Interaction energy between two identical hemispherical surfaces with uniform surface charge density

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Obtaining the interaction energy between two uniformly charged hemispherical surfaces in compact analytical form seems to be an impossible task to achieve under arbitrary conditions. However, we show in this work that one can obtain the interaction energy between two identical hemispherical surfaces with uniform surface charge density for the special condition of them touching each other along the "equator". The mathematical solution method that we apply is remarkable in that the bulk of the treatment is analytic with the only drawback of having to rely on knowledge of certain special functions and their properties.

Keywords: Electrostatic energy, Hemispherical surface, Uniform surface charge density, Legendre polynomials.

I. INTRODUCTION

From the point of view of geometry, a sphere has rotational symmetry around any axis through its center and reflection symmetry across any plane through its center. Any system which has these properties is considered spherically symmetric. For instance, solid spheres and spherical shells are spherically symmetric. Now let us assume that some charge is contained in a spherically symmetric body in such a way that its density at a given point does not depend on direction. For instance, assume that a spherical surface is uniformly charged with constant surface charge density or a solid sphere contains charge with constant volume charge density[1]. This is the scenario of a charge distribution with spherical symmetry. However, if the same spherical surface is charged so that the "northern" hemispherical surface has uniform surface charge density, σ_1 while its "southern" counterpart has a different value, $\sigma_2 \neq \sigma_1$ then this system lacks spherical symmetry. Another example of a spherical surface with surface charge density dependent on polar coordinates is provided by the standard case study of the surface charge density induced on a grounded sphere as a result of the presence of a point charge outside the sphere[2]. This problem illustrates well the application of the method of images.

If the charge distribution has spherical symmetry, its electric field must have spherical symmetry and be a radial vector. A second repercussion of the spherical symmetry is that the electric field's magnitude only depends on the distance from the center of the distribution. As a result, a system of spherical coordinates with origin corresponding to the center of symmetry is ideally suited for the calculation of the electric field at an arbitrary point in a rather easy way. For instance, one can use Gauss's law as illustrated by many such results widely available in the literature[3–7]. In spherical coordinates, one can write the volume charge density as $\rho(r, \theta, \phi)$ and the surface charge density as $\sigma(r, \theta, \phi)$ where, in spherical coordinates, $r \geq 0$ is radial distance, $0 \leq \theta \leq \pi$ is polar angle and $0 \leq \phi < 2\,\pi$ is azimuthal (longitudinal) angle. If we have $\rho(r)$ or $\sigma(r)$, then the system has spherical symmetry.

It is well known that the interaction energy between any two charged arbitrarily shaped bodies cannot be calculated analytically. For this reason, analysis of structures with certain symmetry is of considerable interest in electrostatics[8–14]. Case studies such as uniformly charged solid spheres or spherical surfaces can be found in many textbooks. The calculation of the electrostatic interaction energy between two uniformly charged spheres or spherical surfaces is straightforward. The reason why this calculation is so easy has to do with the spherical symmetry of each of bodies and their charge distributions. From these examples, many undergraduate students get the wrong impression that any system with symmetry can be elegantly handled through various "tricks".

The current work tries to dispel this "misunderstanding" and aims to point out that "tricks" (such as Gauss's law, etc.) have no application even for a relatively simple-looking system such as that of two interacting uniformly charged hemispherical surfaces touching each other along the "equator". Absence of spherical symmetry for this case in point forces one to use more refined approaches that rely on non-trivial properties of special mathematical functions. It is fair to say that a good number of undergraduate students find such mathematical approaches difficult to grasp and somehow unrelated to what they encounter in a physics class. This mindset is part of a wider problem where undergraduate students, often, have difficulty in grasping the connection of such mathematics to real life situations. Therefore, the current problem that we solve is a very good example of the connection of abstract mathematical methods to a real physics problem.

Finding the electrostatic interaction energy for such a system is not an easy task. The process involves trans-

formations and expansions that rely on special functions such as the Legendre polynomials. This means that presence of so many interconnecting mathematical tools makes this problem fascinating from both a scientific and pedagogical perspective. From the point of view of pedagogical aspects, this work shows clearly how complicated mathematical concepts, methods and functions arise from the framework of an appealing physics problem. Since our contribution deals with a basic problem in electrostatics, the present material, which is self-contained, may be of interest to instructors teaching courses in electromagnetism, as well as to undergraduate students. These audiences have the opportunity to discover that, sometimes, simple-looking problems are a great source to uncover the beauty of mathematics at work.

Knowledge of the electric field or potential is an important pre-requisite to calculate the electrostatic interaction energy of a system of two or more bodies as well as the energy stored in each of them[15–19]. There are some simple results that apply to a hemispherical surface and/or solid hemisphere at special points[20] but there is no general exact analytical result that we are aware of. This means that the problem of calculating the interaction energy between two identical hemispherical surfaces with uniform surface charge density at an arbitrary orientation relative to each other seems not to have an exact analytic solution. In this work we show that, despite the difficulty of the problem, the electrostatic interaction energy between two identical hemispherical surfaces with uniform surface charge density can be obtained exactly in analytical form for the special case of the two hemispherical surfaces touching each other along the "equator". The mathematical method that we employed is effective, because its application hinges crucially on the presence of axial symmetry in the system.

The paper is organized in the following form: In Section II we lay out the general theory that applies to such a model. In Section III we display the main results obtained. In Section IV we briefly summarize the outcomes of the work and provide some concluding remarks.

II. THEORY

The halves of a spherical surface as divided by the "equator" represent the respective hemispherical surfaces. The spherical surface's center is called center of the two respective hemispherical surfaces. Let us view this setup as a system consisting of two identical hemispherical surfaces with uniform surface charge density touching each other along the "equator". The hemispherical surfaces have radius, R and charge, Q. Such an amount of charge is uniformly distributed on the respective surfaces. As a result, each of the hemispherical surfaces has a uniform surface charge density of:

$$\sigma = \frac{Q}{2\,\pi\,R^2}\ . \tag{1}$$

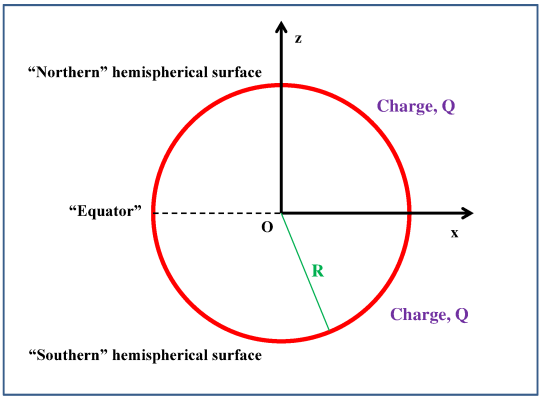

FIG. 1: Schematic view (projected on the $y = 0$ plane) of a system of two identical hemispherical surfaces touching each other along the "equator". The two hemispherical surfaces denoted as "northern" and "southern" have radius, R and charge, Q. Charge is uniformly distributed on the respective surfaces.

We choose spherical coordinates to study this problem. The origin of such a system is chosen to correspond to the coinciding centers of the two hemispherical surfaces and the axes of the system of coordinates are picked in such a way that the "equator" plane is the $z = 0$ plane. The first hemispherical surface denoted as the "northern" hemispherical surface stands above the $z = 0$ plane. The second hemispherical surface is the "southern" one and is positioned below the $z = 0$ plane. A view of the system projected on the $y = 0$ plane is shown in Fig. 1.

For the given setup, all the charge of the "northern" hemispherical surface is contained in the domain:

$$S_1\ :\ \left\{r_1 = R\ ;\ 0 \le \theta_1 \le \frac{\pi}{2}\ ;\ 0 \le \phi_1 < 2\,\pi\right\}\ . \tag{2}$$

where θ_1 and ϕ_1 are, respectively, the polar and the azimuthal (longitudinal) angles. On the other hand, the "southern" counterpart occupies the domain:

$$S_2\ :\ \left\{r_2 = R\ ;\ \frac{\pi}{2} \le \theta_2 \le \pi\ ;\ 0 \le \phi_1 < 2\,\pi\right\}\ . \tag{3}$$

Let us now consider an elementary charge, dQ_1 located on the "northern" hemispherical surface at some position vector, $\vec{r}_1$. Similarly, an elementary charge, dQ_2 is located on the "southern" hemispherical surface at some position vector, $\vec{r}_2$. The elementary surfaces occupied by such charges are, respectively, $dS_1 = R^2\,\sin\theta_1\,d\theta_1\,d\phi_1$ and $dS_2 = R^2\,\sin\theta_2\,d\theta_2\,d\phi_2$. As a result, $dQ_1 = \sigma\,dS_1$ and $dQ_2 = \sigma\,dS_2$. The electrostatic interaction energy between these two hemispherical surfaces is then written as:

$$U_{12} = k_e\,\sigma^2 \iint_{S_1} dS_1 \iint_{S_2} dS_2\ \frac{1}{|\vec{r}_1 - \vec{r}_2|}\ , \tag{4}$$

where k_e is Coulomb's electric constant, S_1 is given from

Eq.(2) and S_2 is its counterpart in Eq.(3). One can write the quantity in Eq.(4) by using spherical coordinates as:

$$U_{12} = k_e\,\sigma^2\,R^4\int_0^{\pi/2} d\theta_1\,\sin\theta_1\int_0^{2\,\pi} d\phi_1\int_{\pi/2}^{\pi} d\theta_2\,\sin\theta_2\int_0^{2\,\pi} d\phi_2\;\frac{1}{|\vec{r}_1-\vec{r}_2|}\;. \tag{5}$$

We attempt the calculation of the above integral by using an approach that combines the generating expression for Legendre polynomials with a "tweak" well-suited for a hemispherical surface with axial symmetry[21].

One knows from the theory of Legendre polynomials[22–26] that one can write $1/|\vec{r}_1-\vec{r}_2|$ as:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\,P_l(\cos\gamma)\;, \tag{6}$$

where $r_<$ denotes the smaller of r_1 and r_2, $r_>$ denotes the larger of r_1 and r_2, $P_l(\cos\gamma)$ are Legendre polynomials and γ is the angle included between vectors $\vec{r}_1$ and $\vec{r}_2$. The case of two hemispherical surfaces is special in that:

$$r_1 = r_2 = R\;, \tag{7}$$

where $r_1 = |\vec{r}_1|$ and $r_2 = |\vec{r}_2|$. It can be shown that, for this very special arrangement, the following expansion is valid (except for the special angles, $\gamma = 0$ and π):

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{R}\sum_{l=0}^{\infty}P_l(\cos\gamma)\quad;\quad r_1 = r_2 = R\;. \tag{8}$$

Details on how the above expression is derived are provided in Appendix A where the discussion relies on the remarks in pg. 740 of Ref.[27]. In a more formal way, one can think of Eq.(8) as originating from Eq.(6) with the understanding that can be seen as a limit of $r_< \to R^-$ and $r_> \to R^+$ (though this approach is not entirely correct and the many subleties explained in Appendix A are lost). It is straightforward to see why the infinite sum over Legendre polynomials in Eq.(8) diverges for $\gamma = 0$ which means $\vec{r}_1 = \vec{r}_2$ ($r_1 = r_2 = R$). Therefore, the truly peculiar case is that of $\gamma = \pi$ which means $\vec{r}_1 = -\vec{r}_2$ (and always $r_1 = r_2 = R$). When $r_1 = r_2 = R$, one can easily calculate that:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{R}\;\frac{1}{2\,\sin(\gamma/2)}\quad;\quad r_1 = r_2 = R\;. \tag{9}$$

Comparison of Eq.(9) to Eq.(8) suggests that $\sum_{l=0}^{\infty}P_l(\cos\gamma) = 1/\Big[2\,\sin(\gamma/2)\Big]$ for all the values of γ where the infinite sum over the Legendre polynomials is convergent. If there is a peculiar case such as $\gamma = \pi$, where the infinite sum over Legendre polynomials is not convergent in the usual sense, it is common practice to define the sum in such a way that $\sum_{l=0}^{\infty}P_l(\cos\gamma)$ is simply assigned the value of $1/2$ consistent with the expected result for $\gamma = \pi$. As explained in Appendix A, in all expressions, the Coulomb factor, $1/|\vec{r}_1-\vec{r}_2|$ for $r_1 = r_2 = R$ is under the sign of integrals, $\iint_{S_1} dS_1$ (and $\iint_{S_2} dS_2$). Therefore, a few "problematic" points when carrying out the integration are not "worrisome".

At this juncture, we proceed to substitute the term, $1/|\vec{r}_1-\vec{r}_2|$ given from Eq.(8) into the result of Eq.(5) and write:

$$U_{12} = k_e\,\sigma^2\,R^4\,\frac{1}{R}\sum_{l=0}^{\infty}\left[\int_0^{\pi/2} d\theta_1\,\sin\theta_1\int_0^{2\,\pi} d\phi_1\;\int_{\pi/2}^{\pi} d\theta_2\,\sin\theta_2\int_0^{2\,\pi} d\phi_2\;P_l(\cos\gamma)\right]\;. \tag{10}$$

By using the addition theorem for spherical harmonics one can express $P_l(\cos\gamma)$ in terms of products of associated Legendre polynomials, for example, see formula (3.68) in pg. 111 of Ref.[2]. Based on this result, one can easily check that:

$$\int_0^{2\,\pi} d\phi_1\;\int_0^{2\,\pi} d\phi_2\;P_l(\cos\gamma) = (2\,\pi)^2\,P_l(\cos\theta_1)\,P_l(\cos\theta_2)\;, \tag{11}$$

where $P_l(\cos\theta_i)$ are Legendre polynomials for the polar angles of vectors $\vec{r}_i$ (angles with respect to the z-axis) and $i = 1$ and 2. The result in Eq.(11) helps us to write the quantity in Eq.(10) as:

$$U_{12} = k_e\,\sigma^2\,R^4\,\frac{(2\,\pi)^2}{R}\,\sum_{l=0}^{\infty}\,\left\{\left[\int_0^{\pi/2} d\theta_1\,\sin\theta_1\,P_l(\cos\theta_1)\right]\left[\int_{\pi/2}^{\pi} d\theta_2\,\sin\theta_2\,P_l(\cos\theta_2)\right]\right\}\,. \tag{12}$$

The expression above can be written more compactly as:

$$U_{12} = k_e\,\sigma^2\,R^4\,\frac{(2\,\pi)^2}{R}\,\sum_{l=0}^{\infty}\,\left\{\left[\int_0^{1} dx\,P_l(x)\right]\left[\int_{-1}^{0} dx\,P_l(x)\right]\right\}\,. \tag{13}$$

Note that the expression in Eq.(13) involves integrals of Legendre polynomials over half a range, namely, with x going from 0 to 1 or from -1 to 0. This is in contrast with all the typical expressions for integrals of Legendre polynomials found in textbooks which are generally written for the full range of integration with x varying from -1 to 1. In addition to that, the quantity in Eq.(13) involves a tedious infinite sum over the index, $l = 0, 1, \ldots \infty$ that must be dealt with.

III. RESULTS

By relying on Eq.(1), one can write:

$$Q^2 = \sigma^2\,(2\,\pi)^2\,R^4\,. \tag{14}$$

As a result, the electrostatic interaction energy in Eq.(13) can be expressed as:

$$U_{12} = \frac{k_e\,Q^2}{R}\,\sum_{l=0}^{\infty}\,\left\{\left[\int_0^{1} dx\,P_l(x)\right]\left[\int_{-1}^{0} dx\,P_l(x)\right]\right\}\,. \tag{15}$$

In order to figure out the integrals appearing in Eq.(15), we calculated exactly some of them for small values of $l = 0, 1, 2, \ldots$. This is doable given that the Legendre polynomials, $P_l(x)$ are relatively simple for small values of l:

$$P_0(x) \;=\; 1\,, \tag{16}$$

$$P_1(x) \;=\; x\,, \tag{17}$$

$$P_2(x) \;=\; \frac{1}{2}\left(3\,x^2 - 1\right)\,, \tag{18}$$

$$P_3(x) \;=\; \frac{1}{2}\left(5\,x^3 - 3\,x\right)\,. \tag{19}$$

The results of the calculations show that the values of the integrals decrease quickly with increase of l. Furthermore, there are recognizable regular patterns of the two sets of the integrals, $\int_0^1 dx\,P_l(x)$ and $\int_{-1}^0 dx\,P_l(x)$ that suggest that these integrals may be calculated analytically. In fact, we were able to obtain some very useful exact analytic results that we explain below. First of all, one can prove by a simple change of variable from x to $-x$, that, quite generally:

$$\int_{-1}^{0} dx\,P_l(x) = (-1)^l\,\int_0^1 dx\,P_l(x) \quad ; \quad l = 0, 1, 2, \ldots\,. \tag{20}$$

This means that one must calculate only the integral, $\int_0^1 dx\,P_l(x)$ for values of $l = 0, 1, 2, \ldots$ when it comes to the expression in Eq.(15). Obviously, the simplest integral, $\int_0^1 dx\,P_l(x)$ to calculate is the one for $l = 0$:

$$\int_0^1 dx\,P_0(x) = 1\,. \tag{21}$$

After that, we noted that the integrals $\int_0^1 dx\,P_l(x)$ are all zero for $l = 2, 4, 6, \ldots$ (nonzero even values of l that exclude $l = 0$). By denoting the index l as $l = 2\,k$ where $k = 1, 2, \ldots$, one writes:

$$\int_0^1 dx\,P_{2\,k}(x) = 0 \quad ; \quad k = 1, 2, \ldots\,. \tag{22}$$

On the other hand, the integrals $\int_0^1 dx\,P_l(x)$ are nonzero for index $l = 1, 3, 5, \ldots$ (odd values of l). By denoting $l = 2\,k + 1$ where $k = 0, 1, \ldots$ we were able to obtain the following result:

$$\int_0^1 dx\,P_{2\,k+1}(x) = \frac{(-1)^k}{2^{2\,k+1}\,(k+1)}\,\frac{(2\,k)!}{(k!)^2} \quad ; \quad k = 0, 1, \ldots\,. \tag{23}$$

Having obtained these important mathematical formulas, we proceed to write the quantity in Eq.(15) as:

$$U_{12} = C\,\frac{k_e\,Q^2}{R}\,, \tag{24}$$

where we use Eq.(20) to express constant, C as:

$$C = \sum_{l=0}^{\infty}\,\left\{(-1)^l\left[\int_0^1 dx\,P_l(x)\right]^2\right\}\,. \tag{25}$$

By using the results in Eq.(21), Eq.(22) and Eq.(23) one can write succintly the constant, C in Eq.(25) as:

$$C = 1 - \sum_{k=0}^{\infty} \left[\int_0^1 dx\, P_{2\,k+1}(x) \right]^2 = 1 - \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{2^{2\,k+1}\,(k+1)} \frac{(2\,k)!}{(k!)^2} \right]^2 . \quad (26)$$

We first calculated the value of the constant from Eq.(25) numerically by truncating the infinite sum over l to values from 0 up to a finite l_{max}. We denote this quantity as C_{num}. To a high degree of accuracy, the results seem to converge to the value of $(2 - 4/\pi) \approx 0.726760$ as l_{max} becomes larger than 200. By using symbolic computation software[28], we were able to verify without any doubt that:

$$C = 1 - \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{2^{2\,k+1}\,(k+1)} \frac{(2\,k)!}{(k!)^2} \right]^2 = 2 - \frac{4}{\pi} . \quad (27)$$

In Table. I we show the results for the numerically calculated constant, C_{num} obtained for several finite values of l_{max} ranging from 10 to 160 (in steps of 10) together with the relative difference, $|\frac{C_{num}-C}{C}|$ where C is the exact value from Eq.(27).

TABLE I: Numerical values, $C_{num} \approx \sum_{l=0}^{l_{max}} \left\{ (-1)^l \left[\int_0^1 dx\, P_l(x) \right]^2 \right\}$ with l_{max} chosen to have values ranging from 10 to 160 in steps of 10 together with the relative difference, $|\frac{C_{num}-C}{C}|$ where C is the exact value from Eq.(27).

l_{max}	C_{num}	$\lvert\frac{C_{num}-C}{C}\rvert$	l_{max}	C_{num}	$\lvert\frac{C_{num}-C}{C}\rvert$
10	0.728195	2.0×10^{-3}	90	0.726780	2.8×10^{-5}
20	0.727139	5.2×10^{-4}	100	0.726776	2.2×10^{-5}
30	0.726931	2.4×10^{-4}	110	0.726773	1.8×10^{-5}
40	0.726857	1.3×10^{-4}	120	0.726771	1.5×10^{-5}
50	0.726823	8.7×10^{-5}	130	0.726770	1.4×10^{-5}
60	0.726804	6.1×10^{-5}	140	0.726769	1.2×10^{-5}
70	0.726792	4.4×10^{-5}	150	0.726767	9.6×10^{-6}
80	0.726785	3.4×10^{-5}	160	0.726767	9.6×10^{-6}

In conclusion, the final exact analytical result for the electrostatic interaction energy in Eq.(24) is:

$$U_{12} = \left(2 - \frac{4}{\pi}\right) \frac{k_e\, Q^2}{R} , \quad (28)$$

where Q is the total amount of charge held in each of the two hemispherical surfaces and R is their radius. We remind the reader that the two hemispherical surfaces are uniformly charged and touch each other along the "equator".

IV. CONCLUSIONS

We considered the problem of two identical hemispherical surfaces of radius R touching each other along the "equator". The two bodies contain the same charge and, thus, are uniformly charged with the same surface charge density. Because of lack of spherical symmetry, the analytic calculation of the interaction energy between such a pair of interacting hemispherical surfaces is a problem yet to be solved under general conditions. Despite the challenging nature of this problem, this work shows that an exact analytic result is possible for such a quantity for at least some special circumstances.

For the scenario in which the two hemispherical surfaces touch each other along the "equator", we know that the charge distribution on each of the bodies does not have spherical symmetry. Nevertheless, the system still retains axial symmetry. Presence of axial symmetry is the key ingredient that leads to a succesful implementation of the current solution method. Absence of axial symmetry, for instance, when the "northern" hemispherical surface is lifted and rotated at some nonzero angle relative to the "southern" hemispherical surface will lead to a problem that, in our opinion, is impossible to solve analytically.

Based on the arguments mentioned above, the model of two uniformly charged hemispherical surfaces considered in this work represents a unique case scenario that is rarely encountered in the mainstream literature. The few examples involving hemispherical surfaces that are available in the literature use special methods or the Maxwell stress tensor approach to compute the net force exerted on the "northern" hemispherical surface of a uniformly charged spherical surface by its "southern" counterpart[20,29]. Another approach that deals with the force of repulsion between the two hemispherical surfaces of a uniformly charged spherical surface is given at Ref.[30]. In few words, the calculation of the interaction energy for such a system is difficult and the objective of the present work is to fill in this gap.

For this reason, this work can be of interest to a wide audience of physics educators[31] as well as some researchers working on the field of electrostatics[32–34] or opto-electronic materials[35,36]. In particular, we believe that the results reported in this work would be of great interest to undergraduate students and university teachers, because this problem illustrates very well key concepts of electrostatics involving a wide range of mathematical tools (special functions, expansions, infinite sums and numerical calculations). One also may envision devices where the described physical situation can be checked

experimentally. For example, imagine a capacitor-like system made up of two electrodes where the typical square/rectangular or circular plates of a conventional capacitor have been replaced by hemispherical ones. Assume that the two electrodes are very close to each other in such a way as to closely resemble the geometry of Fig. 1 with the only difference of charges being $+Q$ and $-Q$. This would be a neat experimental realization of the present model where the experimentally measured capacitance can be compared with the one expected which can calculated by following established recipies[37,38]. Lastly, the result obtained in Eq.(28) can be useful to computational physicists to gauge the efficacy of numerical methods. The numerical calculation of the four-dimensional integral in Eq.(5) is not at all simple but, we believe, can be done numerically via various schemes. Knowing its precise value can help those involved with computational work to identify schemes and/or algorithms that have the best accuracy and stability in a numerical computation.

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APPENDIX A: SPECIAL EXPANSION

The equation for the generating function of Legendre polynomials is:

$$g(t,x) = \frac{1}{\sqrt{1-2\,x\,t+t^2}} = \sum_{l=0}^{\infty} P_l(x)\,t^l \;\;;\;\; |t|<1\;. \quad \text{(A1)}$$

Let us write the Coulomb potential term, $1/|\vec{r}_1-\vec{r}_2|$ as:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{\sqrt{r_1^2-2\,r_1\,r_2\,\cos\gamma+r_2^2}}\;, \quad \text{(A2)}$$

where $r_1=|\vec{r}_1|$, $r_2=|\vec{r}_2|$ and γ is the angle between the two vectors. It is obvious how to obtain the expansion in Eq.(6) by starting from Eq.(A2) and assuming that $r_1 \neq r_2$.

However, let us now not make any assumption whether r_1 is larger/smaller than r_2 or equal to r_2. As a first step one may recast the expression in Eq.(A2) as:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{r_1}\,\frac{1}{\sqrt{1-2\,\cos\gamma\,\frac{r_2}{r_1}+(\frac{r_2}{r_1})^2}}\;. \quad \text{(A3)}$$

Without any loss of generality, one may also extract r_2 out of the square root sign if so one desires. By comparing the right-hand-side of Eq.(A3) to the definition of $g(t,x)$ in Eq.(A1) one has no difficulty to identify:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{r_1}\,g\Big(t=\frac{r_2}{r_1}, x=\cos\gamma\Big)\;. \quad \text{(A4)}$$

Regarding the series expansion in Eq.(A1), it is noted in pg. 740 of Ref.[27] that the series is convergent for $|t|=1$ except for $|x|=1$. This is precisely the special case that we are interested in, because $r_1=r_2=R$ means $t=1$. We know from Eq.(A4) that $t=r_2/r_1$ and $x=\cos\gamma$. Therefore, the series expansion in Eq.(A4) will still converge for $r_1=r_2=R$ except for the value of $\cos(\gamma)=\pm 1$ that implies $|x|=|\cos\gamma|=1$. In other words, for the special case of $r_1=r_2=R$, one can write the quantity in Eq.(A4) as:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{R}\,g(t=1,x=\cos\gamma) \;\;;\;\; \cos\gamma\neq\pm 1\;. \quad \text{(A5)}$$

The above expression is equivalent to:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{R}\sum_{l=0}^{\infty} P_l(\cos\gamma) \;\;;\;\; r_1=r_2=R \;\;;\;\; \gamma\neq 0,\pi\;. \quad \text{(A6)}$$

In a nutshell, the expansion in Eq.(A6) is valid for all angles, γ between two vectors with identical lengths except the special angles, $\gamma=0$ and $\gamma=\pi$ where the infinite sum over Legendre polynomials is not convergent.

The first angle, $\gamma=0$ is easy to understand, because it represents a situation in which two vectors with same length are parallel to each other. Hence, it is obvious that $1/|\vec{r}_1-\vec{r}_2|$ should diverge when $\vec{r}_1=\vec{r}_2$. Note that, for $\gamma=0$, one has: $\sum_{l=0}^{\infty}P_l(1)=1+1+1+1+\ldots\to\infty$ where one may still argue that the formula in Eq.(A6) is valid (in the sense that we have $\infty=\infty$).

The second angle $\gamma=\pi$ means that the two vectors with same length are oriented anti-parallel to each other, $\vec{r}_1=-\vec{r}_2$. This is a very peculiar situation where the expression in Eq.(A6) would give:

$$\frac{1}{2\,R} = \frac{1}{R}\sum_{l=0}^{\infty} P_l(-1) = 1-1+1-1+\ldots\;, \quad \text{(A7)}$$

where the infinite sum would not converge in the usual sense.

To sum it up, only $\gamma=\pi$ is, indeed, a peculiar case if we express $1/|\vec{r}_1-\vec{r}_2|$ as in Eq.(A6). However, note that the Coulomb potential factor, $1/|\vec{r}_1-\vec{r}_2|$ for the case of $r_1=r_2=R$ is under the sign of integrals, $\iint_{S_1} dS_1$ (and $\iint_{S_2} dS_2$) in all expressions. Therefore, it is common knowledge in calculus when calculating integrals that exclusion of few "problematic" points (more precisely, replacing $1/|\vec{r}_1-\vec{r}_2|$ by the quantity involving the infinite sum in the right-hand-side of Eq.(A6) introduces only one "problematic" point, $\gamma=\pi$) when carrying out the integration (that, in principle, involves an infinite sum of points) should not be "worrisome".

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