Competition in Electric Autonomous Mobility on Demand Systems

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Abstract—**This paper investigates the impacts of competition in autonomous mobility-on-demand systems. By adopting a network-flow based formulation, we first determine the optimal strategies of profit-maximizing platform operators in monopoly and duopoly markets, including the optimal prices of rides. Furthermore, we characterize the platform operator's profits and the consumer surplus. We show that for the duopoly, the equilibrium prices for rides have to be symmetric between the firms. Then, in order to study the benefits of introducing competition in the market, we derive universal theoretical bounds on the ratio of prices for rides, aggregate demand served, profits of the firms, and consumer surplus between the monopolistic and the duopolistic setting. We discuss how consumers' firm loyalty affects each of the aforementioned metrics. Finally, using the Manhattan network and demand data, we quantify the efficacy of static pricing and routing policies and compare it to real-time model predictive policies.**

I. Introduction

In the past decade, growing popularity of mobility-on-demand (MoD) platforms has extensively altered the paradigm of urban mobility. Owing to the rapid evolution of enabling technologies for autonomous driving and advancements in eco-friendly electric vehicles (EVs), it is possible for MoD platforms to employ self-driving electric vehicles and therefore preserve the benefits of private automobiles while reducing the consumption of non-renewable energy resources. Given this, the vision of an electric and autonomous mobility-on-demand (AMoD) fleet serving urban customers' mobility needs is gaining traction within the transportation industry, with multiple companies now heavily investing in AMoD technology [1].

Unlike modern ride-sharing platforms that establish a two-sided market between drivers and customers such as Uber and Lyft and rely only on pricing schemes to manage the demand-supply balance, self-driving technology allows AMoD systems to operate in a single-sided market in which the customers are directly served by the platform operator rather than the drivers. This allows the platforms to centrally manage their fleets of vehicles and hence efficiently dispatch them to where they are needed the most without the need to incentivize drivers, including *rebalancing* the idle passenger-free vehicles throughout the network to match supply and demand. Furthermore, autonomous vehicles can better exploit diversity in electricity prices for cheaper operation.

In this paper, we study the effects of competition in electric AMoD systems that are operated by profit-maximizing platform operators. Owing to the opportunities that autonomous electric vehicles create for efficient control schemes and cost-effective operation, it is possible for a single platform operator

to provide cheap rides through optimizing the prices of rides for geographical load balancing as well as optimally routing and charging the fleet of electric vehicles. However, a monopolistic market with a single AMoD provider is in general disadvantageous for customer welfare. Therefore, introduction of another AMoD service provider to the market results in firms competing over the customers, hence forcing them to charge fairer prices and provide a higher quality of service. Our primary goal is to investigate the optimal behaviour of the firms in a monopoly and duopoly and quantify the impacts of competition on the customers as well as the firms.

Our contributions can be summarized as follows:

- We formalize the platform operator's profit maximization problem by adopting a static network-flow based model that captures the characteristics of an AMoD fleet, and derive expressions for the ride prices, profits, and consumer surplus under the optimal static policy.
- We prove that if the competitors have identical costs, then the duopoly equilibrium prices have to be symmetric. We show that under a mild sufficient condition on maximum travel costs that can be met with electric vehicles, the duopoly prices in equilibrium are never larger than the optimal monopoly prices. Furthermore, we derive theoretical bounds for the ratio of prices, induced demand, profits, and consumer surplus in the monopoly and the duopoly equilibrium.
- We study a real-time pricing and fleet management policy using model predictive control, and demonstrate the performance numerically on real network and demand data.

Related work: Research on AMoD systems concentrates on optimal fleet control, with a particular focus on rebalancing. Scholars have tackled the rebalancing problem using queueing theoretical [2], fluidic [3], network flow [4], and Markovian models [5]. These works develop time-invariant policies by relying on the steady-state of the system, which may not be able to address real-world challenges such as variability in customer arrivals or integrality requirements in real-time dispatch. As a consequence, studies aiming to develop efficient real-time control policies using model predictive control (MPC) have emerged in the literature, e.g., [6]–[8]. The authors of [7] design a data-driven MPC algorithm by predicting the future demand, whereas the authors of [8] develop a stochastic MPC algorithm that leverages uncertain travel demand forecasts. In [6], the authors also consider a fleet of EVs and hence propose an MPC approach that optimizes vehicle routing and scheduling subject to energy constraints. EV charging problem in electric AMoD systems has also been studied using dynamic programming [9] and online heuristics [10]. None of these studies however adopt a price-responsive demand model and exploit pricing to further optimize the system. Lately, benefits

of joint pricing and rebalancing in AMoD systems have been demonstrated using macroscopic steady-state models [11], [12] as well as microscopic dynamic models [12].

Research on competition in ride-sharing markets is also relevant to ours. In terms of a broader scope on platform competition in two-sided markets, [13] and [14] introduce general frameworks and provide in-depth analysis. The impacts of single/multi-homing users on the market equilibria have been investigated in [15]. Theoretical studies on dynamic platform competition [16] and spatial platform competition [17] in two sided markets further provide insights towards competition in ride-sharing markets. Besides these, scholars examine the competition between ride-sharing and taxis [18], [19], where Uber is considered to be a monopoly. These works however do not capture the competition among ride-sharing platforms, yet ride-sharing markets are rather oligopolies in many countries [20]. Accordingly, a recent work [21] presents a head-to-head comparison of Uber, Lyft, and taxis using statistical methods. Another line of work related to ours focuses on the benefits of spatial price discrimination [22] and dynamic pricing in ride-sharing networks [23], [24]. These however do not study a competitive market. Closest to our work is [25], which studies the effects of thickness (i.e., the mass of drivers) and competition on the equilibria of ride-sharing markets. It shows that competition always increases the welfare of the drivers, whereas it decreases the welfare of the customers if the market is not sufficiently thick.

To the best of our knowledge, there is no existing work on competition in electric AMoD systems. Our study aims to form the bridge between AMoD and competition literature with our theoretical findings. We hope that the closed form bounds quantifying the impacts of competition would help investors make informed policy decisions about competing AMoD platforms and investing in efficient AMoD technologies.

II. SYSTEM MODEL AND PROBLEM DEFINITION

Network and Demand Models: We consider two fleets of AMoD EVs operated by two competitors within a transportation network characterized by a complete graph consisting of $\mathcal{N} = \{1, \ldots, n\}$ nodes. Each of these nodes can serve as a trip origin or destination.

We study a discrete-time system with time periods normalized to integral units $t \in \{0, 1, 2, \ldots\}$. In each period, potential riders of mass θ_{ij} seek rides between origin-destination (OD) pair (i, j), where $\theta_{ii} = 0$. We assume that customers have different valuations for riding with each firm, represented by the tuple (v_1, v_2) where v_f is the customer's valuation for firm f. To capture customer heterogeneity, we let $(v_1, v_2) \sim \mathcal{V}$, where $\mathcal{V}$ denotes the PDF of the joint distribution with support $[0, \ell_{\max}]^2$. Here, $\ell_{\max}$ is the maximum valuation of the customers for both firms, i.e. the maximum willingness to pay[1]. To characterize the distribution $\mathcal{V}$, we adopt the model proposed by [25] and assume that the distribution of the random variables (v_1, v_2) is defined implicitly through:

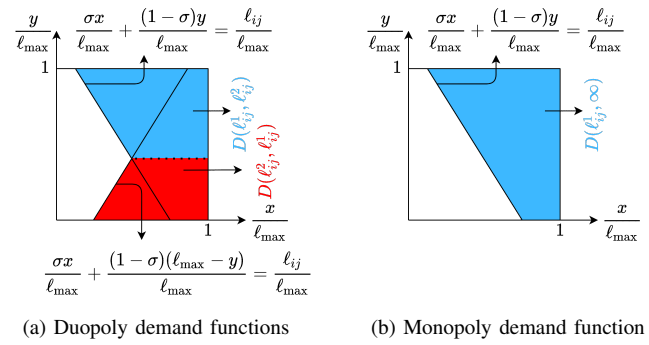(a) Duopoly demand functions   (b) Monopoly demand function

Fig. 1: Graphical illustration of the demand functions for (a) duopoly, and (b) monopoly. The axes correspond to the uniform random variables x and y scaled by $1/\ell_{\max}$. In duopoly, the line $\sigma x + (1-\sigma)y = \ell_{ij}^1$ corresponds to the customers who earn 0 pay-off buying a ride from firm 1, and the line $\sigma x + (1-\sigma)(\ell_{\max} - y) = \ell_{ij}^2$ corresponds to the customers who earn 0 pay-off buying a ride from firm 2. As such, for the price tuple $(\ell_{ij}^1, \ell_{ij}^2)$, the blue shaded area corresponds to the demand function for firm 1, whereas the red corresponds to the demand function for firm 2. Monopoly is the special case of duopoly, where the prices for rides set by firm 2 are set to infinity: $\ell_{ij}^2 = \infty$.

$$v_1 = \sigma x + (1 - \sigma)y, \qquad (1)$$
$$v_2 = \sigma x + (1 - \sigma)(\ell_{\max} - y), \qquad (2)$$

where x and y are iid uniform random variables with support $[0, \ell_{\max}]$ and $\sigma \in [0, 1]$. We refer to x as the *common value component* and y as the *idiosyncratic* component, with σ as the measure of correlation over customers' preferences[2]. In particular, x can be viewed as a customer's valuation of the ride itself and y (or $\ell_{\max} - y$) can be viewed as a customer's valuation of firm 1 itself (or firm 2 itself). A customer is identified by the draws from distributions of x and y, which are then mapped to that customer's valuations for riding with firms 1 and 2 via (1) and (2). A customer with valuations (v_1, v_2) makes a decision upon observing the prices for rides. If the prices for rides between OD pair (i, j) in period are set to be ℓ_{ij}^1 and ℓ_{ij}^2 by firm 1 and 2, respectively, the customer buys a ride from firm f if $v_f - \ell_{ij}^f > 0$ and $v_f - \ell_{ij}^f > v_{-f} - \ell_{ij}^{-f}$ (given firm f, $-f$ denotes the other firm), i.e., the customer gains a positive pay-off for purchasing a ride from firm f and this pay-off is higher than the pay-off that the customer would gain by buying from the other firm. Otherwise they do not buy a ride from either of the firms and leave the system. Hence, for a price tuple $(\ell_{ij}^1, \ell_{ij}^2)$ for OD pair (i, j), the induced mass of arrivals for firm f is given by $\Theta_{ij}^f := \theta_{ij} D(\ell_{ij}^f, \ell_{ij}^{-f})$, where $D : [0, \ell_{\max}]^2 \to [0, 1]$ is the *demand function* of customers which determines the fraction of customers that would buy a ride from firm f upon observing the prices. This function has a simple geometric interpretation depicted in Figure 1. We plot the demand function and the willingness to pay distribution as a function of ride prices for several values of σ in the monopolistic setting in Figure 2. Note that the demand function is concave if $\ell_{ij}^1 < (1-\sigma)\ell_{\max}$, is linear if $(1-\sigma)\ell_{\max} \le \ell_{ij}^1 < \sigma\ell_{\max}$, and is convex if $\sigma\ell_{\max} \le \ell_{ij}^1$.
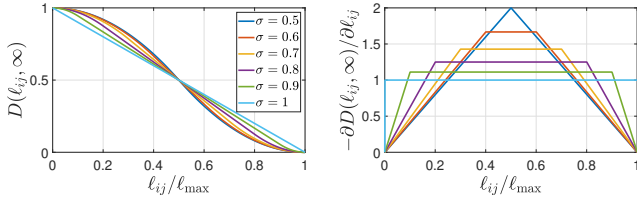
Fig. 2: Demand function (left) and willingness to pay distribution (right) as a function of ride prices for several values of $\sigma \in [0.5, 1]$.

Vehicle Model: In order to best serve its customers and maximize its profits, each operator needs to dispatch its fleet, including vehicle routing and charging. To implicitly capture the effect of trip demand and the associated charging and routing decisions on the fleet size and hence the operational costs incurred by each operator, we assume that each vehicle in charging or trip-making mode has a per period operational cost of β_c and β_t, respectively. A trip-making vehicle can either be occupied by a customer, which we refer to as a customer carrying vehicle; or can be empty, which we refer to as a rebalancing vehicle. We note that in this work, we set the capacity of a vehicle to be one passenger. Furthermore, as the vehicles are electric, they have to sustain charge in order to operate, which needs to be purchased from the power grid. Without loss of generality, we assume there is a charging station placed at each node $i \in \mathcal{N}$. To charge at node i, the operator pays a price of electricity p_i per unit of energy. We assume that all EVs in the fleet have a battery capacity denoted as $e_{\max} \in \mathbb{Z}^+$; therefore, each EV has a discrete battery energy level $e \in \mathcal{E}$, where $\mathcal{E} = \{e \in \mathbb{N} | 0 \le e \le e_{\max}\}$. In our discrete-time model, we assume each vehicle takes one period to charge one unit of energy and τ_{ij} periods to travel between OD pair (i, j), while consuming e_{ij} units of energy.

Platform Operator's Problem: We consider a profit-maximizing AMoD operator that manages a fleet of EVs that make trips to provide transportation services to customers. The operator's goal is to maximize profits by 1) setting prices for rides and hence managing customer demand at each node; 2) optimally operating the AMoD fleet (i.e., charging and routing) to minimize operational and charging costs. Next, we study the static planning problem for both the monopoly and the duopoly settings in order to characterize the optimal static prices and to examine the effects of competition in electric AMoD systems.

III. ANALYSIS OF THE STATIC PROBLEM

In this section, we establish and discuss the static planning problems considering a single operator (i.e., monopoly) and two competing operators (i.e., duopoly) in order to study the effect of competition in an electric AMoD system. We consider the fluid scaling of the network and characterize the static planning problem via a network flow formulation. The static problem is convenient for determining the optimal static pricing, routing, and charging policy of the platform operator.

A. Monopoly Static Planning Problem

We define the monopoly to be the setting where the firm 2 is removed. In order to make the comparison between the monopoly and the duopoly consistent, we keep the customer

behaviour and the demand function D same. Hence, removing firm 2 from the system is equivalent to setting prices for rides posted by firm 2 to be ∞, and the induced demand for rides for OD pair (i, j) to be $D(\ell_{ij}^1, \infty)$ for a given ℓ_{ij}^1.

The goal of the platform operator is to maximize its profits by setting prices for rides and making routing and charging decisions such that the induced demand is served. Let x_{ij}^e be the number of vehicles at node i with energy level e being routed to node j and x_{ic}^e be the number of vehicles charging at node i and currently at energy level e. We state the platform operator's problem as follows:

$$\max_{x_{ic}^e, x_{ij}^e, \ell_{ij}^1} \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{ij} \ell_{ij}^1 D(\ell_{ij}^1, \infty)$$

$$- \sum_{i=1}^{n} \sum_{e=0}^{e_{\max}-1} (\beta_c + p_i) x_{ic}^e - \beta_t \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{e=e_{ij}}^{e_{\max}} x_{ij}^e \tau_{ij} \quad (3a)$$

subject to $\quad \theta_{ij} D(\ell_{ij}^1, \infty) \le \sum_{e=e_{ij}}^{e_{\max}} x_{ij}^e \quad \forall i, j \in \mathcal{N}, \quad (3b)$

$$x_{ic}^e + \sum_{j=1}^{n} x_{ij}^e = x_{ic}^{e-1} + \sum_{j=1}^{n} x_{ji}^{e+e_{ji}},$$
$$\forall i \in \mathcal{N}, \ \forall e \in \mathcal{E}, \quad (3c)$$

$$x_{ic}^{e_{\max}} = 0, \quad \forall i \in \mathcal{N}, \quad (3d)$$

$$x_{ij}^e = 0, \quad \forall e < e_{ij}, \ \forall i, j \in \mathcal{N}, \quad (3e)$$

$$x_{ic}^e \ge 0, \ x_{ij}^e \ge 0, \quad \forall i, j \in \mathcal{N}, \ \forall e \in \mathcal{E}, \quad (3f)$$

$$x_{ic}^e = x_{ij}^e = 0, \quad \forall e \notin \mathcal{E}, \ \forall i, j \in \mathcal{N}. \quad (3g)$$

The objective function (3a) corresponds to the profits earned by the firm per period. In particular, the first term in (3a) accounts for the aggregate revenue the platform generates by providing rides for $\theta_{ij} D(\ell_{ij}^1, \infty)$ number of riders with a price of ℓ_{ij}^1. The second term is the operational and charging costs incurred by the charging vehicles, and the last term is the operational costs of the trip-making vehicles.

The constraint (3b) requires the platform to operate at least as many vehicles to serve all the induced demand between any two nodes i and j (The rest are the vehicles travelling without passengers, i.e., rebalancing vehicles). We will refer to this as the *demand satisfaction constraint*. We let λ_{ij} be the dual variable associated with (3b) and λ_{ij}^m be the optimal dual variable. The constraint (3c) is the *flow balance constraint* for each node and each battery energy level, which restricts the number of available vehicles at node i and energy level e to be the sum of arrivals from all nodes and vehicles that are charging with energy level $e - 1$. The constraint (3d) ensures that the vehicles with full battery do not charge further, and the constraint (3e) ensures the vehicles sustain enough charge to travel between OD pair (i, j).

It is worthwhile to mention that unlike traditional minimum-cost flow problems, where the objective is to minimize total travel cost, the objective of (3) is to maximize the total revenue minus the costs, i.e., profits. Furthermore, in traditional minimum-cost flow problems, demand elasticity in response to price is not explicit and the elasticity is often modeled in response to travel times [26], [27], whereas the explicit

dependency of the induced demand to prices via $D(\ell_{ij}^1, \infty)$ results in a more challenging task. The prices affect the induced demand, which affects the routing decisions and this causes a complex interplay between the decision variables.

Optimal Pricing: The prices for rides are a crucial component of the profits generated. The next proposition highlights how the optimal prices $\ell_{ij}^m := \ell_{ij}^{1*}$ for rides are related to the network parameters, prices of electricity, and the operational costs. In the following results, we investigate this interconnection by providing upper bounds on the prices that a profit-maximizing monopolist may charge customers, as well the corresponding profits generated. We highlight the fact that the monopolist's profits are in fact a decreasing function of the optimal prices for rides. The higher the monopolist has to charge its customers, the lower its generated profits. This could be a motivation for the monopolist to invest in efficient vehicle technology and cheap charging solutions.

Proposition 1. *Define*
$$\overline{\lambda}_{ij} := \beta_t(\tau_{ij} + \tau_{ji}) + e_{ij}(p_j + \beta_c) + e_{ji}(p_i + \beta_c).$$
Let λ_{ij}^m be the optimal dual variable corresponding to the demand satisfaction constraint (3b) for OD pair (i,j). The optimal monopoly prices ℓ_{ij}^m are:

$$\ell_{ij}^m = \begin{cases} \frac{\lambda_{ij}^m + \sqrt{(\lambda_{ij}^m)^2 + 6\sigma(1-\sigma)\ell_{\max}^2}}{3} & , \frac{\lambda_{ij}^m}{\ell_{\max}} < \frac{3-5\sigma}{2} \\ \frac{(1+\sigma)\ell_{\max} + 2\lambda_{ij}^m}{4} & , \frac{3-5\sigma}{2} \le \frac{\lambda_{ij}^m}{\ell_{\max}} < \frac{3\sigma-1}{2} \\ \frac{2\lambda_{ij}^m + \ell_{\max}}{3} & , \frac{3\sigma-1}{2} \le \frac{\lambda_{ij}^m}{\ell_{\max}} \le 1. \end{cases} \quad (4)$$

These prices can be upper bounded by:

$$\ell_{ij}^m \le \begin{cases} \frac{\overline{\lambda}_{ij} + \sqrt{(\overline{\lambda}_{ij})^2 + 6\sigma(1-\sigma)\ell_{\max}^2}}{3} & , \frac{\overline{\lambda}_{ij}}{\ell_{\max}} < \frac{3-5\sigma}{2} \\ \frac{(1+\sigma)\ell_{\max} + 2\overline{\lambda}_{ij}}{4} & , \frac{3-5\sigma}{2} \le \frac{\overline{\lambda}_{ij}}{\ell_{\max}} < \frac{3\sigma-1}{2} \\ \frac{2\overline{\lambda}_{ij} + \ell_{\max}}{3} & , \frac{3\sigma-1}{2} \le \frac{\overline{\lambda}_{ij}}{\ell_{\max}} \le 1. \end{cases} \quad (5)$$

The proof can be found in Appendix A. We can interpret the dual variables λ_{ij}^m as the cost of providing a single ride between i and j to the platform. In the worst case scenario, every single requested ride from node i requires rebalancing and charging both at the origin and the destination. Hence the upper bounds on (5) include the operational costs of passenger-carrying, rebalancing and charging vehicles (both at the origin and the destination); and the energy costs of both passenger-carrying and rebalancing trips multiplied by the price of electricity at the trip destinations (This is exactly what $\overline{\lambda}_{ij}$ consists of).

Similar to the taxes applied on products, whose burden is shared among the supplier and the customer; the costs associated with rides are shared among the platform operator and the riders (which is why the price paid by the riders include some fraction of the cost of the ride).

We note that if the optimal dual variables λ_{ij}^m fall in the region $[(3 - 5\sigma)\ell_{\max}/2, (3\sigma - 1)\ell_{\max}/2]$, then the optimal prices given by (4) fall in the region $[(1-\sigma)\ell_{\max}, \sigma\ell_{\max}]$. In this region, the demand function $D(\ell_{ij}^m, \infty)$ is linear. Hence, the optimization problem (3) (with the additional constraint $(1 - \sigma)\ell_{\max} \le \ell_{ij}^1 \le \sigma\ell_{\max}, \ \forall i,j \in \mathcal{N}$, without losing global optimality) becomes a convex quadratic program and

can be solved in polynomial time. The following assumptions guarantee this:

Assumption 1. *Assume that $\sigma \ge 3/5$, i.e., the customers' preferences over the two firms are highly correlated.*

Assumption 2. *We assume $\max_{i,j} \overline{\lambda}_{ij} \le \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{\max}$ as an upper bound on the maximum cost of a ride in the network.*

Remark 1. *Assumption 1 implies that at least $3/5$ (60%) of the customers' valuations between the firms are correlated. Higher correlation implies that riders' valuations of the rides provided by a firm depend less on the identity of the firm. This is reasonable for ride-sharing platforms, where majority of the customers decide depending heavily on the price rather than the identity of the firm.*
Assumption 2 imposes an upper bound on the maximum cost of a ride. This can be satisfied in practice, especially with electric vehicles. Observe that the bound is increasing with σ, hence it is tightest when $\sigma = 3/5$. To give numbers with a simple calculation, consider a network with farthest OD pair of 15 miles and 30 minutes away (with average speed 30mph), $\sigma = 3/5$ and $\ell_{\max} = \$50$. An average EV consumes 34kWh energy to drive for 100 miles. For an average price of electricity of $\$0.11$ per kWh and a charger with $20kW$ charging speed, the EV charges $10kWh$ in 30 minutes for $\$1.1$, that allows for 30 miles of range. If we amortize the cost of a very expensive EV of $\$100k$ over 5 years, we get per minute operational cost of $\$0.04$. In total, to do the trip and the rebalancing, the vehicle drives for 30 miles for 1 hour and charges for 30 minutes. In total, this yields a cost of $90 \times \$0.04 + \$1.1 = \$4.7 = \max_{i,j} \overline{\lambda}_{ij} \le \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)} = \$7.5$. Whereas the fuel for gasoline vehicles costs about 4 times more (around $\$0.16$ per mile), which would yield $\max_{i,j} \overline{\lambda}_{ij} = \$8.00$.

Next, we relate the optimal prices ℓ_{ij}^m to the profits generated by the operator and the consumer surplus. The profits are defined by the objective function in (3a). The consumer surplus is defined as the difference between the price that customers pay and the price that they are willing to pay, i.e., the aggregate pay-off of the customers.

Proposition 2. *Suppose that Assumptions 1 and 2 hold. With the optimal monopoly prices ℓ_{ij}^m, the profits per period are:*

$$P^m = \sum_{i=1}^n \sum_{j=1}^n \frac{\theta_{ij}}{4\sigma\ell_{\max}}(\ell_{\max}(1+\sigma) - 2\ell_{ij}^m)^2. \quad (6)$$

The consumer surplus with the optimal prices is:

$$CS^m = \sum_{i=1}^n \sum_{j=1}^n \theta_{ij} \frac{\ell_{\max}(\sigma^2+\sigma+1) - 3\ell_{ij}^m(1+\sigma - \frac{\ell_{ij}^m}{\ell_{\max}})}{6\sigma}. \quad (7)$$

The proof can be found in Appendix B. Notice that the profits in (6) are decreasing as the prices for rides increase. Thus expensive rides generate less profits compared to the cheaper rides and it is more beneficial if the optimal dual variables λ_{ij}^m are small and prices are close to $\ell_{\max}(1+\sigma)/4$. Thus, the operator has incentive to use more efficient routing and charging policies so they can lower ride prices as much

as possible. Moreover, by computing $\frac{\partial \mathrm{CS}^m}{\partial \ell_{ij}^m}$ using (7), one identifies that lower prices generate higher consumer surplus, which is an intuitive result.

B. Duopoly Static Planning Problem

We study the duopoly as a game between two firms. At a high level, the game is described by firm f observing firm $-f$'s prices and solving the optimization problem (3) (by considering firm $-f$'s prices to be ℓ_{ij}^{-f} rather than ∞ for the demand function). We consider two competitors with identical operational costs β_t and β_c, and study the optimal pricing strategy when the firms are at an equilibrium. In an equilibrium, no firm benefits from unilaterally changing the prices for any number of OD pairs (and as a result the optimal solution to their static planning problem). Given $\{\ell_{ij}^{-f}\}_{\forall i,j \in \mathcal{N}}$, the best response of firm f is the best pricing, routing and charging strategy of f, which is the solution of (3) (with ℓ_{ij}^{-f} instead of ∞ in the demand function). Since the operational costs and the prices of electricity are identical for both of the firms, their best response to their competitor's prices are the same. As such, it is intuitive that there exists an equilibrium in which both firms set the prices equal ($\ell_{ij}^f = \ell_{ij}^{-f}, \forall i,j \in \mathcal{N}$), and we show that this is in fact the case. Such an equilibrium is commonly referred to as a *symmetric* duopoly equilibrium. Furthermore, we show that no asymmetric equilibria can exist under this setting, i.e., identical firms will not set different prices for the same OD pair at equilibrium.

Let the following static planning problem characterize the state in which both firms serve equal number of customers for all OD pairs and have identical pricing strategies:

$$\max_{x_{ic}^e, x_{ij}^e, \ell_{ij}^1} \sum_{i=1}^n \sum_{j=1}^n \theta_{ij} \ell_{ij}^1 D(\ell_{ij}^1, \ell_{ij}^2)\Big|_{\ell_{ij}^1 = \ell_{ij}^2}$$

$$- \sum_{i=1}^n \sum_{e=0}^{e_{\max}-1} (\beta_c + p_i) x_{ic}^e - \beta_t \sum_{i=1}^n \sum_{j=1}^n \sum_{e=e_{ij}}^{e_{\max}} x_{ij}^e \tau_{ij} \quad (8a)$$

$$\text{subject to } \theta_{ij} D(\ell_{ij}^1, \ell_{ij}^2)\Big|_{\ell_{ij}^1 = \ell_{ij}^2} \leq \sum_{e=e_{ij}}^{e_{\max}} x_{ij}^e, \quad \forall i,j \in \mathcal{N}, \quad (8b)$$

$$(3c) - (3g).$$

We note that the optimization problem (8) is in general non-convex due to $D(\ell_{ij}^1, \ell_{ij}^2)$. Since there are no constraints on the fleet size and furthermore prices that control the demand are decision variables, a feasible solution to the above optimization problem always exists. Moreover, the optimal solution to (8) specifies an equilibrium of the duopoly.

Proposition 3. *Suppose that $\sigma \geq 1/2$ and Assumption 2 holds. The firms are in an equilibrium when their routing, charging, and symmetric pricing strategy follows the solution of (8).*

Proof outline: We first determine the optimal pricing strategy $\{\ell_{ij}^d\}_{i,j \in \mathcal{N}}$ of (8) using the first and second order optimality conditions (similar to proof of Proposition 1). Then, by stating the first order optimality condition for firm f we show that when firm $-f$ sets prices as $\ell_{ij}^{-f} = \ell_{ij}^d, \forall i,j \in \mathcal{N}$, then the best response of firm f is to set $\ell_{ij}^f = \ell_{ij}^d, \forall i,j \in \mathcal{N}$. Hence, they are in an equilibrium. □

The complete proof can be found in Appendix C, which is provided in the online version [28]. Accordingly, there exists a duopoly equilibrium characterized as the optimal solution of (8), in which the firms set identical prices. The optimal solution to (8) is however not necessarily unique and there can be many solutions yielding the same profits. For instance, if $p_i = p_j, \forall i,j \in \mathcal{N}$, then the optimal charging strategy is not unique. We let $\{\ell_{ij}^d\}_{i,j \in \mathcal{N}}$ to be the equilibrium prices determined as an optimal solution of (8) and say that the firms are in a symmetric duopoly equilibrium as long as $\ell_{ij}^1 = \ell_{ij}^2 = \ell_{ij}^d, \forall i,j \in \mathcal{N}$. Furthermore, in the next proposition, we state that if both firms serve all OD pairs, equilibrium prices can not be asymmetric.

Proposition 4. *Suppose that $\sigma \geq 1/2$ and Assumption 2 holds. There exists no asymmetric equilibrium prices, in which both firms serve nonzero demand for all OD pairs with nonzero potential riders.*

Proof outline: We let $\ell_{ij}^1 = \ell_{ij}^2 + \delta$ for some $\delta > 0$ and show by contradiction that the first-order optimality condition can not simultaneously be satisfied for both firms. Since the demand function $D(\ell_{ij}^1, \ell_{ij}^2)$ has different expressions for $\ell_{ij}^1 \leq (1 - \sigma)\ell_{\max}$ and $\ell_{ij}^1 > (1 - \sigma)\ell_{\max}$, we separately study three cases: (i) $\ell_{ij}^1, \ell_{ij}^2 \leq (1 - \sigma)\ell_{\max}$, (ii) $\ell_{ij}^1, \ell_{ij}^2 > (1 - \sigma)\ell_{\max}$, and (iii) $\ell_{ij}^1 > (1 - \sigma)\ell_{\max}$, $\ell_{ij}^2 \leq (1-\sigma)\ell_{\max}$. For all cases, we first assume that the first-order optimality condition hold for both firms and bound the difference between the dual variables leading to ℓ_{ij}^1 and ℓ_{ij}^2 in terms of δ. For cases (i) and (ii), we show by using the bound on the dual variables that if the first-order condition for firm 2 is satisfied (i.e., is equal to 0), then the first-order condition for firm 1 is always less than 0, which is a contradiction. For case (iii), we show that with first-order condition satisfying prices, $\ell_{ij}^2 + \delta$ is always less than ℓ_{ij}^1, which is a contradiction. □

The complete proof is provided in Appendix D of the online version [28]. As we have identified that the duopoly can only be in a symmetric equilibrium, we analyze the effects of competition in state of a symmetric equilibrium.

The next set of results characterize the effects of competition on the ride prices, the operators' profits, the total societal ride demand served, and the consumer surplus. In the first result, we provide lower and upper bounds on the price reduction the customers will see with the introduction of the second firm and moving from a monopoly to a symmetric duopoly equilibrium.

Proposition 5. *Suppose that Assumptions 1 and 2 hold. Let $\overline{\lambda}_{ij}$ be defined as in Proposition 1. Define*

$$\Delta_1(\lambda_{ij}) := 4\ell_{\max}^2 + (2\lambda_{ij} + (15\sigma - 3)\ell_{\max})(2\lambda_{ij} + (1-\sigma)\ell_{\max}),$$

$$\Delta_2(\lambda_{ij}) := 2(\sigma\ell_{\max} - \lambda_{ij})^2 + 2(\ell_{\max} - \lambda_{ij})^2 + 11(\sigma - 1)^2 \ell_{\max}^2.$$

Let λ_{ij}^d be the optimal dual variable corresponding to the demand satisfaction constraint (8b). The symmetric duopoly equilibrium prices are determined as:

$$\ell_{ij}^d = \begin{cases} \frac{(3-5\sigma)\ell_{\max} + 2\lambda_{ij}^d + \sqrt{\Delta_1(\lambda_{ij}^d)}}{8} & , \frac{\lambda_{ij}^d}{\ell_{\max}} \leq \frac{3(1-\sigma)^2}{2(\sigma+1)} \\ \frac{(5-3\sigma)\ell_{\max} + 2\lambda_{ij}^d - \sqrt{\Delta_2(\lambda_{ij}^d)}}{4} & , \text{o.w.,} \end{cases} \quad (9)$$

Moreover, denote the difference between optimal monopoly and symmetric duopoly equilibrium prices for OD pair (i,j)

as $\Delta\ell_{ij} := \ell_{ij}^m - \ell_{ij}^d$. *Then:*

$$\Delta\ell_{ij} \geq \begin{cases} \frac{(7\sigma-1)\ell_{\max}-2\overline{\lambda}_{ij}-\sqrt{\Delta_1(\overline{\lambda}_{ij})}}{8} & , \frac{\overline{\lambda}_{ij}}{\ell_{\max}} \leq \frac{3(1-\sigma)^2}{2(\sigma+1)} \\ \frac{(4\sigma-4)\ell_{\max}-2\overline{\lambda}_{ij}+\sqrt{\Delta_2(\overline{\lambda}_{ij})}}{4} & , \text{o.w.,} \end{cases} \quad (10)$$

and

$$\Delta\ell_{ij} \leq \frac{(7\sigma-1)\ell_{\max}+4\overline{\lambda}_{ij}-\ell_{\max}\sqrt{-15\sigma^2+18\sigma+1}}{8}. \quad (11)$$

Proof outline: We state the first and the second order optimality conditions on (8) to get the duopoly equilibrium prices. To lower bound the price difference, we evaluate the monopoly prices at $\lambda_{ij}^m = 0$ and the duopoly equilibrium prices at $\lambda_{ij}^d = \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{\max}$ (and to upper bound, vice versa). □

The complete proof can be found in Appendix E, which is provided in the online version [28]. An interesting observation is how σ affects the prices. For the optimal monopoly prices, $\partial\ell_{ij}^m/\partial\sigma > 0$, i.e., the monopolist serving a population with higher σ charges more for the rides with identical costs (i.e., identical λ_{ij}^m). The reason is that larger σ shifts the distribution of customers' valuations for the monopolist from intermediate to extreme values (as σ increases from $1/2$ to 1, the distribution shifts from triangular to uniform). This shift in the distribution modifies the demand function $D(\ell_{ij}^1, \infty)$, which leads to an increase on the optimal prices. Simply put, larger σ, i.e., lack of firm loyalty, leads to an increase in the prices for the monopoly. On the contrary for the duopoly equilibrium prices, $\partial\ell_{ij}^d/\partial\sigma < 0$. That is, the duopoly serving a population with higher σ charges less for the rides with identical costs (i.e., identical λ_{ij}^d). The intuition behind is that larger σ indicates a lack of firm loyalty (when $\sigma = 1$, the customers buy from the firm that offers lower prices). Hence, higher σ strengthens the competition and causes the firms to charge less. The reader can observe that when $\sigma = 1$, $\ell_{ij}^d = \lambda_{ij}^d$, i.e., the equilibrium prices are equal to the costs of providing the rides to the platform, which is the lowest the firms can go without losing money but make no profit.

Observe that the lower bounds in (10) are decreasing functions of $\overline{\lambda}_{ij}$. Given the maximum value of $\overline{\lambda}_{ij}$ equal to $\overline{\lambda}_{ij} = \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{\max}$, the lower bound on the price difference is 0. Hence, we can conclude that the duopoly prices are never higher than the monopoly prices, for all OD pairs.

Proposition 5 characterizes the effect of competition on the prices depending on the network parameters and therefore the dual variables. The next series of results aim to determine universal bounds on the ratio of prices, induced demand, profits and consumer surplus in the monopoly and the duopoly, *independent of the network parameters.*

Proposition 6 (Price Bounds). *Suppose that Assumptions 1 and 2 hold. For all OD pairs, the optimal monopoly prices obey the following:*

$$2\ell_{\max}/5 \leq \underline{\ell}^m \leq \ell_{ij}^m \leq \overline{\ell}^m \leq 3\ell_{\max}/4, \quad (12)$$

where $\underline{\ell}^m := \frac{1+\sigma}{4}\ell_{\max}$ and $\overline{\ell}^m := \frac{7+14\sigma-9\sigma^2}{40-24\sigma}\ell_{\max}$. Furthermore, the symmetric duopoly equilibrium prices obey:

$$0 \leq \underline{\ell}^d \leq \ell_{ij}^d \leq \overline{\ell}^d \leq \ell_{\max}/2, \quad (13)$$

where $\underline{\ell}^d := \frac{3-5\sigma+\sqrt{-15\sigma^2+18\sigma+1}}{8}\ell_{\max}$ and $\overline{\ell}^d := \frac{1+\sigma}{4}\ell_{\max}$.

Moreover for all OD pairs (i, j), the ratio between the symmetric duopoly equilibrium prices and the optimal monopoly prices obey the following:

$$\frac{\underline{\ell}^d}{\overline{\ell}^m} \leq \frac{\ell_{ij}^d}{\ell_{ij}^m} \leq \frac{\overline{\ell}^d}{\underline{\ell}^m} = 1. \quad (14)$$

Proof outline: The proof is done by evaluating the optimal monopoly prices given by (4) at $\lambda_{ij}^m = 0$ and $\lambda_{ij}^m = \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{\max}$ as well as the duopoly equilibrium prices given by (9) at $\lambda_{ij}^d = 0$ and $\lambda_{ij}^d = \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{\max}$ to get the bounds on the prices in terms of σ. Then, we impose the condition $\sigma \in [3/5, 1]$ to get the uniform bounds.

The complete proof can be found in Appendix F, which is provided in the online version [28]. An observation is that increasing σ increases both the upper and the lower bounds for the optimal monopoly prices, whereas decreases the lower bound on the duopoly equilibrium prices and increases the upper bound. This is because for the optimal monopoly prices, $\partial\ell_{ij}^m/\partial\sigma > 0$. However, because it strengthens the competition between the firms in the duopoly, it can cause the prices to go much lower, hence decreasing the lower bound (when $\sigma = 1$: if $\lambda_{ij}^d = 0$, then $\ell_{ij}^d = 0$). The upper bound on the duopoly equilibrium prices still increases, because according to Assumption 2 a larger σ permits a larger λ_{ij}^d and hence higher prices. Consequently, the upper bound on the price ratio is always 1 independent of σ while the lower bound is decreasing with σ.

The next result characterizes the effect of competition on the total customer demand for rides that are served by either firm. We show that the aggregate demand served by the duopoly is at least equal to and can be up to 4 times higher than the demand served by the monopoly.

Proposition 7 (Demand Bounds). *Suppose that Assumptions 1 and 2 hold. For all OD pairs (i, j), the monopoly demand functions evaluated at the optimal monopoly prices obey:*

$$1/4 \leq \underline{D}^m \leq D(\ell_{ij}^m, \infty) \leq \overline{D}^m \leq 2/3, \quad (15)$$

where $\underline{D}^m := \frac{13-3\sigma^2-6\sigma}{40\sigma-24\sigma^2}$ and $\overline{D}^m := \frac{1+\sigma}{4\sigma}$. The duopoly demand functions at the duopoly equilibrium prices obey:

$$1/4 \leq \underline{D}^d \leq D(\ell_{ij}^d, \ell_{ij}^d) \leq \overline{D}^d \leq 1/2, \quad (16)$$

where $\underline{D}^d := \frac{1}{4\sigma}$ and $\overline{D}^d := \frac{1}{2} - \frac{\left(-(1+\sigma)+\sqrt{-15\sigma^2+18\sigma+1}\right)^2}{128\sigma(1-\sigma)}$. Furthermore, the ratio between the total demand served between any OD pair $\frac{2D(\ell_{ij}^d, \ell_{ij}^d)}{D(\ell_{ij}^m, \infty)}$ obeys the following:

$$1 \leq \frac{2}{1+\sigma} = \frac{2\underline{D}^d}{\overline{D}^m} \leq \frac{2D(\ell_{ij}^d, \ell_{ij}^d)}{D(\ell_{ij}^m, \infty)} \leq \frac{2\overline{D}^d}{\underline{D}^m} \leq 4 \quad (17)$$

Proof outline: The proof is done by evaluating the demand functions for the monopoly and the duopoly at the price bounds given by (12) and (13), and then imposing the condition $\sigma \in [3/5, 1]$ to get uniform bounds.

The complete proof can be found in Appendix G, which is provided in the online version [28]. Taking into account that induced demand is inversely proportional to prices, the impact of σ on the demand function bounds is in accordance with price bounds in Proposition 6.

Remark 2. *The upper bound in* (17) *is achieved when* $\sigma = 1$, $\lambda_{ij}^m = \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{\max}$ *and* $\lambda_{ij}^d = 0$. *Although it is achievable for some OD pairs, it is not possible to achieve it for all OD pairs simultaneously. This is because for* λ_{ij}^d *to be 0, constraint* (8b) *has to be slack, meaning node* i *has excess supply of vehicles that are being rebalanced to node* j. *This however can not hold simultaneously for all OD pairs, since that would mean there are empty vehicles being routed between all OD pairs, which would not be optimal.*

Interestingly, we see that this potential increase in the aggregate demand never translates into a profit increase for the firms because of the competition. As expected, profits decrease in the presence of competition. According to the next result, the profits generated by a single firm in duopoly is always less than 85% of the profits generated by the monopoly.

Proposition 8 (Profit Bounds). *Suppose that Assumptions 1 and 2 hold. Let profits earned by serving the induced demand between OD pair* (i, j) *in the monopoly be* P_{ij}^m. *With the optimal monopoly prices,* P_{ij}^m *for all* (i, j) *obey the following:*

$$\theta_{ij}\ell_{\max}/16 \leq \theta_{ij}\underline{P}^m \leq P_{ij}^m \leq \theta_{ij}\overline{P}^m \leq \theta_{ij}\ell_{\max}/4, \quad (18)$$

where

$$\underline{P}^m = \frac{(3\sigma^2 + 6\sigma - 13)^2}{64\sigma(5-3\sigma)^2}\ell_{\max}, \quad \overline{P}^m = \frac{(1+\sigma)^2}{16\sigma}\ell_{\max}.$$

Similarly, let profits earned by serving the induced demand between OD pair (i, j) *by a single firm in the duopoly be* P_{ij}^d. *With the duopoly equilibrium prices,* P_{ij}^d *for all* (i, j) *obey:*

$$0 \leq \theta_{ij}\underline{P}^d \leq P_{ij}^d \leq \theta_{ij}\overline{P}^d \leq (4 + \sqrt{10})\ell_{\max}\theta_{ij}/48 \quad (19)$$

where

$$\underline{P}^d = \left(\overline{\ell}^d - \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{\max}\right) \times \underline{D}^d = \frac{1-\sigma}{2\sigma(5-3\sigma)}\ell_{\max},$$

$$\overline{P}^d = \underline{\ell}^d\overline{D}^d.$$

Furthermore, for all OD pairs, the ratio $\frac{P_{ij}^d}{P_{ij}^m}$ *obeys:*

$$\frac{8(1-\sigma)}{(\sigma+1)^2(5-3\sigma)} = \frac{\underline{P}^d}{\overline{P}^m} \leq \frac{P_{ij}^d}{P_{ij}^m} \leq \frac{\overline{P}^d}{\underline{P}^m} \lessapprox 0.85. \quad (20)$$

Proof outline: The proof is done by evaluating the profits for the monopoly given by (6) at the price bounds given by (12). For the duopoly, we first derive the dual objective, show that it decreases with ℓ_{ij}^d, and evaluate at the duopoly equilibrium price bounds given by (13). Then, we impose the condition $\sigma \in [3/5, 1]$ to get the uniform bounds.

The complete proof can be found in Appendix H, which is provided in the online version [28]. Since lower prices generate more profits in the monopoly and the price bounds are increasing with σ, the profit bounds of the monopoly are decreasing with σ. Similarly, the duopoly profit bounds are decreasing with σ too. Since σ increases the upper bound on prices, the lower bound on the profits decrease. However, although σ decreases the lower bound on the prices, the upper bound on the profits still decrease. This is because competition in the duopoly is a downward driving force on the prices. Consequently, lower prices in the duopoly do not only result from lower λ_{ij}^d, but also stronger competition. Hence, although lower prices increase the aggregate demand, because the firms

are now competing over the customers, neither of the firms serve enough customers to compensate for the decrease in the prices. Hence, the profits decrease.

The upper bound in (20) is achieved when $\sigma = 3/5$, $\lambda_{ij}^m = \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{\max}$, and $\lambda_{ij}^d = 0$. Due to the same argument in Remark 2, it can not be achieved simultaneously by all the OD pairs. Consequently, the ratio of total profits can not achieve this upper bound with equality.

How do the customers benefit from the introduction of competition? We saw that a reduction in ride prices is expected. Next, we show that the consumer surplus in the duopolistic setting is at least equal to and can be up to 16 times the consumer surplus in the monopoly.

Proposition 9 (Consumer Surplus Bounds). *Suppose that Assumptions 1 and 2 hold. Let the consumer surplus of customers requesting a ride between OD pair* (i, j) *in the monopoly be* CS_{ij}^m. *With the optimal monopoly prices,* CS_{ij}^m *for all* (i, j) *obey:*

$$\theta_{ij}\frac{\ell_{\max}}{32} \leq \theta_{ij}\underline{\mathrm{CS}}^m \leq \mathrm{CS}_{ij}^m \leq \theta_{ij}\overline{\mathrm{CS}}^m \leq \theta_{ij}\frac{13}{90}\ell_{\max}, \quad (21)$$

where

$$\underline{\mathrm{CS}}^m = \frac{171\sigma^4 - 660\sigma^3 + 1378\sigma^2 - 1748\sigma + 907}{384\sigma(5-3\sigma)^2}\ell_{\max},$$

$$\overline{\mathrm{CS}}^m = (7\sigma^2 - 2\sigma + 7)\ell_{\max}/(96\sigma).$$

Similarly, let the consumer surplus of customers requesting a ride between OD pair (i, j) *in the duopoly be* CS_{ij}^d. *With the duopoly equilibrium prices,* CS_{ij}^d *for all* (i, j) *obey:*

$$\theta_{ij}\frac{\ell_{\max}}{8} \leq \theta_{ij}\underline{\mathrm{CS}}^d \leq \mathrm{CS}_{ij}^d \leq \theta_{ij}\overline{\mathrm{CS}}^d \leq \theta_{ij}\frac{\ell_{\max}}{2}, \quad (22)$$

where

$$\underline{\mathrm{CS}}^d = (\sigma^2 - 2\sigma + 13)\ell_{\max}/(96\sigma),$$

$$\overline{\mathrm{CS}}^d = \frac{\ell_{\max}}{24\sigma(1-\sigma)}\left((2\sigma)^3 - (\sigma + 1 - 2\frac{\underline{\ell}^d}{\ell_{\max}})^3\right.$$

$$\left. - 24\sigma(1 - \frac{\underline{\ell}^d}{\ell_{\max}})(\sigma - 1 + \frac{\underline{\ell}^d}{\ell_{\max}})\right).$$

Furthermore, for all OD pairs, the ratio $\frac{\mathrm{CS}_{ij}^d}{\mathrm{CS}_{ij}^m}$ *obeys:*

$$1 \leq \frac{\sigma^2 - 2\sigma + 13}{7\sigma^2 - 2\sigma + 7} \leq \frac{\underline{\mathrm{CS}}^d}{\overline{\mathrm{CS}}^m} \leq \frac{\mathrm{CS}_{ij}^d}{\mathrm{CS}_{ij}^m} \leq \frac{\overline{\mathrm{CS}}^d}{\underline{\mathrm{CS}}^m} \leq 16. \quad (23)$$

Proof outline: The proof is done by evaluating the consumer surplus for the monopoly given by (7) at the price bounds given by (12). For the duopoly, we compute the consumer surplus at the price bounds given by (13) in a similar fashion to the the proof of Proposition 2. Then, we impose the condition $\sigma \in [3/5, 1]$ to get the uniform bounds.

The complete proof can be found in Appendix I, which is provided in the online version [28]. Considering the fact that lower prices (both in the duopoly and the monopoly) increase the consumer surplus by inducing more customers and increasing the surplus per customer, the dependency of the price bounds on σ reflects to the consumer surplus bounds.

Remark 2 applies for the upper bound in (23) too, and thus it can not be achieved for all OD pairs simultaneously. Therefore,

the ratio of total consumer surplus cannot achieve this upper bound with equality.

So far, we have studied the effects of competition in an electric AMoD system by adopting a static network-flow formulation. Although very convenient for analysis, this formulation does not reflect the randomness in arrivals nor constrains vehicles dispatch decisions to be integer valued (e.g., 0.25 customer may be served). To address these discrepancies with the real environment, in the next section, we modify our model to account for the randomness in arrivals and furthermore design a control policy that can be implemented in real-time.

IV. REAL-TIME CONTROL

To accommodate for the stochastic nature of the arrivals, we model the arrival of the potential customers OD pair (i, j) as a Poisson process with an arrival rate of θ_{ij}. Moreover, we allow the firms to set prices real-time and use the same price-responsive demand model. In particular, during period t, for a price tuple $(\ell_{ijt}^1, \ell_{ijt}^2)$ for OD pair (i, j), the induced arrival rate for firm f is given by $\Theta_{ijt}^f = \theta_{ij} D(\ell_{ijt}^f, \ell_{ijt}^{-f})$. Thus, the number of new ride requests in time period t for firm f is $A_{ijt}^f \sim \text{Pois}(\Theta_{ijt}^f)$ for OD pair (i, j). As a consequence of this randomness in the customer arrivals, the platform operator might not be able to assign every customer to a ride immediately (if the number of induced arrivals exceed the number of available vehicles). In order to address this nuance, we adopt the following ride-sharing model:

Ride Hailing Model: Customers that purchase a ride during period t are not immediately matched with a ride, but enter the queue for OD pair (i, j) to be served at the beginning of period $t + 1$. After the platform operator executes routing decisions for the fleet at the beginning of period $t + 1$, the customers in the queue for OD pair (i, j) are matched with rides and served on a first-come, first-served basis.

Under these additional modeling modifications, our goal is to establish a real-time pricing and fleet management policy that can be implemented in a real environment and provides stability of the queues[3]. In fact, the model studied in Section III is the static planning problem associated with this real environment, where we ignored the stochasticity of the arrivals and used the expected values, while allowing the vehicle routing decisions to be flows (real numbers) rather than integers. For the monopoly (or the symmetric duopoly), the solution to this static planning problem in (3) (or (8)) is the optimal static policy that consists of optimal prices as well as optimal vehicle routing and charging decisions. This policy can not directly be implemented in a real environment because it does not yield integer valued solutions. In an earlier work [12], it was proven that randomizing the vehicle decisions according to the optimal solution of the static problem to get integer-valued actions guarantees stability of the queues. However, considering random arrivals, this method may not execute the most profitable actions since it does not take the real-time queue lengths into consideration. Although it guarantees stability of the queues, it does not seek to minimize

[3]The stability condition that we are interested in is rate stability of all queues. A queue for OD pair (i, j) is rate stable if $\lim_{t \to \infty} q_{ij}(t)/t = 0$.

the queue lengths and hence the wait time of the passengers, which would negatively affect the business.

Instead of using the randomized solution to implement real-time actions, it is possible to realize a real-time policy that acknowledges the queue lengths and hence aims to maximize the profits while minimizing the total wait time of the customers. To achieve this, we propose to apply finite-horizon model predictive control (MPC) in our numerical experiment (albeit with no performance guarantee).

MPC Procedure: The idea of finite-horizon MPC is to observe the current state of the environment and determine the best control strategy for a planning horizon of T by predicting the state path of the environment. Then, only the control strategy at the initial time period is implemented and the process is repeated. Specifically, let $\mathcal{S}$ be the state of the vehicles (locations, energy levels) and $\{Q_{ij}\}_{i,j \in \mathcal{N}}$ be the outstanding customer demand (i.e., people who have requested a ride but not yet served) at the beginning of planning time. The MPC Algorithm is summarized as follows:

Algorithm 1: MPC Procedure

1: $\mathcal{S} \leftarrow$ Get vehicle states (locations, energy levels)
2: $Q_{ij} \leftarrow$ Count outstanding customers
3: $\{x_{ijt}^e, x_{ict}^e, \ell_{ijt}\}_{\forall i,j,e,t} \leftarrow$ Solve (24)
4: Execute $\{x_{ij0}^e, x_{ic0}^e, \ell_{ij0}\}_{\forall i,j,e}$

At each period, Algorithm 1 is run and the system state is observed. Using this information, the optimal fleet management and pricing strategy is computed for the next T periods by solving (24). Vehicle routing/charging and pricing decisions are executed for the initial time period and the environment transitions into next state. Then, Algorithm 1 is re-run and this process is repeated during the entire operation of the system.

Next, we state the optimization problem (24) for the controller using a dynamic pricing scheme in monopoly. Let the decision variable ℓ_{ijt}^1 be the price for rides between OD pair (i, j) in period t, x_{ijt}^e be the number of vehicles at node i with energy level e being routed to node j in period t, x_{ict}^e be the number of vehicles charging at node i starting with energy level e in period t, and q_{ijt} be the people waiting in the queue for OD pair (i, j) in period t. We state the problem as follows:

$$\max_{x_{ict}^e, x_{ijt}^e, q_{ijt}, \ell_{ijt}^1} \sum_{ijt} \ell_{ijt}^1 \theta_{ij} D(\ell_{ijt}^1, \infty) - \sum_{ijt} w_{ijt} q_{ijt}$$
$$- \beta_t \sum_{ijet} \tau_{ij} x_{ijt}^e - \sum_{iet} (\beta_c + p_i) x_{ic}^t \quad (24a)$$

$$\text{s.t.} \quad q_{ijt_0} \geq Q_{ij} - \sum_e x_{ijt_0}^e, \quad \forall i, j \in \mathcal{N} \quad (24b)$$

$$q_{ijt} \geq q_{ijt-1} + \theta_{ij} D(\ell_{ijt-1}^1, \infty) - \sum_e x_{ijt}^e,$$
$$\forall i, j \in \mathcal{N}, \ \forall t > t_0, \quad (24c)$$

$$\sum_j x_{ijt}^e + x_{ict}^e - \sum_j x_{jit-\tau_{ji}}^{e+e_{ji}} - x_{ict-1}^{e-1} = s_{it}^e,$$
$$\forall i \in \mathcal{N}, \ \forall e \in \mathcal{E}, \ \forall t \geq t_0 \quad (24d)$$

$$x_{ict}^{e_{\max}} = 0, \quad \forall i \in \mathcal{N}, \ \forall t \geq t_0, \quad (24e)$$

$$x_{ijt}^e = 0, \quad \forall e < e_{ij}, \forall i,j \in \mathcal{N}, \forall t \geq t_0, \qquad (24f)$$

$$x_{ijt}^e, x_{ict}^e, q_{ijt} \geq 0, \ x_{ijt}^e, x_{ict}^e \in \mathbb{N},$$
$$\forall i,j \in \mathcal{N}, \ \forall e \in \mathcal{E}, \ \forall t \geq t_0 \quad (24g)$$

$$x_{ijt}^e = x_{ict}^e = 0, \forall e \notin \mathcal{E}, \forall t < t_0, \forall i,j \in \mathcal{N}. \qquad (24h)$$

The first term in the objective function (24a) corresponds to the expected revenue gained by setting prices ℓ_{ijt}^1. The second term assigns a cost to the queue lengths, where w_{ijt} is the cost per person in the queue for OD pair (i,j) at the time period t. The third term is the operational costs of the trip-making vehicles, and the last term is the operational and the charging costs of the charging vehicles. Hence, the objective is to maximize the profits minus the queue penalty.

The state variable s_{it}^e denotes the number of vehicles at node i with energy level e, at the beginning of time period t. At the beginning of the planning time $t = t_0$, $s_{it_0}^e$ is simply the number of available vehicles at node i with energy level e. For $t > t_0$, s_{it}^e denotes the number of vehicles that will be available at the beginning of time period t, at node i with energy level e. These are the vehicles that are en route to another node at the time of planning. Hence, (24d) is the vehicle balance constraint. The constraints (24c) along with the non-negativity constraint (24g), implement the queue length transition $q_{ijt} = \max\{0, q_{ijt-1} + \theta_{ij}D(\ell_{ijt-1}^1, \infty) - \sum_e x_{ijt}^e\}$ as two linear inequalities. For $t = t_0$, the queue length is modified via (24b), where Q_{ij} denotes the number of passengers waiting to be served at the planning time.

The MPC controller using a dynamic pricing scheme for the duopoly can be stated in a similar way to the monopoly. Due to space limitations, we exclude it here and refer the reader to the Appendix J of the online version [28].

We end this section by noting that it is possible to implement a model predictive controller with static prices in monopoly simply by adding the constraint $\ell_{ijt}^1 = \ell_{ij}^m, \forall t \geq t_0$ to (24). For the duopoly, we replace $D(\ell_{ijt}^1, \infty)$ with $D(\ell_{ijt}^1, \ell_{ijt}^2)$ and add the constraint $\ell_{ijt}^1 = \ell_{ijt}^2 = \ell_{ij}^d, \forall t \geq t_0$.

V. NUMERICAL STUDY

In this section, we discuss the effects of competition and the performances of the real-time controllers via numerical examples. To solve the optimization problems we used the Gurobi Optimizer [29].

In our discrete-time system, we chose one period to be equal to $\Delta t = 5$ minutes, which is equal to the time it takes to deliver one unit of battery energy. We chose operational costs of $\beta_t = \$0.2$ and $\beta_c = \$0.1$ (by taking the amortized average price of an electric car over 5 years [30] as a reference), maximum willingness to pay $\ell_{\max} = \$50$, and $\sigma = 3/5$. We chose a battery capacity of 24kWh, and discretized the battery energy into $e_{\max} = 6$ units, where one unit of battery energy is 4kWh. Price of electricity per unit of energy (4kWh) ranges from $0.32 to $1.2 [31], and we randomly sampled p_i for all locations uniformly from this range.

For the network and demand data, we divided Manhattan into 20 regions as in Figure 3. Using the yellow taxi data from the New York City Taxi and Limousine Commission dataset [32] for May 09, 2019, Thursday between 15.00-17.00, we extracted the average arrival rates for rides,

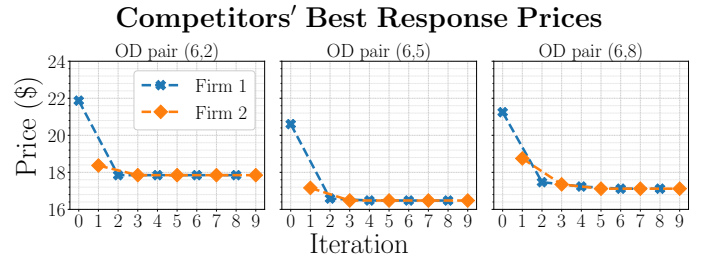### Competitors' Best Response Prices

Fig. 4: Best response prices for some rides originating from node 6.

average trip durations, and average distances between the regions (we excluded the rides occurring in the same region). Note that the demand data used is not the data of potential riders, but the data of realized rides. Although it is not ideal to impose a demand function on the data of realized rides, this is the best data we could use due to lack of available data on potential riders. This is a common approach in the literature of pricing schemes in ride-sharing platforms [11], [22], as the realized rides leaving a location can be seen as a reasonable proxy for the potential riders at that location.

Fig. 3: Manhattan divided into $n = 20$ regions.

A. Effects of Competition Under Static Setting

In this study, we analyze the effects of competition using prices for rides, induced demand, profits, and consumer surplus as metrics. To get the values of the aforementioned metrics in the monopoly, we solved (3). For the duopoly, we can not solve (8) since the problem is non-convex. Therefore, we implemented best-response dynamics to see empirically whether this process would converge to an equilibrium of the duopoly so that we could numerically compare the monopoly and the duopoly. Although we do not have a theoretical guarantee for convergence of best response dynamics, we know that only symmetric equilibria exist according to Proposition 4. Fortunately, our experiment converged to a symmetric equilibrium in a couple of iterations as demonstrated in Figure 4.

In Table I, we display the ratios of performance metrics in the monopoly and the symmetric duopoly equilibrium. Moreover, we compute the theoretic upper and lower bounds derived in Section III for $\sigma = 3/5$ for comparison. To summarize the table, competition results in a 20% decrease in the average prices of rides, a 44% increase in the total induced demand, a 43% decrease in the profits of a single firm, and a 100% increase in the consumer surplus.

Metrics	Empirical	Theoretic LB	Theoretic UB
$\ell_{\mathrm{avg}}^d/\ell_{\mathrm{avg}}^m$	0.80	0.67	1
D^d/D^m	1.44	1.25	2.26
P^d/P^m	0.57	0.39	0.85
$\mathrm{CS}^d/\mathrm{CS}^m$	2.00	1.46	5.89

TABLE I: Ratios of average prices, induced demand, profits, and consumer surplus in the monopoly and the symmetric duopoly equilibrium for $\sigma = 3/5$.
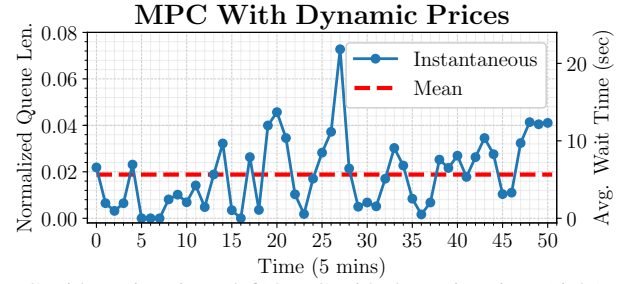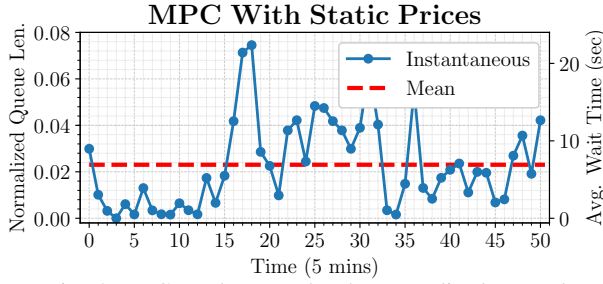
Fig. 5: MPC results. We plot the normalized queue length for the MPC with static prices (left)/MPC with dynamic prices (right).

Impact of σ: The correlation over customers' preferences is measured by σ, and the effects of competition depend on the value of σ. To study how σ influences the effects of competition, we present the ratios of performance metrics in the monopoly and the symmetric duopoly equilibrium for $\sigma = 0.8$ and $\sigma = 1$ in Table II.

Metrics	Empirical		Theoretic LB		Theoretic UB	
	$\sigma = 0.8$	$\sigma = 1$	$\sigma = 0.8$	$\sigma = 1$	$\sigma = 0.8$	$\sigma = 1$
$\ell^d_{avg}/\ell^m_{avg}$	0.42	0.11	0.29	0	1	1
D^d/D^m	1.73	2.04	1.11	1	2.55	4
P^d/P^m	0.32	0	0.19	0	0.74	0
CS^d/CS^m	2.95	4.18	1.22	1	9.22	16

TABLE II: Ratios of average prices, induced demand, profits, and consumer surplus in the monopoly and the symmetric duopoly equilibrium for $\sigma = 0.8$ and $\sigma = 1$.

The results in Tables I and II indicate that the higher the σ, the stronger the competition between the firms. A larger σ indicates higher correlation over customers' preferences, which means that the customers care less about the identity of the firm and more about lower prices when buying a ride ($\sigma = 1$ means they buy from the firm that offers the lower price). Hence, a stronger competition requires the firms to drop their prices further, which in turn decreases their profits more. This is in favor of the customers, since lower prices induce more demand while generating higher consumer surplus.

B. Real-Time Control

In this study, we demonstrate the performances of the model predictive controllers utilizing static and dynamic pricing schemes using profits (minus the queue penalty) and the average wait time of the customers as metrics. To quantify the queue penalty, we set queue penalty per person to be $w_{ijt} = \$4$ (by doubling the average hourly wage of \$24 in the U.S. [33]). We computed the instantaneous profits in one period as:

$$\text{Profits} = \text{Revenue} - (\text{Operational} + \text{Charging Costs}), \quad (25)$$

the queue penalty in one period as:

$$\text{Queue Penalty} = w \times \text{Outstanding Customers}, \quad (26)$$

and used the objective value of (3) as an upper bound on the average profits for comparison. We define

$$\text{Normalized Queue Length} := \frac{\text{Outstanding Customers}}{\text{Induced Demand}} \quad (27)$$

and compute the instantaneous average wait time of customers in one period as:

$$\text{Avg. Wait Time} = \text{Normalized Queue Length} \times \Delta t. \quad (28)$$

We implemented the MPC with $T = 10 \times \Delta t$ as the planning horizon, and ran the environment for $50 \times \Delta t$.
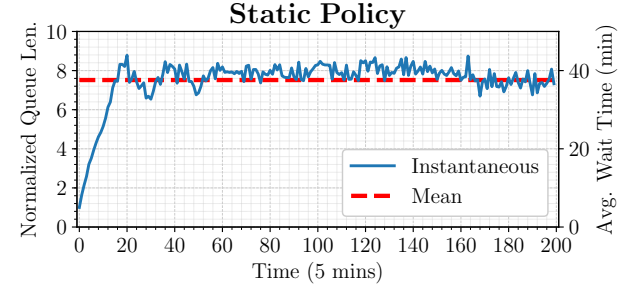

Fig. 6: Monopoly Static Policy Queues

1) Monopoly: We plot the instantaneous average wait time for MPC with static prices (MPC-SP) and dynamic prices (MPC-DP) in Figure 5, and summarize the results in Table III.

Metrics	MPC-SP	MPC-DP	% Impr.
Mean Profits-Queue Penalty (\$)	11700.86	11778.13	0.66%
% of static	98.36%	99.02%	
Mean Avg. Wait Time (sec)	6.91	5.64	18.38%
Var. Avg. Wait Time (sec)	32.58	20.95	35.7%

TABLE III: MPC results in the monopoly. Mean and variance are computed over time. The static objective value is 11894.9.

We observe that both controllers are able to keep the queue lengths very short (around 2% of the induced demand), and still generate substantial amount of profits that is close to the static objective. In particular, MPC-SP generates 98.36% and MPC-DP generates 99.02% of the static profits, including the queue penalty. Although the marginal benefits of using dynamic pricing might seem low, a 0.66% increase in average profits would make a considerable difference in the long run (e.g., from Table III, a \$77 increase in profits per period adds up to more than an increase of \$900 per hour). Moreover, we observe that the mean of average wait time for MPC-SP is 6.91 seconds, while that of MPC-DP is 5.64 seconds which is an improvement of 18.38%. Lastly, a dynamic pricing scheme reduces the variance of the average wait time by 35.7%, which indicates a more robust system with predictable wait times.

We furthermore generated integer actions by randomizing according to the flows of the static solution and implemented the static policy in the real environment to compare its performance. In Figure 6 we plot the average wait time using the static policy. Although it provides stability of the queues, it results in bad wait times with a mean of 36.9 minutes, which is more than 300 times longer than both MPC-SP and MPC-DP.

2) Duopoly: We computed the mean value of the metrics over both firms to get the performances of the controllers. The results are summarized in Table IV.

Similar to the monopoly, both controllers are able to keep

Metrics	MPC-SP	MPC-DP	% Impr.
Mean Profits-Queue Penalty ($)	6670.89	6729.2	0.87%
% of static	98.56%	99.42%	0.87%
Mean Avg. Wait Time (sec)	7.27	5.01	31.08%
Var. Avg. Wait Time (sec)	44.68	17.92	59.89%

TABLE IV: MPC results in the duopoly. Mean and variance are computed over time. The static objective value is 6768.2.

the queues short while generating profits close to the static objective, with dynamic pricing scheme increasing the efficiency.

VI. CONCLUSION

In this paper, we studied the impacts of competition on electric AMoD systems by comparing the monopoly and the duopoly in equilibrium. By formalizing the optimal strategies of profit-maximizing platform operators, we show that the identical competitors can only be in a symmetric equilibrium. In state of a symmetric duopoly equilibrium, the prices for rides and the profits of the firms are always less than those in the monopolistic setting, whereas the aggregate demand served and the consumer surplus are always higher. The closed-form universal bounds we provide quantify the amount of increase/reduction on the said metrics. These bounds depend heavily on correlation between customers' preferences and therefore the strength of the competition. The numerical studies using network and demand data of Manhattan indicate that stronger competition boosts the amount of increase/reduction on the metrics. Lastly, we experimentally demonstrate that it is possible to implement a real-time control policy for fleet management using model predictive control, and show that a real-time pricing policy further improves the performance.

REFERENCES

[1] (2020) 40+ Corporations Working On Autonomous Vehicles. [Online]. Available: https://www.cbinsights.com/research/autonomous-driverless-vehicles-corporations-list/

[2] R. Zhang and M. Pavone, "Control of robotic Mobility-on-Demand systems: A queueing-theoretical perspective," In *Int. Journal of Robotics Research*, vol. 35, no. 1–3, pp. 186–203, 2016.

[3] M. Pavone, S. L. Smith, E. Frazzoli, and D. Rus, "Robotic load balancing for Mobility-on-Demand systems," *Int. Journal of Robotics Research*, vol. 31, no. 7, pp. 839–854, 2012.

[4] F. Rossi *et al.*, "Routing autonomous vehicles in congested transportation networks: Structural properties and coordination algorithms," *Autonomous Robots*, vol. 42, no. 7, pp. 1427–1442, 2018.

[5] M. Volkov, J. Aslam, and D. Rus, "Markov-based redistribution policy model for future urban mobility networks," *IEEE Conference on Intelligent Transportation Systems*, pp. 1906–1911, 09 2012.

[6] R. Zhang, F. Rossi, and M. Pavone, "Model predictive control of autonomous mobility-on-demand systems," in *IEEE ICRA*, May 2016.

[7] F. Miao *et al.*, "Data-driven distributionally robust vehicle balancing using dynamic region partitions," in *2017 ACM/IEEE 8th International Conference on Cyber-Physical Systems*, April 2017, pp. 261–272.

[8] M. Tsao, R. Iglesias, and M. Pavone, "Stochastic model predictive control for autonomous mobility on demand," in *IEEE Conference on Intelligent Transportation Systems*, 2018, pp. 3941–3948.

[9] B. Turan, N. Tucker, and M. Alizadeh, "Smart charging benefits in autonomous mobility on demand systems," in *2019 IEEE Intelligent Transportation Systems Conference (ITSC)*. IEEE, 2019, pp. 461–466.

[10] N. Tucker, B. Turan, and M. Alizadeh, "Online charge scheduling for electric vehicles in autonomous mobility on demand fleets," in *2019 IEEE ITSC*. IEEE, 2019, pp. 226–231.

[11] S. Wollenstein-Betech, I. C. Paschalidis, and C. G. Cassandras, "Joint pricing and rebalancing of autonomous mobility-on-demand systems," *arXiv preprint arXiv:2003.13614*, 2020.

[12] B. Turan, R. Pedarsani, and M. Alizadeh, "Dynamic pricing and fleet management for electric autonomous mobility on demand systems," *Transp. Res. Part C Emerg. Technol.*, vol. 121, p. 102829, 2020.

[13] E. Weyl *et al.*, "Imperfect platform competition: A general framework," 2010.

[14] J.-C. Rochet and J. Tirole, "Platform competition in two-sided markets," *J. Eur. Econ. Assoc.*, vol. 1, no. 4, pp. 990–1029, 2003.

[15] M. Armstrong and J. Wright, "Two-sided markets, competitive bottlenecks and exclusive contracts," *Economic Theory*, vol. 32, no. 2, pp. 353–380, 2007.

[16] Y. Dou and D. Wu, "Dynamic platform competition: Optimal pricing and piggybacking under network effects," *SSRN Electronic Journal*, 2016.

[17] T. Kodera *et al.*, "Spatial competition among multiple platforms," *Economics Bulletin*, vol. 30, no. 2, pp. 1561–1525, 2010.

[18] J. Cramer and A. B. Krueger, "Disruptive change in the taxi business: The case of uber," *Am Econ Rev*, vol. 106, no. 5, pp. 177–82, 2016.

[19] M. McGregor, B. Brown, and M. Glöss, "Disrupting the cab: Uber, ridesharing and the taxi industry," *J. Peer Prod.*, no. 6, 2015.

[20] J. Mayer. (2016) Uber Is Not A Monopoly. [Online]. Available: https://www.forbes.com/sites/jaredmeyer/2016/02/15/uber-guardian-not-monopoly-ridesharing/

[21] S. Jiang, L. Chen, A. Mislove, and C. Wilson, "On ridesharing competition and accessibility: Evidence from uber, lyft, and taxi," in *Proceedings of the 2018 World Wide Web Conference*, 2018, pp. 863–872.

[22] K. Bimpikis, O. Candogan, and D. Saban, "Spatial pricing in ride-sharing networks," *Oper. Res.*, vol. 67, no. 3, pp. 744–769, 2019.

[23] J. C. Castillo, D. Knoepfle, and G. Weyl, "Surge pricing solves the wild goose chase," in *Proceedings of the 2017 ACM Conference on Economics and Computation*, 2017, pp. 241–242.

[24] S. Banerjee, C. Riquelme, and R. Johari, "Pricing in ride-share platforms: A queueing-theoretic approach," *Avail. at SSRN 2568258*, 2015.

[25] A. Nikzad, "Thickness and competition in ride-sharing markets," *Available at SSRN 3065672*, 2017.

[26] S. Dafermos, "The general multimodal network equilibrium problem with elastic demand," *Networks*, vol. 12, no. 1, pp. 57–72, 1982.

[27] H. Yang, "Sensitivity analysis for the elastic-demand network equilibrium problem with applications," *Transportation Research Part B: Methodological*, vol. 31, no. 1, pp. 55–70, 1997.

[28] B. Turan and M. Alizadeh, "Competition in electric autonomous mobility on demand systems," 2020. [Online]. Available: https://arxiv.org/pdf/2007.06051.pdf

[29] L. Gurobi Optimization, "Gurobi optimizer reference manual," 2020. [Online]. Available: http://www.gurobi.com

[30] The average electric car in the US is getting cheaper. [Online]. Available: https://qz.com/1695602/the-average-electric-vehicle-is-getting-cheaper-in-the-us/.

[31] Edmunds. (2019) The True Cost of Powering an Electric Car. [Online]. Available: https://www.edmunds.com/fuel-economy/the-true-cost-of-powering-an-electric-car.html

[32] [Online]. Available: https://www1.nyc.gov/site/tlc/about/tlc-trip-record-data.page

[33] United States Average Hourly Wages. [Online]. Available: https://tradingeconomics.com/united-states/wages.

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APPENDIX

A. Proof of Proposition 1

For brevity of notation, let $\beta_c + p_i = P_i$. Let λ_{ij} be the dual variables corresponding to the demand satisfaction constraints and μ_i^e be the dual variables corresponding to the flow balance constraints. We can state the dual problem as:

$$\min_{\lambda_{ij},\mu_i^e} \max_{\ell_{ij}^1} \quad \sum_{i=1}^n \sum_{j=1}^n \theta_{ij} D(\ell_{ij}^1,\infty)\left(\ell_{ij}^1 - \lambda_{ij}\right) \quad (29a)$$

$$\text{subject to} \quad \lambda_{ij} \geq 0, \quad (29b)$$

$$\lambda_{ij} + \mu_i^e - \mu_j^{e-e_{ij}} - \beta_t \tau_{ij} \leq 0, \quad (29c)$$

$$\mu_i^e - \mu_i^{e+1} - P_i \leq 0 \quad \forall i,j,e. \quad (29d)$$

For fixed λ_{ij} and μ_i^e, the first order optimality condition is:

$$\frac{\partial D(\ell_{ij}^1,\infty)}{\partial \ell_{ij}^1}(\ell_{ij}^1 - \lambda_{ij}) + D(\ell_{ij}^1,\infty) = 0 \quad (30)$$

Depending on the region ℓ_{ij}^1 is in, the demand function $D(\ell_{ij}^1,\infty)$ has different forms:

$$D(\ell_{ij}^1,\infty) = \begin{cases} 1 - \frac{(\ell_{ij}^1)^2}{2\ell_{max}^2 \sigma(1-\sigma)} & , \frac{\ell_{ij}^1}{\ell_{max}} < (1-\sigma) \\ \frac{1+\sigma - \frac{2\ell_{ij}^1}{\ell_{max}}}{2\sigma} & , (1-\sigma) \leq \frac{\ell_{ij}^1}{\ell_{max}} < \sigma \\ \frac{(1-\frac{\ell_{ij}^1}{\ell_{max}})^2}{2\sigma(1-\sigma)} & , \sigma \leq \frac{\ell_{ij}^1}{\ell_{max}} \leq 1 \end{cases} \quad (31)$$

First, suppose that $\frac{\ell_{ij}^1}{\ell_{max}} < (1-\sigma)$. Solving for ℓ_{ij}^1 in (30) using (31), we get:

$$\ell_{ij}^m = \left(\lambda_{ij} + \sqrt{\lambda_{ij}^2 + 6\ell_{max}^2\sigma(1-\sigma)}\right)/3. \quad (32)$$

Furthermore, the second order optimality condition satisfies:

$$\frac{\partial^2 D(\ell_{ij}^1,\infty)}{\partial(\ell_{ij}^1)^2}(\ell_{ij}^m - \lambda_{ij}) + 2\frac{\partial D(\ell_{ij}^1,\infty)}{\partial\ell_{ij}^1}\bigg|_{\ell_{ij}^1=\ell_{ij}^m} < 0. \quad (33)$$

Hence, KKT conditions are satisfied and the optimal primal solution satisfies the dual solution with optimal dual variables λ_{ij}^m. By checking the condition $\ell_{ij}^m \leq (1-\sigma)\ell_{max}$ using (32), we get the condition that $\lambda_{ij}^m \leq \frac{3-5\sigma}{2}$. The optimal prices for the regions where $\frac{\ell_{ij}^m}{\ell_{max}} \in [1-\sigma,\sigma)$ and $\frac{\ell_{ij}^m}{\ell_{max}} \in [\sigma,1]$ are derived in a similar fashion using the demand functions in those regions given in (31).

To get the upper bound on prices, we go through the following algebraic calculations using the constraints. The inequality (29d) gives:

$$\mu_i^{e-e_{ji}} \leq e_{ji}P_i + \mu_i^e, \quad (34)$$

and equivalently:

$$\mu_j^{e-e_{ij}} \leq e_{ij}P_j + \mu_j^e. \quad (35)$$

The inequalities (29c) and (29b) yield:

$$\mu_i^e - \mu_j^{e-e_{ij}} - \beta_t\tau_{ij} \leq 0,$$

and equivalently:

$$\mu_j^e - \mu_i^{e-e_{ji}} - \beta_t\tau_{ji} \leq 0, \quad (36)$$

Inequalities (34) and (36):

$$\mu_j^e \leq \mu_i^e + \beta_t\tau_{ji} + e_{ji}P_i. \quad (37)$$

And finally, the constraint (29c):

$$\lambda_{ij} \leq \beta\tau_{ij} + \mu_j^{e-e_{ij}} - \mu_i^e \overset{(35)}{\leq} \beta_t\tau_{ij} + e_{ij}P_j + \mu_j^e - \mu_i^e$$

$$\overset{(37)}{\leq} \beta_t\tau_{ij} + e_{ij}P_j + \beta\tau_{ji} + e_{ji}P_i.$$

Replacing $P_i = p_i + \beta_c$ and rearranging the terms:

$$\lambda_{ij} \leq \beta_t(\tau_{ij} + \tau_{ji}) + e_{ij}(p_j + \beta_c) + e_{ji}(p_i + \beta_c) = \overline{\lambda}_{ij}, \quad (38)$$

where the last equality follows from the definition provided in the proposition. Hence, we get the desired upper bound on the prices using the upper bound on the dual variables.

B. Proof of Proposition 2

Using Assumption 2, we see that $\frac{(3-5\sigma)}{2} \leq 0$ and $\frac{(3-5\sigma)}{2}\ell_{max} \leq \lambda_{ij}^m \leq \max_{i,j}\overline{\lambda}_{ij} \leq \frac{(3\sigma-1)(3-\sigma)}{4(5-3\sigma)}\ell_{max} \leq \frac{3\sigma-1}{2}\ell_{max}$. Hence, the optimal prices fall in the region $[(1-\sigma)\ell_{max}, \sigma\ell_{max})$, and are given by:

$$\ell_{ij}^m = ((1+\sigma)\ell_{max} + \lambda_{ij}^m)/4. \quad (39)$$

The dual problem with optimal prices in (39) can be stated as:

$$\min_{\lambda_{ij},\mu_i^e} \quad \sum_{i=1}^n \sum_{j=1}^n \frac{\theta_{ij}}{4\sigma\ell_{max}}\left(\frac{(1+\sigma)\ell_{max} - 2\lambda_{ij}}{2}\right)^2 \quad (40a)$$

$$\text{subject to} \quad (29b)-(29d). \quad (40b)$$

The objective function in (40a) with optimal dual variables, along with (39) suggests:

$$P^m = \sum_{i=1}^n \sum_{j=1}^n \frac{\theta_{ij}}{4\sigma\ell_{max}}((1+\sigma)\ell_{max} - 2\ell_{ij}^m)^2,$$

where profits P^m is the optimal value of the objective function of both primal and dual problems (Since the demand function is linear in the specified region, the problem is convex and KKT conditions are satisfied. Hence, strong duality holds).

Consumer surplus is given by the difference between the price that customers pay and the price that they are willing to pay. For OD pair (i,j) the customers with $v_1 > \ell_{ij}^m$ have a positive surplus of $v_1 - \ell_{ij}^m$ and the customers with $v_1 \leq \ell_{ij}^m$ have a zero surplus since they either do not take the ride or have exactly a valuation of ℓ_{ij}^m. Since $v_1 = \sigma x + (1-\sigma)y$ and x and y are iid uniform random variables in $[0,\ell_{max}]$, the consumer surplus for a single unit of potential riders between OD pairs (i,j) is computed as:

$$\int_0^{\ell_{max}}\int_{\frac{\ell_{ij}^m-(1-\sigma)y}{\sigma}}^{\ell_{max}} \frac{1}{\ell_{max}^2}(\sigma x + (1-\sigma)y - \ell_{ij}^m)dxdy =$$
$$\frac{\ell_{max}(\sigma^2 + \sigma + 1) - 3\ell_{ij}^m(1 + \sigma - \frac{\ell_{ij}^m}{\ell_{max}})}{6\sigma}. \quad (41)$$

The total consumer surplus is then:

$$CS^m \sum_{i=1}^n \sum_{j=1}^n \theta_{ij} \frac{\ell_{max}(\sigma^2 + \sigma + 1) - 3\ell_{ij}^m(1 + \sigma - \frac{\ell_{ij}^m}{\ell_{max}})}{6\sigma}. \quad (42)$$