New solution method for the problem of a uniformly charged straight wire

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A common problem in electrostatics is determining the electrostatic potential due to a uniformly charged straight wire. The solution of this problem illustrates well the types of calculations that one must perform in order to obtain the electrostatic potential or field of a given continuous charge distribution. In this work, we reconsider and solve the problem of a uniformly charged straight wire via a new method that is different from the popular direct integration approach found in the majority of physics textbooks. The outcomes of the two methods are compared and the results suggest several interesting mathematical formulas involving special functions.

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I. INTRODUCTION

The study of electrostatics starts with the properties of charged point particles and corresponding discrete systems made up of them. Very often a large system of point charges is so closely-packed that the average separation distance between them is much smaller than the distance from the system to some point of interest in space. In such a situation, the system of charges can be modeled as a continuous charge distribution. The definition of the electrostatic potential and field can be easily generalized if a charge distribution is continuous rather than discrete. For instance, to evaluate the electrostatic potential created by a continuous charge distribution, we simply divide the charge distribution into infinitesimal small elements and treat each piece as a point charge. We then use the laws of electrostatics to calculate the electrostatic potential due to these elements at some point in space. Finally, we evaluate the total electrostatic potential due to the charge distribution at that point by summing the contributions of all the elementary charges.

It is well known that there are three types of continuous charge distribution systems: (i) Linear charge distribution; (ii) Surface charge distribution; and (iii) Volume charge distribution. With very few exceptions that apply to regular bodies with very high symmetry (such as a conducting spherical shell), the equilibrium charge distribution (that makes the body an equipotential) is not uniform. However, finding the precise way how a total charge is distributed over the length, area or volume of an arbitrarily-shaped body is one of the most difficult problems in potential theory. It suffices to say that even the simple-looking problem of finding the exact equilibrium charge distribution on a finite straight conducting wire does not have an entirely clear answer[1–5]

To avoid these difficulties, one usually resorts to the assumption of a uniformly distributed charge. Books of electrostatics generally contain several examples, in which it is assumed that the charge is uniformly distributed on a line, on a surface, or throughout a volume. One of the most common examples found in the literature is that of calculating the electrostatic potential due to a uniformly charged straight wire (also known as the problem of a uniform line of charge, a uniformly charged rod, and similar names). The standard solution method for this problem is direct integration. The approach leads to the desired result in straightforward way. We believe that this is the reason why this approach is adopted by the absolute majority of the textbooks in circulation[6–11].

The purpose of this work is to present a solution of this problem by a new method that is different from the commonly used direct integration approach. The idea to explore other possibilities and solutions is rooted in the belief that such an approach might lead to interesting new mathematical and physical insights. Comparison of the solutions of the same problem obtained via two different methods may allow one to "uncover" novel (uncommon) identities of interest. Furthermore, this approach may allow one to expand the mathematical treatment of the problem (for instance, by including special functions or integral formulas that are seldomly seen in standard textbooks). This is precisely the situation that we encounter when we solve this problem via the method reported in this work. The treatment leads us to a class of special functions known as error functions and certain integral formulas involving error functions that are not encountered when the problem is solved via direct integration. In addition, we also report a quite interesing integral formula involving the difference of two divergent integrals that, under certain conditions, is non-divergent and can be expressed in terms of a natural logarithmic function.

The paper is organized as follows: In Section II we present a quick derivation of known results obtained by using the direction integration method. This provides a fast track introduction for the reader to understand the notation and various formulas that appear in the literature. In Section III we explain our new solution method of the problem. In Section IV we compare various results obtained from the two methods and highlight some interesting mathematical identities that result from this process. In Section V we provide some concluding remarks and summarize the findings.

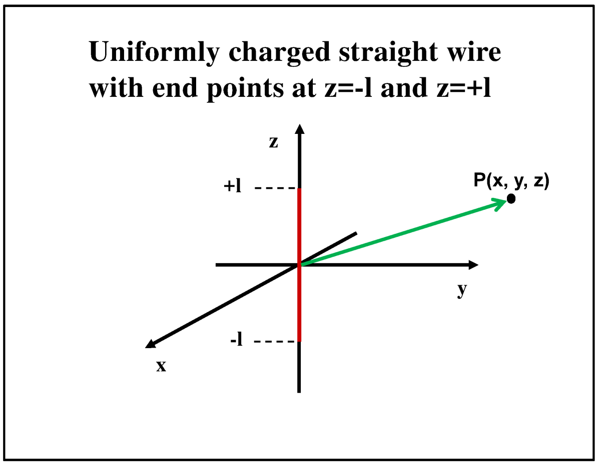

FIG. 1: A uniformly charged straight wire lies along the z axis. The end points of the wire are located at $z = -l$ and $z = +l$ ($l \geq 0$). One searches for the electrostatic potential at an arbitrary point P with coordinates (x, y, z).

II. DIRECT INTEGRATION

The coordinate system is set up in such a way that a uniformly charged straight wire, with linear charge density (charge per unit length) λ, lies on the z axis and extends from $z = -l$ to $z = +l$ where $l \geq 0$ is a non-negative parameter. For this specific setup, one can write:

$$\lambda = \frac{Q}{L} \ , \tag{1}$$

where Q is the total charge and $L = 2\,l$ is the length of the wire. A schematic drawing of the straight wire is shown in Fig. 1. At the point $P(x, y, z)$ the electrostatic potential may be written as:

$$V(\rho, z) = k_e\,\lambda \int_{-l}^{+l} \frac{dz'}{\sqrt{\rho^2 + (z - z')^2}} \ , \tag{2}$$

where k_e is Coulomb's electric constant, $\lambda\,dz'$ is the element of charge on the wire at z' (note that $-l \leq z' \leq +l$) and $\rho^2 = x^2 + y^2 \geq 0$. The specific form of the potential $V(\rho, z)$ can be obtained by direct integration[12]. For the sake of completeness and to help the reader follow the discussions with ease we provide some key derivation steps of this problem in Appendix A. Below, we report two resulting expressions denoted as $V_a(\rho, z)$ and $V_b(\rho, z)$ which can be obtained in a straightforward fashion via direct integration:

$$V_a(\rho, z) = k_e\,\lambda\, \ln\left[\frac{\sqrt{\rho^2 + (z+l)^2} + (z+l)}{\sqrt{\rho^2 + (z-l)^2} + (z-l)}\right] \ , \tag{3}$$

and

$$V_b(\rho, z) = k_e\,\lambda\, \ln\left[\frac{\sqrt{\rho^2 + (z-l)^2} - (z-l)}{\sqrt{\rho^2 + (z+l)^2} - (z+l)}\right] \ . \tag{4}$$

Details of the calculations are readily available in the literature and also include calculations for the more general setup[13] where the end points of the wire are located at arbitrary points z_1 and z_2 ($z_2 \geq z_1$). The expressions $V_a(\rho, z)$ or $V_b(\rho, z)$ for the electrostatic potential show up in one or another form in various textbooks. For instance, the potential is written in the $V_a(\rho, z)$ form in Eq.(3.73), pg. 71 of the textbook in Ref.[14]. On the other hand, a different textbook uses the $V_b(\rho, z)$ form of the potential which appears as Eq.(3.30) in pg. 66 of Ref.[15].

Based on the above-mentioned expressions, one can easily verify that, as expected:

$$V_a(\rho, z) = V_b(\rho, z) \ . \tag{5}$$

Similarly, it can be rigorously proven starting from Eq.(2) that:

$$V(\rho, z) = V(\rho, -z) \ . \tag{6}$$

The fact that the electrostatic potential $V(\rho, z)$ is an even function of z is self-evident from the geometric symmetry of the problem. It is also a nice little exercise, to verify that one may start, let's say from the expression of $V_a(\rho, z)$ in Eq.(3), and obtain:

$$V_a(\rho, -z) = V_b(\rho, z) = V_a(\rho, z) \ . \tag{7}$$

The last step in Eq.(7) is self-evident from Eq.(5). The result in Eq.(7) suggests that one can rewrite the expression for $V_a(\rho, z)$ (if one chooses that) in the following way:

$$V_a(\rho, z) = k_e\,\lambda\, \ln\left[\frac{\sqrt{\rho^2 + (|z|+l)^2} + (|z|+l)}{\sqrt{\rho^2 + (|z|-l)^2} + (|z|-l)}\right] \ . \tag{8}$$

The expression in Eq.(8) is very convenient because it incorporates in a very simple way the inherent $-z$ to $+z$ even parity of the potential and allows one to deal only with the non-negative quantity $|z| \geq 0$. Two special results can be immediately derived from the formula in Eq.(8) and regard the value of the electrostatic potential along the z axis (when $\rho = 0$):

$$V_a(\rho = 0, |z| > l) = k_e\,\lambda\, \ln\left(\frac{|z| + l}{|z| - l}\right) \tag{9}$$

and

$$V_a(\rho = 0, |z| \leq l) = \infty \ . \tag{10}$$

The value of the electrostatic potential on the $z = 0$ plane can also easily be calculated.

III. NEW METHOD

When applying this new solution method to this problem, we first write the expression for the electrostatic potential in general form as:

$$V(x,y,z) = k_e\,\lambda \int_{-l}^{+l} \frac{dz'}{|\vec{r}-\vec{r}'|}\ , \qquad (11)$$

where $\vec{r}=(x,y,z)$ is an arbitrary point in space and $\vec{r}'=(x'=0,y'=0,z')$ is the vector position of the element of charge along the the wire. The new method of calculation hinges upon the use of the following transformation for the Coulomb term $1/|\vec{r}-\vec{r}'|$ which we write as:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} du\, e^{-u^2\,(\vec{r}-\vec{r}')^2}\ . \qquad (12)$$

We subsitute the result from Eq.(12) into Eq.(11) and change the order of integration. After some straightforward algebra the final result reads:

$$V(\rho,z) = k_e\,\lambda\,\frac{2}{\sqrt{\pi}} \int_0^{\infty} du\, e^{-u^2\,\rho^2}\, f(u,z,l)\ , \qquad (13)$$

where $\rho^2 = x^2+y^2 \geq 0$ and $f(u,z,l)$ represents the following auxiliary function:

$$f(u,z,l) = \int_{-l}^{+l} dz'\, e^{-u^2\,(z-z')^2}\ . \qquad (14)$$

Note that we denote the electrostatic potential as $V(\rho,z)$ in Eq.(13) in order to highlight the cylindrical symmetry of the problem. Earlier, we denoted such a potential more generally as $V(x,y,z)$ in Eq.(11). The explicit calculation of $f(u,z,l)$ is done in Appendix B. The expression that we use is the one from Eq.(B7) and is given below:

$$f(u,z,l) = \frac{\sqrt{\pi}}{2\,u}\Big\{erf\Big[u\,\Big(|z|+l\Big)\Big] - erf\Big[u\,\Big(|z|-l\Big)\Big]\Big\}\ . \qquad (15)$$

By subsituting $f(u,z,l)$ from Eq.(15) to Eq.(13) one has:

$$V(\rho,z) = k_e\,\lambda\,\left\{\int_0^{\infty} \frac{du}{u}\, e^{-u^2\,\rho^2}\, erf\Big[u\,\Big(|z|+l\Big)\Big] - \int_0^{\infty} \frac{du}{u}\, e^{-u^2\,\rho^2}\, erf\Big[u\,\Big(|z|-l\Big)\Big]\right\}\ . \qquad (16)$$

The integrals in Eq.(16) can be carried out by using the following integral formula:

$$\int_0^{\infty} \frac{dx}{x}\, e^{-a^2\,x^2}\, erf(c\,x) = \sinh^{-1}\left(\frac{c}{|a|}\right)\ , \qquad (17)$$

where it is assumed that a and c are real parameters. In the above expression, $\sinh^{-1}(x)$ is an inverse hyperbolic sine function:

$$\sinh^{-1}(x) = \ln\Big(x+\sqrt{x^2+1}\Big)\ . \qquad (18)$$

The formula in Eq.(17) is a "tweaked" version of formula 6.293, page 650 of Ref.[16] and can be found in specialized published literature, for instance, see Eq.(B6) and Eq.(D7) of Ref.[17] that applies to the $c=1$ case.

Use of the formula in Eq.(17) for the two integrals in Eq.(16) leads to the following result:

$$V(\rho,z) = k_e\,\lambda\,\left[\sinh^{-1}\left(\frac{|z|+l}{\rho}\right) - \sinh^{-1}\left(\frac{|z|-l}{\rho}\right)\right]\ . \qquad (19)$$

The final result in Eq.(19) represents the final expression for the electrostatic potential due to a uniformly charged straight wire obtained through this method. One should expect the expression in Eq.(19) to be fully equivalent to the one in Eq.(8) that was obtained via direct integration. We leave it as an exercise to the reader to show that the expression in Eq.(19) is equivalent to the one in Eq.(8). The check is straightforward. One starts from Eq.(19) and rewrites it carefully in terms of the natural logarithmic function by using the defining formula of the inverse hyperbolic sine function in Eq.(18). What is appealing to us is the fact that the equivalence of the final results obtained with two different methods may suggest some interesting mathematical formulas that otherwise are not clearly visible. Indeed, it will be shown in the following discussions that this is the case.

IV. DISCUSSIONS

Let's first discuss the form of the electrostatic potential for few special cases. Such would be the expression of the electrostatic potential along the z axis. The calculation when $V(\rho=0,z)$ should be done with care since we already know that the potential is infinite (divergent) for $|z| \leq l$. One strategy is to work directly with the expression in Eq.(19) and obtain carefully its $\rho \to 0$ limit. This can be done by going back to expressions involving natural logarithmic functions. However, as we will show below, it turns out to be a much better choice for the wealth of mathematical transformations uncovered if we start the calculation of $V(\rho = 0, z)$ from the expression in Eq.(16). By subsituting $\rho = 0$ in Eq.(16) one has:

$$V(\rho = 0, z) = k_e\,\lambda\, \left\{ \int_0^\infty \frac{du}{u}\, erf\left[u\,\left(|z| + l\right)\right] - \int_0^\infty \frac{du}{u}\, erf\left[u\,\left(|z| - l\right)\right] \right\} \, . \tag{20}$$

Each of the two integrals in Eq.(20) individually diverges, however, their difference may or may not diverge. In fact, we were able to prove in Appendix C that the following mathematical formula applies:

$$\int_0^\infty \frac{dx}{x}\, \Big[erf\,(a\,x) - erf\,(b\,x) \Big] = \begin{cases} +\ln\!\left(\frac{a}{b}\right) \; ; \; a > 0 \text{ and } b > 0 \, , \\ -\ln\!\left(\frac{|a|}{|b|}\right) \; ; \; a < 0 \text{ and } b < 0 \, , \\ +\infty \; ; \; a > 0 \text{ and } b < 0 \, , \\ -\infty \; ; \; a < 0 \text{ and } b > 0 \, . \end{cases} \tag{21}$$

We also checked the correctness of the formula in Eq.(21) with Wolfram's Mathematica software[18]. The result in Eq.(21) basically states that the difference of two diverging integrals involving parameters a and b (where a and b are assumed real) turns out to be a well-behaved function if a and b have both the same sign. The case $a > 0$ and $b > 0$ applies to our case in Eq.(20). If such a condition is violated, for example if $a > 0$ and $b < 0$, the quantity in Eq.(21) is singular ($\pm\infty$).

By observing the form of the integrals in Eq.(20) one concludes that the conditions in Eq.(21) leading to a non-divergent logarithmic result are met when $|z| > l$. Note that the integrals in Eq.(20) correspond to parameters $a = |z| + l$ and $b = |z| - l$ in Eq.(21). Having simultaneously both $a > 0$ and $b > 0$ leads to the condition $|z| > l$ being satisfied. For such a case:

$$V(\rho = 0, |z| > l) = k_e\,\lambda\, \ln\left(\frac{|z| + l}{|z| - l}\right) \, . \tag{22}$$

This result is in agreement with the previously derived expression in Eq.(9). By the same token, for $|z| < l$ the formula in Eq.(21) would lead to $V(\rho = 0, |z| \leq l) = \infty$ consistent with the result in Eq.(10).

Another compact formula that one can obtain from Eq.(19) is that for the electrostatic potential on the $z = 0$ plane that bisects the straight wire:

$$V(\rho, z = 0) = k_e\,\lambda\; 2\,\sinh^{-1}\left(\frac{l}{\rho}\right) \, . \tag{23}$$

Overall, the new solution method highlighted in this work turns out to be a very fertile ground to derive various mathematical identities and integral formulas that are not often dealt with in the mainstream literature. In this process, we encountered interesting definite integrals involving the error function in combination with other functions as well as a rather peculiar integral formula shown in Eq.(21) that we have not seen before in popular books of integrals.

V. CONCLUSIONS

Only in relatively simple cases can the expression for the electrostatic potential of a charged body be obtained in a closed analytic form. Such is the case study of a uniformly charged straight wire for which the exact expression of the electrostatic potential can be be obtained by carrying out the required integrations. A direct integration approach is the method of choice to solve this problem and appears in the majority of textbooks dealing with electrostatics and/or electromagnetism. The purpose of this work that lays out a novel solution method to this problem is not to suggest that the direction integration approach should be avoided. On the contrary, the direct integration approach should be the method of choice to solve this problem on account of its simplicity.

The key message that we would like to transmit is that we can gain a lot by solving a given problem by a different method. This approach is very beneficial because it allows one to have a fresh new look on a well known problem from a different perspective. Additionally, this approach can be very enriching from a pedagogical point of view since it may allow one to uncover novel mathematical expressions or identities that otherwise are not routinely seen. For instance, the current method that we used to solve the problem allowed us to obtain several one-dimensional integral expressions involving a special class of functions known as error functions. In the process, we also had the opportunity to deal with certain type of integrals involving products of error functions, exponential functions and power functions that generally are found only in specialized textbooks or papers. One such rather peculiar formula is the one reported in Eq.(21) which we have not seen it before in standard handbooks of integrals. In a nutshell, we believe that the approach considered in this work has clear pedagogical values.

Furthermore, the employed method can also be of interest from the perspective of addressing similar problems that arise in disciplines such as electrostatics or mathematical physics. For example, this method can be used to solve more complicated problems that go beyond a uniformly charged straight wire such as the calculation of the electrostatic potential created by a uniformly charged two-dimensional (2D) rectangular plate or by a uniformly charged three-dimensional (3D) cuboid object. Let's illustrate the application of the method by considering the most difficult case of a uniformly charged cuboid. We consider a cuboid that has a volume $L_x\, L_y\, L_z$ and contains a total amount of charge Q which is uniformly spread over its volume. This situation results in a volume charge density, $\rho_0 = Q/(L_x\, L_y\, L_z)$. Assume that the origin of a Cartesian system of coordinates is taken at the center of the cuboid and the axes are parallel to its edges. By following the same notation as for the case of the uniformly charged straight wire, we say that a point in the 3D cuboid region has coordinates such that: $-l_x \leq x' \leq +l_x$, $-l_y \leq y' \leq +l_y$, $-l_z \leq z' \leq +l_z$ where $L_x = 2\,l_x$, $L_y = 2\,l_y$ and $L_z = 2\,l_z$. The electrostatic potential created by the uniformly charged cuboid at some arbitrary point in space can be written as:

$$V(x,y,z) = k_e\,\rho_0 \int_{-l_x}^{+l_x} dx' \int_{-l_y}^{+l_y} dy' \int_{-l_z}^{+l_z} dz' \frac{1}{|\vec{r}-\vec{r}\,'|}\ , \tag{24}$$

where $\vec{r} = (x, y, z)$ is an arbitrary point in space and $\vec{r}\,' = (x\,', y\,', z\,')$ is the vector position of the element of charge within the cuboid region. It is straightforward to use the current method which relies on the transformation of $1/|\vec{r}-\vec{r}\,'|$ according to Eq.(12) and obtain the following result:

$$V(x,y,z) = k_e\,\rho_0\,\frac{2}{\sqrt{\pi}} \int_0^{\infty} du\, f(u,x,l_x)\, f(u,y,l_y)\, f(u,z,l_z)\ . \tag{25}$$

This time, there are three auxiliary functions under the sign of the integral. Although an explicit analytic calculation of the resulting integrals might be challenging, the expression obtained in Eq.(25) as a one-dimensional integral is very easy to handle from the perspective of numerical methods. To conclude, this example illustrates well the power and elegance of this method when extended to more realistic uniformly charged bodies that occupy a more complicated multi-dimensional space.

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APPENDIX A: CALCULATION OF $V(\rho,z)$ BY DIRECT INTEGRATION

The quantity to calculate is:

$$V(\rho,z) = k_e\,\lambda \int_{-l}^{+l} \frac{dz'}{\sqrt{\rho^2+(z-z')^2}}\ , \tag{A1}$$

where in cylindrical coordinates $\rho^2 = x^2+y^2 \geq 0$ and $-\infty < z < +\infty$. We use direct integration to obtain the explicit expression for $V(\rho,z)$. Two slightly different approaches denoted as (a) and (b) are followed to complete the integration.

1. Approach (a)

In approach (a) we introduce the following auxiliary variable,

$$u = z - z\,'\ . \tag{A2}$$

This allows us to rewrite the expression in Eq.(A1) as:

$$V(\rho,z) = k_e\,\lambda \int_{z-l}^{z+l} \frac{du}{\sqrt{\rho^2+u^2}}\ . \tag{A3}$$

The following integration formula applies:

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)\ . \tag{A4}$$

As a result:

$$V(\rho,z) = k_e\,\lambda\,\ln\left(u+\sqrt{u^2+\rho^2}\right)\Bigg|_{u=z-l}^{u=z+l}\ . \tag{A5}$$

We denote the final result as $V_a(\rho,z)$ where (a) implies that this expression was obtained via approach (a) and write it as:

$$V_a(\rho,z) = k_e\,\lambda\,\ln\left[\frac{\sqrt{\rho^2+(z+l)^2}+(z+l)}{\sqrt{\rho^2+(z-l)^2}+(z-l)}\right]\ . \tag{A6}$$

2. Approach (b)

In approach (b) we introduce the following auxiliary variable,

$$u = z' - z \ . \tag{A7}$$

This allows us to rewrite the expression in Eq.(A1) slightly differently as:

$$V(\rho, z) = k_e \, \lambda \int_{-l-z}^{+l-z} \frac{du}{\sqrt{\rho^2 + u^2}} \ . \tag{A8}$$

We use again the integration formula in Eq.(A4). This leads to:

$$V(\rho, z) = k_e \, \lambda \, \ln\Big(u + \sqrt{u^2 + \rho^2}\Big)\Bigg|_{u=-l-z}^{u=+l-z} \tag{A9}$$

We denote the result as $V_b(\rho, z)$ where (b) implies this expression was obtained via approach (b). After minor arrangements we write it as:

$$V_b(\rho, z) = k_e \, \lambda \, \ln\left[\frac{\sqrt{\rho^2 + (z-l)^2} - (z-l)}{\sqrt{\rho^2 + (z+l)^2} - (z+l)}\right] \ . \tag{A10}$$

APPENDIX B: FUNCTION $f(u, z, l)$

The following function is defined by the integral:

$$f(u, z, l) = \int_{-l}^{+l} dz' \, e^{-u^2 \, (z-z')^2} \ , \tag{B1}$$

where all quantities are real. We introduce the new dummy variable $v = z - z'$. This change of variables allows us to write the expression in Eq.(B1) as:

$$f(u, z, l) = \int_{z-l}^{z+l} dv \, e^{-u^2 \, v^2} \ . \tag{B2}$$

The following integral formula is used:

$$\int dx \, e^{-a^2 \, x^2} = \frac{\sqrt{\pi}}{2\, a} \, erf\,(a\, x) \ , \tag{B3}$$

where $erf(x) = 2/\sqrt{\pi} \int_0^x dt \, e^{-t^2}$ is an error function and a is a real constant. Straightforward algebra leads to:

$$f(u, z, l) = \frac{\sqrt{\pi}}{2\, u}\Big\{erf\Big[u\,\Big(z+l\Big)\Big] - erf\Big[u\,\Big(z-l\Big)\Big]\Big\} \ . \tag{B4}$$

One can easily conclude that $f(u, z, l)$ is an even function of z by looking at Eq.(B1) as well as Eq.(B4):

$$f(u, z, l) = f(u, -z, l) \ . \tag{B5}$$

Recall that the error function is an odd function:

$$erf(x) = -erf(-x) \ . \tag{B6}$$

This means that, without any loss of generality, one can rewrite the function $f(u, z, l)$ in Eq.(B4) as:

$$f(u, z, l) = \frac{\sqrt{\pi}}{2\, u}\Big\{erf\Big[u\,\Big(|z| + l\Big)\Big] - erf\Big[u\,\Big(|z| - l\Big)\Big]\Big\} \ . \tag{B7}$$

Writing the function in terms of the absolute value of $|z|$ is quite convenient from the perspective of not having to deal with negative values of z separately.

APPENDIX C: INTEGRAL $I(a, b) = \int_0^\infty \frac{dx}{x}\Big[erf(a\,x) - erf(b\,x)\Big]$

We want to calculate the following integral:

$$I(a, b) = \int_0^\infty \frac{dx}{x}\Big[erf(a\,x) - erf(b\,x)\Big] \ , \tag{C1}$$

where a and b are assumed real. We know that integrals of the form, $\int_0^\infty \frac{dx}{x} erf(c\,x)$ diverge (c assumed real). Obviously, the sign of c determines whether the integral goes to $+\infty$ or $-\infty$ given that $erf(x)$ is an odd function of x. This means that, intuitively speaking, $I(a, b)$ may have a chance of being finite (different from $\pm\infty$) when: (i) both $a > 0$ and $b > 0$ or (ii) both $a < 0$ and $b < 0$. To facilitate the precise calculation of $I(a, b)$ in Eq.(C1), let us introduce a convergence factor, $\epsilon > 0$ and write:

$$I(a, b) = \lim_{\epsilon \to 0} \int_0^\infty \frac{dx}{x} e^{-\epsilon^2 \, x^2} \left[erf(a\,x) - erf(b\,x)\right] \ . \tag{C2}$$

The convergence factor make the integrals non-singular and each of them can be calculated from the formula below:

$$\int_0^\infty \frac{dx}{x} \, e^{-\epsilon^2 \, x^2} \, erf(c\,x) = \sinh^{-1}\left(\frac{c}{|\epsilon|}\right) \ , \tag{C3}$$

which applies for any parameter ϵ and c assumed real. Note that the formula in Eq.(C3) is an iteration of the formula in Eq.(17) with a slightly different notation. Application of this formula leads immediately to:

$$I(a, b) = \lim_{\epsilon \to 0}\Big[\sinh^{-1}\Big(\frac{a}{\epsilon}\Big) - \sinh^{-1}\Big(\frac{b}{\epsilon}\Big)\Big] \ , \tag{C4}$$

where $\epsilon = |\epsilon| > 0$ in our case. At this juncture, we revert back to expressions in terms of natural logarithmic functions by recalling that $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$. After some straightforward algebra, we have:

$$I(a, b) = \lim_{\epsilon \to 0} \ln\left(\frac{a + \sqrt{a^2 + \epsilon^2}}{b + \sqrt{b^2 + \epsilon^2}}\right) \ . \tag{C5}$$

The limit in Eq.(C5) should be calculated with care. The conclusion is obvious when $a > 0$ and $b > 0$ where one obtains:

$$I(a, b) = \ln\left(\frac{a}{b}\right) \quad ; \quad a > 0 \quad ; \quad b > 0 \ . \tag{C6}$$

Likewise, it is not difficult to see that $I(a,b)$ diverges to $\pm\infty$ when a and b have opposite signs:

$$I(a,b) = \ln\left(\frac{a+|a|}{b+|b|}\right) = \begin{cases} +\infty \ ; \ a>0 \text{ and } b<0 \\ -\infty \ ; \ a<0 \text{ and } b>0 \ . \end{cases} \quad \text{(C7)}$$

However, one still should be careful when calculating the limit in Eq.(C5) when both $a<0$ and $b<0$. In our view, the simplest way to understand this case is to rewrite the expression in Eq.(C1) for $a<0$ and $b<0$ as:

$$I(a<0,b<0) = -\int_0^{\infty} \frac{dx}{x}\Big[erf(|a|\,x) - erf(|b|\,x)\Big] \ , \quad \text{(C8)}$$

where we wrote $a=-|a|<0$, $b=-|b|<0$ and used the fact that $erf(x)$ is an odd function of x. With help from the formula in Eq.(C6) that applies to the quantity in Eq.(C1) when the arguments of the error functions are both positive, one has:

$$I(a<0,b<0) = -\ln\left(\frac{|a|}{|b|}\right) \quad ; \quad a=-|a|<0 \quad ; \quad b=-|b|<0 \ . \quad \text{(C9)}$$

[1] J. Batle, O. Ciftja, S. Abdalla, M. Elhoseny, M. Alkhambashi, and A. Farouk, Eur. J. Phys. **38**, 055202 (2017).

[2] J. D. Jackson, Am. J. Phys. **70**, 409 (2002).

[3] J. D. Jackson, Am. J. Phys. **68**, 789 (2000).

[4] R. H. Good, Am. J. Phys. **65**, 155 (1997).

[5] D. J. Griffiths and Y. Li, Am. J. Phys. **64**, 706 (1996).

[6] D. C. Giancoli, Physics for Scientists and Engineers, Fourth Edition, Prentice Hall, Upper Saddle River, New Jersey (2008).

[7] H. D. Young and R. A. Freeman, Sears and Zemansky's University Physics with Modern Physics, Fourteenth Edition, Pearson, New York City, New York (2016).

[8] R. A. Serway and J. W. Jewett, Jr, Physics for Scientists and Engineers with Modern Physics, Sixth Edition, Brooks/Cole-Thomson Learning, Belmont, California (2004).

[9] R. H. Good, Classical Electromagnetism, Saunders College Publishing, Orlando, Florida (1999).

[10] D. J. Griffiths, Introduction to Electrodynamics, Third Edition, Prentice Hall, Upper Saddle River, New Jersey (1999).

[11] W. M. Saslow, Electricity, Magnetism and Light, Academic Press, Cambridge, Massachusetts (2002).

[12] G. L. Pollack and D. R. Stump, Am. J. Phys. **75**, 485 (2007).

[13] O. Ciftja, K. Beharie, and T. Beharie, Eur. J. Phys. **40**, 015501 (2019).

[14] G. L. Pollack and D. R. Stump, Electromagnetism, Addison Wesley, San Francisco, CA (2002).

[15] A. Zangwill, Modern Electrodynamics, Cambridge University Press, New York, NY (2012).

[16] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Fourth Edition, Academic Press, New York, NY (1965).

[17] O. Ciftja, J. Nanosci. Nanotechnol. **16**, 9964 (2016).

[18] Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, IL (2012).