected block power constraint for each block of length $N$ (where $\mathsf{E}[\cdot]$ denotes expectation):

$$\mathsf{E}\left[\sum_{k=1}^{N} X_{i,k}^2\right] \leq NP_i. \tag{2}$$

2. Two decoding functions: $\phi_{i,k} : \mathcal{Y}_i^k \to \mathcal{W}_{3-i}$.

3. A non-negative transmission time $\Delta$ (a random variable) satisfying $\mathsf{E}[\Delta] \leq N$, defined as the slot at which both messages are decoded (and transmitters can move on to the next message).

Let the average rate in the $i \to (3-i)$ direction be: $\bar{R}_{i,(3-i)} = \frac{\log|\mathcal{W}_i|}{\mathsf{E}[\Delta]}$. Next, let the signal-to-noise ratio for each direction be $\text{SNR}_{i,(3-i)} = P_i/\sigma_{3-i}^2$, and the maximum error probability attained in each direction by a two-way $\mathcal{C}_{\text{vl}}\left(|\mathcal{W}_1|, |\mathcal{W}_2|, P_1, P_2, \sigma_1^2, \sigma_2^2, N\right)$ variable length code at an average rate-pair $(\bar{R}_{12}, \bar{R}_{21})$ under power constraint (2) be:

$$\begin{aligned}&\mathsf{P}_{\text{error}}^{i\to(3-i)}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}, \Delta\right)\\ &\quad := \max_{m_i \in \mathcal{W}_i} \mathsf{P}\left(\phi_i^{\Delta} \neq m_{3-i} \mid M_i = m_i, M_{3-i} = m_{3-i}\right)\end{aligned}$$

The two-way capacity region is known [16] to equal all rate-pairs $(R_1, R_2)$ inside the rectangle bounded by $R_1 \leq C_{12} = \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma_2^2}\right)$ and $R_2 \leq C_{21} = \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma_1^2}\right)$.

*Definition 1:* An error exponent pair, $(E_{12}, E_{21})$, is achievable if simultaneously, for $\mathsf{E}[\Delta] \leq N$:

$$\begin{aligned}&E_{12}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}\right) \geq\\ &\qquad \frac{-\ln \mathsf{P}_{\text{error}}^{1\to2}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}, N\right)}{\mathsf{E}[\Delta]}\\ &E_{21}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}\right) \geq\\ &\qquad \frac{-\ln \mathsf{P}_{\text{error}}^{1\leftarrow2}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}, N\right)}{\mathsf{E}[\Delta]}\end{aligned}$$

*Definition 2:* The error exponent region (EER) of a two-way AWGN channel transmitting at an average rate-pair $(\bar{R}_{12}, \bar{R}_{21})$ under an expected block power constraint corresponds to the union of all achievable error exponent pairs $E_{12}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}\right)$ and $E_{21}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}\right)$.

We first present a proposition that involves the use of block codes under an average power constraint $\sum_{k=1}^{N} X_{i,k} \leq NP$ in the absence of terminal interaction / feedback (i.e. the encoding functions are functions of the messages alone):

*Proposition 1:* An achievable error exponent pair for the two-way AWGN channel for the rate pair $(R_{12}, R_{21})$ under an average power constraint is:

$$\begin{aligned}E_{12}(R_{12}, R_{21}, \text{SNR}_{12}, \text{SNR}_{21}) &\geq E_{\text{AWGN}}^{\text{rc}}(R_{12}, \text{SNR}_{12}),\\ E_{21}(R_{12}, R_{21}, \text{SNR}_{12}, \text{SNR}_{21}) &\geq E_{\text{AWGN}}^{\text{rc}}(R_{21}, \text{SNR}_{21}),\end{aligned}$$

where $E_{\text{AWGN}}^{\text{rc}}(R, \text{SNR})$ corresponds to the random coding error exponent lower bound for a one-way AWGN channel of signal to noise ratio SNR at rate $R$, see [17, Section 7.4].

Our main results correspond to two achievable EERs defined for any average rate-pair in the capacity region. One uses compression to send the feedback signals and the other does not. The former is useful for rates close to capacity, whereas the latter for lower rates. Our results are both based on a variable length coding scheme under power constraint (2) that exploits interaction to facilitate error detection and correction. We will show how the scheme operates for the case with compression (the one without compression can be easily obtained from the one with compression). Let $R := \max\{\bar{R}_{12}, \bar{R}_{21}\}$ and $C := \min\{C_{12}, C_{21}\}$.

*Theorem 1: Uncompressed feedback:* An achievable error exponent pair for the two-way AWGN channel under variable-length coding and an expected block power constraint at an average rate-pair $(\bar{R}_{12}, \bar{R}_{21})$, for $0 < R < 0.5C$ is determined as the union over all $0 \leq \lambda \leq 1$, $R_{\text{FB}} = R$, and satisfying $\bar{R}_{12}/\lambda \leq C_{12}$, $\bar{R}_{21}/\lambda \leq C_{21}$ and $R_{\text{FB}}/(1-\lambda) \leq C$, of

$$\begin{aligned}E_{12}(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}, N) &\geq (3)\\ E_{21}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}, N\right) &\geq (4).\end{aligned}$$

*Theorem 2: Compressed feedback:* An achievable error exponent pair for the two-way AWGN channel under variable-length coding and an expected block power constraint at an average rate-pair $(\bar{R}_{12}, \bar{R}_{21})$, is determined as the union over all $\lambda$ and $R_{\text{FB}}$ in $0 \leq \lambda \leq 1$, $\bar{R}_{12}/\lambda \leq C_{12}$, $\bar{R}_{21}/\lambda \leq C_{21}$, $0 \leq \bar{R}_{\text{FB}} < \min\{(1-\lambda)C, R\}$ of

$$\begin{aligned}&E_{12}(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}, N)\\ &\qquad \geq \min\left\{(3), R_{\text{FB}}\ln 2 + \lambda E_{\text{AWGN}}^{\text{rc}}\left(\frac{\bar{R}_{12}}{\lambda}, \text{SNR}_{12}\right)\right\}\\ &E_{21}\left(\bar{R}_{12}, \bar{R}_{21}, \text{SNR}_{12}, \text{SNR}_{21}, N\right)\\ &\qquad \geq \min\left\{(4), R_{\text{FB}}\ln 2 + \lambda E_{\text{AWGN}}^{\text{rc}}\left(\frac{\bar{R}_{21}}{\lambda}, \text{SNR}_{21}\right)\right\}.\end{aligned}$$

Note that we have excluded the zero-rate regime, since this scheme is outperformed by the one derived in [18], which results from a generalization of a one-way scheme with noisy feedback [15] to the two-way AWGN channel under the same power constraint.

The following section presents a coding scheme that achieves Theorems 1 and 2.

## III. Two-way interactive coding scheme

Both Theorems employ a coding scheme in which terminals first exchange their messages, and then initiate a cooperative feedback stage aiming to detect errors at both receivers. The only difference between the two coding schemes is that one sends feedback uncompressed, which can only be done for small enough rates, and the other uses hashing to compress the feedback signal, which allows transmission at higher rates. If an error is detected at any terminal, an alarm signal is triggered during the final stage, otherwise both transmitters remain silent. The occurrence of an alarm forces both terminals to

$$E_{12}(\bar{R}_{12},\bar{R}_{21},\mathrm{SNR}_{12},\mathrm{SNR}_{21},N)\geq\min\left\{(1-\lambda)E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{R_{\mathrm{FB}}}{1-\lambda},\mathrm{SNR}_{12}\right)\right. \tag{3}$$
$$\left.+(1-\lambda)E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{R_{\mathrm{FB}}}{1-\lambda},\mathrm{SNR}_{21}\right)+\lambda E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{\bar{R}_{12}}{\lambda},\mathrm{SNR}_{12}\right),\lambda E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{\bar{R}_{12}}{\lambda},\mathrm{SNR}_{12}\right)+\lambda E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{\bar{R}_{21}}{\lambda},\mathrm{SNR}_{21}\right)\right\}.$$

$$E_{21}(\bar{R}_{12},\bar{R}_{21},\mathrm{SNR}_{12},\mathrm{SNR}_{21},N)\geq\min\left\{(1-\lambda)E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{R_{\mathrm{FB}}}{1-\lambda},\mathrm{SNR}_{21}\right)\right. \tag{4}$$
$$\left.+(1-\lambda)E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{R_{\mathrm{FB}}}{1-\lambda},\mathrm{SNR}_{12}\right)+\lambda E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{\bar{R}_{21}}{\lambda},\mathrm{SNR}_{21}\right),\lambda E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{\bar{R}_{21}}{\lambda},\mathrm{SNR}_{21}\right)+\lambda E^{\mathrm{rc}}_{\mathrm{AWGN}}\left(\frac{\bar{R}_{12}}{\lambda},\mathrm{SNR}_{12}\right)\right\}.$$

retransmit their messages using a new block of length $N$. Since alarm events occur with exponentially small probability, retransmissions are very rare and thus the power constraint (2) is satisfied. If no alarm is triggered, both transmitters move to the transmission of a new message.

During the feedback stage terminals exchange a special message that is a function of the true message sent out in the first stage and the one received from the other terminal. This message should be the same for both directions. Once this message is exchanged, both terminals can compare the decoded special message and trigger an alarm if they are not equal. By means of this cooperation each terminal may become aware of decoding errors made locally or at the other end during the first stage. As we will see, we may still have decoding errors in which no alarm is triggered.

## A. Scheme operation

We present the coding scheme which employs compression (Theorem 2), but note that the no compression Theorem 1 may be easily obtained from this scheme by simply omitting the hashing function used to reduce the feedback rate. We first introduce some notation that will be useful in the upcoming sections. Let $\mathcal{C}^{\mathrm{RC}}(2^{NR},P,N)$ denote a randomly generated code for the transmission for $2^{NR}$ messages using a block of length $N$ under average power $P$. An achievable error exponent for this code is determined by the random coding error exponent lower bound $E^{\mathrm{RC}}_{\mathrm{AWGN}}(R,\mathrm{SNR})$ as shown in [17].

Figure 2 shows a block diagram of our scheme comprising three stages whose durations are parameterized by $\lambda\in[0,1]$:

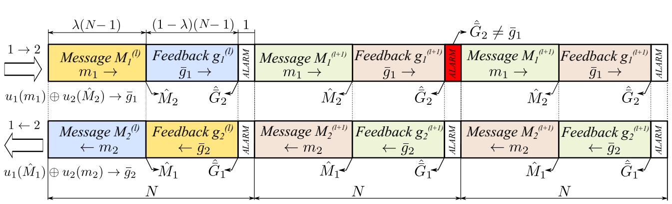


Fig. 2. Block diagram for the two-way coding scheme under a expected block power constraint. Note that the special message $g_i$ or its corresponding hash $\bar{g}_i$ is fed back depending on whether compression is used or not.

**1. Transmission:** This stage lasts for $\lambda(N-1)$ channel uses, where terminal $T_i$ transmits message $M_i=m_i$, uniformly selected from $\mathcal{W}_i$ utilizing a random code in each direction, which are respectively denoted by $\mathcal{C}^{\mathrm{RC}}(2^{N\bar{R}_{12}},P_1,\lambda N)$ and $\mathcal{C}^{\mathrm{RC}}(2^{N\bar{R}_{21}},P_2,\lambda N)$. By the end of this stage, each terminal has a preliminary estimate of the received message $\hat{M}_{3-i}$.

**2. Feedback:** This stage lasts for $(1-\lambda)(N-1)$ channel uses. Here, each terminal $T_i$ generates a special feedback message we denote by $g_i$ that are exchanged during this stage and used later to detect errors. To generate $g_i$, the $i$-th encoder combines the true known message $m_i$ and the estimate $\hat{M}_{3-i}$ as follows: Let $\mathbb{F}_q^N$ be a finite field of size $q^N$, where $q$ is chosen as the smallest prime for which $q^N>\max\left\{\lceil 2^{N\bar{R}_{12}}\rceil,\lceil 2^{N\bar{R}_{21}}\rceil\right\}$, where $\lceil x\rceil$ denotes to the smallest integer larger than $x$. Next, let $u_i(m_i)$ be an injective mapping $m_i\mapsto\mathbb{F}_q^N$ where $m_i\in\mathcal{W}_i$. Then, for terminal $T_1$, $g_1=u_1(m_1)\oplus u_2(\hat{M}_2)$, whereas for $T_2$, $g_2=u_1(\hat{M}_1)\oplus u_2(m_2)$, where $\oplus$ denotes modulo addition over the finite field $\mathbb{F}_q^N$. Message $g_i$ is an element of the set $\mathcal{G}=\{0,1,...,\max\{2^{N\bar{R}_{12}},2^{N\bar{R}_{21}}\}-1\}$ , whose cardinality is determined by the direction transmitting at a higher rate. Note that in the absence of errors during the first stage, the messages decoded at each terminal $T_{3-i}$ are $\hat{M}_i=m_i$ for $i=1,2$, and we must have $g_1=g_2$.

Since both directions transmit at the same rate and only $(1-\lambda)(N-1)$ channel uses remain from the first stage, we consider the compression method introduced in [19] in which the $|\mathcal{G}|=\max\{2^{N\bar{R}_{12}},2^{N\bar{R}_{21}}\}$ messages are randomly assigned to $2^{NR_{\mathrm{FB}}}$ bins, where $R_{\mathrm{FB}}$ is a design parameter. Thus, the feedback message becomes the bin number (or hash) that contains $g_i$, which we denote by $\bar{g}_i\in\{1,...,2^{N\bar{R}_{\mathrm{FB}}}\}$. It follows that $\bar{R}_{\mathrm{FB}}\leq(1-\lambda)\min\{C_{12},C_{21}\}$. If $\bar{R}_{\mathrm{FB}}=\max\{2^{N\bar{R}_{12}},2^{N\bar{R}_{21}}\}$ then each bin contains exactly one message, and $\bar{g}_i=g_i$. Messages $\bar{g}_i$ are exchanged using a $\mathcal{C}^{\mathrm{RC}}\left(2^{N\bar{R}_{\mathrm{FB}}},P_i,(1-\lambda)N\right)$ code in each direction and respectively decoded as $\hat{\bar{G}}_{3-i}$. Observe as well that the compression following the generation of messages $g_i$, may cause binning (or hash) collisions in which a $g_i$ containing an error may result in the same bin as the $g_i$ of an error free transmission. We consider this and other possibilities when we analyze the probability of error of the scheme in Section IV.

**3. Alarm:** For this stage, each terminal compares the locally generated message bin index $\bar{g}_i$ with the estimate $\hat{\bar{G}}_{3-i}$

obtained in the second stage. An alarm event is declared in case of a mismatch. The result of this operation is sent to the other terminal using the single channel use signaling (5):

$$X_{i,N} = \begin{cases} 0, & \text{if } \bar{g}_i = \hat{\hat{G}}_{3-i}, \\ \sqrt{\frac{P_i}{\mathsf{P}(\text{Alarm})}}, & \text{otherwise.} \end{cases} \tag{5}$$

Thus, an alarm corresponds to a very high amplitude transmission since, as we show later in the Appendix of [1], $\mathsf{P}(\text{Alarm})$ is exponentially small (and also corresponds to the probability of a retransmission $\mathsf{P}(\text{Rtx})$). This transmission is decoded at the $(3-i)$-th terminal by comparing the received signal $Y_{3-i,N}$ with a threshold $\Upsilon = N$, as in [15], where this signaling is introduced for the AWGN channel with active noisy feedback and the transmission of two messages. Moreover, it can be shown that the probability of error in decoding $Y_{3-i,N}$ decays to zero faster than any exponential.

When an alarm occurs, both terminals discard their preliminary estimates and initiate a retransmission, which means a repetition of the three stages using a new block of length $N$ for the same message. Figure 2 illustrates three of these consecutive blocks. Stages have been colored to identify what message they are associated with and for which direction. We have depicted the transmission of a stream of messages indexed by $(l)$. The first block corresponds to the $l$-th messages being sent from both terminals and successfully decoded since no alarms are triggered. The second block corresponds to the transmission of the $(l+1)$-th messages. Note that terminal $T_1$ triggers an alarm (colored in red), therefore, a retransmission is necessary for both directions and occurs in the next block, where the three stages are repeated for messages $(l+1)$. This time, transmission is successful since no alarms are triggered, and both terminals can move to message $(l+2)$ in the next block (not shown).

**Decoding rule:** Once the three stages have concluded, the $i$-th receiver declares that the message sent by the other terminal corresponds to the preliminary decision $\hat{M}_{3-i}$ if no alarm is detected, otherwise, it awaits until the end of a new block of length $N$ that conveys a retransmission. The final decoding decision occurs only in the absence of alarms, hence, multiple retransmissions may happen until both terminals can move to a new message.

## IV. Proof of Theorem 2

This section presents a short version of the proof of Theorem 2. We refer the reader to the Appendix A for the complete analysis. The proof consists of three parts: the analysis of the probability of error, the expected transmission time, and the error exponents. Here, we consider compressed feedback, since as we show in the Appendix of the extended version [1], this is related to the uncompressed one by the inclusion of an extra term in the overall probability of error.

### A. Probability of error analysis

In the following, the feedback stage uses compression. We analyze the $1 \to 2$ direction only, since the other follows by symmetry. Let the probability of error of the first and second stages, meaning that a message sent (without feedback) is incorrectly decoded, be denoted as $\overrightarrow{\mathsf{P}_{e_1}}$ for the first stage, and $\overrightarrow{\mathsf{P}_{e_2}}$ for the second stage, where arrows indicate the communication direction. Compression in the feedback stage is performed by randomly assigning messages into $2^{NR_{\text{FB}}}$ bins whose index number / hash is transmitted instead. The probability of a hash-collision is $p_h = \frac{1}{2^{NR_{\text{FB}}}}$. Note that when no compression is used, $p_h = 0$. Then,

$$\begin{aligned} \mathsf{P}^{1\to 2}_{\text{error}} &= \mathsf{P}\left(\hat{M}_1 \neq m_1, \text{No-Alarm} \mid M_1 = m_1, M_2 = m_2\right) \\ &= \mathsf{P}\left(\text{No-Alarm}; \hat{M}_1 \neq m_1; \hat{M}_2 = m_2 \mid M_1 = m_1, M_2 = m_2\right) \\ &\quad + \mathsf{P}\left(\text{No-Alarm}; \hat{M}_1 \neq m_1; \hat{M}_2 \neq m_2 \mid M_1 = m_1, M_2 = m_2\right) \end{aligned} \tag{6}$$

The event $\{\text{No-Alarm}\} \equiv \{(\hat{\hat{G}}_1 = \bar{g}_2) \cap (\hat{\hat{G}}_2 = \bar{g}_1)\}$, means that no alarm occurs if both terminals declare that their *feedback message* $\bar{g}_i$ matches the one received from the other terminal. As shown in the Appendix of [1], this probability can be upper bounded as:

$$\mathsf{P}^{1\to 2}_{\text{err}} \leq \max\left\{\overrightarrow{\mathsf{P}_{e_2}}\overleftarrow{\mathsf{P}_{e_2}}\overrightarrow{\mathsf{P}_{e_1}}, p_h\overrightarrow{\mathsf{P}_{e_1}}, \overrightarrow{\mathsf{P}_{e_1}}\overleftarrow{\mathsf{P}_{e_1}}\right\} \tag{7}$$

$$\mathsf{P}^{1\leftarrow 2}_{\text{err}} \leq \max\left\{\overleftarrow{\mathsf{P}_{e_2}}\overrightarrow{\mathsf{P}_{e_2}}\overleftarrow{\mathsf{P}_{e_1}}, p_h\overleftarrow{\mathsf{P}_{e_1}}, \overleftarrow{\mathsf{P}_{e_1}}\overrightarrow{\mathsf{P}_{e_1}}\right\} \tag{8}$$

### B. Expected transmission time:

Recalling that a retrasmission occurs when an alarm is declared at either terminal, the alarm event corresponds to: $\{\text{Alarm}\} = \{(\hat{\hat{G}}_2 \neq \bar{g}_1) \cup (\hat{\hat{G}}_1 \neq \bar{g}_2)\}$. Hence, a retransmission happens with probability $\mathsf{P}(\text{Rtx}) = \mathsf{P}(\text{Alarm})$:

$$\begin{aligned} \mathsf{P}(\text{Rtx}) &= \mathsf{P}\left((\hat{\hat{G}}_2 \neq \bar{g}_1) \cup (\hat{\hat{G}}_1 \neq \bar{g}_2) \mid M_1 = m_1, M_2 = m_2\right) \\ &\leq \mathsf{P}\left(\hat{\hat{G}}_2 \neq \bar{g}_1 \mid M_1 = m_1, M_2 = m_2\right) \\ &\quad + \mathsf{P}\left(\hat{\hat{G}}_1 \neq \bar{g}_2 \mid M_1 = m_1, M_2 = m_2\right) \end{aligned}$$

As we show in the Appendix of [1], $\mathsf{P}(\text{Rtx}) \to 0$ as the block length $N \to \infty$. It follows that the expected transmission time is determined by the probability of retransmission, given as:

$$\mathsf{E}[\Delta] = N \cdot \sum_{k=0}^{\infty} \cdot \mathsf{P}(\text{Rtx})^k = N \cdot \frac{1}{1 - \mathsf{P}(\text{Rtx})}$$

Thus, $\mathsf{E}[\Delta] \approx N$ when $\mathsf{P}(\text{Rtx}) \to 0$.

### C. Error exponents

Equations (7) and (8) describe the probability of error in terms of the following probabilities.

$$\begin{aligned} \overrightarrow{\mathsf{P}_{e_1}} &\leq \exp\left\{-(N-1)\lambda E^{\text{rc}}_{\text{AWGN}}\left(\tfrac{\bar{R}_{12}}{\lambda}, \text{SNR}_{12}\right)\right\}, \\ \overrightarrow{\mathsf{P}_{e_2}} &\leq \exp\left\{-(N-1)(1-\lambda)E^{\text{rc}}_{\text{AWGN}}\left(\tfrac{\bar{R}_{12}}{1-\lambda}, \text{SNR}_{12}\right)\right\}, \\ \overleftarrow{\mathsf{P}_{e_1}} &\leq \exp\left\{-(N-1)\lambda E^{\text{rc}}_{\text{AWGN}}\left(\tfrac{\bar{R}_{21}}{\lambda}, \text{SNR}_{21}\right)\right\}, \\ \overleftarrow{\mathsf{P}_{e_2}} &\leq \exp\left\{-(N-1)(1-\lambda)E^{\text{rc}}_{\text{AWGN}}\left(\tfrac{\bar{R}_{21}}{1-\lambda}, \text{SNR}_{21}\right)\right\}. \end{aligned}$$

Note that in each of the probability of error terms shown above, the error exponent is scaled down by either $\lambda$ or $(1-\lambda)$ depending on whether the term corresponds to the first or second stage of the scheme. Moreover, the instantaneous

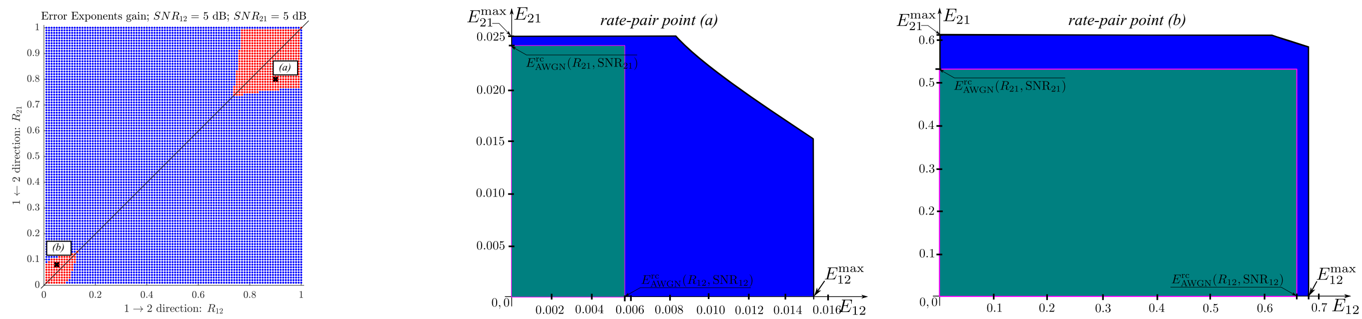


Fig. 3. Consider a two-way channel with $\text{SNR}_{12} = \text{SNR}_{21} = 5$dB. Left: Capacity region, where blue dots represent rate-pairs for which the random coding error exponent can be achieved. The red dots represent rate-pair for which the random coding error exponent can be exceeded for both directions. Center-Right: EER for the rate-pair point (a) $(0.9C_{12}, 0.8C_{21})$, and rate-pair point (b) $(0.02C_{12}, 0.08C_{21})$.

transmission rate is scaled up in order to compensate for the shorter block length (determined by duration of each stage) and to guarantee that the target operating rate-pair is achieved. Finally, from (7) and an expected transmission time $\mathsf{E}[\Delta] \approx N$, we have that for very large $N$:

$$\frac{-1}{\mathsf{E}[\Delta]}\ln \mathsf{P}_{\text{error}}^{1\to 2} \geq \frac{-1}{\mathsf{E}[\Delta]}\min\left\{\ln\left(\overrightarrow{\mathsf{P}_{e_2}}\overleftarrow{\mathsf{P}_{e_2}}\overrightarrow{\mathsf{P}_{e_1}}\right), \ln\left(p_h\overleftarrow{\mathsf{P}_{e_1}}\right), \ln\left(\overrightarrow{\mathsf{P}_{e_1}}\overleftarrow{\mathsf{P}_{e_1}}\right)\right\},$$

from which (3) is obtained. The result for the other direction follows by symmetry.

## V. Numerical Simulations

This section presents numerical evaluations of our results. Figure 3-left presents the capacity region of a two-way AWGN channel where red color denotes the rate-pair regimes in which our schemes outperform the random coding error exponent simultaneously in both directions. In the center/right plots, we present the achievable error exponent regions for the rate-pair points marked as (a) and (b) in the capacity region. As a comparison reference, these plots also show the achievable EER by means of point-to-point transmissions and no cooperation, corresponding to the darker rectangle resulting from Proposition 1. Observe an interesting trade off between the error exponents of both directions. This is more dramatic for point (a), which is in the higher rate-pair regime. Also, note that for both points (a) and (b) it is possible to attain error exponents larger than Proposition 1 in both directions simultaneously. Next, we evaluate Theorems 1 and 2 for the rate-pairs along the line that connects the points $(0, 0)$ and $(C_{12}, C_{21})$ of the capacity region of the two-way AWGN channel (as shown in Figure 3-left with a solid black line). We considered a symmetric two-way channel in which both directions are of similar SNR. Figure 4 shows the largest error exponent achieved by the scheme in the $1 \to 2$ direction. A similar plot would result for the opposite direction. The solid blue line presents the random coding error exponent lower bound, achievable when terminals do not interact. The dashed red line results by evaluating both theorems and choosing the largest achievable error exponent. There exists important error exponent gains in two regimes: lower (close to zero) and higher (close to capacity) rate regimes. In the remaining rate-pairs the scheme achieves the random coding error exponent.

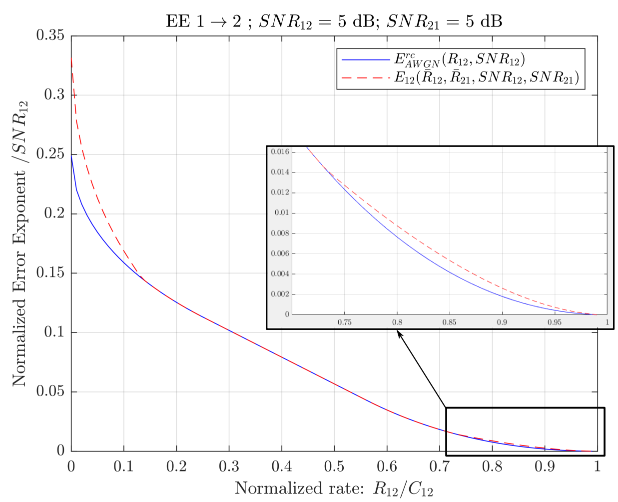


Fig. 4. Achievable error exponents for the $1 \to 2$ direction for rates $0 < \bar{R}_{12} \leq C_{12}$ for a Two-Way AWGN channel with $\text{SNR}_{12} = \text{SNR}_{21} = 5$dB. Error exponents are normalized over the SNR and evaluated for rate-pairs along the line connecting points $(0, 0)$ and $(C_{12}, C_{21})$, see Figure 3 (left).

## VI. Conclusion

The coding scheme we presented suggests that in a two-way AWGN channel, interaction may be exploited to improve (over non-feedback one-way error exponents under block coding) error exponents in both directions simultaneously – even when both directions have similar channel strength. Our feedback strategy correlates the errors in the two directions, and any terminal may trigger an alarm when the received feedback message does not match the one sent. This cooperation increases the error detection capabilities in both terminals. Moreover since we use variable length coding, a detected error can be corrected by the message retransmission that follows the occurrence of an alarm.

The expressions in Theorems 1 and 2 are mathematically involved. We have left analytically optimizing the parameters $\lambda$ and $R_{\text{FB}}$ for future work.

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