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Achievable error exponents for the two-way parallel DMC

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***Abstract*—We investigate error exponent regions for the parallel two-way DMC in which each terminal sends its own message and provides feedback to the other terminal. Various error exponents are presented in different rate-region regimes based on the relative rates and zero-error capacities of both directions. The schemes employed are extensions of error exponents for one-way DMCs with noiseless, rate-limited and noisy feedback[1].**

I. Introduction

Shannon [1] introduced the two-way discrete memoryless channel (DMC), and derived inner and outer bounds to its capacity region. We focus on the two-way *parallel* DMC, whose capacity region is a rectangle determined by the one-way capacity of each link. Adaptation / interaction, or using the feedback present in two-way channels, cannot increase this capacity region, but may be exploited to attain larger error exponents. We consider that either one or both directions have a positive zero-error capacity C_0, and improve the reliability at rate-pairs in the small error regime. $C_0 > 0$ also alleviates synchronization issues in variable length coding (VLC), since the beginning / end of a message is signaled without error[2].

In the two-way setting, a terminal may transmit messages with small error at all rates below the one-way capacity, with zero-error at all rates below C_0, provide noiseless feedback for the other terminal limited to a certain rate below C_0, provide noisy feedback, or any combination of the above. We present achievable schemes and error exponents for the two-way parallel DMC based on coding schemes for the one-way DMC with feedback using VLC. The one-way reliability function is defined for VLC as:

$$E(\bar{R}) = \lim_{\mathsf{E}[\Delta]\to\infty} \frac{-1}{\mathsf{E}[\Delta]} \log \mathsf{P}_e(\bar{R}, \Delta),$$

for $0 \leq \bar{R} \leq C$ (for C the small error capacity), transmission time Δ, and probability of error $\mathsf{P}_e(\bar{R}, \Delta)$. Next we present some fundamental results:

Burnashev's reliability: for any DMC of capacity C, zero-error capacity of zero, and *noiseless output feedback* using VLC, Burnashev [4] demonstrated that $E(R) \leq E_{\text{Burn}}(\bar{R})$:

$$E_{\text{Burn}}(\bar{R}) := C_1 \left(1 - \frac{\bar{R}}{C}\right), \quad \text{for } 0 \leq \bar{R} \leq C, \tag{1}$$

where $C_1 = \max_{x_1,x_2} D\left(p(y|x_1)||p(y|x_2)\right)$ is the Kullback-Leibler divergence of the distributions induced by the two most distinguishable symbols of the forward direction alphabet.

Yamamoto-Itoh's scheme [5]: utilizes noiseless feedback in a two-stage VLC scheme to achieve Burnashev's upper bound. In the message stage, a capacity achieving code is used to send a message whose preliminary estimate is fed back without error. In the control stage, the encoder indicates whether the receiver should accept the decision (ACK), or await a retransmission (NACK). The control message estimate is also fed back to keep synchronization. Errors result from a wrong preliminary decision and a missed NACK. Retransmissions occur if the decoder declares a NACK, which happens with an exponentially small probability and the expected number of transmissions for a message tends to one.

Forney's error exponent [6]: is attained by using a *single bit of noiseless feedback* to request a retransmission when decoding leads to an erasure. Then, $E(\bar{R}) \geq E_{\text{Forn}}\left(\bar{R}\right)$, where:

$$E_{\text{Forn}}\left(\bar{R}\right) := E_{\text{sp}}(\bar{R}) + C - \bar{R}, \text{ for } \bar{R}_\infty \leq \bar{R} \leq C. \tag{2}$$

Above, $E_{\text{sp}}(\bar{R})$ corresponds to the sphere packing bound for a DMC without feedback, and $\bar{R}_\infty$ to the smallest rate for which the sphere packing upper bound tends to infinity [7, Sec. 5.8].

Rate-limited noiseless feedback: this interesting regime has seen limited work – [8] characterized the noiseless feedback rate needed to attain Burnashev's bound. In Section III, we extend these results to obtain achievable error exponents with noiseless rate-limited feedback.

Noisy feedback: this more complicated case, due to synchronization issues, was studied for one-way DMCs using VLC in [2], [9], and for the two-way parallel DMC in [3].

Contributions: We present achievable error exponents of the one-way DMC with limited-rate noiseless feedback in Section III. We use these results in Section IV for two-way systems, where either direction may have $C_0 > 0$. Depending on the availability and amount of C_0, a terminal may provide rate-limited noiseless feedback to the other direction in addition to the transmission of its own messages (either with zero or small error). The operating rate-pair determines if this noiseless feedback can be exploited to either exceed Forney's reliability, achieve Burnashev's bound, or attain infinite reliability. Due to space constraints, all proofs are relegated to the Appendix of the extended version of this manuscript, available at [10].

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[2]Synchronization over channels with noisy feedback has been addressed in [2], and extended to two-way channels in [3].

II. Problem statement

Let $(\mathcal{X}, W, \mathcal{Y})$ denote a DMC characterized by finite input and output alphabets $\mathcal{X}, \mathcal{Y}$ and transition probability $W(y|x)$, for the transmission of equally likely messages from set $\mathcal{M}$. Let W^n denote n uses of the channel, then $W^n(y^n|x^n) = \prod_{k=1}^{n} W(y_k|x_k)$ for $x^n \in \mathcal{X}^n$ and $y^n \in \mathcal{Y}^n$. For systems with active noiseless feedback, let $R_{\text{FB}} \in \mathbb{R}^+$ be the available rate of the feedback channel. For two-way channels, each terminal is denoted by T_i for $i = 1, 2$. Let a two-way $(\mathcal{X}_1, \mathcal{Y}_1, W(y_1y_2|x_1x_2), \mathcal{Y}_2, \mathcal{X}_2)$ DMC be characterized by a set of transition probability mass functions $W(y_1y_2|x_1x_2)$, finite input and output alphabets $\mathcal{X}_i$, $\mathcal{Y}_i$, and message sets $\mathcal{M}_1$ and $\mathcal{M}_2$. In the two-way parallel DMC, $W(y_1y_2|x_1x_2) = W_{12}(y_2|x_1) \cdot W_{21}(y_1|x_2)$, where subscripts denote the communication direction from T_i to T_{3-i}. This is equivalent to two independent links operating in parallel and opposite directions.

A. The one-way DMC

The concepts of small-error capacity C, and zero-error capacity C_0, for a one-way $(\mathcal{X}, W, \mathcal{Y})$ DMC were introduced by Shannon in [11] and [12] respectively.

Definition 1: A variable length block code $\mathcal{C}(\mathcal{M}, R_{\text{FB}}, N)$ for a one-way $(\mathcal{X}, W, \mathcal{Y})$ DMC with noiseless rate-limited R_{FB} active feedback and block length N, comprises:

- A set of equally likely messages $\mathcal{M}$.
- A set of forward channel encoding functions: $x_n : \mathcal{M} \times \mathcal{Z}^{n-1} \to \mathcal{X}_n$, where Z^n is the sequence received through the rate-limited noiseless feedback link to produce channel inputs $X_n = x_n(M, Z^{n-1})$.
- A set of feedback channel encoding functions: $z_n : \mathcal{Y}^{n-1} \to \mathcal{Z}$, which produce feedback inputs $Z_n = z_n(Y^{n-1})$, with $|Z^N| \leq 2^{NR_{\text{FB}}}$ per block of length N.
- A set of forward decoding functions: $\phi_n : \mathcal{Y}^n \to \mathcal{M}$,

for $n = 1, 2, ..., \Delta$, where Δ corresponds to the transmission time (a random variable), and is a stopping time for which $E[\Delta] \leq N$. Let $\bar{R} = \frac{\log|\mathcal{M}|}{E[\Delta]}$ define the average transmission rate, and let $\mathsf{P}_e(\bar{R}, \Delta, R_{\text{FB}})$ (argument R_{FB} is present according to the availability of feedback) be the maximum error probability attained among all messages at rate $\bar{R}$ and decoding (not erasure) occurring at time Δ, with a noiseless feedback of rate R_{FB}. Then, $\mathsf{P}_e(\bar{R}, \Delta, R_{\text{FB}}) = \max_{M \in \mathcal{M}} P[\phi_\Delta(Y^\Delta) \neq m | M = m \text{ sent}]$.

Definition 2: An error exponent is achievable at an expected rate $\bar{R}$ over a one-way DMC with rate-limited feedback if there exists a sequence of VLC codes such that:

$$E(\bar{R}, R_{\text{FB}}) \geq \lim_{\mathsf{E}[\Delta] \leq N, N \to \infty} \frac{-1}{\mathsf{E}[\Delta]} \log \mathsf{P}_e(\bar{R}, \Delta, R_{\text{FB}}),$$

for $C_0 < \bar{R} \leq C$. $E(\bar{R}, R_{\text{FB}}) = \infty$ for $0 \leq \bar{R} \leq C_0, \forall R_{\text{FB}}$.

B. The two-way parallel DMC

A parallel two-way DMC is formed by terminals T_i for $i = 1, 2$, and channels $\left(\mathcal{X}_i, W_{i,(3-i)}(y_{3-i}|x_i), \mathcal{Y}_{3-i}\right)$. Let $\bar{R}_{i,(3-i)}$ be the expected rate in the $i \to (3-i)$ direction, and $\Delta_{i,(3-i)}$ the transmission time[3] at which decoding decision about message M_i is made at T_{3-i}.

[3]Each direction has its own transmission time.

Definition 3: Terminals T_1 and T_2 *interact* if their corresponding channel inputs at time n adapt to past outputs as $X_{i,n} = x_{i,n}\left(M_i, y_i^{n-1}\right)$.

Definition 4: A two-way variable length code $\mathcal{C}(\mathcal{M}_1, \mathcal{M}_2, N)$ for a two-way parallel $(\mathcal{X}_1, \mathcal{Y}_1, W(y_1y_2|x_1x_2), \mathcal{Y}_2, \mathcal{X}_2)$ DMC comprises:

- Two sets of equally likely messages $\mathcal{M}_i$.
- Two sets of encoding functions $x_{i,n} : \mathcal{M}_i \times \mathcal{Y}_i^{n-1} \to \mathcal{X}_{i,n}$, producing channel inputs $X_{i,n} = x_{i,n}\left(M_i, Y_i^{n-1}\right)$,
- Two sets of decoding functions $\phi_{i,n} : \mathcal{Y}_i^n \to \mathcal{M}_i$,

for $n = 1, 2, ..., \Delta_{i,(3-i)}$, where $\Delta_{i,(3-i)}$ corresponds to the transmission time in which a message is decoded at T_{3-i} (as in one-way case, a random variable with $E[\Delta_{i,(3-i)}] \leq N$.

Let an average rate-pair $(\bar{R}_{12}, \bar{R}_{21})$ be defined by the communication rates: $\bar{R}_{i,(3-i)} = \frac{\log|\mathcal{M}_i|}{\mathsf{E}[\Delta_{i,(3-i)}]}$ for $i = 1, 2$, and let the error probability in each direction be denoted as $\mathsf{P}_{e_{i,(3-i)}}\left(\bar{R}_{12}, \bar{R}_{21}, \Delta_{i,(3-i)}\right)$.

Definition 5: An error exponent pair $E_{i,(3-i)}\left(\bar{R}_{12}, \bar{R}_{21}\right)$ is achievable for a rate-pair $\left(\bar{R}_{12}, \bar{R}_{21}\right)$, over a two-way parallel DMC if there exists a sequence of two-way variable length codes such that $\mathsf{E}[\Delta_{i,(3-i)}] \leq N$ for $i = 1, 2$, and for very large N simultaneously:

$$\frac{-\log \mathsf{P}_{e_{i,(3-i)}}\left(\bar{R}_{12}, \bar{R}_{21}, \Delta_{i,(3-i)}\right)}{\mathsf{E}[\Delta_{i,(3-i)}]} \geq E_{i,(3-i)}\left(\bar{R}_{12}, \bar{R}_{21}\right).$$

Definition 6: The achievable error exponent region (EER) is the union over all achievable error exponent pairs at rate-pair $(\bar{R}_{12}, \bar{R}_{21})$.

III. Main Results: one-way

Consider a one-way DMC with $C_0 = 0$ and noiseless active feedback with rate-limited to R_{FB}[4]. Let any attainable error exponent for this channel at rate R in the absence of feedback be $E_{1w}(R)$. When noiseless feedback is used, improvements on the achievable error exponents depend on how R_{FB} compares to the forward expected rate $\bar{R}$, and how feedback is used to detect and correct errors. With Yamamoto-Itoh's [5] scheme, Burnashev's reliability is attained by feeding back the message decoding decision made at the receiver. However, the rate of the noiseless feedback transmission must equal that of the forward link only up to a critical rate $\bar{R}^*$, beyond which, as shown in [8], compressed noiseless feedback may be transmitted instead in the message mode of the Yamamoto-Itoh scheme in two forms: i) *random-hashing*: independently and uniformly assigning each of the messages into $2^{NR_{\text{FB}}}$ bins, whose index is fed back to the transmitter as a hash; and, ii) a *joint channel-code / hash-function* design where an erasure decoding rule takes into account the bins containing messages and is used to form a lower rate code. These approaches result in the following two propositions, as extensions of [8] that characterize achievable error exponents for a given noiseless feedback rate[5]:

[4]The noiseless feedback link has a capacity of $C_{0_{\text{FB}}}$, thus $R_{\text{FB}} \leq C_{0_{\text{FB}}}$.

[5]Error exponents are defined for the regime $R_\infty < \bar{R} \leq C$ if they depend on $E_{\text{sp}}(\cdot)$. In the regime $C_0 < \bar{R} \leq R_\infty$, $E_{1w}(R)$ is achievable without feedback using block codes. The reliability is unbounded for rates below C_0.

Proposition 1: An achievable error exponent for a one-way DMC with rate-limited R_{FB} noiseless active feedback, utilizing *random hashing* and VLC is given by $E(\bar{R}, R_{\text{FB}}) \geq E^{\text{RH}}_{\text{RL-FB}}(\bar{R}, R_{\text{FB}})$, for $\bar{R}_\infty \leq \bar{R} \leq C$, where:

$$E^{\text{RH}}_{\text{RL-FB}}(\bar{R}, R_{\text{FB}}) := \begin{cases} \max_{\bar{R} \leq R_{\text{data}} \leq C} \frac{\bar{R}}{R_{\text{data}}} E_{\text{Forn}}(R_{\text{data}}) \\ \quad + \min\left\{R_{\text{FB}}, \left(1 - \frac{\bar{R}}{R_{\text{data}}}\right) C_1\right\}, & \text{if } R_{\text{FB}} < \bar{R}, \\ E_{\text{Burn}}(\bar{R}), & \text{if } R_{\text{FB}} \geq \bar{R}, \end{cases} \quad (3)$$

where $E_{\text{Forn}}(\cdot)$ corresponds to (2), and $E_{\text{Burn}}(\cdot)$ to (1). The maximization above applies for $\bar{R} < \bar{R}^*_c = \frac{CC_1}{C+C_1}$; when $\bar{R} \geq R^*_c$ then $R_{\text{data}} = C$.

Proof: See Appendix A in [10].

When the joint channel-coding / hashing-function method from [8, Sections 3.4-5] is used instead, we have the following:

Proposition 2: An achievable error exponent for a one-way DMC with rate-limited R_{FB} noiseless feedback, and *joint design channel-coding / hashing-function* and VLC is given by: $E(\bar{R}, R_{\text{FB}}) \geq E^{\text{Joint}}_{\text{RL-FB}}(\bar{R}, R_{\text{FB}})$ for $\bar{R}_\infty \leq \bar{R} \leq C$ as:

$$E^{\text{Joint}}_{\text{RL-FB}}(\bar{R}, R_{\text{FB}}) := \begin{cases} \min\left\{R_{\text{FB}} + \frac{\bar{R}}{C} E_{\text{sp}}\left(\frac{C}{\bar{R}}(\bar{R} - R_{\text{FB}}), Q^*, W\right),\right. \\ \quad \left.\left(1 - \frac{\bar{R}}{C}\right) C_1\right\}, & \text{if } R_{\text{FB}} < \bar{R}, \\ E_{\text{Burn}}(\bar{R}), & \text{if } R_{\text{FB}} \geq \bar{R}, \end{cases} \quad (4)$$

where $E_{\text{sp}}(R, Q, W)$ corresponds to the sphere packing[6] error exponent for rate R, input distribution Q and channel law W, and Q^* is the capacity achieving input distribution.

Proof: Equation (4) results from [8, Equations (21-22)]. See Appendix B in [10].

Propositions 1 and 2 are based on the Yamamoto-Itoh scheme but using compressed noiseless feedback and allowing erasure decoding. Feedback is also exploited to maintain synchronization: the error-free control message informs the transmitter of whether the preliminary decision was accepted or not, regardless of correctness. The largest error exponent is attained by a hybrid system that chooses the scheme to use based on the rate-pair $(\bar{R}, R_{\text{FB}})$:

$$E_{\text{RL-FB}}\left(\bar{R}, R_{\text{FB}}\right) \geq \max\left\{E^{\text{RH}}_{\text{RL-FB}}\left(\bar{R}, R_{\text{FB}}\right), E^{\text{Joint}}_{\text{RL-FB}}\left(\bar{R}, R_{\text{FB}}\right)\right\}.$$

Figure 1 shows this error exponent (vertical axis) for different values of forward and feedback rate-pairs (mapped on the horizontal plane). Note that for a fixed $\bar{R}$, as $0 \leq R_{\text{FB}} < \bar{R}$ the first line in the expressions of either Proposition 1 (green area) or 2 (gray area) are achievable. Once R_{FB} reaches $\bar{R}$, the reliability jumps to Burnashev's, which for low $\bar{R}$ occurs at an edge.

If the forward channel has $C_0 > 0$, the operation of the Yamamoto-Itoh scheme is simplified since the control stage is free of errors in both directions, thus we have:

Proposition 3: An achievable error exponent for a one-way DMC with rate-limited R_{FB} noiseless active feedback, using random hashing, VLC, and satisfying $0 < C_0 < \bar{R}$, is given by $E(\bar{R}, R_{\text{FB}}) \geq E^{\text{RH-C0}}_{\text{RL-FB}}(\bar{R}, R_{\text{FB}})$, where:

$$E^{\text{RH-C0}}_{\text{RL-FB}}(\bar{R}, R_{\text{FB}}) := \begin{cases} R_{\text{FB}} + E_{\text{Forn}}(\bar{R}), & \text{if } R_{\text{FB}} < \bar{R}, \\ & \text{for } \bar{R}_\infty \leq \bar{R} \leq C, \\ \infty, & \text{if } R_{\text{FB}} \geq \bar{R}, \\ & \text{for } 0 \leq \bar{R} \leq C. \end{cases} \quad (5)$$

Note that $E_{1w}(R)$ is achievable in the regime $C_0 < \bar{R} < R_\infty$, and recall that infinite reliability is attainable for $0 \leq \bar{R} \leq C_0$.

This proposition illustrates how $C_0 > 0$ in the forward channel may be exploited to boost reliability in the small error regime as a consequence of having perfect knowledge of the receiver's control mode decisions.

Proof: See Appendix C in [10].

For channels with noisy feedback, VLC strategies must use additional synchronization recovery techniques. In Yamamoto-Itoh like schemes, feedback control messages may be incorrectly decoded at the encoder, causing terminals to lose track of what message is being transmitted. In contrast, a single bit transmitted with zero-error in either direction suffices to maintain synchronization: i.e. a terminal may use this bit to signal the termination of its own message, or alternatively, that it accepted the current message sent from the other terminal.

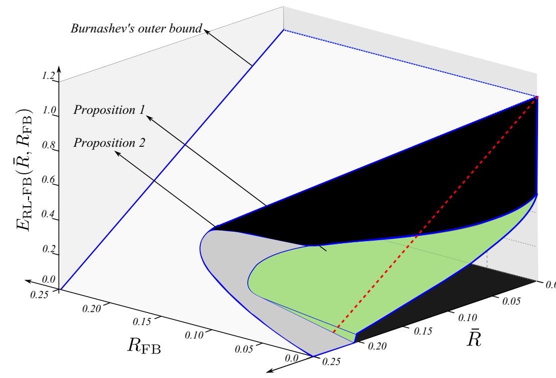

Fig. 1. Achievable Error exponent with rate-limited noiseless feedback and forward $C_0 = 0$. We evaluated all rate pairs satisfying $0 \leq \bar{R} \leq C$ and $0 \leq R_{\text{FB}} \leq C$ for a BSC(0.215).

IV. Main resuslts: two-way

1. Non-interacting terminals: Any error exponent achievable for a one-way DMC without feedback, $E_{1w}(R)$, is attainable in each direction of a two-way parallel DMC: i.e., $E_{12}(R_{12}, R_{21}) \geq E_{1w}(R_{12})$ and $E_{21}(R_{12}, R_{21}) \geq E_{1w}(R_{21})$.

2. Interacting terminals: when terminals employ feedback/ interaction, this affects the error exponents:

Proposition 4: An achievable error exponent pair for the two-way parallel DMC, in the rate-pair regime $0 < C_{0_{12}} < \bar{R}_{12} < C_{12}$ and $0 < C_{0_{21}} < \bar{R}_{21} < C_{21}$, using VLC is:

$$E_{12}(\bar{R}_{12}, \bar{R}_{21}) = E_{\text{Forn}}\left(\bar{R}_{12}\right)$$
$$E_{21}(\bar{R}_{12}, \bar{R}_{21}) = E_{\text{Forn}}\left(\bar{R}_{21}\right)$$

[6] $E_{\text{sp}}(R, Q, W)$ is used under the assumption of totally symmetric channels. See the discussion in [8, Equation (20), Sec. 3.4].

Proof: When $C_{0_{12}} = C_{0_{21}} = 0$, this is shown in [3, Prop. 3(i)]. Alternatively when $C_{0_{12}} > 0$ and $C_{0_{21}} > 0$, each terminal has access to at least a zero-rate noiseless feedback link, and Forney's (2) reliability can be directly achieved.

Next, we consider special cases where Proposition 4 can be further improved when noiseless feedback at positive rate is used in either direction depending on the zero-error capacity of each link. We first recall that Shannon [13] showed that in DMCs $R_\infty > 0$ if and only if every output cannot be reached from at least one channel input. Moreover, $0 \le C_0 \le R_\infty \le C$. Thus, depending on the transition probability matrix of a DMC, the following four cases (denoted by c_j for $j = 1, 2, 3, 4$) are possible in the two-way parallel DMC:

$$\begin{aligned} &c_1 : (C_0 = 0, R_\infty = 0), \quad c_2 : (C_0 = 0, R_\infty > 0), \\ &c_3 : (C_0 > 0, R_\infty > C_0), \quad c_4 : (C_0 = R_\infty > 0). \end{aligned} \tag{6}$$

There exist channels satisfying $R_\infty = C$, though we focus on the cases above. Thus, a two-way parallel DMC may result from the $1 \to 2$ link satisfying c_j, and the $1 \leftarrow 2$, c_k. We denote this by $(c_j; c_k)$ for $j, k \in \{1, \cdots, 4\}$. There are ten possible scenarios resulting from combinations of (6), two shown in Figure 2. Each may employ distinct schemes in different rate-pair regimes. We do not enumerate all possible cases; rather we present three examples that show how propositions of Section III can be used in the two-way setting.

In the first example, we show how the direction with $C_0 = 0$ benefits when the other has positive zero-error capacity. In the second and third examples, both channels have positive zero-error capacity. In the former the zero error capacities are equal and both directions benefit from it, exceeding Forney's reliability in subregion III. In the latter, zero-error capacities are different, yielding a small region outside rectangular regime $C_{0_{12}} \times C_{0_{21}}$ where infinite reliability is attainable.

Example 1. $(c_1; c_4)$: Consider that direction $1 \to 2$ is a symmetric channel with all probability entries strictly positive, (i.e. a row of the matrix is $[1-\epsilon, \frac{\epsilon}{2}, \frac{\epsilon}{2}]$ for $\epsilon > 0$) and $C_{0_{12}} = R_{\infty_{12}} = 0$. Direction $1 \leftarrow 2$ is a noisy typewriter channel [12] of 4 inputs with $C_{0_{21}} = R_{\infty_{21}} = 1$, and crossover probability $\epsilon < 1/2$. Thus $(c_1; c_4)$ results in the rate-pair regimes III and V as shown in Figure 2 (right). Regime V is further divided into two sub-regimes, V_I and V_{II} (see [10, Eq. (10)]), and a small portion labeled Ω. Region Ω is included in V_I or V_{II} depending on whether Proposition 1 or 2 is used for the $1 \to 2$ direction. For all rate-pairs in V, the $1 \leftarrow 2$ direction attains infinite reliability, thus we focus next on the $1 \to 2$ direction and illustrate how to take advantage of $C_{0_{21}} > 0$. We formalize this in the following:

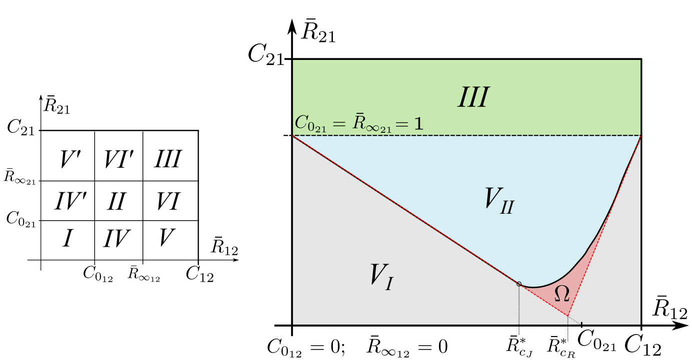

Fig. 2. Regimes in the capacity region of the two-way parallel DMC. Left: channels $(c_3; c_3)$, Right: channels $(c_1; c_4)$ –Example 1–.

Proposition 5: An achievable error exponent region for the two-way parallel DMC with $C_{0_{21}} = R_{\infty_{21}} > 0$ and $C_{0_{12}} = R_{\infty_{12}} = 0$ is determined for the following rate-pair regimes:

a. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in V_I$:

$$\begin{aligned} E_{12}(\bar{R}_{12}, \bar{R}_{21}) &= E_{\text{Burn}}(\bar{R}_{12}), \\ E_{21}(\bar{R}_{12}, \bar{R}_{21}) &= \infty, \end{aligned}$$

b. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in V_{II}$:

$$\begin{aligned} E_{12}(\bar{R}_{12}, \bar{R}_{21}) &\ge E^{\text{Joint}}_{\text{RL-FB}}(\bar{R}_{12}, C_{0_{21}} - \bar{R}_{21}), \\ E_{21}(\bar{R}_{12}, \bar{R}_{21}) &= \infty, \end{aligned}$$

when Proposition 2 is used, and if Proposition 1 is used:

$$\begin{aligned} E_{12}(\bar{R}_{12}, \bar{R}_{21}) &\ge E^{\text{RH}}_{\text{RL-FB}}(\bar{R}_{12}, C_{0_{21}} - \bar{R}_{21}), \\ E_{21}(\bar{R}_{12}, \bar{R}_{21}) &= \infty, \end{aligned}$$

c. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in III$:

$$E_{12}(\bar{R}_{12}, \bar{R}_{21}) = \begin{cases} \max_{\bar{R}_{12} \le R_{\text{data}_{12}} \le C_{12}} \frac{\bar{R}_{12}}{R_{\text{data}_{12}}} E_{\text{Forn}}(R_{\text{data}_{12}}) \\ \quad + \min\left\{ C_{0_{21}}\left(1 - \frac{\bar{R}_{21}}{R_{\text{data}_{21}}}\right), \left(1 - \frac{\bar{R}_{12}}{R_{\text{data}_{12}}}\right) C_1 \right\}, \\ \qquad\qquad \text{if } C_{0_{21}}\left(1 - \frac{\bar{R}_{21}}{R_{\text{data}_{21}}}\right) < \bar{R}_{12}, \\ E_{\text{Burn}}(\bar{R}_{12}), \\ \qquad\qquad \text{if } C_{0_{21}}\left(1 - \frac{\bar{R}_{21}}{R_{\text{data}_{21}}}\right) \ge \bar{R}_{12}. \end{cases}$$

$$E_{21}\left(\bar{R}_{12}, \bar{R}_{21}\right) \ge \max_{\bar{R}_{21} \le R_{\text{data}_{21}} \le C_{21}} \frac{\bar{R}_{21}}{R_{\text{data}_{21}}} E_{\text{Forn}}\left(R_{\text{data}_{21}}\right).$$

Proof: See Appendix D in [10].

Example 2. $(c_4; c_4)$: Consider a two-way parallel DMC formed by two identical channels with positive zero error capacity. Each direction corresponds to a noisy typewriter channel of 4 inputs, with $C_{0_{12}} = R_{\infty_{12}} = C_{0_{21}} = R_{\infty_{21}} = 1$ and crossover probability $\epsilon < 1/2$. The resulting rate-regimes are subregions I, V, V' and III, from Figure 2 (left). Error exponent for regions V and V' follow similarly.

Proposition 6: An achievable error exponent region for the two way-parallel DMC with both directions having the same zero-error capacity is:

a. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in I$: $E_{12}(\bar{R}_{12}, \bar{R}_{21}) = \infty, E_{21}(\bar{R}_{12}, \bar{R}_{21}) = \infty$.
b. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in III$: See Proposition 7e in Example 3.
c. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in V$:

$$\begin{aligned} E_{12}(\bar{R}_{12}, \bar{R}_{21}) &\ge E_{\text{Forn}}(\bar{R}_{12}) + (C_{0_{21}} - \bar{R}_{21}), \\ E_{21}(\bar{R}_{12}, \bar{R}_{21}) &= \infty. \end{aligned}$$

Proof: See Appendix E in [10]. Note that analogous results apply $\forall(\bar{R}_{12}, \bar{R}_{21}) \in V'$.

Example 3. $(c_3; c_4)$: Consider a two-way parallel DMC, both directions with positive zero-error capacity, one larger than the other. Let the $1 \to 2$ direction be a 4 input noisy typewriter channel with $C_{0_{12}} = R_\infty = 1$ and crossover probability $\epsilon < 1/2$; and let the $1 \leftarrow 2$ direction to be a pentagon channel [14] with $C_{0_{21}} = \log\sqrt{5} \approx 1.16$, and $R_\infty = \log\left(\frac{5}{2}\right)$. Figure 3 shows the capacity region and rate-pair regimes. The small triangle Γ achieves the same reliability as in I. Error exponents for V' result by flipping those of V, but Γ.

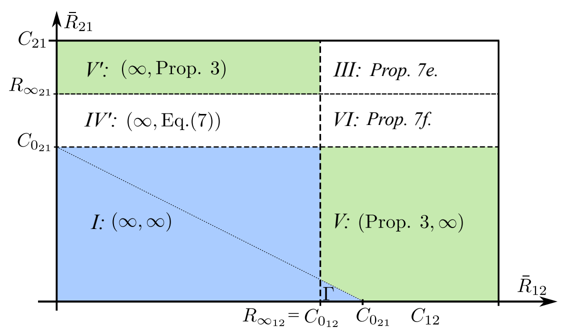

Fig. 3. Two-way parallel DMC of channel combination $(c_4; c_3)$.

Proposition 7: An achievable error exponent region for the two-way parallel DMC satisfying: $0 < C_{021} < R_{\infty_{21}} < C_{21}$, and $0 < C_{0_{12}} = R_{\infty_{12}} < C_{12}$ and $C_{0_{21}} > C_{0_{12}}$ is:

a. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in I$: $E_{12}(\bar{R}_{12}, \bar{R}_{21}) = \infty, E_{21}(\bar{R}_{12}, \bar{R}_{21}) = \infty$.

b. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in IV'$:

$$E_{12}(\bar{R}_{12}, \bar{R}_{21}) = \infty$$
$$E_{21}(\bar{R}_{12}, \bar{R}_{21}) \geq E_r(\bar{R}_{21}) + (C_{0_{12}} - \bar{R}_{12}), \quad (7)$$

where (7) results from Proposition 3 with the random coding error exponent $E_r(R)$ instead of Forney's.

c. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in V$:

$$E_{12}(\bar{R}_{12}, \bar{R}_{21}) \geq E_{\text{Forn}}(\bar{R}_{12}) + (C_{0_{21}} - \bar{R}_{21}),$$
$$E_{21}(\bar{R}_{12}, \bar{R}_{21}) = \infty.$$

d. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in \Gamma$: $E_{12}(\bar{R}_{12}, \bar{R}_{21}) = \infty, E_{21}(\bar{R}_{12}, \bar{R}_{21}) = \infty$.

e. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in III$: An achievable error exponent region results as Figure 4 (refer to Appendix F).

f. $\forall(\bar{R}_{12}, \bar{R}_{21}) \in VI$: This regime is essentially similar to III, with the distinction that the $1 \leftarrow 2$ direction cannot achieve Forney's reliability, thus, a general $E_{1w}(\cdot)$ error exponent should be used instead of $E_{\text{Forn}}(\cdot)$ since ML decoding is used.

Proof: See Appendix F in [10].

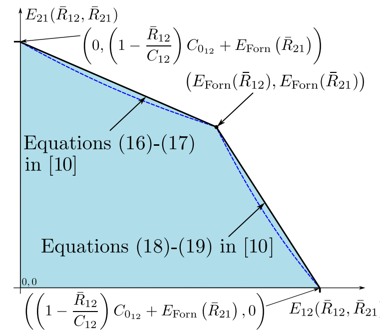

Fig. 4. Achievable EER for the subregion III of Proposition 7.

V. Discussion and Open Problems

There exist multiple open problems in two-way channels, including a) the two-way variable length zero-error capacity region; b) outer bounds for the two-way DMC EER for all rate-pair regimes; and c) how messages and feedback can be transmitted without invoking a time-sharing argument. Our initial characterization of EER aims to illustrate how a positive zero-error capacity can be exploited to not only resolve synchronization but also simplify coding schemes and increase reliability in the small error regime.

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