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A Convex Optimization Framework for the Inverse Problem of Identifying a Random Parameter in a Stochastic Partial Differential Equation*

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Dedicated to Professor Joachim Gwinner on his 70th birthday

Abstract. The primary objective of this work is to study the inverse problem of identifying a stochastic parameter in partial differential equations with random data. In the framework of stochastic Sobolev spaces, we prove the Lipschitz continuity and the differentiability of the parameter-to-solution map and provide a new derivative characterization. We introduce a new energy-norm based modified output least-squares (OLS) objective functional and prove its smoothness and convexity. For stable inversion, we develop a regularization framework and prove an existence result for the regularized stochastic optimization problem. We also consider the OLS based stochastic optimization problem and provide an adjoint approach to compute the derivative of the OLS-functional. In the finite-dimensional noise setting, we give a parameterization of the inverse problem. We develop a computational framework by using the stochastic Galerkin discretization scheme and derive explicit discrete formulas for the considered objective functionals and their gradient. We provide detailed computational results to illustrate the feasibility and efficacy of the developed inversion framework. Encouraging numerical results demonstrate some of the advantages of the new framework over the existing approaches.

Key words. stochastic parameter identification, stochastic inverse problem, partial differential equations with random data, stochastic Galerkin method, regularization, infinite-dimensional noise

AMS subject classifications. 35R30, 49N45, 65J20, 65J22, 65M30

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1. Introduction.

Numerous models in applied and social sciences employ the broad spectrum of partial differential equations (PDEs) involving parameters that characterize the physical features of the model. For instance, the diffusion coefficient in the second-order PDEs,

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the rigidity coefficient in fourth-order PDEs emerging from plate models, and the Lamé parameters in linear elasticity describe characteristics of the underlying medium. In the applied models, these parameters are estimated based on experiments that involve noise. Indeed the commonly assumed properties of the parameters, which provide a convenient analytical framework, are simplifications of the experimental feedback of these parameters. Unfortunately, in many cases, the simplifications dilute certain essential features of the material parameters. A sensible way is to treat these parameters as random variables. However, it would make the solutions of the PDEs also random, giving rise to severe mathematical and theoretical challenges. Although, even in the early sixties, several authors raised related important questions (see [6, 36]), it is only in recent years that significant advancements in the numerical treatment of PDEs with random data have been made. Improvements in high-performance computational capabilities have substantially enhanced these developments.

This work focuses on the inverse problem of identifying a random coefficient in a PDE with random data. Assume that $(\Omega, \mathcal{F}, \mu)$ is a probability space, that is, Ω is a nonempty set whose elements are termed as elementary events, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and μ is a probability measure. Assume that $D \subset \mathbb{R}^n$ is a bounded domain and ∂D is its sufficiently smooth boundary. Given random fields $a : \Omega \times D \to \mathbb{R}$ and $f : \Omega \times D \to \mathbb{R}$, the direct problem in this work consists of finding a random field $u : \Omega \times D \to \mathbb{R}$ that almost surely satisfies the following PDE with random data:

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(\omega, x) \text{ in } D, \tag{1.1a}$$

$$u(\omega, x) = 0 \text{ on } \partial D. \tag{1.1b}$$

The above PDE models interesting real-world phenomena and has been studied in great detail. For example, in (1.1), u may represent the steady-state temperature at a given point of a body; then a would be a variable thermal conductivity coefficient and f the external heat source. The system (1.1) also models underground steady-state aquifers in which the parameter a is the aquifer transmissivity coefficient, u is the hydraulic head, and f is the recharge.

A natural interpretation of (1.1) is that realizations of the data lead to deterministic PDEs. That is, for a fixed $\omega \in \Omega$, PDE (1.1), under appropriate conditions, admits a weak solution $u(\omega, \cdot) \in H_0^1(D)$.

The inverse problem of identifying stochastic parameters in a PDE from a noisy measurement of the PDE solution has attracted a great deal of attention in the last few years. The most commonly adopted approach for inverse problems is a Bayesian formulation, which conditions a prior distribution on the coefficient function on observations of the PDE solution. The so-called variational approach that attracted quite a bit of attention in recent years was inspired by some of the Bayesian approach's challenges; see [25]. The variational approach is appropriate for identifying distributed and spatially correlated parameters in PDEs. It consists of posing a stochastic optimization problem whose solution can provide information concerning the unknown parameter's stochasticity/statistics. The variational approach's key advantages include access to a wide-ranging arsenal of efficient and reliable optimization algorithms, a rigorous functional analytic framework for convergence analysis, and easy amalgamation of the parameter's structural characteristics into the inversion framework. There are mainly two approaches to obtaining a stochastic optimization formulation in the variational approach: either defining an unconstrained stochastic optimization problem or introducing a constrained stochastic optimization problem in which the PDE itself is the constraint. The variational approach minimizes the following output least-squares (OLS) objective functional:

$$\widehat{J}_0(a) := \frac{1}{2}\int_{\Omega}\int_{D} |u_a(\omega, x) - z(\omega, x)|^2 dx\, d\mu(\omega), \tag{1.2}$$

where $u_a(\omega, x)$ is the solution of (1.1) for $a(\omega, x)$ and $z(\omega, x)$ is the data.

One of the main motivations of this work is to circumvent the significant deficiency of the OLS functional of being nonconvex, in general. However, before describing our approach, we briefly discuss some of the related developments. We begin by noting that Narayanan and Zabaras [5] investigate the inverse problem in the presence of uncertainties in the material data and develop an adjoint-approach based identification process by employing the spectral stochastic finite element method. They compute the gradient of the OLS-type objective functional and use a conjugate gradient strategy to provide promising numerical results. In [48], the authors develop a scalable methodology for the stochastic inverse problem using a sparse grid collocation approach. The inverse problem, posed as a stochastic optimization problem, is converted into a deterministic optimization in a high-dimensional space. Numerical examples are given to illustrate various aspects of the study. In [42], the authors develop a robust and efficient approach by employing generalized polynomial chaos expansion to identifying uncertain elastic parameters from experimental modal data. In [32], the authors give an implicit sampling for parameter identification. In [46], the authors develop a general framework for solving inverse problems under uncertainty using stochastic reduced-order models. They study the inverse problem as a constrained stochastic optimization problem. As an example, the authors identify random material parameters in elasto-dynamical systems. In [40], the authors study the parameter identification in a Bayesian setting for the elastoplastic problem. In [37], the authors focus on the inverse problem of parameter identification, where the parameters are random. They develop a sampling method that exploits the sensitivity derivatives of the control variable with respect to the random parameters. In [35], the authors study the optimal control problem for the stochastic diffusion equation. Using the Karhunen–Loève (KL) expansion, they separate the stochastic and the deterministic components and couple the finite element method and the polynomial chaos expansion for a numerical solution of the problem. In [26], the authors focus on determining the optimal thickness of a cylindrical shell subjected to stochastic forcing. The authors pose the problem as a stochastic optimization problem and derive necessary optimality conditions. For the numerical computation of a cylindrical shell's optimal thickness, they develop a gradient-based numerical scheme and provide numerical examples. In [1], the authors investigate the impact of errors and uncertainties of the conductivity on the electrocardiography imaging solution. They conduct the study in a stochastic optimization framework by using an OLS-type function. They use the stochastic Galerkin method for the numerical treatment of the direct and the inverse problem. Some of the related developments are available in [2, 3, 39, 8, 7, 9, 10, 11, 16, 21, 13, 14, 15, 17, 22, 28, 31, 30, 33, 34, 38, 43, 44, 27, 41, 47] and the cited references therein.

We note that whereas only the OLS approach is available for the stochastic inverse problems, other formulations exist for the identification of deterministic parameters. For example, the equation error approach, which results in a quadratic optimization problem (cf. [12]), and the coefficient-dependent OLS (see [18, 19, 24]), which leads to a convex minimization problem.

The primary objective of this paper is to propose a new energy least-squares (ELS) formulation for identifying stochastic parameters. The main contributions of this work are as follows:

1. We study the topological properties of the parameter-to-solution map. In particular, we establish its Lipschitz continuity and give a new Fréchet derivative characterization. We propose a new objective functional and prove its smoothness and convexity by using the derivative characterization. We devise a regularization framework and give an existence result for the regularized ELS-based stochastic optimization problem. For comparison, we also study the OLS-based stochastic optimization problem. We develop an adjoint approach to obtain the derivative of the OLS-functional. We emphasize that the derivative of the ELS objective does not involve the derivative of the parameter-to-solution map.
2. Under the finite-dimensional noise assumption, we obtain a parametrization of the stochastic variational problem and the associated stochastic optimization problems.
3. We give a stochastic Galerkin based discretization scheme for the continuous inverse problem. We provide explicit discrete formulas for the OLS and the ELS functionals and their gradients. We provide detailed computational results.

This work is mainly concerned with the inverse problem's mathematical aspects and how the discrete formulas from deterministic cases can be extended to stochastic cases. There is one critical issue that we have not addressed in this work, and that is the estimation of the noise distribution. We note that the unknown noise distribution can be dealt with in practice as follows: From the data for the inverse problem, which would be in the form of a random sample, one computes the covariance operator's eigenpairs. The decay of the eigenvalues suggests the size of the discrete KL expansion at the sample points, leading to the finite-noise vector $\{Y_i\}$ in the KL expansion. For the computation of the density function from $\{Y_i\}$, it is then possible to employ the so-called parametric or nonparametric methods; see [45] for more details.

We divide the contents of this paper into seven sections. Section 2 describes the variational formulation of the PDE with random data and gives the derivative characterization for the stochastic parameter-to-solution map. We study the new ELS approach in section 3. We present the parameterized stochastic inverse problem and the adjoint approach in section 4. We develop the computational framework in section 5 and give the numerical examples in section 6. The paper concludes with some general remarks and future research goals.

2. Variational problem and derivative characterization for the parameter-to-solution map.

An appropriate functional setting to study variational problems emerging from stochastic PDEs is provided by Bochner spaces of random variables; see [29]. Given a real Banach space X, a probability space $(\Omega, \mathcal{F}, \mu)$, and an integer $p \in [1, \infty)$, the Bochner space $L^p(\Omega; X)$ consists of Bochner integrable functions $u : \Omega \to X$ with finite pth moment, that is,

$$\|u\|_{L^p(\Omega;X)} := \left(\int_\Omega \|u(\omega)\|_X^p d\mu(\omega) \right)^{1/p} = \mathbb{E}\left[\|u(\omega)\|_X^p \right]^{1/p} < \infty.$$

If $p = \infty$, then $L^\infty(\Omega; X)$ is the space of Bochner measurable functions $u : \Omega \to X$ such that

$$\operatorname{ess\,sup}_{\omega\in\Omega} \|u(\omega)\|_X < \infty.$$

Intrinsic features of $L^p(D)$ spaces of Lebesgue integrable functions translate naturally to Bochner spaces $L^p(\Omega; X)$. It is known that (see [23]) $L^\infty(\Omega; L^\infty(D)) \subset L^\infty(\Omega \times D)$, but $L^\infty(\Omega; L^\infty(D)) \neq L^\infty(\Omega \times D)$, in general. Furthermore, the space $L^p(\Omega; L^q(D))$, for $p, q \in [1, \infty)$, is isomorphic to

$$\left\{ v : \Omega \times D \to \mathbb{R}^n | \int_\Omega \left(\int_D |v(\omega, x)|^q \, dx \right)^{p/q} d\mu(\omega) < \infty \right\}.$$

The variational formulation of (1.1) seeks $u \in V := L^2(\Omega; H_0^1(D))$ such that

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x)dx\right] = \mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\,dx\right] \text{ for all } v \in V. \tag{2.1}$$

In the following, we will assume that there are constants k_0 and k_1 such that

$$0 < k_0 \le a(\omega, x) \le k_1 < \infty \text{ almost everywhere in } \Omega \times D. \tag{2.2}$$

In particular, $a \in L^\infty(\Omega \times D)$.

By the aid of a bilinear form $s : V \times V \mapsto \mathbb{R}$ and a functional $\ell : V \mapsto \mathbb{R}$ given by

$$s(u, v) := \mathbb{E}\left[\int_D a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x)dx\right], \tag{2.3}$$

$$\ell(v) := \mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\,dx\right], \tag{2.4}$$

variational problem (2.1) can be written as a problem of finding $u \in V$ such that

$$s(u, v) = \ell(v) \quad \text{for every } v \in V. \tag{2.5}$$

It follows that

$$\begin{aligned}
|s(u,v)| &= \left|\mathbb{E}\left[\int_D a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x)dx\right]\right| \\
&\le \int_{\Omega\times D} |a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x)|dx\,d\mu(\omega) \\
&\le \|a(\omega,x)\|_{L^\infty(\Omega\times D)} \int_{\Omega\times D} |\nabla u(\omega,x)\cdot\nabla v(\omega,x)|dx\,d\mu(\omega) \\
&\le \|a(\omega,x)\|_{L^\infty(\Omega\times D)} \|u\|_V \|v\|_V,
\end{aligned}$$

which establishes the continuity of the bilinear form $s(\cdot, \cdot)$.

Furthermore, the bilinear form $s(\cdot,\cdot)$ is coercive as well because

$$\begin{aligned} s(v,v) &= \mathbb{E}\left[\int_D a(\omega,x)|\nabla v(\omega,x)|^2 dx\right] \\ &= \int_{\Omega\times D} a(\omega,x)|\nabla v(\omega,x)|^2 dx\, d\mu(\omega) \\ &\geq k_0 \int_{\Omega\times D} |\nabla v(\omega,x)|^2 dx\, d\mu(\omega) \\ &= \alpha\|v(\omega,x)\|_V^2, \end{aligned}$$

where α is a positive constant involving the Poincaré constant of the domain D.

For the given $f \in L^2(\Omega; H^1(D)^*)$ and for any $v \in V$, for the functional $\ell(\cdot)$, we have

$$|\ell(v)| = \left|\mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\, dx\right]\right| \leq \|f(\omega,x)\|_{L^2(\Omega;H^1(D)^*)}\|v(\omega,x)\|_V,$$

which proves the continuity of ℓ.

Consequently, by the Lax–Milgram lemma, variational problem (2.5) is uniquely solvable. Furthermore, it can be shown that there is a constant $c_1 > 0$, involving the Poincaré constant, such that

$$\|u(\omega,x)\|_V \leq c_1\|f(\omega,x)\|_{L^2(\Omega;H^1(D)^*)}. \tag{2.6}$$

Remark 2.1. Implicitly in the existence of the solution of variational problem (2.5), the measurability of the ω-wise solutions has to be verified; otherwise u does not necessarily belong to V.

At the core of inverse problems are the continuity and the differentiability properties of the parameter-to-solution map $a \mapsto u_a(\omega,x)$, which assigns to a the unique solution $u_a(\omega,x)$ of (2.1). For this, let $A \subset B := L^\infty(\Omega; L^\infty(D))$ be the set of feasible parameters with a nonempty interior.

The following result proves the Lipschitz continuity of the parameter-to-solution map.

Proposition 2.2. *For any $a(\omega,x) \in A$, the map $a(\omega,x) \mapsto u_a(\omega,x)$ is Lipschitz continuous.*

Proof. Let $u_a(\omega,x) \in V$ be the solution of (2.1) corresponding to $a(\omega,x) \in A$ and $u_b(\omega,x) \in V$ be the solution of (2.1) corresponding to $b(\omega,x) \in A$. The definitions of $u_a(\omega,x)$ and $u_b(\omega,x)$ yield

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla u_a(\omega,x)\cdot\nabla v(\omega,x)dx\right] = \mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\, dx\right] \text{ for every } v \in V,$$
$$\mathbb{E}\left[\int_D b(\omega,x)\nabla u_b(\omega,x)\cdot\nabla v(\omega,x)dx\right] = \mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\, dx\right] \text{ for every } v \in V,$$

and by subtracting the second equation from the first, for every $v \in V$, we obtain

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla u_a(\omega,x)\cdot\nabla v(\omega,x)dx\right] - \mathbb{E}\left[\int_D b(\omega,x)\nabla u_b(\omega,x)\cdot\nabla v(\omega,x)dx\right] = 0,$$

which, after a rearrangement of the terms, leads to the following identity:

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla\ (u_a(\omega,x)-u_b(\omega,x))\cdot\nabla v(\omega,x)dx\right]$$
$$+\,\mathbb{E}\left[\int_D (a(\omega,x)-b(\omega,x))\nabla u_b(\omega,x)\cdot\nabla v(\omega,x)dx\right]=0,\ \text{for every}\, v\in V.$$

We set $v(\omega,x)=u_a(\omega,x)-u_b(\omega,x)$ in the above equation to obtain

$$\begin{aligned}
&k_0\mathbb{E}\left[\|\nabla(u_a(\omega,\cdot)-u_b(\omega,\cdot))\|^2_{L^2(D)}\right]\\
&\quad\le\mathbb{E}\left[\int_D a(\omega,x)|\nabla(u_a(\omega,x)-u_b(\omega,x))|^2dx\right]\\
&\quad=-\mathbb{E}\left[\int_D (a(\omega,x)-b(\omega,x))\nabla u_b(\omega,x)\cdot\nabla(u_a(\omega,x)-u_b(\omega,x))dx\right]\\
&\quad\le\mathbb{E}\left[\int_D |(a(\omega,x)-b(\omega,x))\nabla u_b(\omega,x)\cdot\nabla(u_a(\omega,x)-u_b(\omega,x))|\,dx\right]\\
&\quad\le\|a(\omega,x)-b(\omega,x)\|_{L^\infty(\Omega\times D)}\mathbb{E}\left[\int_D |\nabla u_b(\omega,x)\cdot\nabla(u_a(\omega,x)-u_b(\omega,x))|\,dx\right]\\
&\quad\le\|a(\omega,x)-b(\omega,x)\|_{L^\infty(\Omega\times D)}\|u_b(\omega,x)\|_V\|u_a(\omega,x)-u_b(\omega,x)\|_V,
\end{aligned}$$

and by using (2.6), we have

$$\|u_a(\omega,x)-u_b(\omega,x)\|_V\le c\|a(\omega,x)-b(\omega,x)\|_{L^\infty(\Omega\times D)}$$

for a constant $c>0$. The proof is complete. ■

The following result gives a derivative characterization (in Fréchet sense) of the parameter-to-solution map.

Theorem 2.3. *For each $a(\omega,x)$ in the interior of A, the map $a(\omega,x)\mapsto u_a(\omega,x)$ is differentiable at $a(\omega,x)$. The derivative $\delta u_a := Du_a(\delta a)$ of $u_a(\omega,x)$ at $a(\omega,x)$ in the direction $\delta a(\omega,x)$ is the unique solution of the variational problem: Find $\delta u_a(\omega,x)\in V$ such that for every $v(\omega,x)\in V$, we have*

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla\delta u_a(\omega,x)\cdot\nabla v(\omega,x)dx\right]=-\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla u_a(\omega,x)\cdot\nabla v(\omega,x)\,dx\right].\tag{2.7}$$

Proof. Note that a proof of the unique solvability of (2.7) is similar to the unique solvability of (2.1). To prove (2.7), for $a(\omega,x)\in A$, let $\delta a(\omega,x)$ be sufficiently small so that $a(\omega,x)+\delta a(\omega,x)\in A$. Therefore, the quantity $\delta w(\omega,x)=u_{a+\delta a}(\omega,x)-u_a(\omega,x)$ is well-defined.

By the definition of $u_a(\omega,x)$ and $u_{a+\delta a}(\omega,x)$, for every $v(\omega,x)\in V$, we have

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla u_a(\omega,x)\cdot\nabla v(\omega,x)dx\right]=\mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\,dx\right],\tag{2.8}$$

$$\mathbb{E}\left[\int_D (a(\omega,x)+\delta a(\omega,x))\nabla u_{a+\delta a}(\omega,x)\cdot\nabla v(\omega,x)dx\right]=\mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\,dx\right].\tag{2.9}$$

We subtract (2.8) from (2.9) to get

$$\mathbb{E}\left[\int_D (a(\omega,x)+\delta a(\omega,x))\nabla\delta w(\omega,x)\cdot\nabla v(\omega,x)dx\right] = -\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla u_a(\omega,x)\cdot\nabla v(\omega,x)dx\right]$$

and subtract (2.7) from the above equation to obtain

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla\left(\delta w(\omega,x)-\delta u_a(\omega,x)\right)\cdot\nabla v(\omega,x)dx\right] = -\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla\delta w(\omega,x)\cdot\nabla v(\omega,x)dx\right].$$

By setting $v(\omega,x)=\delta w(\omega,x)-\delta u_a(\omega,x)$, we have

$$\mathbb{E}\left[\int_D a(\omega,x)|\nabla\left(\delta w(\omega,x)-\delta u_a(\omega,x)\right)|^2dx\right]$$
$$= -\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla\delta w(\omega,x)\cdot\nabla\left(\delta w(\omega,x)-\delta u_a(\omega,x)\right)dx\right].$$

As before, the above identity implies that for a constant $c>0$, we have

$$\|\delta w(\omega,x)-\delta u_a(\omega,x)\|_V \le c\|\delta w(\omega,x)\|_V\|\delta a(\omega,x)\|_{L^\infty(\Omega\times D)} \le c\|\delta a(\omega,x)\|^2_{L^\infty(\Omega\times D)},$$

where we used the Lipschitz continuity of the solution map, and consequently

$$\frac{\|u_{a+\delta a}(\omega,x)-u_a(\omega,x)-\delta u(\omega,x)\|_V}{\|\delta a(\omega,x)\|_{L^\infty(\Omega\times D)}} = o\left(\|\delta a(\omega,x)\|_{L^\infty(\Omega\times D)}\right),$$

which by taking $\|\delta a(\omega,x)\|_{L^\infty(\Omega\times D)}\to 0$ confirms that $\delta u(\omega,x)$ is the sought derivative. ■

3. A new convex inversion framework.

We propose the following new ELS objective functional:

$$J_0(a) = \frac{1}{2}\mathbb{E}\left[\int_D a(\omega,x)|\nabla(u_a(\omega,x)-z(\omega,x))|^2dx\right], \tag{3.1}$$

where $u_a(\omega,x)$ is the solution of (2.1) for $a(\omega,x)$ and $z(\omega,x)\in L^2(\Omega;H_0^1(D))$ is the data. We recall that the commonly known optimization formulation for the stochastic inverse problem of parameter identification is the OLS:

$$\widehat{J}_0(a) := \frac{1}{2}\mathbb{E}\left[\|u_a(\omega,x)-z(\omega,x)\|^2\right], \tag{3.2}$$

where $u_a(\omega,x)$ is the solution of (2.1) for $a(\omega,x)$, $z(\omega,x)\in L^2(\Omega;L^2(D))$ is the measured data, and $\|\cdot\|$ is a suitable norm, For example, $L^2(D)$-norm was considered in [1], whereas $H^1(D)$-norm was employed in [7]; $H^1(D)$-seminorm is another possibility.

One of the significant deficiencies of the OLS formulation is its inherited nonconvexity, which causes severe theoretical and computational challenges and poses the risk of locating only local solutions of the OLS-based stochastic optimization problem. The ELS functional, on the other hand, is convex, as shown by the following result.

Theorem 3.1. *The ELS functional given in* (3.1) *is convex in the interior of the set* A.

Proof. We compute the first derivative of J_0 in any direction $\delta a(\omega, x)$ by the chain rule as follows:

$$\begin{aligned} DJ_0(a)(\delta a) = &\frac{1}{2}\mathbb{E}\left[\int_D \delta a(\omega,x)|\nabla(u_a(\omega,x) - z(\omega,x))|^2 dx\right] \\ &+ \mathbb{E}\left[\int_D a(\omega,x)\nabla\delta u_a(\omega,x)\cdot\nabla(u_a(\omega,x)-z(\omega,x))dx\right], \end{aligned}$$

where $\delta u_a(\omega,x)$ is the derivative of $u_a(\omega,x)$ in the direction $\delta a(\omega,x)$.

Since

$$\begin{aligned} &\mathbb{E}\left[\int_D a(\omega,x)\nabla\delta u_a(\omega,x)\cdot\nabla(u_a(\omega,x)-z(\omega,x))dx\right] \\ &\qquad = -\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla u_a(\omega,x)\cdot\nabla(u_a(\omega,x)-z(\omega,x))dx\right], \end{aligned}$$

we obtain

$$\begin{aligned} DJ_0(a)(\delta a) &= \frac{1}{2}\mathbb{E}\left[\int_D \delta a(\omega,x)|\nabla(u_a(\omega,x) - z(\omega,x))|^2 dx\right] \\ &\quad - \mathbb{E}\left[\int_D \delta a(\omega,x)\nabla u_a(\omega,x)\cdot\nabla(u_a(\omega,x)-z(\omega,x))dx\right] \\ &= -\frac{1}{2}\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla(u_a(\omega,x)+z(\omega,x))\cdot\nabla(u_a(\omega,x)-z(\omega,x))dx\right]. \end{aligned}$$

The second-order derivative can be computed as follows:

$$\begin{aligned} D^2J_0(a)(\delta a,\delta a) &= -\frac{1}{2}\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla\delta u_a(\omega,x)\cdot\nabla(u_a(\omega,x)-z(\omega,x))dx\right] \\ &\quad -\frac{1}{2}\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla(u_a(\omega,x)+z(\omega,x))\cdot\nabla\delta u_a(\omega,x)dx\right] \\ &= -\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla u(a)(\omega,x)\cdot\nabla\delta u_a(\omega,x)dx\right] \\ &= \mathbb{E}\left[\int_D a(\omega,x)|\nabla\delta u_a(\omega,x)|^2 dx\right], \end{aligned}$$

where we used Theorem 2.3. Hence, there is a constant $\alpha > 0$ involving the Poincaré constant such that the inequality holds for all $a(\omega,x)$ in the interior of A:

$$D^2J_0(a)(\delta a,\delta a) \geq \alpha\|\delta u_a(\omega,x)\|_V^2; \tag{3.3}$$

consequently J_0 is a convex functional. ∎

The inverse problem of identifying stochastic parameters in PDEs is ill-posed, and for a stable identification process, some type of regularization is essential. For this, we tailor a general setting by defining the following admissible set:

$$A := \left\{a \in H = L^2(\Omega; H(D)) :\ 0 < k_0 \leq a(\omega,x) \leq k_1 \text{ a.s. } \Omega\times D\right\},$$

where H is a separable Hilbert space compactly embedded into $B := L^\infty(\Omega; L^\infty(\Omega))$, and $H(D)$ is continuously embedded in $L^\infty(\Omega)$

We consider the following regularized ELS functional:

$$\min_{a\in A} J_\kappa(a) := \frac{1}{2}\mathbb{E}\left[\int_D a(\omega,x)|\nabla(u(\omega,x)-z(\omega,x))|^2 dx\right] + \frac{\kappa}{2}\|a(\omega,x)\|_H^2, \tag{3.4}$$

where $u_a(\omega, x)$ is the solution of (2.1) for $a(\omega, x)$, $z(\omega, x) \in L^2(\Omega; L^2(D))$ is the measured data, $\kappa > 0$ is a fixed regularization parameter, and $\|\cdot\|_H^2$ is the regularizer. We note that the norm $\|\cdot\|_H$ already includes the expectation operator.

We have the following existence result.

Theorem 3.2. *For each $\kappa > 0$, the ELS-based problem (3.4) has a unique solution.*

Proof. Since $J_\kappa(a) \geq 0$, for every $a \in A$, there is a minimizing sequence $\{a_n\}$ in A such that

$$\lim_{n\to\infty} J_\kappa(a_n) = \inf\{J_\kappa(a) |\, a \in A\}.$$

Therefore, $\{J_\kappa(a_n)\}$ is bounded, and consequently $\{a_n\}$ is bounded in $\|\cdot\|_H$. Since H is compactly embedded, $\{a_n\}$ has a subsequence, which converges in norm to some $\bar{a} \in A$. Retaining the same notation for subsequences, let u_n be the solution of the variational problem that corresponds to a_n. That is,

$$\mathbb{E}\left[\int_D a_n(\omega,x)\nabla u_n(\omega,x)\cdot\nabla v(\omega,x)dx\right] = \mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\,dx\right] \text{ for all } v \in V.$$

We set $v = u_n$ and obtain

$$\mathbb{E}\left[\int_D a_n(\omega,x)|\nabla u_n(\omega,x)|^2 dx\right] = \mathbb{E}\left[\int_D f(\omega,x)u_n(\omega,x)\,dx\right],$$

which leads to the boundedness of $\{u_n\}$. Therefore, $\{u_n\}$ has a subsequence that converges weakly to some $\bar{u} \in V$. We claim that $\bar{u} = u_{\bar{a}}$. Since

$$\mathbb{E}\left[\int_D a_n(\omega,x)\nabla u_n(\omega,x)\cdot\nabla v(\omega,x)dx\right] = \mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\,dx\right] \quad \text{for every } v \in V,$$

after a simple rearrangement of terms, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_D [\bar{a}(\omega,x)\nabla\bar{u}(\omega,x)\cdot\nabla v(\omega,x) - f(\omega,x)v(\omega,x)]\,dx\right]\\
&\quad= -\mathbb{E}\left[\int_D (a_n(\omega,x)-\bar{a}(\omega,x))\nabla u_n(\omega,x)\cdot\nabla v(\omega,x))\,dx\right]\\
&\qquad - \mathbb{E}\left[\int_D \bar{a}(\omega,x)\nabla(u_n(\omega,x)-\bar{u}(\omega,x))\cdot\nabla v(\omega,x)\,dx\right].
\end{aligned}\tag{3.5}$$

Notice that

$$
\begin{aligned}
&\left|\mathbb{E}\left[\int_D (a_n(\omega,x)-\bar{a}(\omega,x))\nabla u_n(\omega,x)\cdot\nabla v(\omega,x)\,dx\right]\right| \\
&\quad\le \left(\mathbb{E}\left[\int_D |a_n(\omega,x)-\bar{a}(\omega,x))|\,|\nabla u_n(\omega,x)|^2\,dx\right]\right)^{\frac{1}{2}} \\
&\qquad\times\left(\mathbb{E}\left[\int_D |a_n(\omega,x)-\bar{a}(\omega,x)|\,\left|\nabla v(\omega,x)\right|^2\,dx\right]\right)^{\frac{1}{2}}\to 0
\end{aligned}
$$

by the dominated convergence theorem. Since the second term on the right-hand side of (3.5) also converges to zero, we have

$$
\mathbb{E}\left[\int_D \left[\bar{a}(\omega,x)\nabla\bar{u}(\omega,x)\cdot\nabla v(\omega,x)-f(\omega,x)v(\omega,x)\right]\,dx\right]=0.
$$

Since $v\in V$ is arbitrary, and (2.1) is uniquely solvable, we get $\bar{u}=u_{\bar{a}}$.

We claim that $J_0(a_n)\to J_0(\bar{a})$. The identities

$$
\begin{aligned}
&\mathbb{E}\left[\int_D a_n(\omega,x)|\nabla(u_n(\omega,x)-z(\omega,x))|^2dx\right] \\
&\quad=\mathbb{E}\left[\int_D f(\omega,x)(u_n(\omega,x)-z(\omega,x))dx\right] \\
&\qquad-\mathbb{E}\left[\int_D a_n(\omega,x)\nabla z(\omega,x)\cdot\nabla(u_n(\omega,x)-z(\omega,x))dx\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\int_D \bar{a}(\omega,x)|\nabla(\bar{u}(\omega,x)-z(\omega,x))|^2dx\right] \\
&\quad=\mathbb{E}\left[\int_D f(\omega,x)(\bar{u}(\omega,x)-z(\omega,x))dx\right] \\
&\qquad-\mathbb{E}\left[\int_D \bar{a}(\omega,x)\nabla z(\omega,x)\cdot\nabla(\bar{u}(\omega,x)-z(\omega,x))dx\right],
\end{aligned}
$$

in view of the rearrangement

$$
\begin{aligned}
&\mathbb{E}\left[\int_D a_n(\omega,x)\nabla z(\omega,x)\cdot\nabla(u_n(\omega,x)-z(\omega,x))dx\right] \\
&\quad-\mathbb{E}\left[\int_D \bar{a}(\omega,x)\nabla z(\omega,x)\cdot\nabla(\bar{u}(\omega,x)-z(\omega,x))dx\right] \\
&=\mathbb{E}\left[\int_D (a_n(\omega,x)-\bar{a}(\omega,x))\nabla z(\omega,x)\cdot\nabla(u_n(\omega,x)-z(\omega,x))dx\right] \\
&\quad+\mathbb{E}\left[\int_D \bar{a}(\omega,x)\nabla z(\omega,x)\cdot\nabla(u_n(\omega,x)-\bar{u}(\omega,x))dx\right],
\end{aligned}
$$

imply that

$$\mathbb{E}\left[\int_D a_n(\omega,x)\nabla(u_n(\omega,x)-z(\omega,x))\cdot(u_n(\omega,x)-z(\omega,x))dx\right]$$
$$\to \mathbb{E}\left[\int_D \bar{a}(\omega,x)\nabla(\bar{u}(\omega,x)-z(\omega,x))\cdot\nabla(\bar{u}(\omega,x)-z(\omega,x))dx\right],$$

and consequently,

$$\begin{aligned}
J_\kappa(\bar{a}) &= \frac{1}{2}\mathbb{E}\left[\int_D \bar{a}(\omega,x)|\nabla(\bar{u}(\omega,x)-z(\omega,x))|^2dx\right]+\frac{\kappa}{2}\|\bar{a}(\omega,x)\|_H^2\\
&\le \lim_{n\to\infty}\frac{1}{2}\mathbb{E}\left[\int_D a_n(\omega,x)|\nabla(u_{a_n}(\omega,x)-z(\omega,x))|^2dx\right]+\liminf_{n\to\infty}\frac{\kappa}{2}\|a_n(\omega,x)\|_H^2\\
&\le \liminf_{n\to\infty}\left\{\frac{1}{2}\mathbb{E}\left[\int_D a_n(\omega,x)|\nabla(u_{a_n}(\omega,x)-z(\omega,x))|^2dx\right]+\frac{\kappa}{2}\|a_n(\omega,x)\|_H^2\right\}\\
&= \inf\left\{J_\kappa(a)\mid a\in A\right\},
\end{aligned}$$

confirming that $\bar{a}$ is a solution of (3.4). The proof is complete. ■

4. Parametrization of the stochastic inverse problem. A vital component of the study of stochastic PDEs and stochastic optimization problems is the representation of the random fields by a finite number of mutually independent random variables. Random fields that are functions of only a finite number of random variables are known as finite-dimensional noise, formally defined in the following (see [4, 29]).

Definition 4.1. *Let $\xi_k : \Omega \mapsto \Gamma_k$, for $k = 1,\dots,M$, be real-valued random variables with $M < \infty$. A function $v \in L^2(\Omega; L^2(D))$ of the form $v(x,\xi(\omega))$ for $x \in D$ and $\omega \in \Omega$, where $\xi = (\xi_1,\xi_2,\dots,\xi_M) : \Omega \mapsto \Gamma \subset \mathbb{R}^M$ and $\Gamma := \Gamma_1 \times \Gamma_2 \cdots \times \Gamma_M$, is called a* finite-dimensional noise.

If a random field $v(x,\xi)$ is finite-dimensional noise, a change of variables can be made for evaluating expectations. For instance, denoting by σ the joint density of ξ, we have

$$\|v\|^2_{L^2(\Omega;L^2(D))} = \mathbb{E}\left[\|v\|^2_{L^2(D)}\right] = \int_\Gamma \sigma(y)\|v(y,\cdot)\|^2_{L^2(D)}dy.$$

Consequently, by defining $y_k := \xi_k(\omega)$ and setting $y = (y_1,y_2,\dots,y_M)$, we associate a random field $v(x,\xi)$ with a finite-dimensional noise by a function $v(x,y)$ in the weighted L^2 space

$$L^2_\sigma(\Gamma;L^2(D)) := \left\{v : \Gamma\times D\to\mathbb{R} : \int_\Gamma \sigma(y)\|v(\cdot,y)\|^2_{L^2(D)}dy < \infty\right\}.$$

In this work, we assume that $a(\omega,x)$ and $f(\omega,x)$ are finite-dimensional noises and given by

$$a(\omega,x) = a_0(x) + \sum_{k=1}^{P} a_k(x)\xi_k(\omega),$$
$$f(\omega,x) = f_0(x) + \sum_{k=1}^{L} f_k(x)\xi_k(\omega),$$

where the real-valued functions a_k and f_k are uniformly bounded.

It follows from the Doob–Dynkin lemma that a solution of (2.1) is finite-dimensional noise and u is a function of ξ where $\xi = (\xi_1, \xi_2, \dots, \xi_M) : \Omega \mapsto \Gamma$ and $M := \max\{P, L\}$; see [29].

Then, the variational problem (2.1) reduces to the following parametric deterministic variational problem: Find $u(y,x) \in V_\sigma := L^2_\sigma(\Gamma; H^1_0(D))$ such that for every $v(y,x) \in V_\sigma$, we have

$$\int_\Gamma \sigma(y) \int_D a(y,x) \nabla u(y,x) \cdot \nabla v(y,x) dx\, dy = \int_\Gamma \sigma(y) \int_D f(y,x) v(y,x)\, dx\, dy. \tag{4.1}$$

For the inverse problem, we will assume that the data z depends, via ξ, on the finite-dimensional noise variables $\{\xi_i\}_{i=1}^M$. Therefore, we will assume that the unknown parameter a is also the function of the variables $\{\xi_i\}_{i=1}^M$. That is,

$$a(x,\xi) = a(x, \xi_1(\omega), \xi_2(\omega), \dots, \xi_M(\omega)) \in \widetilde{H}(D) := L^2_\sigma(\Gamma; H(D)).$$

The following finite-dimensional noise variants of the OLS and the ELS objectives read

$$\min_{a \in A} \widehat{J}_0(a) := \frac{1}{2} \int_\Gamma \sigma(y) \int_D |\, (u_a(y,x) - z(y,x))\,|^2 dx\, dy, \tag{4.2}$$

$$\min_{a \in A} J_0(a) := \frac{1}{2} \int_\Gamma \sigma(y) \int_D a(y,x) |\nabla\, (u_a(y,x) - z(y,x))\,|^2 dx\, dy, \tag{4.3}$$

where $u_a(y,x)$ solves (4.1) for $a(y,x)$ and $z(y,x)$ is the finite-dimensional noise data.

Following Theorem 2.3, we obtain a derivative characterization of the finite-dimensional noise parameter-to-solution map and the derivative formula of the ELS functional.

Theorem 4.2. *Let a be in the interior of A. Then, the derivative $\delta u_a := Du_a(\delta a)$ of $u_a(y,x)$ at $a(y,x)$ in the direction $\delta a(y,x)$ is the unique solution of the following parameterized variational problem:*

$$\int_\Gamma \sigma(y) \int_D a(y,x) \nabla \delta u_a(y,x) \cdot \nabla v(y,x) dx\, dy$$
$$= - \int_\Gamma \sigma(y) \int_D \delta a(y,x) \nabla u_a(y,x) \cdot \nabla v(y,x)\, dx\, dy \quad \textit{for every}\, v \in V_\sigma.$$

Furthermore, the derivative of the finite-dimensional noise ELS (4.3) reads

$$DJ_0(a)(\delta a) = \frac{1}{2} \int_\Gamma \sigma(y) \int_D \delta a(y,x) \nabla (u_a(y,x) + z(y,x)) \cdot \nabla (u_a(y,x) - z(y,x)) dx\, dy.$$

To compute the derivative of the OLS objective, we will now devise an adjoint approach. For this, we note that from

$$\widehat{J}_0(a) = \frac{1}{2} \int_\Gamma \sigma(y) \int_D (u_a(y,x) - z(y,x))^2 dx\, dy,$$

by a direct computation, we have

$$D\widehat{J}_0(a)(\delta a) = \int_\Gamma \sigma(y) \int_D \delta u_a(y,x)(u_a(y,x) - z(y,x)) dx\, dy,$$

where the derivative $\delta u_a(y,x) = Du_a(\delta a(y,x))$ can be computed by Theorem 4.2.

To devise an adjoint approach, for $v \in V_\sigma$, we define

$$L(a,v) = \widehat{J}_0(a) + \int_\Gamma \sigma(y) \int_D a(y,x)\nabla u_a(y,x) \cdot \nabla v(y,x) dx\, dy - \int_\Gamma \sigma(y) \int_D f(y,x) v(y,x) dx\, dy$$

and note that by definition, we have

$$\frac{\partial}{\partial a} L(a,v) = D\widehat{J}_0(a)(\delta a) \quad \text{for every } v \in V_\sigma. \tag{4.4}$$

We consider the adjoint equation of finding $w = w(y,x) \in V_\sigma$ such that for every $v \in V_\sigma$,

$$\int_\Gamma \sigma(y) \int_D a(y,x)\nabla w(y,x) \cdot \nabla v(y,x) dx\, dy = \int_\Gamma \sigma(y) \int_D (z(y,x) - u_a(y,x)) v(y,x) dx\, dy. \tag{4.5}$$

Then, for every $v \in V_\sigma$, we have

$$\begin{aligned}\frac{\partial}{\partial a} L(a,v)(\delta a) = D\widehat{J}_0(a)(\delta a) &+ \int_\Gamma \sigma(y) \int_D \delta a(y,x) \nabla u_a(y,x) \cdot \nabla v(y,x) dx\, dy \\ &+ \int_\Gamma \sigma(y) \int_D a(y,x) \nabla \delta u_a(y,x) \cdot \nabla v(y,x) dx\, dy,\end{aligned}$$

and for the choice $v = w$, we have

$$\begin{aligned}\frac{\partial}{\partial a} L(a,w)(\delta a) &= \int_\Gamma \sigma(y) \int_D \delta u(y,x))(u_a(y,x) - z(y,x)) dx\, dy \\ &\quad + \int_\Gamma \sigma(y) \int_D \delta a(y,x) \nabla u_a(y,x) \cdot \nabla w(y,x) dx\, dy \\ &\quad + \int_\Gamma \sigma(y) \int_D a(y,x) \nabla \delta u_a(y,x) \cdot \nabla w(y,x) dx\, dy,\end{aligned}$$

which due to the definition of adjoint variable in (4.5) yields

$$\begin{aligned}\frac{\partial}{\partial a} L(a,w)(\delta a) &= \int_\Gamma \sigma(y) \int_D \delta u_a(y,x))(u_a(y,x) - z(y,x)) dx\, dy \\ &\quad + \int_\Gamma \sigma(y) \int_D \delta a(y,x) \nabla u_a(y,x) \cdot \nabla w(y,x) dx\, dy \\ &\quad - \int_\Gamma \sigma(y) \int_D (z(y,x) - u_a(y,x)) \delta u(y,x) dx\, dy \\ &= \int_\Gamma \sigma(y) \int_D \delta a(y,x) \nabla u_a(y,x) \cdot \nabla w(y,x) dx\, dy,\end{aligned}$$

and consequently,

$$\frac{\partial}{\partial a} L(a,w)(\delta a) = \int_\Gamma \sigma(y) \int_D \delta a(y,x) \nabla u_a(y,x) \cdot \nabla w(y,x) dx\, dy,$$

and from (4.4), we deduce

$$D\widehat{J}_0(a)(\delta a) = \int_\Gamma \sigma(y) \int_D \delta a(y,x) \nabla u_a(y,x) \cdot \nabla w(y,x) dx\, dy. \tag{4.6}$$

Summarizing, we obtain the following scheme to compute $D\widehat{J}_0(a)(\delta a)$:

1. Compute $u_a(y,x)$ by solving variational problem (4.1).
2. Compute $w(y,x)$ by solving adjoint problem (4.5).
3. Compute $D\widehat{J}_0(a)(\delta a)$ by using (4.6).

5. Computational framework. We now proceed to derive discrete formulas for the direct problem, the OLS and the ELS objective functionals, and their gradients. Recall that the variational problem that needs to be discretized reads, Find $u \in V_\sigma = L^2_\sigma(\Gamma; H^1_0(D))$ such that

$$\int_\Gamma \sigma(y) \int_D a(y,x)\nabla u(y,x)\cdot\nabla v(y,x)dx\,dy = \int_\Gamma \sigma(y)\int_D f(y,x)v(y,x)dx\,dy \text{ for all } v \in V_\sigma.$$

Let V_{hk} be a finite-dimensional subspace of V_σ. An element $u_{hk} \in V_{hk}$ is the stochastic Galerkin solution if

$$\int_\Gamma \sigma(y) \int_D a(y,x)\nabla u_{hk}(y,x)\cdot\nabla v(y,x)dx\,dy = \int_\Gamma \sigma(y)\int_D f(y,x)v(y,x)dx\,dy \text{ for all } v \in V_{hk}.$$

Let V_h be an N-dimensional subspace of $H^1_0(D)$ and S_k be a Q-dimensional subspace of $L^2_\sigma(\Gamma)$ with

$$V_h = \operatorname{span}\{\phi_1,\phi_2,\ldots,\phi_N\},$$
$$S_k = \operatorname{span}\{\psi_1,\psi_2,\ldots,\psi_Q\}.$$

We assume that the basis $\{\psi_1,\psi_2,\ldots,\psi_Q\}$ is orthonormal with respect to σ, that is,

$$\int_\Gamma \sigma(y)\psi_n(y)\psi_m(y)dy=\delta_{nm},$$

where δ_{nm} is the Kronecker delta: $\delta_{nm} = 1$ for $n = m$, $\delta_{nm} = 0$ for $n \neq m$. We construct a finite-dimensional subspace of V_σ by tensorizing the basis functions ϕ_i and ψ_j. That is, the following NQ-dimensional subspace will be the trial and test space for solving the discrete variational problem:

$$V_{hk} := V_h \otimes S_k := \operatorname{span}\{\phi_i\psi_j |\ i = 1,\ldots,N,\ j = 1,\ldots,Q\}.$$

Therefore, any $v \in V_h \otimes S_k$ has the representation

$$v(y,x) = \sum_{i=1}^{N}\sum_{j=1}^{Q} V_{ij}\phi_i(x)\psi_j(y) = \sum_{j=1}^{Q}\left[\sum_{i=1}^{N} V_{ij}\phi_i(x)\right]\psi_j(y) = \sum_{j=1}^{Q} V_j(x)\psi_j(y),$$

where

$$V_j(x) \equiv \sum_{i=1}^{N} V_{ij}\phi_i(x).$$

It is convenient to introduce the following vectorized notation:

$$V = \mathrm{vec}(V_{ij}) = \begin{pmatrix} V_{11} \\ \vdots \\ V_{N1} \\ V_{12} \\ \vdots \\ V_{N2} \\ \vdots \\ V_{1Q} \\ \vdots \\ V_{NQ} \end{pmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_Q \end{bmatrix}, \tag{5.1}$$

where

$$V_j := \begin{bmatrix} V_{1j} \\ \vdots \\ V_{Nj} \end{bmatrix} \in \mathbb{R}^N.$$

Following the use of the KL expansion, we will assume that the unknown random field is expressed as a finite linear expansion:

$$a(y,x) = a_0(x) + \sum_{s=1}^{M} y_s a_s(x) = \sum_{s=0}^{M} y_s a_s(x), \tag{5.2}$$

where, by convention, we denote $y_0 = 1$. The spatial components a_s are discretized by using another P-dimensional space,

$$A_h = \mathrm{span}\{\varphi_1, \ldots, \varphi_P\}.$$

By following the same vectorial notation, we have

$$a(y,x) = \sum_{i=1}^{P} A_{i0}\varphi_i(x) + \sum_{s=1}^{M}\left(\sum_{i=1}^{P} A_{is}\varphi_i(x)\right) y_s = \sum_{s=0}^{M} A_s y_s, \tag{5.3}$$

where the vectors $A_s(x) \equiv (A_{is}) \in \mathbb{R}^P$ for $s = 0\ldots, M$,

$$A = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_M \end{bmatrix} \in \mathbb{R}^{P(M+1)\times 1}.$$

Evidently, the discrete variational problem seeks $u_{hk}(y,x) \in V_h \otimes S_Q$ such that

$$\int_\Gamma \sigma(y)\psi_n(y)\left(\int_D a(y,x)\nabla u_{hk}(y,x)\nabla\phi_i(x)dx\right)dy = \int_\Gamma \sigma(y)\psi_n(y)\left(\int_D f(y,x)\phi_i(x)dx\right)dy$$

for every $i = 1,\ldots,N,\ n = 1,\ldots,Q$.

By using the representation

$$u_{hk} = \sum_{k=1}^{N} \sum_{m=1}^{Q} U_{km} \phi_k(x) \psi_m(y),$$

we obtain

$$\int_\Gamma \sigma(y)\psi_n(y) \left(\int_D a(y,x) \nabla \left(\sum_{k=1}^{N} \sum_{m=1}^{Q} U_{km} \phi_k(x) \psi_m(y) \right) \nabla \phi_i(x) dx \right) dy$$
$$= \int_\Gamma \sigma(y)\psi_n(y) \left(\int_D f(y,x) \phi_i(x) dx \right) dy,$$

or equivalently,

$$\sum_{k=1}^{N} \sum_{m=1}^{Q} U_{km} \int_\Gamma \sigma(y)\psi_n(y)\psi_m(y) \left(\int_D a(y,x) \nabla \phi_k(x) \nabla \phi_i(x) dx \right) dy$$
$$= \int_\Gamma \sigma(y)\psi_n(y) \left(\int_D f(y,x) \phi_i(x) dx \right) dy,$$

for every $i = 1, \ldots, N$, $n = 1, \ldots, Q$.

By using the expansion (5.3) in the above identity, we obtain

$$\begin{aligned}
&\sum_{k=1}^{N} \sum_{m=1}^{Q} U_{km} \int_\Gamma \sigma(y)\psi_n(y)\psi_m(y) \left(\int_D a(y,x) \nabla \phi_k(x) \nabla \phi_i(x) dx \right) dy \\
&= \sum_{k=1}^{N} \sum_{m=1}^{Q} \left(\int_\Gamma \sigma(y)\psi_n(y)\psi_m(y) dy \right) \left(\int_D A_0(x) \nabla \phi_k(x) \nabla \phi_i(x) dx \right) U_{km} \\
&\quad + \sum_{s=1}^{M} \sum_{k=1}^{N} \sum_{m=1}^{Q} \left(\int_\Gamma \sigma(y)\psi_n(y)\psi_m(y) y_s dy \right) \left(\int_D A_s(x) \nabla \phi_k(x) \nabla \phi_i(x) dx \right) U_{km} \\
&= \sum_{k=1}^{N} \sum_{m=1}^{Q} \delta_{nm} K(A_0)_{ik} U_{km} + \sum_{s=1}^{M} \sum_{k=1}^{N} \sum_{m=1}^{Q} \left(\int_\Gamma \sigma(y)\psi_n(y)\psi_m(y) y_s dy \right) K(A_s)_{ik} U_{km} \\
&= \sum_{k=1}^{N} K(A_0)_{ik} U_{kn} + \sum_{s=1}^{M} \sum_{k=1}^{N} \sum_{m=1}^{Q} g^s_{nm} K(A_s)_{ik} U_{km} \\
&= \left(K(A_0) + \sum_{s=1}^{M} g^s_{nn} K(A_s) \right) U_n + \sum_{m \neq n} \sum_{s=1}^{M} g^s_{nm} K(A_s) U_m,
\end{aligned}$$

where for every $s \in \{0, \ldots, M\}$, we define $K(A_s) \in \mathbb{R}^{n \times n}$ and $g^s_{nm} \in \mathbb{R}$ by

$$K(A_s)_{i,k} = \int_D A_s(x) \nabla \phi_k(x) \nabla \phi_i(x) dx,$$
$$g^s_{nm} = \int_\Gamma \sigma(y)\psi_n(y)\psi_m(y) y_s dy.$$

Now, for $s \in \{0, \dots, M\}$, we set

$$G^s = (g^s_{nm}) \in \mathbb{R}^{Q\times Q},$$

where the case $s = 0$, by orthonormality, corresponds to the identity matrix as follows:

$$G^0 = \left(\int_\Gamma \sigma(y)\psi_n(y)\psi_m(y)dy\right) = I.$$

On the other hand, we discretize the right-hand side as follows:

$$(F_n)_i = \int_\Gamma \sigma(y)\psi_n(y)\int_D f(y,x)\phi_i(x)dx\,dy \quad \text{for every} \quad n = 1,\dots,Q.$$

Summarizing, the discrete variational problem reads

$$\left(K(A_0) + \sum_{s=1}^M g^s_{nn}K(A_s)\right)U_n + \sum_{m\neq n}\sum_{s=1}^M g^s_{nm}K(A_s)U_m = F_n \text{ for every } n = 1,\dots,Q,$$

which corresponds to solving the following linear system:

$$\begin{pmatrix} K(A_0)+\sum_{s=1}^M g^s_{11}K(A_s) & \sum_{s=1}^M g^s_{12}K(A_s) & \cdots & \sum_{s=1}^M g^s_{1Q}K(A_s) \\ \sum_{s=1}^M g^s_{21}K(A_s) & K(A_0)+\sum_{s=1}^M g^s_{22}K(A_s) & & \sum_{s=1}^M g^s_{2Q}K(A_s) \\ \vdots & & \ddots & \vdots \\ \sum_{s=1}^M g^s_{Q1}K(A_t) & \cdots & \cdots & K(A_0)+\sum_{s=1}^M g^s_{QQ}K(A_s) \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_Q\end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_Q\end{pmatrix}.$$

By using Kronecker product $\otimes$, we can express this system in a compact form,

$$\left[\sum_{s=0}^M G^s \otimes K(A_s)\right] U = F. \tag{5.4}$$

5.1. Discrete ELS. Recall that the continuous ELS functional reads

$$\min_{a\in A} J_\kappa(a) = \frac{1}{2}\int_\Gamma \sigma(y)\int_D a(y,x)\left|\nabla(u_a - z)\right|^2 dx\,dy + \frac{\kappa}{2}\int_\Gamma \sigma(y)\left\|a(y,x)\right\|^2_{H^1(D)}dy, \tag{5.5}$$

where $u_a \equiv u_a(y,x)$ is the solution of the finite-dimensional noise variational problem

$$\int_\Gamma \sigma(y)\int_D a(y,x)\nabla u(y,x)\nabla v(y,x)dx\,dy = \int_\Gamma \sigma(y)\int_D f(y,x)v(y,x)dx\,dy \text{ for every } v \in V_\sigma.$$

Assuming finite linear expansion (5.2) for the unknown coefficient, we have

$$\begin{aligned} J_0(a) &:= \frac{1}{2}\int_\Gamma \sigma(y)\int_D a(y,x)\left|\nabla(u(y,x) - z(y,x))\right|^2\, dx\,dy \\ &= \frac{1}{2}\sum_{s=0}^M \int_\Gamma \sigma(y)y_s\int_D a_s(x)\left|\nabla v(x,y)\right|^2 dx\,dy, \end{aligned}$$

where we set $v(y,x) = u(y,x) - z(y,x)$.

Setting, $v(y,x) = \sum_{k=1}^{N}\sum_{m=1}^{Q} V_{km}\phi_k(x)\psi_m(y)$, we have

$$\sum_{s=0}^{M}\int_\Gamma \sigma(y)y_s\int_D A_s(x)\nabla\left(\sum_{k=1}^{N}\sum_{m=1}^{Q} V_{km}\phi_k(x)\psi_m(y)\right)\cdot\nabla\left(\sum_{k=1}^{N}\sum_{m=1}^{Q} V_{km}\phi_k(x)\psi_m(y)\right)dx\,dy$$
$$=\sum_{s=0}^{M}\int_\Gamma \sigma(y)y_s\int_D A_s(x)\sum_{k_1,k_2=1}^{N}\sum_{m_1,m_2=1}^{Q} V_{k_1m_1}V_{k_2m_2}\nabla\phi_{k_1}(x)\nabla\phi_{k_2}(x)\psi_{m_1}(y)\psi_{m_2}(y)dx\,dy$$
$$=\sum_{s=0}^{M}\sum_{k_1,k_2=1}^{N}\sum_{m_1,m_2=1}^{Q} V_{k_1m_1}V_{k_2m_2}\left(\int_\Gamma\sigma(y)\psi_{m_1}(y)\psi_{m_2}(y)y_s dy\right)\left(\int_D A_s(x)\nabla\phi_{k_1}(x)\nabla\phi_{k_2}(x)dx\right)$$
$$=\sum_{s=0}^{M}\sum_{k_1,k_2=1}^{N}\sum_{m_1,m_2=1}^{Q} V_{k_1m_1}g^s_{m_1m_2}K(A_s)_{k_1k_2}V_{k_2m_2}$$
$$=V^\top\left(\sum_{s=0}^{M}G^s\otimes K(A^s)\right)V$$

by using the known properties of the Kronecker product $\otimes$, and the vectorial notation (5.1). Hence,

$$J_0(A)=\frac{1}{2}(U-Z)^\top\left(\sum_{s=0}^{M}G^s\otimes K(A^s)\right)(U-Z).$$

Analogously, we obtain the following discrete form for the regularization term:

$$R(A)=\frac{\kappa}{2}\int_\Gamma\sigma(y)\,\|a(y,x)\|^2_{H^1(D)}\,dy$$
$$=\frac{\kappa}{2}\sum_{s=0}^{M}\int_\Gamma\sigma(y)\left(\int_D\left(\sum_{t=0}^{M}y_tA_t(x)\right)\left(\sum_{s=0}^{M}y_sA_s(x)\right)\right.$$
$$\left.+\nabla\left(\sum_{t=0}^{M}y_tA_t(x)\right)\cdot\nabla\left(\sum_{s=0}^{M}y_sA_s(x)\right)dx\right)dy$$
$$=\frac{\kappa}{2}\sum_{s,t=0}^{M}\left(\int_\Gamma\sigma(y)y_sy_tdy\right)\left(\int_D A_s(x)A_t(x)dx+\int_D\nabla A_s(x)\cdot\nabla A_t(x)dx\right),$$

and hence

$$R(A)=\frac{\kappa}{2}\,A^\top(\Psi\otimes(Q_A+K_A))A,$$

where $\Psi\in\mathbb{R}^{(M+1)\times(M+1)}$, and $K_A,\ Q_A\in\mathbb{R}^{P\times P}$ are given by

$$\Psi_{s,t}=\int_\Gamma\sigma(y)y_sy_tdy\text{ for every }s,t=0,\dots,M,$$
$$(Q_A)_{i,j}=\int_D\varphi_j(x)\varphi_i(x)dx\text{ for every }i,j=1,\dots,P,$$
$$(K_A)_{i,j}=\int_D\nabla\varphi_j(x)\nabla\varphi_i(x)dx\text{ for every }i,j=1,\dots,P.$$

Summarizing,

$$J_\kappa(A) = \frac{1}{2}(U - Z)^\top \left[\sum_{s=0}^{M} G^s \otimes K(A^s)\right](U - Z) + \frac{\kappa}{2} A^\top (\Psi \otimes (Q_A + K_A)) A.$$

We also recall that the continuous derivative formula is given by

$$DJ_0(a)(b) = -\frac{1}{2}\int_\Gamma \sigma(y) \int_D b(y,x)\nabla(u+z)\cdot\nabla(u-z)dx\,dy,$$

and consequently,

$$DJ_0(A)(B) = -\frac{1}{2}(U + Z)^\top \left[\sum_{s=0}^{M} G^s \otimes K(B_s)\right](U - Z).$$

To obtain an explicit formula for the gradient $\nabla J_0(A)$, we recall the notion of the adjoint stiffness matrix $L(\cdot) \in \mathbb{R}^{N\times P}$, satisfying

$$L(V)B = K(B)V \text{ for every } B \in \mathbb{R}^P,\ V \in \mathbb{R}^N.$$

Let us define

$$\mathcal{A} = \left[\sum_{s=0}^{M} G^s \otimes K(B_s)\right](U - Z).$$

By definition

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_Q \end{bmatrix} \in \mathbb{R}^{PQ\times 1},$$

where

$$\mathcal{A}_k = \sum_{s=0}^{M}\sum_{j=1}^{Q} g^s_{kj} K(B_s)(U_j - Z_j) = \sum_{s=0}^{M}\left(\sum_{j=1}^{Q} g^s_{kj} L(U_j - Z_j)\right) B_s.$$

Consequently

$$\begin{aligned} DJ_0(A)(B) &= -\frac{1}{2}(U+Z)^\top \mathcal{A} = -\frac{1}{2}\sum_{i=1}^{Q}(U_i + Z_i)^\top \left(\sum_{s=0}^{M}\left(\sum_{j=1}^{Q} g^s_{ij} L(U_j - Z_j)\right)\right) B_s \\ &= \sum_{s=0}^{M}\left[-\frac{1}{2}\sum_{i,j=1}^{Q} g^s_{ij}(U_i + Z_i)^\top L(U_j - Z_j)\right] B_s, \end{aligned}$$

implying

$$DJ_0(A)(B) = \sum_{s=0}^{M}\left(-\frac{1}{2}\sum_{i,j=1}^{Q} g^s_{ij}(U_i + Z_i)^\top L(U_j - Z_j)\right) B_s, \tag{5.6}$$

and from this we obtain an explicit block matrix expression for the gradient

$$\nabla J_0(A) = -\frac{1}{2}\left[\begin{array}{ccc} \sum_{i,j=1}^{Q} g_{ij}^{0}(U_i+Z_i)^\top L(U_j-Z_j) & \dots & \sum_{i,j=1}^{Q} g_{ij}^{M}(U_i+Z_i)^\top L(U_j-Z_j) \end{array}\right].$$

Alternatively, using the Kronecker product, we have

$$\nabla J_0(A) = -\frac{1}{2}\left[\begin{array}{ccc} (U+Z)^\top \left(G^0\otimes I_N\right) L(U-Z) & \dots & (U+Z)^\top \left(G^M\otimes I_N\right)(U-Z) \end{array}\right],$$

where, following vectorial notation, we have

$$L(U-Z) = \begin{bmatrix} L(U_1-Z_1) \\ L(U_2-Z_2) \\ \vdots \\ L(U_Q-Z_Q) \end{bmatrix}.$$

Finally,

$$\nabla J_\kappa(A) = -\frac{1}{2}\left[\begin{array}{ccc} (U+Z)^\top \left(G^0\otimes I_N\right) L(U-Z) & \dots & (U+Z)^\top \left(G^M\otimes I_N\right)(U-Z) \end{array}\right] + \kappa A^\top \left(\Psi\otimes(Q_A+K_A)\right).$$

5.2. Discrete OLS. Recall that the finite-dimensional OLS reads

$$\min_{a\in A} \widehat{J}_\kappa(a) = \frac{1}{2}\int_\Gamma \sigma(y)\int_D (u(y,x)-z(y,x))^2 dx\, dy + \frac{\kappa}{2}\int_\Gamma \sigma(y)\,\|a(y,x)\|^2_{H^1(D)}\, dy. \tag{5.7}$$

As before, setting $v = u(y,x)-z(y,x)$, the discrete version takes the form

$$\begin{aligned}
\widehat{J}_0(A) &= \frac{1}{2}\int_\Gamma \sigma(y)\int_D \left(\sum_{k=1}^{N}\sum_{m=1}^{Q} V_{km}\phi_k(x)\psi_m(y)\right)\left(\sum_{k=1}^{N}\sum_{m=1}^{Q} V_{km}\phi_k(x)\psi_m(y)\right) dx\, dy \\
&= \frac{1}{2}\sum_{k_1,k_2=1}^{N}\sum_{m_1,m_2=1}^{Q} V_{k_1m_1}V_{k_2m_2}\left(\int_\Gamma \sigma(y)\psi_{m_1}(y)\psi_{m_2}(y)dy\right)\left(\int_D \phi_{k_1}(x)\phi_{k_2}(x)dx\right) \\
&= \frac{1}{2}\sum_{k_1,k_2=1}^{N}\sum_{m_1,m_2=1}^{Q} V_{k_1m_1}V_{k_2m_2}\delta_{m_1m_2}\int_D \phi_{k_1}(x)\phi_{k_2}(x)dx \\
&= V^\top \left(I_Q\otimes Q_U\right)V,
\end{aligned}$$

where

$$(Q_U)_{i,j} = \int_D \phi_j(x)\phi_i(x)dx,$$

and consequently

$$\widehat{J}_k(A) = \frac{1}{2}(U-Z)^\top \left(I_Q\otimes Q_U\right)(U-Z) + \frac{\kappa}{2}A^\top(\Psi\otimes(K_A+Q_A))A.$$

We also recall that by the adjoint approach, the continuous derivative is given by

$$D\widehat{J}_0(a)(b) = \int_\Gamma \sigma(y) \int_D b(y,x)\nabla u_a(y,x) \cdot \nabla w(y,x) dx\, dy, \tag{5.8}$$

where $w \in V$ satisfies, for every $v \in V$, the adjoint equation:

$$\int_\Gamma \sigma(y) \int_D a(y,x)\nabla w(y,x) \cdot \nabla v(y,x) dx\, dy = \int_\Gamma \sigma(y) \int_D (z(y,x) - u(y,x))v(y,x) dx\, dy.$$

By following the same line of arguments for discretization, we have

$$\left[\sum_{s=0}^{M} G^s \otimes K(A_s)\right] W = P,$$

where

$$P = \begin{pmatrix} P_1 \\ \vdots \\ P_Q \end{pmatrix} \in \mathbb{R}^{NQ}$$

is given by

$$(P_n)_i = \int_\Gamma \sigma(y)\psi_n(y) \int_D (z(y,x) - u(y,x))\phi_i(x) dx\, dy \quad \text{for every} \quad n \in \{1, \ldots, Q\}.$$

Therefore, for every $i \in \{1, \ldots, Q\}$, we have

$$\begin{aligned}
(P_n)_i &= \int_\Gamma \sigma(y)\psi_n(y) \int_D \left[\sum_{k=1}^{N}\sum_{m=1}^{Q} (Z_{km} - U_{km})\, \phi_k(x)\psi_m(y)\right] \phi_i(x) dx\, dy \\
&= \sum_{k=1}^{N}\sum_{m=1}^{Q} \left(\int_\Gamma \sigma(y)\psi_n(y)\psi_m(y) dy\right)\left(\int_D \phi_k(x)\phi_i(x) dx\right)(Z_{km} - U_{km}) \\
&= \sum_{k=1}^{N}\sum_{m=1}^{Q} \delta_{nm}\left(\int_D \phi_k(x)\phi_i(x) dx\right)(Z_{km} - U_{km}) \\
&= \sum_{k=1}^{N} \left(\int_D \phi_k(x)\phi_i(x) dx\right)(Z_{kn} - U_{kn}),
\end{aligned}$$

which implies

$$P_n = Q_U(Z_n - U_n),$$

with

$$(Q_U)_{i,j} = \int_D \phi_j(x)\phi_i(x) dx \text{ for every } i, j = 1 \ldots, P.$$

Since, $P = (I_N \otimes Q_U)(Z - U)$, we obtain the discrete adjoint equation:

$$\left[\sum_{s=0}^{M} G^s \otimes K(A_s)\right] W = (I_N \otimes Q_U)(Z - U).$$

From this, the discrete version of the derivative of the OLS reads

$$D\widehat{J}_0(A)(B) = \sum_{s=0}^{M} \left(\sum_{i,j=1}^{Q} g_{ij}^{s} (U_i^{\top} L(Wj) \right) B_s,$$

where the corresponding gradient is given by

$$\nabla \widehat{J}_0(A) = \left[\; U^{\top} \left(G^0 \otimes I_N \right) L(W) \quad \dots \quad U^{\top} \left(G^M \otimes I_N \right) L\left(W\right) \; \right].$$

Summarizing, we have the following discrete formula for the regularized OLS:

$$\nabla \widehat{J}_{\kappa}(A) = \left[\; U^{\top} \left(G^0 \otimes I_N \right) L(W) \quad \dots \quad U^{\top} \left(G^M \otimes I_N \right) L\left(W\right) \; \right] + \kappa A^{\top} \left(\Psi \otimes (Q_A + K_A) \right).$$

In numerical experimentation, we will also use the OLS

$$\min_{a \in A} \widehat{J}_{\kappa}(a) = \frac{1}{2} \int_{\Gamma} \sigma(y) \left\| u(y,x) - z(y,x) \right\|_{H^1(D)}^2 dy + \frac{\kappa}{2} \int_{\Gamma} \sigma(y) \left\| a(x,y) \right\|_{H^1(D)}^2 dy, \tag{5.9}$$

where instead of the $L^2(D)$, we use the $H^1(D)$ norm as the fitting term in the physical space.

6. Computational experiments.

In the two considered examples, which are inspired by [45, Examples 1 and 2], following (5.2), we consider that the unknown parameter admits the following form of a finite linear combination:

$$a(\omega, x) = a_0(x) + \sum_{i=1}^{M} a_i(x) Y_i(\omega). \tag{6.1}$$

We incorporate one degree of stochasticity (that is, $M = 1$ in (6.1)) in the first example and two degrees of stochasticity (that is, $M = 2$ in (6.1)) in the second example. In both examples, we assume that the distribution of random variables $\{Y_i(\omega)\}_{i=1}^{M}$ is known a priori. We test three different optimization formulations, the ELS objective, and the OLS objective with the L^2 and the H^1 data-fitting in the physical space as defined in (5.5), (5.7), and (5.9), respectively. Since the experiments are synthetic, the data are computed by solving the direct problem, not measured. All the computational experiments were carried out on a computer with an Intel Core i5-8250U CPU at 1.60 GHz and 8 GB of memory by using MATLAB. The optimization problems were solved using the trust-region-reflective algorithm implementation supplied by MATLAB through `fmincon`.

Example 6.1. We set $D = (0, 1)$, and for $Y_1(\omega) \sim U[0, 1]$ uniformly distributed over $[0, 1]$, define

$$\begin{aligned} \bar{a}(\omega, x) &= 1 + Y_1(\omega), \\ \bar{u}(\omega, x) &= x(1 - x) + Y_1(\omega) \sin(\pi x), \\ f(\omega, x) &= (1 + Y_1(\omega))(2 + \pi^2 Y_1(\omega) \sin \pi x). \end{aligned}$$

Table 1
Stochastic Galerkin discretization error for Example 6.1.

$\dim V_h$	$\frac{\mathbb{E}\left(\int_D (\bar{u}(\omega,x)-\bar{u}_h(y,x))^2 dx\right)}{\mathbb{E}\left(\int_D (\bar{u}(\omega,x)^2 dx\right)}$	$\frac{Var\left(\int_D (\bar{u}(\omega,x)-\bar{u}_h(\omega,x)^2 dx)\right)}{Var\left(\int_D (\bar{u}(\omega,x))^2 dx)\right)}$
50	2.3292e-04	3.2248e-05
100	5.9521e-05	8.2273e-06
150	2.6648e-05	3.6812e-06
200	9.6498e-06	1.3324e-06

We use piecewise linear finite elements and the same nodal basis for both V_h and A_h (we need two more degrees of freedom for the representation of the coefficient A_h as we do not enforce a homogeneous boundary condition). Since we have $M = 1$, we consider $\sigma(y) = 1$ and orthonormal Legendre polynomials defined on $[0, 1]$. We solve the direct problem by the stochastic Galerkin method. In Table 1, we check its accuracy by solving direct discrete problem (5.4) for $\bar{a}$ and f.

We measure the expectation and the variance of the identification error via the (relative) error functional. For example, for the ELS objective functional, we estimate the identification error by the quantities

$$\varepsilon^M_{\text{mean}}(a) = \frac{\sqrt{\int_D (\mathbb{E}[a(\omega,x)] - \mathbb{E}[a^M_h(\omega,x)])^2 dx}}{\sqrt{\int_D \mathbb{E}\left[a(\omega,x)\right]^2 dx}},$$

$$\varepsilon^M_{\text{var}}(a) = \frac{\sqrt{\int_D (\text{Var}[a(\omega,x)] - \text{Var}[a^M_h(\omega,x)])^2 dx}}{\sqrt{\int_D \text{Var}\left[a(\omega,x)\right]^2 dx}},$$

where a^M_h is the estimated coefficient by the ELS approach. Similarly, we measure the simulated data error by the quantities

$$\varepsilon^M_{\text{mean}}(u) = \frac{\sqrt{\int_D (\mathbb{E}[u(\omega,x)] - \mathbb{E}[u_h(a^M_h)(\omega,x)])^2 dx}}{\sqrt{\int_D \mathbb{E}\left[u(\omega,x)\right]^2 dx}},$$

$$\varepsilon^M_{\text{var}}(u) = \frac{\sqrt{\int_D (\text{Var}[u(\omega,x)] - \text{Var}[u_h(a^M_h)(\omega,x)])^2 dx}}{\sqrt{\int_D \text{Var}\left[u(\omega,x)\right]^2 dx}},$$

where $u_h(a^M_h)(\omega, x)$ corresponds to solving stochastic Galerkin system (5.4) for estimated coefficient a^M_h. Based on several test-runs, we fix κ = 1e-05, which seems to render a stable reconstruction for the considered discretization levels. The numerical results, given in Tables 2, 3, and 4, are quite satisfactory for the three optimization formulations. Both the ELS formulation and the H^1-OLS formulation give a better reconstruction than the L^2-OLS. Moreover, in terms of computational time, the ELS formulation completely outperforms its

Table 2
Numerical errors for the ELS approach recorded using $\kappa =$ 1-e05 *in Example* 6.1.

$\dim V_h$	$\varepsilon^M_{\text{mean}}(a)$	$\varepsilon^M_{\text{var}}(a)$	$\varepsilon^M_{\text{mean}}(u)$	$\varepsilon^M_{\text{var}}(u)$	CPU time
50	8.4782e-04	1.0577e-02	4.8165e-06	4.6978e-06	0.97 s.
100	2.0969e-04	2.7276e-03	1.1645e-06	1.7138e-06	4.09 s.
150	1.0585e-04	3.0194e-04	2.5107e-07	1.2807e-07	11.5 s.
200	1.0681e-04	1.9378e-03	1.7999e-06	4.7615e-07	31.3 s.

Table 3
Numerical errors for the L^2*-OLS approach recorded using* $\kappa =$ 1-e05 *in Example* 6.1.

$\dim V_h$	$\varepsilon^{LO}_{\text{mean}}(a)$	$\varepsilon^{LO}_{\text{var}}(a)$	$\varepsilon^{LO}_{\text{mean}}(u)$	$\varepsilon^{LO}_{\text{var}}(u)$	CPU time
50	3.5288e-03	4.6838e-02	1.7136e-04	6.3924e-05	3.09 s.
100	4.5333e-03	2.3041e-02	1.8143e-04	3.1305e-05	13.5 s.
150	3.4689e-03	2.1957e-02	1.3924e-04	4.8058e-05	46.5 s.
200	3.0372e-03	1.9054e-02	1.3423e-04	4.3304e-05	185 s.

Table 4
Numerical errors for the H^1*-OLS approach recorded using* $\kappa =$ 1-e05 *in Example* 6.1.

$\dim V_h$	$\varepsilon^{HO}_{\text{mean}}(a)$	$\varepsilon^{HO}_{\text{var}}(a)$	$\varepsilon^{HO}_{\text{mean}}(u)$	$\varepsilon^{HO}_{\text{var}}(u)$	CPU time
50	7.1777e-03	1.9618e-03	4.4760e-06	9.4688e-07	2.73 s.
100	3.2125e-04	7.2055e-04	4.9516e-06	3.6578e-07	18.8 s.
150	2.2223e-04	2.2117e-03	4.7213e-06	7.1059e-07	58.3 s.
200	1.9965e-04	2.3618e-03	4.4808e-06	5.5407e-07	157 s.

two OLS analogues. Reconstructions of the parameter a (Figure 1) and the corresponding simulated data (u_h computed by using the identified coefficient a_h) are excellent for all three optimization formulations (Figure 2). The samples of the estimated coefficient a in the figure are randomly generated by taking the representation (6.1) into account.

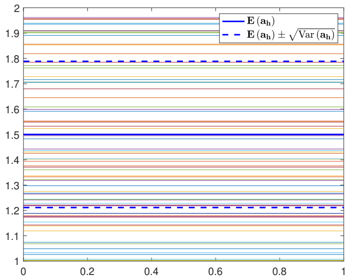
Example 6.2. For $D = (0,1)$ and for $Y_1(\omega), Y_2(\omega) \sim U[0,1]$ uniformly distributed over $[0,1]$, we define the random fields

$$\bar{a}(\omega,x) = 3 + x^2 + Y_1(\omega)\cos(\pi x) + Y_2(\omega)\sin(2\pi x),$$
$$\bar{u}(\omega,x) = x(1-x)Y_1(\omega),$$

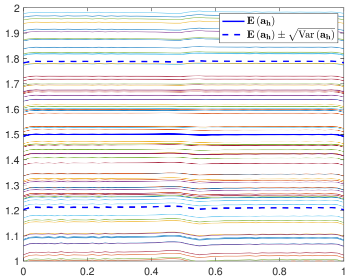
and compute the right-hand side accordingly.

We adhere to the discretization scheme of the first example with the stochastic domain given by $\Gamma = [0,1]\times[0,1]$. Here $\sigma(y_1,y_2) = 1$ and orthornormal Legendre polynomials on $[0,1]\times[0,1]$ are defined as a tensorial product of the one-dimensional ones. In Table 5, we show the accuracy of stochastic Galerkin for this data set.

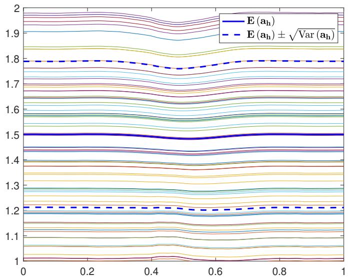
For the numerical results given in Tables 6, 7, and 8, the regularization parameter is κ =1e-05. The ELS approach in this case gives a very good reconstruction, while the



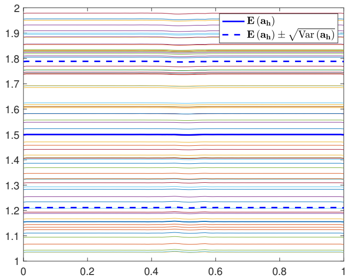
(a) Exact parameter a.



(b) Estimated parameter a_h^{ELS} by ELS.

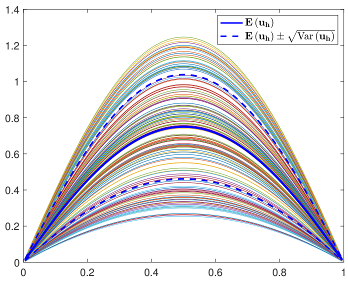


(c) Estimated parameter $a_h^{L^2-OLS}$ by L^2-OLS.

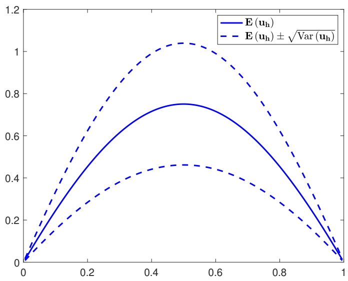


(d) Estimated parameter $a_h^{H^1-OLS}$ by H^1-OLS.

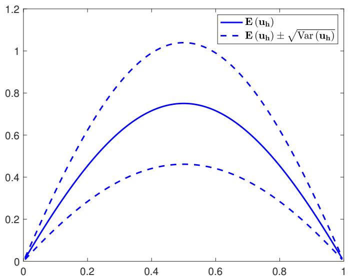
Figure 1. *Exact and estimated parameters (75 realizations) for Example 6.1.*



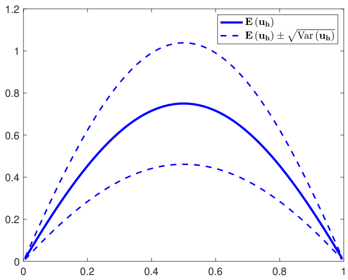
(a) Real data u_h (75 realizations).



(b) Simulated data $u_h(a_h^{ELS})$ obtained by ELS.



(c) Simulated data $u_h(a_h^{L^2-OLS})$ obtained by L^2-OLS.



(d) Simulated data $u_h(a_h^{H^1-OLS})$ obtained by H^1-OLS.

Figure 2. *Real and simulated data for Example 6.1.*

Table 5

Stochastic Galerkin discretization error for Example 6.2.

$\dim V_h$	$\frac{\mathbb{E}\left(\int_D (\bar{u}(\omega,x)-\bar{u}_h(y,x))^2 dx\right)}{\mathbb{E}\left(\int_D (\bar{u}(\omega,x)^2 dx\right)}$	$\frac{Var\left(\int_D (\bar{u}(\omega,x)-\bar{u}_h(\omega,x)^2 dx)\right)}{Var\left(\int_D (\bar{u}(\omega,x))^2 dx)\right)}$
50	6.6266e-05	3.8455e-06
100	1.6972e-05	9.7537e-07
150	7.6045e-06	4.3558e-07
200	4.2950e-06	2.4560e-07

Table 6

Numerical errors obtained by ELS with $\kappa =$ 1e-06 *for Example 6.2.*

$\dim V_h$	$\varepsilon^M_{\text{mean}}(a)$	$\varepsilon^M_{\text{var}}(a)$	$\varepsilon^M_{\text{mean}}(u)$	$\varepsilon^M_{\text{var}}(u)$	`CPU time`
50	4.5268e-03	4.0556e-02	2.6610e-05	4.4452e-06	4.43 s.
100	6.2400e-03	1.9205e-02	1.7895e-05	1.5850e-06	27.4 s.
150	6.3291e-03	2.8512e-02	1.8012e-05	1.5886e-06	95.3 s.
200	6.9930e-03	3.2212e-02	1.7258e-05	1.3371e-06	212.3 s.

Table 7

Numerical errors obtained by H^1*-OLS with* $\kappa =$ 1e-06 *for Example 6.2.*

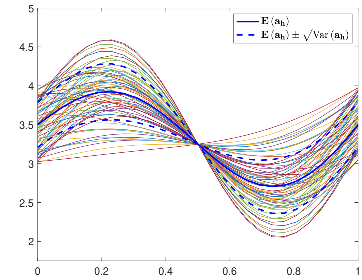
$\dim V_h$	$\varepsilon^{HO}_{\text{mean}}(a)$	$\varepsilon^{HO}_{\text{var}}(a)$	$\varepsilon^{HO}_{\text{mean}}(u)$	$\varepsilon^{HO}_{\text{var}}(u)$	`CPU time`
50	1.1893e-02	3.7707e-02	5.5440e-05	3.6486e-06	6.11 s.
100	1.2621e-02	4.3191e-02	5.6433e-05	3.3801e-06	45.9 s.
150	1.4543e-02	6.5231e-02	7.2858e-05	4.7766e-06	126 s.
200	1.2355e-02	5.0199e-02	5.6855e-05	4.1495e-06	326 s.

Table 8

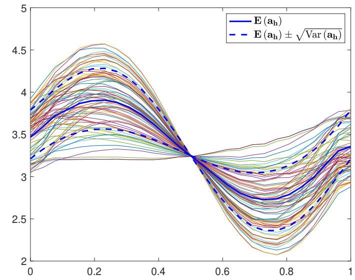
Numerical errors obtained by L^2*-OLS with* $\kappa =$ 1e-06 *for Example 6.2.*

$\dim V_h$	$\varepsilon^{LO}_{\text{mean}}(a)$	$\varepsilon^{LO}_{\text{var}}(a)$	$\varepsilon^{LO}_{\text{mean}}(u)$	$\varepsilon^{LO}_{\text{var}}(u)$	`CPU time`
50	7.7033e-02	3.9343e-01	8.9804e-04	5.3416e-05	4.98 s.
100	7.5412e-02	5.0431e-01	9.2701e-04	6.4441e-05	37.5 s.
150	7.6292e-02	4.3639e-01	9.1230e-04	5.8506e-05	151 s.
200	7.6757e-02	4.6106e-01	9.2280e-04	6.0029e-05	825 s.

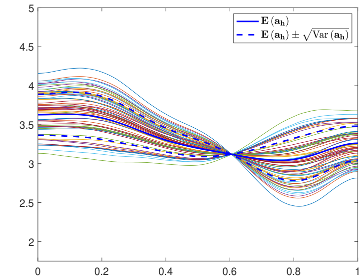
H^1-OLS objective functional also gives a reasonable quality reconstruction of the coefficient (see Figure 3). As shown in part (c) of Figure 3, the L^2-OLS approach, however, doesn't give a good quality reconstruction (poor reconstructions are observed for various values of the regularization parameter κ). Simulated data $u_h(a_h)$ shown in Figure 4 are all good matches in all three cases. Comparisons of the errors and computational times are shown in Tables 6, 7, and 8, and the ELS approach is the most efficient.



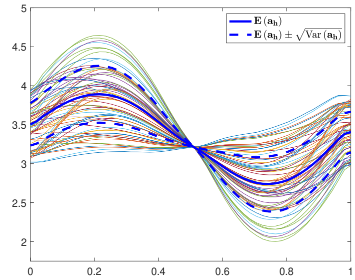
(a) Exact parameter a_h.



(b) Estimated parameter a_h^{ELS} by ELS.

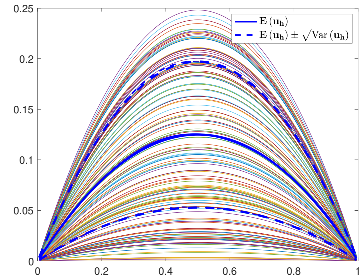


(c) Estimated parameter $a_h^{L^2-OLS}$ by L^2-OLS.

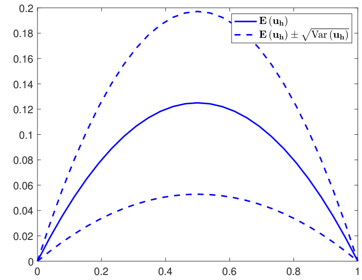


(d) Estimated parameter $a_h^{H^1-OLS}$ by H^1-OLS.

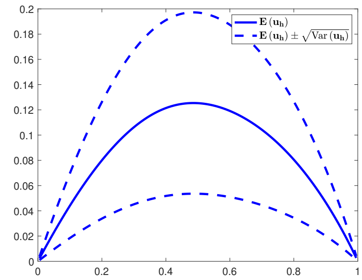
Figure 3. *Exact and estimated parameters (75 realizations) for Example 6.2.*



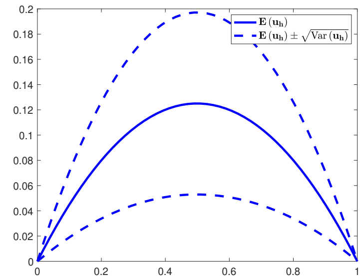
(a) Real data u (75 realizations).



(b) Simulated data $u_h(a_h^{ELS})$ obtained by ELS.



(c) Simulated data $u_h(a_h^{L^2-OLS})$ obtained by L^2-OLS.



(d) Simulated data $u_h(a_h^{H^1-OLS})$ obtained by H^1-OLS.

Figure 4. *Real and simulated data for Example 6.2.*

7. Concluding remarks.

We studied the inverse problem of estimating the stochasticity of a random parameter in stochastic partial differential equations. One of the main contributions is a new convex stochastic optimization formulation for the inverse problem, which employs a new energy-type norm. We provide a derivative characterization of the parameter-to-solution map and existence results for the optimization problem in the setting of stochastic Sobolev spaces. By employing the Karhunen–Loève expansion, we separate the stochastic and the deterministic components. However, as another novelty of our study, we develop the discretization framework by focusing on coefficients in the finite-dimensional expansion, which permits us to obtain all the discrete formulas, including the two considered objective functionals and their gradients as Kronecker product of the known matrices. Besides the convexity of the proposed functional, it has an additional advantage that its derivative does not involve the result of the parameter-to-solution map, which allows for fast numerical computations. This is evident in the two given numerical examples where the ELS-based framework outperforms the two OLS-based approaches in terms of the reconstruction quality and computational efficiency. We emphasize that the presented numerical results are quite simple, and more detailed and thorough numerical testing needs to be done. It would also be advantageous to test various optimization solvers, a wide range of regularization parameters, and take into account the data regularity. Deriving error estimates for the inverse problem is also of evident importance. Inverse problems of parameter identification have recently been extended to variational and quasi-variational inequalities; see [20]. To develop stochastic counterparts of such studies will significantly enhance the applicability of the inversion framework for many more applied models, such as random obstacle problems, among others.

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