Modified forward and inverse Born series for the Calderon and diffuse-wave problems

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Abstract

We propose a new direct reconstruction method based on series inversion for Electrical Impedance Tomography (EIT) and the inverse scattering problem for diffuse waves. The standard Born series for the forward problem has the limitation that the series requires that the contrast lies within a certain radius for convergence. Here, we instead propose a modified Born series which converges for the forward problem unconditionally. We then invert this modified Born series and compare reconstructions with the usual inverse Born series. We also show that the modified inverse Born series has a larger radius of convergence.

1 Introduction

In this article, we propose a modified version of the Born series for inversion of conductivity and diffusion type problems. These problems, related to the Calderon problem as formulated by A. P. Calderon [7], consists of recovering the electrical conductivity or absorption of a medium based on measurements on the boundary. The Calderon problem can be stated as follows: Let Ω be a bounded medium. Corresponding to the conductivity σ of the medium, there exists a Dirichlet-to-Neumann map, which we denote by Λ_σ. Can one determine σ from Λ_σ? This problem is very well studied, and has numerous applications in fields as varied as medical imaging, geophysics and non-destructive testing. Several of these applications have for example been considered in the articles [2, 5, 6, 22].

Various reconstruction methods for the inverse conductivity problem have been proposed along with rigorous theoretical analysis. These include the $\bar{\partial}$ method (see e.g. [11, 13, 17]), regularized Newton type schemes (see e.g. [9, 14]) and linearization methods (see e.g. [8, 20]). Using the approach of inversion of the Born series, a reconstruction method for the Calderon problem was proposed in [1] , where in the convergence, stability and approximation error of the method was analyzed. In that article, the forward Born series for the Calderon problem was shown to converge if the perturbation to the background conductivity satisfies a certain "smallness condition". The modified forward Born series that we propose here, which exploits a modified version of the relevant volume integral equation originally given in [4], has the advantage of being unconditionally convergent in the sense that it converges for any assumed perturbation of the background medium which preserves the ellipticity of the PDE in the model, and hence the underlying physics of the problem. Based on this modified series, we propose the corresponding modified version of the inverse Born series and analyze its convergence, stability and approximation error. We show in particular that the modified inverse series has a larger radius of convergence than the usual inverse series. We note that an alternative approach for deriving unconditionally convergent modified forward Born series for the forward Helmholtz equation is proposed in [18].

The inverse scattering problem (ISP) for diffuse waves consists of recovering the spatially varying absorption coefficient in the interior of some bounded domain, which we assume to be $\Omega \subset \mathbb{R}^d$ again based solely on measurements taken on the boundary $\partial\Omega$. The problem of ISP arises in the study of optical tomography which is widely used in the area of biomedical imaging, see e.g. [3, 19]. Indeed, in [23], the authors have shown that it is a general feature of wave propagation in random media. In the past, authors in [12, 15, 16] have analyzed the convergence and stability of the series solutions for diffuse waves and have developed fast image reconstruction algorithms based on such methods. Just as the usual Born series for the Calderon problem, the Born series for the diffuse waves is also conditionally convergent. In this article, we will describe the modified series for diffuse waves which will be unconditionally convergent, and use it in an accordingly modified version of the inverse Born series. Convergence, stability and approximation error results obtained for the Calderon problem essentially carry over to diffuse waves thanks to the formal similarity of the respective modified inverse series.

The organization of this paper is as follows: in Section 2, we describe the mathematical set up of the Calderon problem and define the corresponding modified forward and inverse series. In Section 3 we do the same for the inverse scattering problem of diffuse waves. In both cases we analyze the convergence of the modified inverse series along with providing stability and approximation error estimates for the modified method. In section 4, we present the results of numerical reconstructions of an unknown conductivity in a two-dimensional medium for the Calderon problem based on the modified inversion method. We compare this with similar reconstructions based on the inversion of the usual Born series.

2 Set up for the inverse conductivity problem.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a smooth boundary represented by $\partial\Omega$. Consider a scalar field $u(x)$ and a function $\sigma(x) \geq \sigma_{min} > 0$ for all $x \in \Omega$ such that $\sigma(x)|_{\partial\Omega} = \sigma_0$. We consider the following elliptic equation with Robin boundary conditions,

$$\nabla \cdot \sigma(x)\nabla u(x) = 0, \quad x \in \Omega \tag{1}$$

$$u + z\sigma_0 \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega. \tag{2}$$

In electrical impedance tomography, where the above mathematical model often appears, $u(x)$ represents the electrical potential, $\sigma(x)$ is identified with the conductivity of the medium and assumed to be equal to a known constant background conductivity σ_0 on the boundary, $z > 0$ is another constant called surface impedance, $\frac{\partial u}{\partial \nu}$ is the normal derivative of u and g is the current density.

Following [1] we will now briefly describe the Born series for the forward solution u. Here we assume that the conductivity is given by $\sigma(x) = \sigma_0(1 + \eta(x))$, where $\eta(x) \in L^\infty(B_a)$ is compactly supported in Ω within a ball B_a of radius a. Equation (1) therefore can be rewritten as

$$-\Delta u = \nabla \cdot \eta(x)\nabla u(x), \quad x \in \Omega. \tag{3}$$

The solution to the equation (3) along with the boundary condition (2) can be written as an integral equation,

$$u = u_0(x) + \int_\Omega G(x, y)\nabla \cdot \eta(y)\nabla u(y)dy, \quad x \in \Omega \tag{4}$$

where $u_0(x)$ solves (3) when $\eta = 0$ with boundary condition (2). Here G is the Green's function for the Laplacian, satisfying a homogeneous Robin boundary condition (2) . The background solution $u_0(x)$ is therefore given by

$$u_0(x) = \frac{1}{z\sigma_0}\int_{\partial\Omega} G(x, y)g(y)dy, \quad x \in \Omega.$$

We then integrate the right hand side of (4) by parts to obtain

$$u(x) = u_0(x) - \int_\Omega \nabla_y G(x, y) \cdot \nabla u(y)\eta(y)dy \quad x \in \Omega. \tag{5}$$

Beginning with $u_0(x)$ and performing a fixed point iteration in (5), we can write the solution as a series

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

where the terms are given by the recursive relation

$$u_{n+1}(x) = -\int_\Omega \nabla_y G(x, y) \cdot \nabla u_n(x)\eta(y)dy.$$

This series solution is called the Born series for the scalar field $u(x)$; truncating the series to the linear term $u_1(x)$ gives rise to the Born approximation. We represent the above series as

$$\phi(x) = u(x) - u_0(x) = K_1\eta + K_2\eta^{\otimes 2} + K_3\eta^{\otimes 3} + \cdots \tag{6}$$

where $\eta^{\otimes n} = \eta \otimes \eta \otimes \cdots \otimes \eta$ is the n-tensor product of η with itself and K_n is a multilinear operator defined as

$$\begin{aligned} K_n\eta^{\otimes n}(x) = (-1)^n \int_\Omega \eta(y_1)\nabla_{y_1} G(y_1, x) \cdot \nabla_{y_1} \int_\Omega \eta(y_2)G(y2, y1) \cdots \\ \cdots \nabla_{y_{n-1}} \int_\Omega \eta(y_n)G(y_{n-1}, y_n) \cdot \nabla_{y_n} u_0(y_n)dy_n \cdots dy_1. \end{aligned} \tag{7}$$

As in [1], we introduce the operators

$$S : L^2[\Omega]^d \to H^1(\Omega)$$
$$T : L^2[\Omega]^d \to L^2[\Omega]^d$$

defined as

$$(Sf)(x) = \int_\Omega \nabla_y G(x,y) \cdot f(y) dy$$
$$Tf = \nabla(Sf) \tag{8}$$

so that we can write

$$K_n(\eta^{\otimes n}) = (-1)^n S(\eta T)^{n-1}(\eta \nabla u_0).$$

We now state the following two theorems concerning the forward Born series for the inverse conductivity problem. The proofs can be found in [1].

Theorem 1. *[1, Lemma 2.3] The operator*

$$K_n : L^\infty(B_a \times \cdots \times B_a) \to L^\infty(\partial\Omega)$$

defined by (7) *is bounded and*

$$||K_n|| < \nu,$$

where

$$\nu = \sup_{x\in\partial\Omega} ||\nabla G(x,\cdot)||_{L^2(B_a)} ||\nabla u_0||_{L^2(B_a)}.$$

Theorem 2. *[1, Proposition 2.1] If the smallness condition* $||\eta||_{L^\infty(B_a)} < 1$ *holds, then the Born series converges in the* $L^\infty(\partial\Omega)$ *norm.*

2.1 Modified Born series for the conductivity problem.

The modified Born series for the Calderon problem that we present here has the advantage of converging unconditionally, i.e. without any requirement of smallness of η. In order to formulate this modified series, let us first introduce the modified operator

$$\widetilde{T} : L^2(B;\mathbb{R}^d) \to L^2(B;\mathbb{R}^d)$$
$$\widetilde{T} := 2T - I \tag{9}$$

Lemma 1. *The operator* $\widetilde{T}$ *is bounded and satisfies* $\|\widetilde{T}\| \leq 1$.

Proof. The proof is similar to the proof of [1, Lemma 2.2]. We will provide it here for the sake of completeness. We will first evaluate $\|\widetilde{T}f\|$ for $f \in C_c^\infty(\Omega)$, and the result will follow from the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$. Consider

$$(Sf)(x) = \int_\Omega \nabla_y G(x,y) \cdot f(y) dy = -\int_\Omega G(x,y) \nabla \cdot f(y) dy \tag{10}$$

where we have integrated by parts. Since G is the Green's function for the Laplace operator with Robin boundary conditions, for $\phi(x) := Sf(x)$ we have

$$\begin{aligned} \Delta\phi &= \nabla \cdot f \quad \text{in} \quad \Omega \\ \phi + z\sigma_0 \frac{\partial \phi}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{11}$$

Recall that $(Tf)(x) = \nabla(Sf)(x) = \nabla\phi(x)$. Thus

$$\|2T(f) - f\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega)} + 4\|\nabla\phi\|^2_{L^2(\Omega)} - 4\langle \nabla\phi, f\rangle_{L^2(\Omega)}.$$

However, we also have

$$\begin{aligned} 0 = \langle -\Delta\phi + \nabla \cdot f, \phi\rangle_{L^2(\Omega)} = \langle \nabla\phi - f, \nabla\phi\rangle_{L^2(\Omega)} - \langle \partial_\nu \phi, \phi\rangle \\ = \|\nabla\phi\|^2_{L^2(\Omega)} - \langle \nabla\phi, f\rangle_{L^2(\Omega)} + \frac{1}{z\sigma_0}\langle \phi, \phi\rangle. \end{aligned}$$

Hence $\|2T(f) - f\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega)} - \frac{4}{z\sigma_0}\langle \phi, \phi\rangle \leq \|f\|^2_{L^2(\Omega)}$, which proves our claim. □

Let us introduce a new function $h(x) = \eta(x)\nabla u(x)$. From now on, wherever clearly understood from the context, we will drop the dependence of these functions on x and simply write h instead of $h(x)$. Equation (5) can then be recast as

$$(I + \eta T)h = \eta\nabla u_0 \text{ in } \Omega. \tag{12}$$

Furthermore, it is easy to check that

$$I + \eta T = \frac{\eta + 2}{2}(I + \frac{\eta}{\eta + 2}\widetilde{T}).$$

Now, since $\eta > -1$ if we define $\xi := \frac{\eta}{\eta+2}$, we have that $|\xi| < 1$. We can then rewrite equation (12) as

$$(I + \xi\widetilde{T})h = 2\xi\nabla u_0 \text{ in } \Omega. \tag{13}$$

From (13) we can generate the formal Neumann series,

$$h = (I - \xi\widetilde{T} + (\xi\widetilde{T})^2 - \dots)(2\xi\nabla u_0) \tag{14}$$

which is guaranteed to converge for $\eta > -1$ since $|\xi| < 1$ and $\|\widetilde{T}\|_{L^2(\Omega)} = 1$. As $h = \eta\nabla u$, $\phi = -S(h)$, and we have from (14):

$$\begin{aligned} \phi = -S(h) &= -S\big[(I - \xi\widetilde{T} + (\xi\widetilde{T})^2 - \dots)\big](2\xi\nabla u_0) \\ &=: 2\widetilde{K}_1\xi + 2\widetilde{K}_2\xi^{\otimes 2} + \dots \end{aligned}$$

where we will call the above series "modified Born series" in analogy with the series representation given by (6). It can be easily checked that

$$\widetilde{K}_n \xi^{\otimes n} = (-1)^n S(\xi \widetilde{T}(\cdots \xi \widetilde{T}(\xi \nabla u_0))).$$

Clearly the modified series converges for all $\eta \geq \eta_{min} > -1$ by the convergence of the Neumann series (14). From now on, we assume that η and hence ξ is compactly supported in Ω within a ball of radius a. The operator $\widetilde{K}_n$ can be extended to act multi-linearly on $L^\infty(B_a)^n$,

$$\widetilde{K}_n(\xi_1 \otimes \cdots \otimes \xi_n) := (-1)^n S(\xi_1 \widetilde{T}(\cdots \xi_{n-1} \widetilde{T}(\xi_n \nabla u_0)))$$

and we have the following estimate on the norm of $\widetilde{K}_n$,

Lemma 2. *The operator $\widetilde{K}_n : L^\infty(B_a \times \cdots \times B_a) \to L^\infty(\partial\Omega)$ is bounded and*

$$\|\widetilde{K}_n\| \leq \nu$$

where

$$\nu = \operatorname*{Sup}_{x \in \partial\Omega} \|\nabla G(x, \cdot)\|_{L^2(B_a)} \|\nabla u_0\|_{L^2(B_a)}$$

Proof. The proof is similar to the proof of [1, Lemma 2.3] and we will skip it here. The only change that we need to make is to substitute $\widetilde{T}$ for T and use the fact that $\|\widetilde{T}\| \leq 1$. □

2.2 Modified inverse series for the conductivity problem.

The inverse problem is to find η from the data ϕ. It is clear however, that this problem is equivalent to finding ξ as $\eta = \frac{2\xi}{1-\xi}$. In this section we will write down the modified inverse Born series and comment about its convergence and stability. Similarly to [1], we write the formal series,

$$\xi = \widetilde{L}_1 \phi + \widetilde{L}_2(\phi \otimes \phi) + \widetilde{L}_3(\phi \otimes \phi \otimes \phi) + \ldots$$

where the inverse operators $\widetilde{L}_1, \ldots, \widetilde{L}_n, \ldots$ are given by

$$\begin{aligned}
\widetilde{L}_1 &= (2\widetilde{K}_1)^\dagger; \quad (2\widetilde{K}_1)^\dagger \text{ is a pseudoinverse of } 2\widetilde{K}_1 \\
\widetilde{L}_2 &= -\widetilde{L}_1 2\widetilde{K}_2(\widetilde{L}_1 \otimes \widetilde{L}_1) \\
\widetilde{L}_3 &= -\left[\widetilde{L}_2(2\widetilde{K}_1 \otimes 2\widetilde{K}_2 + 2\widetilde{K}_2 \otimes 2\widetilde{K}_1) + \widetilde{L}_1 2\widetilde{K}_3\right](\widetilde{L}_1 \otimes \widetilde{L}_1 \otimes \widetilde{L}_1) \\
\widetilde{L}_n &= -\sum_{m=1}^{n-1} \widetilde{L}_m \left(\sum_{i_1 + \cdots + i_m = n} 2\widetilde{K}_{i_1} \otimes \cdots \otimes 2\widetilde{K}_{i_m} \right)(\widetilde{L}_1 \otimes \cdots \otimes \widetilde{L}_1).
\end{aligned} \tag{15}$$

As in [12, 15], we view the imposed boundary data g as corresponding to a source at a boundary point x_1, and we read the solution u at another boundary point x_2. In this manner, we can view the measured data ϕ as lying in $L^\infty(\partial\Omega \times \partial\Omega)$. Since the inverse operators act on tensor products of data space functions, we instead view ϕ as lying in $L^2(\partial\Omega \times \partial\Omega)$. In this manner the tensor product spaces are more straightforwardly defined, and

$$\widetilde{L}_n : L^2(\partial\Omega \times \cdots \times \partial\Omega) \to L^2(B_a)$$

where the domain space contains $2n$ copies of $\partial\Omega$. The analysis here follows [12, 15]. First, using these L^2 based norms, we find an estimate on the operator norm of $\widetilde{L}_j$.

$$\begin{aligned}\|\widetilde{L}_j\| &\leq \sum_{m=1}^{j-1} \sum_{i_1+\cdots+i_m=j} \|\widetilde{L}_m\| \cdot \|2\widetilde{K}_{i_1}\| \ldots \|2\widetilde{K}_{i_m}\| \cdot \|\widetilde{L}_1\|^j \\ &\leq \|\widetilde{L}_1\|^j \sum_{m=1}^{j-1} \|L_m\| \cdot \Pi(j,m)(2\nu)^m \end{aligned} \tag{16}$$

where

$$\Pi(j,m) = \binom{j-1}{m-1}$$

and is the number of ordered partitions of the integer j in to m parts. Next, similar calculations as done in [12, 15] then give us

$$\|\widetilde{L}_j\| \leq (1+2\nu)^j \|\widetilde{L}_1\|^j \sum_{m=1}^{j-1} \|\widetilde{L}_m\|. \tag{17}$$

The above estimate for $\|\widetilde{L}_j\|$ has a recursive structure and it can be shown , subject to the assumption $(1+2\nu)\|\widetilde{L}_1\| < 1$, that

$$\|\widetilde{L}_j\| \leq C(1+2\nu)^j \|\widetilde{L}_1\|^j, \tag{18}$$

where $C = C(\nu, \|L_1\|)$ is a constant independent of j. Now we are ready to show the convergence of the inverse Born series. The series $\sum_j \widetilde{L}_j \phi^{\otimes j}$ converges in the norm if $\sum_j \|\widetilde{L}_j \phi^{\otimes j}\|_{L^2(B_a)}$ converges.

A variation on (16) and (17) gives

$$\|\widetilde{L}_j \phi\|_{L^2(B_a)} \leq \left(\sum_{m=1}^{j-1} \|\widetilde{L}_m\|\right) (1+2\nu)^j \, \|\widetilde{L}_1 \phi\|^j_{L^2(B_a)} \leq C(1+2\nu)^j \|\widetilde{L}_1 \phi\|^j_{L^2(B_a)},$$

with the last inequality obtained using (18) and the assumption $(1+2\nu)\|\widetilde{L}_1\| < 1$. Hence,

$$\sum_j \|\widetilde{L}_j \phi^{\otimes j}\|_{L^2(B_a)} \leq C \sum_j \Big((1+2\nu)\|\widetilde{L}_1 \phi\|_{L^2(B_a)}\Big)^j, \tag{19}$$

and the right-hand side of (19) converges when $(1+2\nu)\|\widetilde{L}_1 \phi\|_{L^2(B_a)} < 1$.

Summarizing, sufficient conditions for convergence of the inverse Born series are

$$(1+2\nu)\|\widetilde{L}_1\| < 1, \qquad (1+2\nu)\|\widetilde{L}_1\phi\|_{L^2(B_a)} < 1.$$

We note that the first of these conditions is not satisfied in practice, while the second condition appears to be a better indicator of inverse series convergence. However, for rigorous analysis we need to assume both are satisfied. If the series limit is denoted by $\widetilde{\xi}$, then the remainder can be estimated as well:

$$\begin{aligned}
\|\widetilde{\xi} - \sum_{j=1}^{N} \widetilde{L}_j \phi^{\otimes j}\| &\leq \sum_{j=N+1}^{\infty} \|\widetilde{L}_j \phi^{\otimes j}\| \\
&\leq C \sum_{j=N+1}^{\infty} \left[(1+2\nu)\|\widetilde{L}_1\phi\|\right]^j \\
&\leq C \frac{\left((1+2\nu)\|\widetilde{L}_1\phi\|\right)^{N+1}}{1-\left((1+2\nu)\|\widetilde{L}_1\phi\|\right)}
\end{aligned}$$

Thus we have the following theorem:

Theorem 3 (Convergence of the modified inverse series)**.** *The modified inverse Born series for the Calderon problem converges in the $L^2(B_a)$ norm if $(1+2\nu)\|L_1\| < 1$ and $(1+2\nu)\|\widetilde{L}_1\phi\| < 1$. Additionally, the following estimate for the series limit $\widetilde{\xi}$ holds:*

$$\|\widetilde{\xi} - \sum_{j=1}^{N} \widetilde{L}_j \phi^{\otimes j}\| \leq C \frac{\left((1+2\nu)\|\widetilde{L}_1\phi\|\right)^{N+1}}{1-\left((1+2\nu)\|\widetilde{L}_1\phi\|\right)}$$

We also have a result on the stability of the modified inverse series (when it converges) with respect to perturbation in the measured data ϕ.

Theorem 4 (Stability)**.** *Let $\widetilde{\xi}_1$ and $\widetilde{\xi}_2$ be the series limit for the modified inverse Born series corresponding to the measured data ϕ_1 and ϕ_2 respectively and $M = \max\{\|\phi_1\|_2, \|\phi_2\|\}$. Furthermore, let us assume $M \cdot \|L_1\| < \frac{1}{1+2\nu}$. Then*

$$\|\widetilde{\xi}_1 - \widetilde{\xi}_2\|_{L^2(B_a)} < C\|\phi_1 - \phi_2\|_{L^2(\partial\Omega\times\partial\Omega)}$$

where $C = C(\nu, L_1, M)$ is otherwise independent of ϕ_1 and ϕ_2.

Proof. Let $\psi = \phi_1 - \phi_2$. Then by following the arguments from [15, Theorem 3.2] , one can first show that

$$\|\widetilde{\xi}_1 - \widetilde{\xi}_2\| < \widetilde{C}\|\psi\| \cdot \|\widetilde{L}_1\| \sum_j j\left((1+2\nu)M\|\widetilde{L}_1\|\right)^j$$

The series on the right hand side converges when $\Big((1+2\nu)M\|\widetilde{L}_1\|\Big) < 1$ and we get

$$\|\widetilde{\xi}_1 - \widetilde{\xi}_2\| < \frac{\widetilde{C}}{\nu\|\widetilde{L}_1\|M}\|\widetilde{L}_1\| \cdot \|\phi_1 - \phi_2\| \qquad \square$$

Finally we have an error bound between the series limit $\widetilde{\xi}$ and true ξ. The proof for the theorem below is exactly the same as that for the proof of [15, Theorem 3.3] and we skip it here for brevity.

Theorem 5 (Error bound)**.** *Let* $\mathcal{M} = \max\{\|\widetilde{\xi}\|_{L^2(B_a)}, \|\widetilde{L}_1\widetilde{K}_1\xi\|_{L^2(B_a)}\}$ *and assume that* $\mathcal{M} < 1/(1+2\nu), \|\widetilde{L}_1\|_2 < 1/(1+2\nu)$ *and* $\|\widetilde{L}_1\phi\|_{L^2(B_a)} < 1/(1+2\nu)$*. Then we have:*

$$\|\widetilde{\xi} - \xi\|_{L^2(B_a)} \le C\|I - \widetilde{L}_1\widetilde{K}_1\xi\|_{L^2(B_a)}$$

where the constant $C = C(\nu, \|\widetilde{L}_1\|, \mathcal{M})$*.*

Remark 1 (Comparison of series radii)**.** Recall that the usual inverse Born series was given in [15] as

$$\phi = \sum_{j=1}^{\infty} \mathcal{K}_j \eta^{\otimes j}$$

where the terms in the usual inverse series $\mathcal{K}_j$ are given by the equation (15) with $\widetilde{L}_j$ replaced with $\mathcal{K}_j$ and $2\widetilde{K}_j$ replaced with K_j. We also note that $\widetilde{K}_1 = K_1$ where K_1 is the first term of the usual Born series defined above and $\widetilde{L}_1 = \frac{1}{2}\mathcal{K}_1$. Furthermore, the convergence of the usual inverse Born series was shown if $\|\mathcal{K}_1\|_2 < \frac{1}{1+\nu}$ and $\|\mathcal{K}_1\phi\| < \frac{1}{1+\nu}$. The quantity $\frac{1}{1+\nu}$ was referred to as the radius of convergence $\mathcal{R}$ there. Analogously, for the modified series, we conclude from the above discussion that the modified inverse series converges if $\|\widetilde{L}_1\|_2 < \frac{1}{1+2\nu}$ and $\|\widetilde{L}_1\phi\| < \frac{1}{1+2\nu}$. In other words, the usual inverse Born series will converge if

$$\|\mathcal{K}_1\| < \frac{1}{1+\nu}, \|\mathcal{K}_1\phi\| < \frac{1}{1+\nu},$$

while the modified inverse series will converge if

$$\|\mathcal{K}_1\| < \frac{2}{1+2\nu}, \|\mathcal{K}_1\phi\| < \frac{2}{1+2\nu}.$$

We will refer to the quantity $\frac{2}{1+2\nu}$ as the radius of convergence $\widetilde{\mathcal{R}}$ of the modified inverse series and we note that clearly

$$\widetilde{\mathcal{R}} > \mathcal{R}.$$

2.3 A simple analytical example

Before moving on to describe the modified series for the diffuse waves, we would first like to compare the convergence of the modified series for the Calderon problem for a simple analytical example. For this illustration, we will consider a Neumann condition instead of the Robin boundary condition. Let Ω be a disk of radius b and consider $\eta(x)$ to be a constant on a ball of radius $a < b$, and equal to zero outside of the disk of radius a. let $\alpha = \frac{a}{b} < 1$. Then the solution to the Neumann problem

$$\begin{aligned} -\Delta u &= \nabla \cdot \eta(x) \nabla u(x), \quad x \in \Omega \\ \frac{\partial u}{\partial \nu} &= g, \quad x \in \partial\Omega \\ \int_{\partial\Omega} u ds &= 0 \end{aligned} \tag{20}$$

with boundary condition $g = \cos(\theta)$ is given by

$$u(r,\theta) = \frac{2r\cos\theta}{2 + (1+\alpha^2)\eta} \quad (0 \le r \le a), \tag{21}$$

$$= \frac{1}{2 + (1+\alpha^2)\eta}\left[(\eta+2)r - \eta\frac{a^2}{r^2}\right]\cos\theta \quad (a < r \le b). \tag{22}$$

The background solution $u_0(r,\theta)$ to the Neumann problem stated above, with the same boundary condition as in (20) is given by

$$u_0(r,\theta) = r\cos\theta.$$

Thus the data measured on the boundary is given in polar coordinates by

$$u(b,\theta) - u_0(b,\theta) = \phi(\eta) b\cos\theta, \qquad \phi(\eta) = -\frac{2\eta\alpha^2}{2 + (1+\alpha^2)\eta}. \tag{23}$$

The goal is now to recover η from the measurement $\phi(\eta)$.

2.3.1 Convergence of the usual series

Let $X := \frac{1}{2}(1+\alpha^2)\eta$. We introduce the following variables $C = 2/(1+\alpha^2)$ and $D = \alpha^2 C$. Thus $X = \eta/C$. The measurement $\phi(\eta)$ given by (23) becomes

$$\phi(\eta) = -D\frac{X}{1+X},$$

which can be expressed as a power series of η if $|X| < 1$, i.e. $|\eta| < C$, in which case we have

$$\varphi(\eta) = -D\big[X - X^2 + X^3 - X^4 \dots\big], \qquad X := \eta/C. \tag{24}$$

Writing an inverse series for η of the form

$$CX = \eta = l_1\phi + l_2\phi^2 + l_3\phi^3 + \cdots ,$$

where ϕ is a given measurement, we find

$$l_n = (-1)^n C D^{-n},$$

i.e.

$$\eta = C\Big[-\Big(\frac{\phi}{D}\Big) + \Big(\frac{\phi}{D}\Big)^2 - \Big(\frac{\phi}{D}\Big)^3 \dots \Big] \tag{25}$$

This series converges provided

$$\Big|\frac{\phi}{D}\Big| = \frac{1+\alpha^2}{2\alpha^2}|\phi| < 1$$

Assuming the measurement ϕ to be exact (i.e. $\phi = \varphi(\eta)$ for some admissible η), the above convergence condition is verified if $\eta \in (-\frac{1}{2}C, +\infty)$. Moreover, the inverse series (25) maps $\varphi \in (-D, D)$ to $\eta \in (-\frac{1}{2}C, +\infty)$. We note that the forward Born series (24) converges for $\eta \in (-C, C)$, whereas the inverse Born series (24) converges for all data associated with $\eta \in (-\frac{1}{2}C, +\infty)$. Therefore, there are values of η that are attained by convergent inverse series but for which the forward series does not converge, and vice versa.

2.3.2 Convergence of the modified series

Expressing the measurement model (23) in terms of the modified contrast $\xi = \eta/(\eta+2)$, i.e. $\eta = 2\xi/(1-\xi)$, we have

$$\phi(\eta) = -\frac{2\alpha^2\xi}{1-\alpha^2\xi} =: \widetilde{\phi}(\xi), \tag{26}$$

Since $|\xi| < 1$ (for all physically relevant contrasts η) and $\alpha < 1$, the measurement model $\widetilde{\phi}(\xi)$ can be expressed as a power series of ξ, which converges unconditionally. Now setting $Y := \alpha^2\xi$, we have:

$$\tilde{\phi}(\xi) = -2\big[Y - Y^2 + Y^3 - Y^4 \dots\big], \qquad Y := \alpha^2\xi, \quad |Y| < 1,$$

Clearly, this series converges unconditionally. Now, the inverse series for ξ in terms of $\widetilde{\phi}$ has the form

$$\alpha^{-2}Y = \xi = \tilde{l}_1\widetilde{\phi} + \tilde{l}_2\widetilde{\phi}^2 + \tilde{l}_3\widetilde{\phi}^3 + \cdots ,$$

which corresponds to (25) with C and D replaced with α^{-2} and 2, respectively, and is therefore given by

$$\xi = \alpha^{-2}\Big[-\Big(\frac{\widetilde{\phi}}{2}\Big) + \Big(\frac{\widetilde{\phi}}{2}\Big)^2 - \Big(\frac{\widetilde{\phi}}{2}\Big)^3 \dots \Big]$$

The series converges provided $|\widetilde{\phi}| < 2$, i.e. on a larger set of measurements than (25). For an exact measurement, i.e if $\phi(\eta) = \widetilde{\phi}(\xi)$ for some $\xi \in (-1, 1)$, the condition $|\widetilde{\phi}| < 2$ is found using (26) to be satisfied whenever

$$|\alpha^2 \xi| < \frac{1}{2}$$

This condition is always true if $\alpha < \frac{1}{\sqrt{2}}$ (i.e. the inverse series is then convergent for all $\phi = \widetilde{\phi}(\xi),\ \xi \in (-1, 1)$). Otherwise, it translates into $|\xi| < \frac{1}{2\alpha^2}$ or equivalently, $\frac{-2}{2\alpha^2+1} < \eta < \frac{2}{2\alpha^2-1}$ and can be shown to be less restrictive than the corresponding condition $\eta > -C/2$ found for the usual Born inverse series. We note that there still are combinations of η and α such that the modified inverse series does not converge, even though the modified forward series is unconditionally convergent.

3 Set up for Diffuse Waves

In this section, we will apply the above method for inversion in an absorbing medium. Let the energy density of diffuse waves in a bounded $\Omega \subset \mathbb{R}^d$ be represented by $u(x)$, wave number be given by $k > 0$, $\eta(x)$ be the spatially varying absorption constant, and x_1 be the location of a point source on the boundary of the medium. We also assume that $supp(\eta(x)) \subset B_a$ where B_a is a ball of radius a contained in Ω and $1 + \eta(x) \geq 1 + \eta_{min} > 0$ for all $x \in \Omega$. The energy density of the wave satisfies the following time-independent diffusion equation:

$$\begin{aligned} -\Delta u(x) + k^2(1 + \eta(x))u(x) &= \delta_{x_1}, \quad x \in \Omega, \\ u(x) + \ell \frac{\partial u}{\partial \nu} &= 0 \in \partial\Omega, \end{aligned} \tag{27}$$

where ℓ is a positive constant. If G is the Green's function for the operator $-\Delta + k^2$, one can show that the energy density $u(x)$ is given by the following integral equation:

$$u(x) = u_{in}(x) - k^2 \int_\Omega G(x, y)u(y)\eta(y)dy \tag{28}$$

where $u_{in}(x)$ is the energy density of the incident wave and satisfies:

$$\begin{aligned} -\Delta u_{in}(x) + k^2 u_{in}(x) &= \delta_{x_1} \quad x \in \Omega \\ u_{in}(x) + \ell \frac{\partial u_{in}}{\partial n} &= 0 \quad x \in \partial\Omega \end{aligned}$$

Let us introduce the following operator:

$$\begin{aligned} T &: L^2(B_a) \to L^2(B_a) \\ (T(f))(x) &= \int_\Omega G(x, y)f(y)dy \end{aligned}$$

and an operator $S : L^2(B) \to H^1(\Omega)$ which can be understood as an extension of the operator T. It is easily seen that the integral equation given by equation (28) can be rewritten as:

$$u(x) = u_{in}(x) - k^2(S(h))(x) \quad x \in \Omega$$

where h is given by

$$(I + k^2\eta T)(h(x)) = \eta u_{in}(x) \quad x \in B_a \tag{29}$$

Applying fixed point iteration to the above, we get the formal Born series for diffuse waves :

$$\phi(x) := u(x) - u_{in}(x) = K_1\eta + K_2\eta^{\otimes 2} + K_3\eta^{\otimes 3} + \dots$$

where $K_j\eta^{\otimes j} = (-1)^j k^{2j} S(\eta T)^{j-1}(\eta u_{in})$. It was shown in [15], that

$$\|K_j\| \le \nu\mu^{j-1}$$

where

$$\nu = k^2|B_a|^{1/2} \sup_{x\in B_a} \|G(x,\cdot)\|_{L^2(\partial\Omega)}$$

and

$$\mu = k^2 \sup_{x\in B_a} \|G(x,\cdot)\|_{L^2(B_a)}.$$

Analogous to the results obtained for the Calderon problem, we have the following

Theorem 6. *[15, Proposition 2.1] If the smallness condition $\|\eta\|_{L^2(B_a)} < 1/\mu$ holds, then the Born series for diffuse waves converges in $L^2(\partial\Omega \times \partial\Omega)$*

Similarly to the Calderon problem, the usual Born series for diffuse waves is conditionally convergent. One can also write a formal inverse Born series for the diffuse waves,

$$\eta = \sum_{j=1}^{\infty} \mathcal{K}_j\phi^{\otimes j}$$

where

$$\mathcal{K}_j : L^2(\partial\Omega \times \cdots \times \partial\Omega) \to L^2(B_a)$$
$$\mathcal{K}_j = -\sum_{m=1}^{j-1} \mathcal{K}_m \Bigg(\sum_{i_1+\cdots+i_m=j} K_{i_1} \otimes \cdots \otimes K_{i_m} \Bigg) (\mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_1)$$

We note that $\mathcal{K}_1$ is a pseudoinverse of K_1. We also have the following results for the usual Born series for the diffuse waves:

Theorem 7. *[15, Theorem 3.1] The operator $\mathcal{K}_j$ that appears in the inverse Born series for the diffuse waves is a bounded operator and satisfies $\|\mathcal{K}_j\| \le C(\mu+\nu)^j\|\mathcal{K}_1\|^j$, where $C = C(\mu, \nu, \|\mathcal{K}_1\|_2)$ is a constant independent of j. Furthermore, if $\|\mathcal{K}_1\|_2 < 1/(\mu+\nu)$ and $\|\mathcal{K}_1\phi\| < 1/(\mu+\nu)$, then the inverse Born series for the diffuse waves converges and the series limit $\widetilde{\eta}$ satisfies:*

$$\|\widetilde{\eta} - \sum_{j=1}^{N} \mathcal{K}_j\phi^{\otimes j}\|_{L^2(B_a)} \le C \frac{\Big((\mu+\nu)\|\mathcal{K}_1\phi\|_{L^2(\partial\Omega\times\partial\Omega)}\Big)^{N+1}}{1 - \Big((\mu+\nu)\|\mathcal{K}_1\phi\|_{L^2(\partial\Omega\times\partial\Omega)}\Big)}$$

3.1 Modified Born series for diffuse waves.

First, we define an operator $\widetilde{T}$ related to T in the following manner:

$$\begin{aligned}\widetilde{T} : L^2(B_a) &\to L^2(B_a)\\ \widetilde{T} &= 2k^2(T) - I\end{aligned}$$

It is easy to check that:

$$I + k^2\eta T = \frac{\eta+2}{2}(I + \frac{\eta}{\eta+2}\widetilde{T})$$

Lemma 3. *The operator $\widetilde{T} : L^2(B_a) \to L^2(B_a)$ is bounded and satisfies $\|\widetilde{T}\| \leq 1$.*

Proof. The proof follows that of Lemma 2.2 in [1]. We proceed by evaluating $\|2k^2T[h]-h\|^2$, considering first the case of $h \in C_0^\infty(\Omega)$. The volume potential $w := S[h]$ satisfies $-\Delta w + k^2 w = h$, together with the boundary conditions as in (27). We then have:

$$\|2k^2T[h] - h\|^2 = \|h\|^2_{L^2(\Omega)} + 4k^4\|w\|^2 - 4k^2\langle h, w\rangle$$

Using the above definition of w and Green's first identity, we have

$$\begin{aligned}0 &= \langle -\Delta w + k^2 w - h, w\rangle\\ &= \|\nabla w\|^2 + k^2\|w\|^2 - \langle h, w\rangle_\Omega - \langle \partial_\nu w, w\rangle\\ &= \|\nabla w\|^2 + k^2\|w\|^2 - \langle h, w\rangle + \ell^{-1}\|w\|^2_{H^{1/2}(\partial\Omega)}\end{aligned}$$

implying that

$$\|2k^2T[h] - h\|^2 = \|h\|^2 - 4k^2(\|\nabla w\|^2 + l^{-1}||w\|^2) \leq \|h\|^2$$

for any $h \in C_0^\infty(\Omega)$. The lemma follows by density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$. □

Similar to what was done above for the modified Born series for the Calderon problem, we introduce a variable $\xi = \eta/(\eta+2)$. Then the integral equation (29) can be recast as:

$$(I + \xi\widetilde{T})h = 2\xi u_{in}$$

The formal Neumann series representation for h given by:

$$\begin{aligned}h &= (I + \xi\widetilde{T})(2\xi u_{in})\\ &= (I - \xi\widetilde{T} + (\xi\widetilde{T})^2 + \dots)(2\xi u_{in})\end{aligned}$$

is guaranteed to converge since $|\xi| < 1$. The measurement ϕ is then given by:

$$\begin{aligned}\phi &= -2k^2 S(I - \xi\widetilde{T} + (\xi\widetilde{T})^2 + \dots)(\xi u_{in})\\ &= 2\widetilde{K}_1\xi + 2\widetilde{K}_2\xi^{\otimes 2} + \dots\end{aligned}$$

In analogy with the modified Born series for the Calderon problem, we call this the modified Born series for diffuse waves. Note that,

$$\widetilde{K}_j(\xi^{\otimes j}) = (-1)^j k^2 S(\xi\widetilde{T})^{j-1}(\xi u_{in}).$$

Thus we have:

Theorem 8. *The modified Born series for the diffuse waves given by*

$$\phi = 2\widetilde{K}_1\xi + 2\widetilde{K}_2\xi^{\otimes 2} + \dots,$$

with $\xi := \eta/(\eta+2)$, converges.

3.2 Modified inverse Born series for diffuse waves

As before the inverse problem is to find ξ given the measurements ϕ. We write down the formal inverse series:

$$\xi = \widetilde{L}_1\phi + \widetilde{L}_2(\phi\otimes\phi) + \widetilde{L}_3(\phi\otimes\phi\otimes\phi) + \dots$$

whose terms $\widetilde{L}_j$ are given by equation (15) wherein $\widetilde{K}_j$ are the terms that appear in the modified Born series for diffuse equations. The convergence and stability results for this inverse series can be proved in a manner similar to what has been done for the modified inverse series for Calderon problem. We will skip the proof and just state the theorems here.

Theorem 9 (Convergence of the modified inverse series)**.** *The operators $\widetilde{L}_j$ of the modified inverse series for diffuse waves are bounded operators and satisfy: $\|L_j\|_2 \le C(\mu+2\nu)^j\|\widetilde{L}_1\|_2^j$. Moreover, the modified inverse Born series for diffuse waves converges in the L^2 norm if $(\mu+2\nu)\|L_1\phi\|_{L^2(\partial\Omega\times\partial\Omega)} < 1$. Additionally, the following estimate for the series limit $\widetilde{\xi}$ holds:*

$$\|\widetilde{\xi} - \sum_{j=1}^{N}\widetilde{L}_j\phi^{\otimes j}\|_{L^2(B_a)} \le C\frac{\Big((\mu+2\nu)\|\widetilde{L}_1\phi\|_{L^2(\partial\Omega\times\partial\Omega)}\Big)^{N+1}}{1-\Big((\mu+2\nu)\|\widetilde{L}_1\phi\|_{L^2(\partial\Omega\times\partial\Omega)}\Big)}$$

Theorem 10 (Stability)**.** *Let $\widetilde{\xi}_1$ and $\widetilde{\xi}_2$ be the series limit for the modified inverse Born series corresponding to the measured data ϕ_1 and ϕ_2 respectively for the diffuse waves and $M = \max\{\|\phi_1\|_2, \|\phi_2\|_2\}$. Furthermore, let us assume $M\|L_1\|_2 < \frac{1}{\mu+2\nu}$. Then*

$$\|\widetilde{\xi}_1 - \widetilde{\xi}_2\|_{L^2(B_a)} < C\|\phi_1-\phi_2\|_{L^2(\partial\Omega\times\partial\Omega)}$$

where $C = C(\nu, L_1, M)$ is otherwise independent of ϕ_1 and ϕ_2.

Theorem 11 (Error bound)**.** *Consider the modified inverse Born series for the scattered waves. Let $\mathcal{M} = \max\{\|\widetilde{\xi}\|_{L^2(B_a)}, \|\widetilde{L}_1\widetilde{K}_1\xi\|_{L^2(B_a)}\}$ and assume that $\mathcal{M} < 1/(\mu+2\nu)$,$\|\widetilde{L}_1\|_2 < 1/(\mu+2\nu)$ and $\|\widetilde{L}_1\phi\|_{L^2(B_a)} < 1/(1+2\nu)$.Then we have:*

$$\|\widetilde{\xi} - \xi\|_{L^2(B_a)} \le C\|I - \widetilde{L}_1\widetilde{K}_1\xi\|_{L^2(B_a)}$$

where the constant $C = C(\nu, \|\widetilde{L}_1\|, \mathcal{M})$.

Remark 2 (Comparison of series radii)**.** Exploiting again the formal similarity of the usual and modified Born series for the diffuse wave problem with their counterparts for the Calderon problem, we find that the usual inverse Born series will converge if $\|\mathcal{K}_1\| < \mathcal{R}$, $\|\mathcal{K}_1\phi\| < \mathcal{R}$ (with $\mathcal{R} = 1/(\mu+\nu)$), while the modified inverse series will converge if $\|\mathcal{K}_1\| < \widetilde{\mathcal{R}}$, $\|\mathcal{K}_1\phi\| < \widetilde{\mathcal{R}}$ (with $\widetilde{\mathcal{R}} = 2/(\mu+2\nu)$). Clearly we again have $\widetilde{\mathcal{R}} > \mathcal{R}$.

4 Numerical results

We will now show the results of numerical simulations carried out to compare the modified Born series approach to the usual Born series approach for the conductivity problem. We note here again that the chief advantage of the modified approach is that the modified forward series converges unconditionally, while the usual forward Born series is guaranteed to converge for $|\eta| < 1$.

4.1 Forward series

The forward data has been generated by solving equation (3) with the boundary condition given by (2) using a Lagrangian FEM of degree 2 on a mesh of size 400×400 on the domain $\Omega = [0,1] \times [0,1]$. Note that the operator S is defined by (10), and as such the function $\phi := Sf$ solves a boundary value problem. Thus S can be implemented in FEniCS as an operator which when applied to a function $f \in L^2(\Omega)$ returns a function $Sf \in H^1(\Omega)$ which is a solution to the BVP given by (11). Similarly the operators T and $\widetilde{T}$ given by equations (8) and (9) can also be implemented in FEniCS. Our data collection follows the model described in [21]. We assume that there are 32 piecewise constant, evenly distributed local current sources attached to the boundary $\partial\Omega$ of the domain (object) Ω. We also assume the background conductivity $\sigma_0 = 1$. For each $i \in \{1, \ldots, 32\}$, we solve the equation (3) with boundary condition (2) and label the corresponding solutions as u_i. Each u_i is then measured at 32 equally spaced points $x_1, x_2, \ldots, x_{32}$ all around the boundary $\partial\Omega$. Similarly the corresponding background solution $(u_0)_i$ can be measured at these points $x_1, x_2, \ldots, x_{32}$, and we use these to construct our discrete data vector ϕ of length 1024 given by $\phi_{ij} = u_i(x_j) - (u_0)_i(x_j) \quad i,j = 1, \ldots, 32$.

Now we compare the convergence of the forward series for the usual versus the modified approach. First, in Figure 1 we show the plots that establish the convergence of the modified series even for values of $|\eta| > 1$. This is done by showing that the L^2 error between the data ϕ and the first few partial sums $\sum_1^N 2\widetilde{K}_j \xi \otimes \cdots \otimes \xi$ goes down with N for the modified Born series while the error for the usual Born series given by $|\phi - \sum_1^N \widetilde{K}_j \eta \otimes \cdots \otimes \eta|$ decreases only for $|\eta| < 1$. Note that the plots have been presented on a semilog scale.

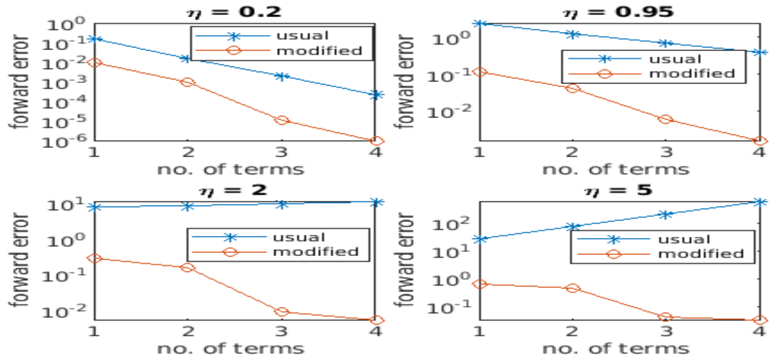

Figure 1: Comparison of the L^2 error for the usual and modified forward series for varying values of the contrast η.

4.2 Inversion

We now present some reconstruction results. The reconstructions have been carried out on a mesh of size 64×64 using a linear finite element method. The inversion operators L_n for the usual Born series are the same as the inversion operators κ_n of [1]. As such these can be implemented recursively in exactly the same fashion as given by [1, Algorithm 2]. The inversion operators $\widetilde{L}_n$ given by (15) are similar to L_n except for a factor of 2 that multiplies each of the terms $\widetilde{K}_{i_n}$. Thus these operators can also be implemented recursively in much the same fashion as L_n. The inversion of both the usual and modified Born series call for the regularized pseudoinverse of the matrix K_1. We evaluate this pseudoinverse using the inbuilt 'pinv' function implemented in the linear algebra package of numpy. For evaluating the pseudoinverse, we set the parameter 'rcond' in 'pinv' to 10^{-5} (meaning that the pseudoinverse is regularized by ignoring all singular values smaller than 'rcond' times the largest one). Finally, for each of the reconstructions of η below for the usual or the modified series, we carry out reconstructions up to the fourth inverse term. We note here that the modified series actually reconstructs ξ, but we can easily reconstruct η using the relation $\eta = \frac{2\xi}{1-\xi}$. Indeed, all plots in the figures show reconstructions of η.

In the reconstructions in Figure 2 we notice that at low contrast ($\eta = 0.5$) both methods perform almost equally well, with the modified method performing slightly better. At medium high contrast ($\eta = 1.5$), we see in Figure 3 that the modified method does better compared to the usual method. The same trend of the modified method performing somewhat better is seen for a still higher contrast ($\eta = 4$) in Figure 4. However, we observed that the two-ellipse phantom required more regularization than the single-circle phantom of the same contrast. This led to more quantitative error in the medium and high contrast reconstructions of the two-ellipse phantoms, especially the appearance of boundary artifacts in these cases. This may be partially due to the lack of resolution inherent in EIT. Authors in [10] have derived a constructive criterion to decide whether a desired resolution can be achieved in a given measurement setup. It would be interesting to apply the ideas developed in [10] to improve the resolution in the reconstructions for both the usual and the modified approach described in this article.

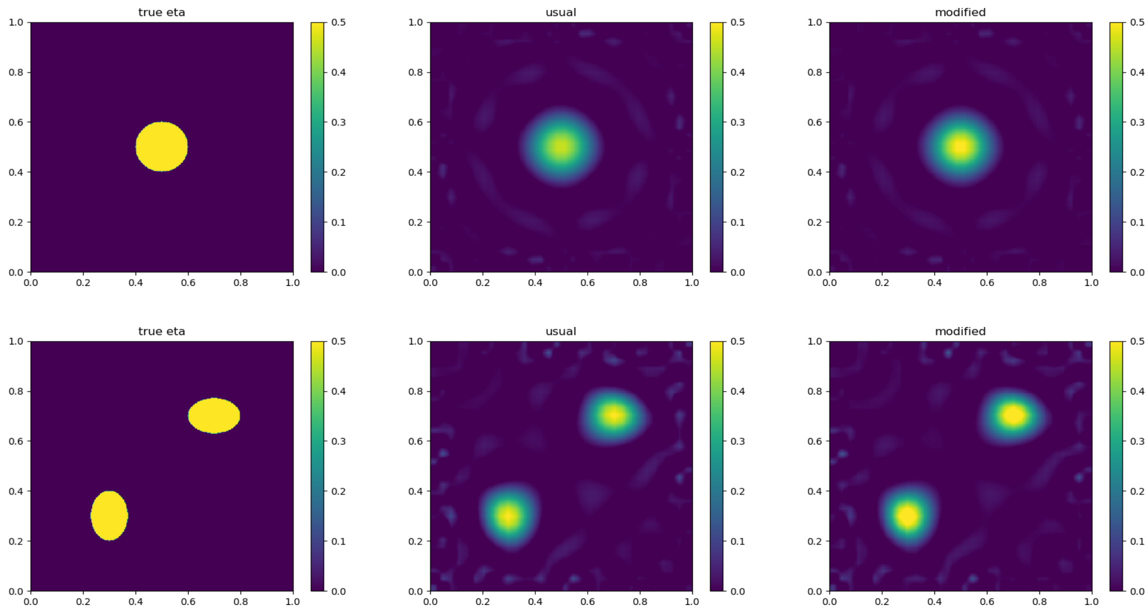

Figure 2: Reconstructions of a low contrast phantom conductivity (left column) using four terms of the inverse Born series (center) and modified inverse Born series (right).

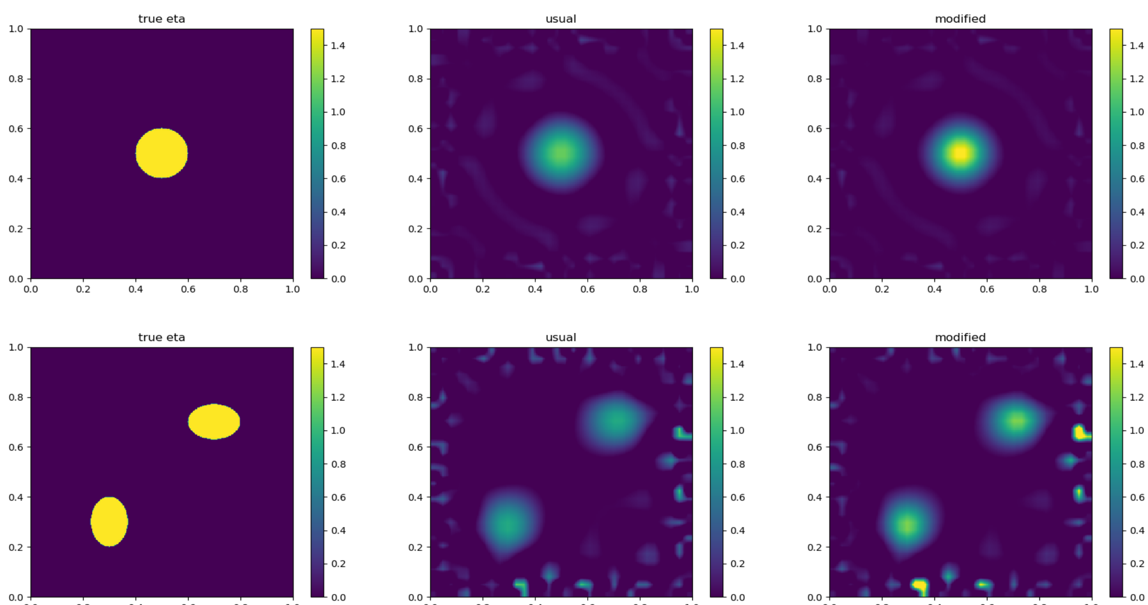

Figure 3: Reconstructions of a medium contrast phantom conductivity (left column) using four terms of the inverse Born series (center) and modified inverse Born series (right).

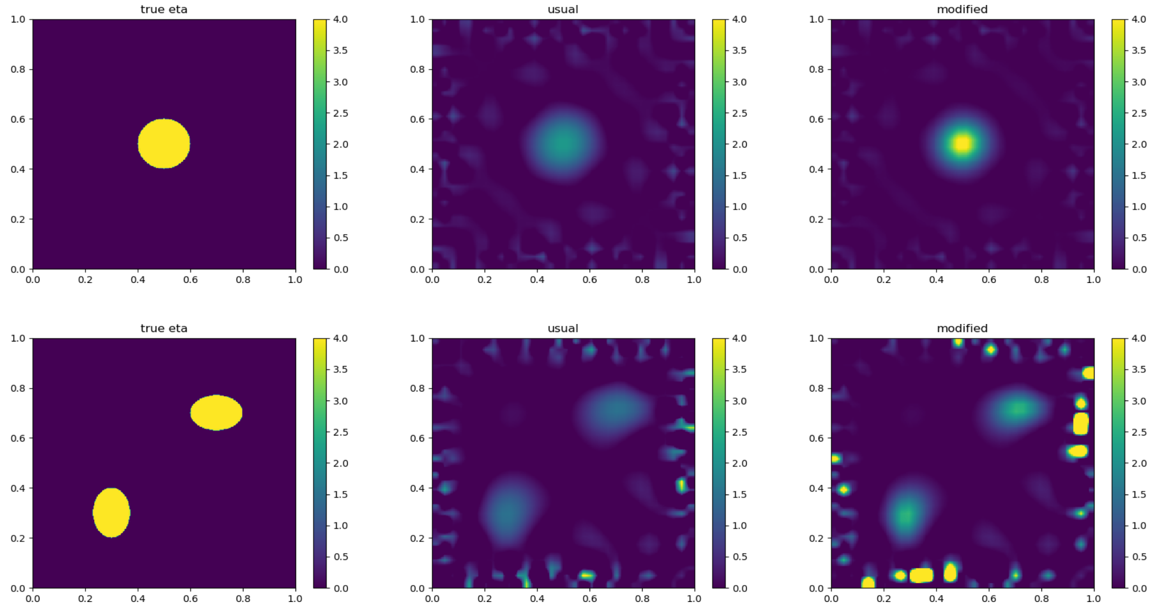

Figure 4: Reconstructions of a high contrast phantom conductivity (left column) using four terms of the inverse Born series (center) and modified inverse Born series (right).

5 Discussion

In this work, we have proposed a modified (forward) Born series for the Calderon problem that has the advantage of being unconditionally convergent, i.e. the forward series converges regardless of the value of the contrast η. Compared to the conditional convergence of the usual (forward) Born series under a smallness condition ($\eta < 1$), this offers us a chance to carry out a Born series based reconstruction for the Calderon problem even in presence of higher contrasts. We have also showed that the radius of convergence of the modified inverse Born series compares favorably with that of the usual Born series. Numerical simulations performed on synthetic data provide further evidence for the fact that the modified approach is better suited for the reconstructions for all values of the contrast η. However, our numerical studies show that the quality of reconstruction deteriorates in both approaches as the contrast η increases. Furthermore, observations from numerical experiments suggest that more regularization (of the pseudoinverse) offers us a way to reconstruct phantoms with higher contrast, in the sense that the reconstructions seem to converge to the same shape as the original phantom. However the penalty that one pays in this case is that the reconstructed images have contrasts much lower than those of the original phantom. To illustrate this we present in Fig. 5 another reconstruction of the two-ellipses phantom at contrast $\eta = 4$, for which the parameter 'rcond' in the package 'pinv' is now set to 10^{-4}. The regularization applied here to the pseudoinverse is thus ten times larger than that used in the reconstructions of Figs.2, 3 and 4. We observe that the modified method again performs better than the usual method.

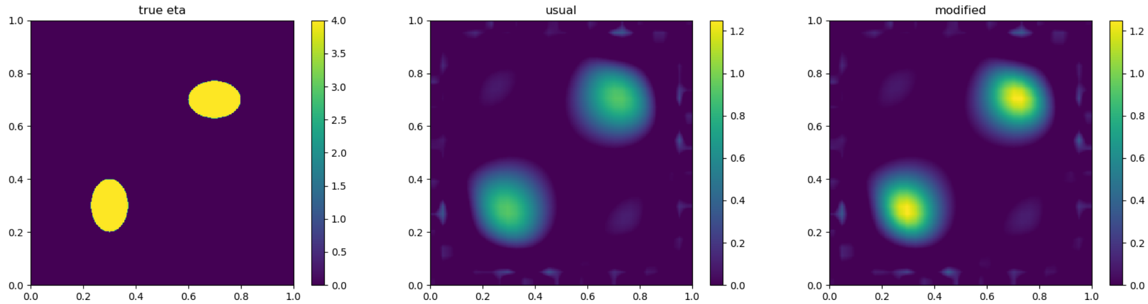

Figure 5: Reconstructions of a high contrast phantom conductivity (left column) using four terms of the inverse Born series (center) and modified inverse Born series (right) with a more regularized pseudoinverse.

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