Results for the electrostatic potential of a uniformly charged hemispherical surface

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We study the form of the electrostatic potential created by a hemispherical surface with uniform surface charge density. We specifically consider a "northern" hemispherical surface and try to obtain analytical expressions for the electrostatic potential at an arbitrary point in space. Some compact analytical results are obtained for special cases and an interesting mathematical integral formula is derived as a by-product of the approach. The results suggest that a general expression for the electrostatic potential at an arbitrary point in space, if possible, would be very hard to obtain in compact analytical form.

Keywords: Electrostatic energy, Hemispherical surface, Uniform surface charge density, Legendre polynomials.

The calculation of the electrostatic potential created by a charged body[1] leads to a better understanding of how charged systems interact with each other[2] and how much energy they can store[3,4]. This information is very important to gain insight into the electrostatic properties[5–8] or opto-electronic features of many materials[9–12]. A compact analytic expression for the electrostatic potential created by a uniformly charged hemispherical surface at some arbitrary point in three-dimensional (3D) space is not readily available in the literature[13,14]. Absence of spherical symmetry is a major factor that makes this calculation quite challenging. Nevertheless, making use of the existence of axial symmetry in the system may serve as a good starting point to reconsider this problem.

The model under consideration is a uniformly charged "northern" hemispherical surface. The hemispherical surface has a radius, R and contains a total arbitrary charge, Q that is uniformly distributed on the surface. The uniform surface charge density of the system is:

$$\sigma = \frac{Q}{2\,\pi\,R^2} \ . \tag{1}$$

The system of coordinates is chosen in such a way that the center of the "northern" hemispherical surface is the origin of a Cartesian system of coordinates, the z axis is oriented towards the "northern" pole and represents the axis of symmetry while the "equatorial" plane lies on the $x-y$ plane ($z = 0$ plane). A view of the system projected on the $y = 0$ plane is shown in Fig. 1. For this choice of the coordinative system, the body has axial symmetry about the z axis. Therefore, based on considerations of symmetry, the electrostatic potential created at some arbitrary point in 3D space, $V(x, y, z) = V(\vec{r})$ can be conveniently expressed as $V(\rho, z)$ in cylindrical coordinates where $\rho^2 = x^2 + y^2$ or $V(r, \theta)$ in spherical coordinates where $r^2 = x^2+y^2+z^2$ and θ is the polar angle (relative to z axis). We believe that it should be straightforward for a reader to understand the notation used for various expressions in different systems of coordinates without much elaborations by looking at the context.

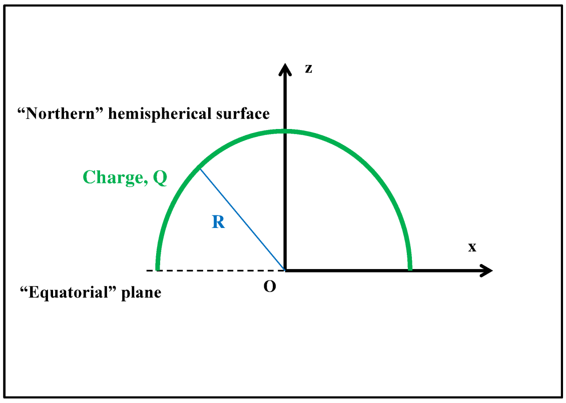

FIG. 1: Schematic view (projected on the $y = 0$ plane) of a "northern" hemispherical surface. The hemispherical surface has a radius, R and is uniformly charged with charge, Q resulting in a constant surface charge density, $\sigma = Q/(2\,\pi\,R^2)$.

The electrostatic potential created by a uniformly charged "northern" hemispherical surface at some arbitrary location, $\vec{r} = (x, y, z)$ can be written as:

$$V(\vec{r}) = k\,\sigma \iint_{S'} dS' \,\frac{1}{|\vec{r} - \vec{r}\,'|} \ , \tag{2}$$

where k is Coulomb's electric constant, σ is the surface charge density from Eq.(1), $dS' = R^2\,d\theta'\,\sin\theta'\,d\phi'$ is an elementary surface (θ' is the polar angle, while ϕ' is the azimuthal angle) and $\vec{r}\,' = (x', y', z')$ is the position of the elementary charge on the "northern" hemispherical surface (with the constraint that $r' = |\vec{r}\,'| = R$). For this choice of the system of coordinates, the domain occupied by the "northern" hemispherical surface, namely the integration region in Eq.(2), is:

$$S' \; : \; \left\{ r' = R \; ; \; 0 \le \theta' \le \frac{\pi}{2} \; ; \; 0 \le \phi' < 2\,\pi \right\} . \quad (3)$$

Axial symmetry helps the calculation of the potential along the axis of symmetry, $\vec{r} = (0,0,z)$. For such a case, one writes:

$$V(0,0,z) = k\,\sigma\,R^2 \int_0^{\pi/2} d\theta' \,\sin\theta' \int_0^{2\,\pi} d\phi' \,\frac{1}{\sqrt{z^2 + R^2 - 2\,z\,R\,\cos\theta'}} \, . \quad (4)$$

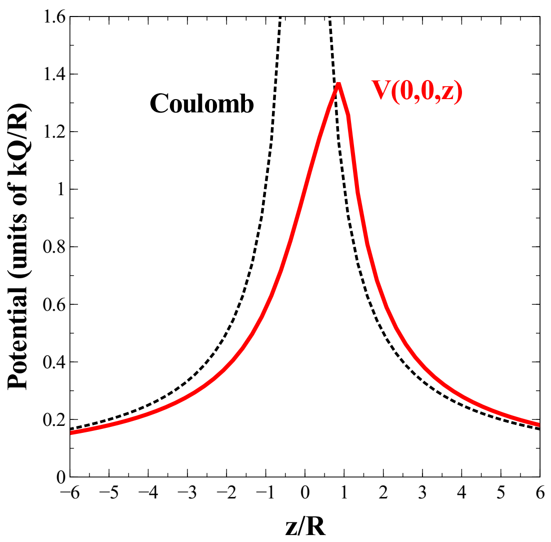

FIG. 2: Plot of $V(0,0,z)$ in units of $k\,Q/R$ as a function of dimensionless distance, z/R (solid line). The result is compared to the Coulomb potential counterpart, $k\,Q/|z|$ (dotted line).

The reason why the integral above can calculated in a straightforward manner by introducing a new variable, $t = z^2 + R^2 - 2\,z\,R\,\cos\theta'$ hinges on the fact that the scalar product (dot product) of the two position vectors $\vec{r} = (0,0,z)$ and $\vec{r}'$ written as $\vec{r}\cdot\vec{r}' = z\,z'$ simplifies in a considerable manner for this special case. For the given system of coordinates, due to the hemispherical shape, $z' = R\,\cos\theta'$, thus $\vec{r}\cdot\vec{r}' = z\,z' = z\,R\,\cos\theta'$. The final result obtained is:

$$V(0,0,z) = \frac{k\,Q}{R}\,\frac{1}{z}\left(\sqrt{z^2+R^2} - |z-R|\right) . \quad (5)$$

A plot of $V(0,0,z)$ is shown in Fig. 2.

The value of the potential at the center ($z=0$), at the "northern" pole ($z=R$) or at any point of interest along the z axis of the "northern" hemispherical surface (for instance, $z=-R$) can be calculated from Eq.(5). The following results apply:

$$V(0,0,z=0) = \frac{k\,Q}{R} \quad ; \quad V(0,0,z=R) = \sqrt{2}\,\frac{k\,Q}{R} \, , \quad (6)$$

and

$$V(0,0,z=-R) = \left(2-\sqrt{2}\right)\frac{k\,Q}{R} \, . \quad (7)$$

The next step to attemp is the calculation of the potential at some point on the "equatorial" line, for example, at $\vec{r} = (x=R,0,0)$. By using the same approach as for the calculation of $V(0,0,z)$, one ends up with the following quantity:

$$V(x=R,0,0) = \frac{1}{\sqrt{2}\,(2\,\pi)}\,\frac{k\,Q}{R}\int_0^{\pi/2} d\theta' \,\sin\theta' \int_0^{2\,\pi} d\phi' \,\frac{1}{\sqrt{1-\sin\theta'\,\cos\phi'}} \, . \quad (8)$$

Efforts to calculate the above integral by using standard integration techniques did not succeed to generate any compact analytical result and we ended up with very complicated terms when reducing the expression to a one-variable integral. We tried to calculate the integral by using symbolic computation software[15] but also this effort was not succesful.

On the other hand, symmetry considerations suggest that the result for the potential should be simple not only for a point on the "equatorial" line, but also for any arbitrary point on the "equatorial" plane ($z=0$). Let us consider an arbitrary point, $(x,y,z=0)$ on the "equatorial" plane. The potential at this point may be written as $V(\rho, z=0)$ and depends only on $\rho = \sqrt{x^2+y^2}$ (which is the same as $r = \sqrt{x^2+y^2+z^2}$ for $z=0$).

Let us now choose a new system of coordinates, (X,Y,Z) with origin at the center as shown in Fig. 3

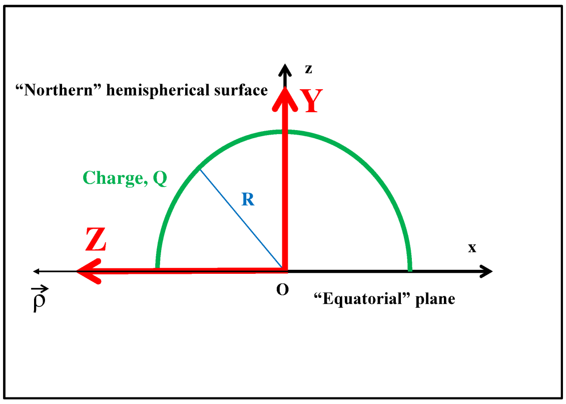

FIG. 3: Schematic view of the new system of coordinates, (X, Y, Z) so that the Z axis is along the vector, $\vec{\rho} = (x, y)$ while the Y axis points towards the "northern" pole. The X axis (not shown) is perpendicular to the $Y - Z$ plane pointing in the direction towards the reader.

so that the polar axis, Z is taken along the vector, $\vec{\rho} = (x, y)$, the Y axis is oriented to point towards the "northern" pole while the X axis is perpendicular to both Y and Z as required (pointing towards the reader). We start from the expression in Eq.(2) where $\vec{r} = \vec{\rho}$. It requires only a little bit of visualization to determine how to express the domain of space occupied by the hemispherical surface in the new system of coordinates depicted in Fig. 3 by noticing that the two relevant angles are such that $\theta' \in [0, \pi]$ and $\phi' \in [0, \pi]$. The conclusion is that one can write:

$$V(\rho, z = 0) = k\,\sigma\,R^2 \int_0^{\pi} d\theta'\,\sin\theta' \int_0^{\pi} d\phi'\,\frac{1}{\sqrt{\rho^2 + R^2 - 2\,\rho\,R\,\cos\theta'}}\ . \tag{9}$$

The same technique as the one used for the calculation of $V(0, 0, z)$ leads to a final result for $V(\rho, z = 0)$ that reads:

$$V(\rho, z = 0) = \frac{k\,Q}{R}\,\frac{1}{2\,\rho}\left(\rho + R - |\rho - R|\right)\ . \tag{10}$$

By further scrutinizing the expression in Eq.(10) one notices that: $V(\rho \geq R, z = 0) = k\,Q/\rho$ while $V(\rho \leq R, z = 0) = k\,Q/R$. This means that the electrostatic potential created by a uniformly charged hemispherical surface in its "equatorial" plane is given by the same expression as the one that applies to a uniformly charged spherical surface with radius, R containing the same total charge, Q. Obviously, on the "equatorial" line we have:

$$V(\rho = R, z = 0) = \frac{k\,Q}{R}\ . \tag{11}$$

Let us return to to the expression in Eq.(8). Although we were unable to calculate the integral in Eq.(8) using standard integration techniques, a comparison of Eq.(11) to Eq.(8) suggests that:

$$\int_0^{\pi/2} d\theta'\,\sin\theta' \int_0^{2\,\pi} d\phi'\,\frac{1}{\sqrt{1 - \sin\theta'\,\cos\phi'}} = \sqrt{2}\,(2\,\pi)\ . \tag{12}$$

This result was checked numerically[15]. Efforts to obtain a compact general expression for the electrostatic potential at an arbitrary point in space by using approaches similar to the ones described above do not appear to lead to feasible integrals.

A general scheme for spherical coordinates that relies on the transformation of the Coulomb term, $1/|\vec{r} - \vec{r}\,'|$ appearing in Eq.(2) as a function of Legendre polynomials may also be attempted. In physical applications of Legendre polynomials[16,17] one routinely can write:

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}\,P_l(\cos\gamma)\ , \tag{13}$$

where $r_<$ denotes the smaller of r and r', $r_>$ denotes the larger of r and r', $P_l(\cos\gamma)$ are Legendre polynomials and γ is the angle included between vectors $\vec{r}$ and $\vec{r}\,'$. Note that $\cos\gamma = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\,\cos(\phi - \phi')$. This approach leads to a an infinite series solution of the potential written as $V(r, \theta)$ but the expression is neither simple, nor compact. This scheme exploits the axial symmetry of the hemispherical surface in a spherical system of coordinates and utilises suitable mathematical transformations[18,19] that eventually reduce the integral problem into a final infinite series. The drawback of the process is that many mathematical transformations rely heavily on properties of various types of special functions such as Legendre polynomials. Furthermore, this infinite series does not appear that can be expressed in terms of simple functions and can be calculated only numerically.

To conclude, in this work we studied the nature of the electrostatic potential created by a "northern" hemispherical surface with uniform surface charge density. This problem is of interest to a broad audience of researchers and educators working on the field of electrostatics or electrodynamics[20]. We derived exact compact analytic expressions for the electrostatic potential that apply to some special cases by using a mathematical approach that optimally utilizes the axial symmetry of the body. We also speculated on the nature of general solution at an arbitrary point in space hinting that it may be calculated as an infinite series, but not in a compact analytical form. An interesting mathematical integral formula is obtained in the form of the expression in Eq.(12) as a simple by-product of the approach used.

CRediT authorship contribution statement

Orion Ciftja: Conceptualization, Methodology, Validation, Writing - original draft , Writing - review and editing, Supervision. Brent Ciftja: Methodology, Validation, Formal analysis, Investigation, Writing - reviewing and editing. Cal Colbert-Pollack: Methodology, Validation, Formal analysis. Lindsey Littlejohn: Methodology, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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