Optimization Letters (2021) 15:2127–2145
https://doi.org/10.1007/s11590-021-01722-3

ORIGINAL PAPER

Stability analysis of conically perturbed linearly constrained least-squares problems by optimizing the regularized trajectories

A. Khan[1] · M. Sama[2]

Received: 6 May 2020 / Accepted: 26 February 2021 / Published online: 18 March 2021

Abstract

This paper studies linearly constrained least-square optimization problems in Hilbert spaces for which the KKT system is not necessarily available to analyze and compute the solution. The primary objective is to develop new qualitative and quantitative stability estimates for the regularization error in the conical regularization approach. To attain this goal, we associate the notion of stability with the solvability of some scalar and vector optimization problems defined in terms of the regularized trajectory on the domain space and the regularized state trajectory on the constraint space. We analyze three optimization formulations. The first formulation minimizes a scalar objective function over the regularized trajectory. The second formulation consists of vector optimizing the regularized trajectory on the domain space for a specific Bishop–Phelps cone. The third formulation results in vector optimizing the regularized state trajectory for the constraint cone. We provide numerical examples to illustrate the efficacy of the developed framework.

Keywords Constrained quadratic optimization · Dilating cones · Set-valued analysis · Vector optimization · Stability · Parametric optimization theory

Mathematics Subject Classification 90C20 · 90C29 · 90C31

✉ M. Sama
msama@ind.uned.es

A. Khan
aaksma@rit.edu

1 Center for Applied and Computational Mathematics, School of Mathematical Sciences, Rochester Institute of Technology, Rochester, NY 14623, USA

2 Departamento de Matemática Aplicada, E.T.S.I.I. Universidad Nacional de Educación a Distancia, Calle Juan del Rosal 12, 28040 Madrid, Spain

1 Introduction

Let U, H, and Y be separable Hilbert spaces. Let $C \subset Y$ be a pointed, closed, and convex cone, inducing a partial ordering $\leq_C$ on Y, that is, $y_1 \leq_C y_2$, if and only if, $y_2 - y_1 \in C$. We denote by Y^* the dual space of Y and by

$$C^* = \left\{\mu^* \in Y^* : \mu^*(y) \geq 0, \text{ for every } y \in C\right\},$$

we denote the positive dual cone of C. In this paper, we study the following convex optimization problem:

$$\text{minimize } J(u) = \frac{1}{2}\|Su - z_d\|_H^2 + \frac{\kappa}{2}\|u - u_d\|_U^2, \quad \text{subject to } Gu \leq_C w, \quad u \in U, \tag{P}$$

where $S : U \to H$ and $G : U \to Y$ are bounded linear operators, $\kappa > 0$ is a given parameter, $u_d \in U$, $z_d \in H$ and $w \in Y$ are given elements. The optimization problem (P), which is uniquely solvable with unique solution denoted by u_0, encompasses a wide variety of problems in applied mathematics. The main obstacle in the adequate treatment of infinite-dimensional problems with linear constraints like (P) is the existence of the dual solutions. In strict contrast to such optimization problems posed in a finite-dimensional setting, in infinite-dimensional optimization problems, the existence of the dual solutions requires quite restrictive conditions. This critical situation has been addressed in various theoretical and applied models, for example, in PDE constrained optimization or in Mathematical Economics models, see [1,2] for more details.

In this work, our interest is on the conical regularization methods for problem (P) which construct a family of optimization problem by replacing the cone C by an approximating family of cones. To be precise, for $\varepsilon \in (0, 1)$, we denote by $\{C_\varepsilon\} \subset Y$, a family of dilating cones associated with C. That is, $\{C_\varepsilon\}$ is a family of closed, convex, and pointed cones with nonempty interior such that, firstly, $C\backslash\{0\} \subset \operatorname{int}(C_\varepsilon)$, for every $\varepsilon \in (0, 1)$, and secondly, $C = \bigcap_{\varepsilon > 0} C_\varepsilon$. The regularized family (P_ε) is then obtained by replacing C in (P) by C_ε as follows:

$$\text{minimize } J(u) = \frac{1}{2}\|Su - z_d\|_H^2 + \frac{\kappa}{2}\|u - u_d\|_U^2, \quad \text{subject to } Gu \leq_{C_\varepsilon} w, \quad u \in U. \tag{P_ε}$$

It was shown in [3] that the regularized solutions u_ε of (P_ε) converge in norm to u_0 under an additional mild condition, see $(A1)$. Moreover, u_ε can be computed using the KKT system: There is a multiplier $\mu_\varepsilon^* \in C_\varepsilon^*$ such that

$$DJ(u_\varepsilon) + \mu_\varepsilon^* \circ G = 0, \tag{1a}$$

$$\mu_\varepsilon^*(Gu_\varepsilon - w) = 0, \tag{1b}$$

$$Gu_\varepsilon - w \leq_{C_\varepsilon} 0, \tag{1c}$$

where $\circ$ denotes the usual composition operator.

In this work, we focus on the case when C_ε is a family of Henig dilating cones, see [4,5]. An essential aspect of the study of the regularization methods is to establish qualitative and quantitative estimates for the regularization error $\|u - u_\varepsilon\|_U$. In this context, in [6] (see also [7]), the following condition was considered: There exists a constant $c > 0$ such that

$$-DJ(u_0)(d) \geq c, \text{ for every } d \in \mathbb{T}, \tag{2}$$

where the set $\mathbb{T}$ is defined by

$$\mathbb{T} := \left\{ d \in U : \exists\, \varepsilon_n \downarrow 0 \text{ such that } \frac{u_{\varepsilon_n} - u_0}{\left\| u_{\varepsilon_n} - u_0 \right\|_X} \rightharpoonup d \text{ as } n \to \infty \right\}.$$

Condition (2) guarantees an improved a priori convergence rate, see [6, Proposition 3.2], independently of the regularity of the problem, and hence it is of evident interest to investigate conditions under which it holds. For instance, if (P) is regular, then (2) is equivalent to the stability at $\varepsilon = 0$ of the regularized optimal trajectory $\Phi : [0, 1) \to U$, defined by $\Phi(\varepsilon) := u_\varepsilon$, that is, $\|\Phi(\varepsilon) - \Phi(0)\|_U = O(\varepsilon)$. This is not true in general. In Example 4.2 below, we show that (2) holds, albeit (P) is not regular. On the other hand, the converse is true, that is, if Φ is stable, then (P) is regular, and hence (2) necessarily holds.

In this paper, we associate (2) with the solvability of one scalar optimization problem and two vector optimization problems, all related to the regularized trajectory. To be precise, for the scalar optimization problem, we minimize $-J$ over (the image of) the regularized optimal trajectory $\Gamma := \Phi([0, 1))$:

$$\text{minimize} - J(u), \text{ subject to } u \in \Gamma. \tag{Q}$$

In Theorem 3.1, we prove that (2) is equivalent to the strict local minimality of u_0. Then, we pose a vector optimization problem to minimize the optimal regularized trajectory with respect to an ordering cone in the space U. More specifically, for $\delta \geq 0$ sufficiently small, we consider the following vector optimization problem

$$D_\delta\text{-minimize } \Phi(x), \quad x \in [0, 1), \tag{$P(\Phi, D_\delta)$}$$

where $D_\delta := \left\{ u \in U : -DJ(u_\delta)(u) \geq c\, \|u\|_U \right\}$.

Under condition (2), D_δ is a Bishop–Phelps cone and problem $(P(\Phi, D_\delta))$ is well-defined and solvable. Moreover, in Theorem 3.3, we prove that condition (2) implies that $\varepsilon = 0$ is a proper minimizer of problem $(P(\Phi, D_\delta))$, and a strongly minimizer, if Φ is pseudoconvex. We then consider another vector optimization problem for the optimal regularized state trajectory $G \circ \Phi$ with respect to the constraint cone C:

$$C\text{-minimize } (G \circ \Phi)(x), \quad x \in [0, 1). \qquad (P(G \circ \Phi, C))$$

In Theorem 3.6, we show that the stability of Φ implies that 0 is a strict local minimizer of $(P(G \circ \Phi, C))$. We also give illustrative examples which support the provided stability results.

Notation and preliminaries

We now briefly describe the notations that will be used in this paper, recalling some notions in optimization and set-valued analysis, see [8]. Given a Hilbert space X, we denote its inner product and norm by $\langle \cdot, \cdot \rangle_X$ and $\|\cdot\|_X$, respectively. We designate the strong and the weak convergence in any space by $\to$ and $\rightharpoonup$. For a sequence $\{(x_n, y_n)\}$ and a element $\{(x_0, y_0)\}$ in $X \times Y$, by $\{(x_n, y_n)\} \underset{s,w}{\to} (x_0, y_0)$, we mean that $x_n \to x_0$ and $y_n \rightharpoonup y_0$. The quasi-interior of the dual cone C^* is defined by $C^\natural = \{\lambda^* \in Y^* : \lambda^*(c) > 0 \text{ for every } c \in C \backslash \{0\}\}$. Given $\lambda^* \in X^*$ with $\|\lambda^*\|_{X^*} = 1$, and $c \in (0, 1)$, by $C(\lambda^*, c) \subset X$ we denote the Bishop–Phelps cone

$$C(\lambda^*, c) = \{v \in X : \lambda^*(v) \geq c \|v\|_X\},$$

which is a closed, convex, and pointed cone (see [9,10]). We note that the notion of Henig dilating cones that we will recall shortly, was introduced by Henig [4,5] in a finite-dimensional setting, and later on extended to general spaces by Borwein and Zhuang [11]. As Y is separable, the cone C has a closed convex base $\Theta \subset Y$ such that

$$C = \bigcup_{\lambda \geq 0} \{\lambda\theta : \theta \in \Theta\} \text{ and } 0 \notin \Theta.$$

Without any loss of generality (see [8, Theorem 2.2.7]), we assume that the base Θ is given by a strictly positive functional $\beta^* \in C^\natural$, that is, $\Theta = \{y \in C : \beta^*(y) = 1\}$, where we normalize to $\|\beta^*\|_{Y^*} = 1$. Given $\varepsilon > 0$, the Henig dilating cone C_ε is then given by

$$C_\varepsilon = \text{cl}\,[\text{cone}\,(\Theta + \varepsilon B_Y)]\,, \tag{3}$$

where $B_Y = \{y \in Y : \|y\|_Y \leq 1\}$ denotes the closed unit ball in Y. It is known that the Henig cone is a solid, pointed, closed, and convex cone such that $C = \bigcap_{\varepsilon > 0} C_\varepsilon$ (see [11, Theorem 1.1]).

Given $S \subset X$ and $\bar{s} \in S$, the contingent cone $T(S, \bar{x})$ of S at $\bar{x}$ is the set of all $v \in X$ such that there are $\{t_n\} \subset \mathbb{R}_+$ and $\{s_n\} \subset S$ such that $s_n \to \bar{x}$ and $t_n(s_n - \bar{s}) \to v$. If we replace the convergence by the weak convergence, we obtain the notion of the weak contingent cone denoted by $T^w(S, \bar{x})$. For $A \subset X \times Y$ and $(x_0, y_0) \in A$, the quasi-weak contingent cone of A at (x_0, y_0), denoted by $T^{sw}(A, (x_0, y_0))$, is defined as the set of all $(v, w) \in X \times Y$ such that there exist $\{t_n\} \subset \mathbb{R}_+$ and $\{(x_n, y_n)\} \subset A$ such that $(x_n, y_n) \underset{s,w}{\to} (x_0, y_0)$ and $t_n(x_n - x_0, y_n - y_0) \underset{s,w}{\to} (v, w)$.

For a set-valued map $F : X \rightrightarrows Y$, we denote by $\text{dom}(F) := \{x \in X : F(x) \neq \emptyset\}$ its effective domain and by $\text{graph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$, $\text{epi}(F) := \{(x, y) \in X \times Y : y \in F(x) + C\}$, $\text{Im}(F) := \{y \in Y : y \in F(x)\}$ its graph, epigraph and image respectively. The contingent derivative of F at $(x_0, y_0) \in \text{graph}(F)$ (see [12]) is the set-valued map $D_c F(x_0, y_0) : X \rightrightarrows Y$ such that $\text{graph}(D_c F(x_0, y_0)) = T(\text{graph}(F), (x_0, y_0))$, while the τ^w-contingent derivative of F at $(x_0, y_0) \in \text{graph}(F)$ (see [13]) is the set-valued map $D_c^w F(x_0, y_0) : X \rightrightarrows Y$ such that

$$\text{graph}(D_c^w F(x_0, y_0)) = T^{sw}(\text{graph}(F), (x_0, y_0)).$$

When Y is finite-dimensional, both derivatives coincide. For a scalar map $F : X \rightrightarrows \mathbb{R}$ by $D_\uparrow F(x_0, y_0)$, we denote the contingent epiderivative by (see [14])

$$\begin{aligned} D_\uparrow F(x_0, y_0)(x) &= \min\{\mu : \mu \in D_c(F + \mathbb{R}_+)(x_0, y_0)(x)\} \\ &\quad \text{for all } x \in \text{dom}(D_c(F + \mathbb{R}_+)(x_0, y_0)). \end{aligned}$$

A set-valued $F : X \rightrightarrows Y$ map is said to be τ^w -pseudoconvex at $(x, y) \in \text{graph}(F)$ if $F(x) \subset y + D_c^w F(x, y)(z - x)$ for every $z \in \text{dom}(D_c^w F(x, y))$. This condition is a natural generalization of convexity and it holds in particular when F is pseudoconvex in the usual sense given in [15] or when $T^{sw}(\text{graph}(F), (x, y))$ is a convex set by following standard arguments (see for example [14,16]).

We review some vector optimization notions, see [10]. Given a map $f : S \subset X \to Y$, we denote a generic vector optimization problem by

$$C\text{-minimize } f(x), \quad \text{subject to } x \in S. \tag{$P(f, C)$}$$

A point $\bar{x} \in S$ is called a local Pareto minimizer of $(P(f, C))$, if there exists a neighborhood A of $\bar{x}$ such that

$$(f(\bar{x}) - C) \bigcap f(S \cap A) = f(\bar{x}),$$

where $f(S \cap A) := \bigcup_{x \in S \cap A} f(x)$. Analogously, a point $\bar{x} \in S$ is called a local strongly minimizer of $(P(f, C))$, if

$$f(S \cap A) \subset f(\bar{x}) + C.$$

When $A = X$, we say simply that $\bar{x}$ is a Pareto or a strongly minimizer of $(P(f, C))$. Furthermore, if $\bar{x}$ is a local Pareto minimizer of $(P(f, C))$ and

$$T(f(S \cap A) + C, \bar{x}) \bigcap -C = \{0\},$$

then $\bar{x}$ is a local proper minimizer of $(P(f, C))$, see [17].

We will also consider the following notion of strict local minimizer introduced in [18]. A point $\bar{x} \in S$ is said to be a strict local minimizer (of order 1) of $(P(f, C))$, if there exists a constant $\alpha > 0$ and a neighborhood U of $\bar{x}$ such that

$$(f(\bar{x}) + C) \bigcap B\left(f(\bar{x}), \alpha \|x - \bar{x}\|_U\right) = \emptyset, \text{ for every } x \in S \cap A \backslash \{\bar{x}\}.$$

In the scalar case, $Y = \mathbb{R}$, $C = \mathbb{R}_+$ collapses to the known notions in scalar optimization (see [19]):

$$f(x) \geq f(\bar{x}) + \alpha \|x - \bar{x}\|_U \text{ for every } x \in S \cap A \backslash \{\bar{x}\}.$$

Proper minimizers and strict minimizers are proper subsets of Pareto minimizer (see [18]), all these optimality notions coincide with the notion strongly minimizers provided that strongly minimizers exist.

Finally, following [6], throughout this paper we impose two conditions which avoid pathological cases:

(A1) $Gu_0 - w \neq 0$.
(A2) $u_\varepsilon \neq u_0$, for ε small enough.

2 Some basic results

In this section, we study some basic properties of the set $\mathbb{T}$ and condition (2) which will be used later. For this, we first recall that the image of the regularized optimal trajectory is given by

$$\Gamma = \Phi([0, 1)) = \{u_\varepsilon : \ u_\varepsilon = \Phi(\varepsilon), \text{ for some } \varepsilon \in [0, 1)\}.$$

We first give the following technical result that can also be derived from Lemma 3.1 and the proof of Theorem 5.1 in [6]. However, for the sake of completeness, we provide a direct proof.

Lemma 2.1 *Given any* $(0, 0) \neq (r, v) \in \operatorname{graph}(D_c^w \Phi(0, u_0))$*, there exists a positive scalar* $\lambda_v > 0$*, with* $\lambda_v \geq \|v\|_U$ *, and* $d_v \in \mathbb{T}$ *such that*

$$(r, v) = \lambda_v (r_v, d_v). \tag{4}$$

In particular, we have $(r_v, d_v) \in \operatorname{graph}(D_c^w \Phi(0, u_0))$*. Furthermore, if* U *is finite-dimensional, then* $\lambda_v = \|v\|_U$*,* $d_v = \|v\|_U^{-1} v$*, for every* $v \neq 0$.

Proof By the definition of $D_c^w \Phi(0, u_0)$, given $(0, 0) \neq (r, v) \in \operatorname{graph}(D_c^w \Phi(0, u_0))$, there are $\{t_n\} \subset \mathbb{R}_+$, $\left\{(\varepsilon_n, u_{\varepsilon_n})\right\} \subset \operatorname{graph}(D_c^w \Phi(0, u_0))$ such that $\left\{t_n(\varepsilon_n, u_{\varepsilon_n} - u_0)\right\} \underset{s,w}{\rightarrow} (r, v)$. Since by hypothesis (A2), $u_{\varepsilon_n} \neq u_0$, , we can define sequences $\{\lambda_n\}$, $\{r_n\}$, and

$\{d_n\}$ by $\lambda_n := \left\|t_n(u_{\varepsilon_n} - u_0)\right\|_U$, $r_n := \frac{\varepsilon_n}{\|u_{\varepsilon_n}-u_0\|_U}$, and $d_n := \dfrac{u_{\varepsilon_n} - u_0}{\left\|u_{\varepsilon_n} - u_0\right\|_U} \subset U$. Then,

$$\{\lambda_n\,(r_n, d_n)\} = \left\{t_n(\varepsilon_n, u_{\varepsilon_n} - u_0)\right\} \underset{s,w}{\to} (r, v). \tag{5}$$

Since $\left\{t_n(u_{\varepsilon_n} - u_0)\right\}$ is weakly convergent, it is bounded, and hence $\{\lambda_n\}$ is bounded. Keeping the same notation for subsequences, let $\{\lambda_n\}$ be a subsequence that converges to some $\lambda_v \geq 0$. We claim that $\lambda_v > 0$. In fact, if $\lambda_v = 0$, then $(r, v) = (0, 0)$ which contradicts the hypothesis. Furthermore, since every norm is weakly lower continuous, and $\left\{t_n(u_{\varepsilon_n} - u_0)\right\} \rightharpoonup v$, we have that $\lambda_v \geq \|v\|_U$. On the other hand, since U is reflexive, by taking subsequences, if necessary, we conclude that $d_n = \dfrac{u_{\varepsilon_n} - u_0}{\left\|u_{\varepsilon_n} - u_0\right\|_U} \rightharpoonup d_v$, for some element $d_v \in \mathbb{T}$. From (5), $\lambda_n r_n \to r$, $\lambda_n d_n \rightharpoonup \lambda_v d_v$. Applying [6, Lemma 3.1], $r_n \to r_v$, for some $r_v \geq 0$, and consequently $(r, v) = \lambda_v(r_v, d_v)$, which proves (4). If U is finite-dimensional, for $v \neq 0$, we can assume that $t_n(u_{\varepsilon_n} - u_0) \to v$, and hence $\lambda_n \to \|v\|_U$ and $d_n \to \|v\|_U^{-1}\, v$, therefore $\lambda_v = \|v\|_U$, $d_v = \|v\|_U^{-1}\, v$. The proof is complete. □

Remark 2.2 Lemma 2.1 implies that $\mathrm{Im} D_c^w \Phi(0, u_0) = \mathrm{cone}(\mathbb{T})$. Since, $T_\Gamma^w(u_0) = \mathrm{Im} D_c^w \Phi(0, u_0)$, we deduce

$$T_\Gamma^w(u_0) = \mathrm{cone}\,(\mathbb{T}) = \mathrm{Im} D_c^w \Phi(0, u_0). \tag{6}$$

If in addition U is finite dimensional, then $T_\Gamma(u_0) = T_\Gamma^w(u_0)$, and hence $\mathbb{T} = \{\, \|v\|_U^{-1}\, v : \; v \in T_\Gamma(u_0)\backslash\{0\}\}$.

Now we review and give some new useful characterizations of property (2). As we said before, (2) implies an improved a priori convergence rate for the regularization error. For this, for each $\varepsilon \in [0, 1)$, we define

$$\beta_\varepsilon = -DJ(u_0)(u_\varepsilon - u_0). \tag{7}$$

Since $u_\varepsilon \to u_0$, $\{\beta_\varepsilon\} \subset \mathbb{R}_+$ is a sequence of positive reals converging to 0. In general, we have $\|u_0 - u_\varepsilon\|_U = O\left(\sqrt{\beta_\varepsilon}\right)$. However, if (2) holds, then $\|u_0 - u_\varepsilon\| = O\left(\beta_\varepsilon\right)$, as we show next. This rate is known when the problem is quasi-regular (see [6, Proposition 3.2]) and regular, where, in this case, $O\left(\beta_\varepsilon\right) = O\left(\varepsilon\right)$ (see [6, Section 3] and [7, Section 3]).

Lemma 2.3 *The following statements are equivalent:*

(a) $\|u_\varepsilon - u_0\|_U = O(\beta_\varepsilon)$, *where* β_ε *is given in* (7).
(b) *There exists a constant* $c > 0$ *such that*

$$-DJ(u_0)\,(d) \geq c \textit{ for every } d \in \mathbb{T}.$$

(c) *The following condition holds:*

$$\mathbb{T} \cap \ker DJ(u_0) = \emptyset. \tag{8}$$

(*d*) $0 \notin \mathbb{T}$ *and the following condition holds:*

$$\left(T_\Gamma^w(u_0)\backslash\{0\}\right) \cap \ker DJ(u_0) = \emptyset. \tag{9}$$

Proof (*a*) $\Rightarrow$ (*b*). Let $d \in \mathbb{T}$ be arbitrary. By definition there exists $\varepsilon_n \to 0$ with $\frac{u_{\varepsilon_n}-u_0}{\|u_{\varepsilon_n}-u_0\|_U} \rightharpoonup d$. For β_{ε_n} as in (7), by hypothesis there is a constant $c > 0$, independent of ε_n, such that $\left\|u_{\varepsilon_n} - u_0\right\|_U \leq \frac{1}{c} DJ(u_0)(u_{\varepsilon_n} - u_0)$. Consequently, $-DJ(u_0)\left(\frac{u_{\varepsilon_n}-u_0}{\|u_{\varepsilon_n}-u_0\|_U}\right) \geq c$, and by passing to limit $n \to \infty$, we get $-DJ(u_0)(d) \geq c$.

(*b*) $\Rightarrow$ (*a*). Assume that (*a*) does not hold. Then, we can define a sequence $\varepsilon_n \to 0$ such that

$$\frac{\left\|u_{\varepsilon_n} - u_0\right\|_U}{\beta_{\varepsilon_n}} = \frac{\left\|u_{\varepsilon_n} - u_0\right\|_U}{-DJ(u_0)(u_{\varepsilon_n} - u_0)} = \frac{1}{-DJ(u_0)\left(\frac{u_{\varepsilon_n}-u_0}{\|u_{\varepsilon_n}-u_0\|_U}\right)} \to \infty,$$

which is equivalent to $\left\{DJ(u_0)\left(\frac{u_{\varepsilon_n}-u_0}{\|u_{\varepsilon_n}-u_0\|_U}\right)\right\} \to 0$. Since U is reflexive, we can assume, taking subsequences if necessary, that $\left\{\frac{u_{\varepsilon_n}-u_0}{\|u_{\varepsilon_n}-u_0\|_U}\right\} \rightharpoonup d$, for some $d \in \mathbb{T}$. Consequently $DJ(u_0)(d) = 0$ which contradicts (*b*).

(*b*) $\Rightarrow$ (*c*). This condition is straightforward.

(*c*) $\Rightarrow$ (*b*). See [7, Proposition 3.1].

(*c*) $\Rightarrow$ (*d*). First of all, as a direct consequence of (8), we have $0 \notin \mathbb{T}$. Now, take any $0 \neq v \in T_\Gamma^w(u_0)$. Since $T_\Gamma^w(u_0) = \text{cone}(\mathbb{T})$ by (6), we can find $\alpha > 0$, $d \in \mathbb{T}$ such that $v = \alpha d$. Furthermore, $d \notin \ker DJ(u_0)$ by hypothesis (8). From this, it is immediate that $v \notin \ker DJ(u_0)$, thus (8) holds.

(*d*) $\Rightarrow$ (*c*). By hypothesis $0 \notin \mathbb{T}$, and $\mathbb{T} \subseteq T_\Gamma^w(u_0)$ by (6), therefore $\mathbb{T} \subseteq T_\Gamma^w(u_0)\backslash\{0\}$ and (8) follows from (9). $\square$

A interesting fact is that the disjunction $\ker DJ(u_0) \bigcap \mathbb{T} = \emptyset$ holds true in a neighborhood of u_0.

Lemma 2.4 *The following statements are equivalent:*

(a) $\ker DJ(u_0) \bigcap \mathbb{T} = \emptyset$.

(b) *There exists small enough* $\bar{\varepsilon} > 0$ *such that*

$$\ker DJ(u_\delta) \bigcap \mathbb{T} = \emptyset, \quad \textit{for every } 0 \leq \delta \leq \bar{\varepsilon}.$$

(c) *There exists small enough* $\bar{\varepsilon} > 0$ *and a constant* $k > 0$ *such that*

$$-DJ(u_\delta)(d) \geq k, \quad \textit{for every } d \in \mathbb{T}, 0 \leq \delta \leq \bar{\varepsilon}. \tag{10}$$

Proof $(b) \Rightarrow (a)$. It is straightforward.

$(a) \Rightarrow (b)$. Assume that (b) does not hold. Then, we can take sequences $\varepsilon_n \to 0$ and $\{d_n\} \subset \mathbb{T}$ such that

$$DJ(u_{\varepsilon_n})(d_n) = 0. \tag{11}$$

Every element of $\mathbb{T}$ is norm bounded by 1, and hence by taking subsequences if necessary, we consider $d_n \rightharpoonup d$, where $d \in \mathbb{T}$ because $\mathbb{T}$ is weakly sequentially closed by definition. Since $u_{\varepsilon_n} \to u_0$, taking limits in (11), we have $DJ(u_0)(d) = 0$, and hence $d \in \ker DJ(u_0) \bigcap \mathbb{T}$, contradicting (a).

$(b) \Rightarrow (c)$. Assume that (10) does not hold. Then we can take sequences $\varepsilon_n \to 0$, and $\{d_n\} \subset \mathbb{T}$ such that

$$DJ(u_{\varepsilon_n})(d_n) \to 0. \tag{12}$$

Reasoning as before $d_n \rightharpoonup d_0$ as $n \to \infty$, for some $d_0 \in \mathbb{T}$. By continuity $DJ(u_0)(d_0) = 0$, contradicting (b).

$(c) \Rightarrow (a)$. It is a consequence of Lemma 2.3. □

Based on the previous result, in the sequel by $D_\delta \subset U$, we denote the cone

$$D_\delta = \{u \in U : -DJ(u_\delta)(u) \geq k \|u\|_U\} \text{ for each } 0 \leq \delta \leq \bar{\varepsilon}.$$

Remark 2.5 Assuming that any condition of Lemma 2.4 holds, constant $k > 0$ in (10) verifies

$$k \leq \inf_{0 \leq \delta \leq \bar{\varepsilon}, d \in \mathbb{T}} - DJ(u_\delta)(d),$$

and, consequently, we have $k \leq \|DJ(u_\delta)\|_{U^*} \|d\|_U \leq \|DJ(u_\delta)\|_{U^*}$. Without loss of generality, we assume that

$$0 < k < \|DJ(u_\delta)\|_{U^*}. \tag{13}$$

This proves that $D_\delta = C\left(\frac{DJ(u_\delta)}{\|DJ(u_\delta)\|_{U^*}}, \frac{k}{\|DJ(u_\delta)\|_{U^*}}\right)$ is indeed a Bishop–Phelps cone.

As a consequence, we have the following result.

Proposition 2.6 *If* (2) *holds, then there exists small enough* $\bar{\varepsilon} > 0$ *such that*

$$T_\Gamma^w(u_0) \subset D_\delta, \quad \textit{for every } 0 \leq \delta \leq \bar{\varepsilon}. \tag{14}$$

Proof Let $v \in T_\Gamma^w(u_0)$ be arbitrary. By Lemma 2.1, there is $\lambda_v \geq \|v\|_U$, $d_v \in \mathbb{T}$ such that $v = \lambda_v d_v$. On the other hand, by the previous two lemmas, we can assume that there exists a small enough $\bar{\varepsilon} > 0$ and $k > 0$ such that

$$-DJ(u_\delta)(d) \geq k, \quad \text{for every } d \in \mathbb{T}, \quad 0 \leq \delta \leq \bar{\varepsilon}.$$

From that $-DJ(u_\delta)(d_v) \geq k$ and by linearity, we get $-DJ(u_\delta)(v) = -DJ(u_\delta)(\lambda_v d_v) \geq \lambda_v k \geq k\,\|v\|_U$. Since $v \in T_\Gamma^w(u_0)$ is arbitrary, (14) follows by the definition of cone D_δ. $\square$

3 Scalar and vector optimization over the optimal regularized trajectory

In this section, we study three optimization problems over the optimal regularized trajectory Γ. We begin with the following scalar optimization problem:

$$\text{minimize } -J(u),\ u \in \Gamma. \tag{Q}$$

Condition (2) is equivalent the strict minimality of $-J$ over the optimal regularized trajectory Γ as we prove next:

Theorem 3.1 *Let $c > 0$. The following statements are equivalent:*

(a) *u_0 is a strict local minimizer of (Q).*
(b) *$-DJ(u_0)(d) \geq c$, for every $d \in \mathbb{T}$.*

Proof $(a)\Rightarrow(b)$. By definition of strict minimality, for sufficiently small ε, we have

$$J(u_0) - J(u_\varepsilon) \geq c\,\|u_0 - u_\varepsilon\|_U\,.$$

By using the Taylor expansion of J at $u = u_0$, we have

$$J(u_0) - \left[J(u_0) + DJ(u_0)(u_\varepsilon - u_0) + \frac{1}{2}\|Su_0 - Su_\varepsilon\|_H^2 + \frac{\kappa}{2}\|u_0 - u_\varepsilon\|_U^2\right] \\ \geq c\,\|u_0 - u_\varepsilon\|_U\,,$$

equivalently

$$-DJ(u_0)\left(\frac{u_0 - u_\varepsilon}{\|u_0 - u_\varepsilon\|_U}\right) + \frac{1}{2}\frac{\|Su_0 - Su_\varepsilon\|_H^2}{\|u_0 - u_\varepsilon\|_U} + \frac{\kappa}{2}\|u_0 - u_\varepsilon\|_U \geq c. \tag{15}$$

Now, for any $d \in \mathbb{T}$, there exists $\varepsilon_n \to 0$ such that $\frac{u_0 - u_{\varepsilon_n}}{\|u_0 - u_{\varepsilon_n}\|_U} \rightharpoonup d$, and since $u_{\varepsilon_n} \to u_0$, by continuity

$$-DJ(u_0)\left(\frac{u_0 - u_{\varepsilon_n}}{\left\|u_0 - u_{\varepsilon_n}\right\|_U}\right) + \frac{1}{2}\frac{\left\|Su_0 - Su_{\varepsilon_n}\right\|_H^2}{\left\|u_0 - u_{\varepsilon_n}\right\|_U} + \frac{\kappa}{2}\left\|u_0 - u_{\varepsilon_n}\right\|_U \to -DJ(u_0)(d)\,.$$

Applying this expression in (15), we finally get $-DJ(u_0)(d) \geq c$.

$(b) \Rightarrow (a)$. Assume that u_0 is not a strict local minimizer of $-J$. Therefore, we can take a sequence $\varepsilon_n \to 0$ and $\frac{J(u_0)-J(u_{\varepsilon_n})}{\|u_0-u_{\varepsilon_n}\|_U} \to 0$. Using a Taylor expansion of J at $u = u_0$ as before, this is equivalent to

$$-DJ(u_0)\left(\frac{u_0 - u_{\varepsilon_n}}{\left\|u_0 - u_{\varepsilon_n}\right\|_U}\right) \to 0.$$

On the other hand, taking subsequences if necessary, we can take $d \in \mathbb{T}$ such that $\frac{u_0-u_{\varepsilon_n}}{\|u_0-u_{\varepsilon_n}\|_U} \rightharpoonup d$. And this implies that $-DJ(u_0)\,(d) = 0$, which contradicts condition (b). □

As a consequence of the previous theorem, Lemma 2.3 and [6, Proposition 3.3], we have the following result.

Corollary 3.2 $\|u_0 - u_\varepsilon\|_U = O(\beta_\varepsilon)$*, if and only if,* u_0 *is a strict local minimizer of* (Q)*. In the particular case when* (P) *is regular, the strict minimality is equivalent to the stability of the regularized optimal trajectory* Φ*.*

Assuming that (2) holds, we now focus on the following vector optimization

$$D_\delta\text{-minimize } \Phi(x), \quad x \in [0, 1), \qquad (P(\Phi, D_\delta))$$

recalling that $D_\delta \subset U$ is defined by $D_\delta = \left\{u \in U : -DJ(u_\delta)(u) \geq k\,\|u\|_U\right\}$, with $k > 0$ and $0 \leq \delta \leq \bar{\varepsilon}$ as stated in condition (c) in Lemma 2.4. Therefore, $(P(\Phi, D_\delta))$ is well defined, and since D_δ is a Bishop–Phelps cone by Remark 2.5, for $(P(\Phi, D_\delta))$, there are local Pareto minimizers as a consequence of Bishop–Phelps lemma, see [10, p.159] and [9]. In fact, we now prove that $\varepsilon = 0$ is a local proper minimizer of $(P(\Phi, D_\delta))$.

Theorem 3.3 *Let any* $0 \leq \delta \leq \bar{\varepsilon}$*. If condition* (2) *holds, then* $\varepsilon = 0$ *is a local proper minimizer of* $(P(\Phi, D_\delta))$*. If in addition* Φ *is* τ^w *-pseudoconvex at* $(0, \Phi(0))$*, then* $\varepsilon = 0$ *is a strongly minimizer of* $(P(\Phi, D_\delta))$*.*

Proof We fix $0 \leq \delta \leq \bar{\varepsilon}$. If (2) holds, following Lemma 2.4 we can take that

$$-DJ(u_\delta)\left(\frac{u_\varepsilon - u_0}{\|u_\varepsilon - u_0\|_U}\right) \geq k,$$

for ε small enough. In particular, this implies that $-DJ(u_\delta)(\Phi(\varepsilon) - \Phi(0)) > 0$ for ε small enough. Now, since $-DJ(u_\delta) \in D_\delta^\natural$ by definition, applying [10, Theorem 5.21], we get that $\varepsilon = 0$ is a local proper minimizer of $(P(\Phi, D_\delta))$.

On the other hand, by Remark 2.2, condition $T_\Gamma^w(u_0) \subset D_\delta$ in Proposition 2.6 can be equivalently rewritten in the following form

$$D_c^w\Phi(0, u_0)(x) \subset D_\delta, \quad \text{for every } x \in \operatorname{dom}(D_c^w\Phi(0, u_0)(x)). \qquad (16)$$

Finally, by pseudoconvexity we have $\Phi\,(\varepsilon) \subset \Phi\,(0) + D_c^w\Phi(0, u_0)(\varepsilon) \subset \Phi\,(0) + D_\delta$ and u_0 is a strongly minimal. □

As a consequence of this result and regularized KKT conditions (1), we get the following.

Corollary 3.4 *Let any* $0 \leq \delta \leq \bar{\varepsilon}$. *If condition* (2) *holds, then* $\varepsilon = 0$ *is a strict local minimizer of the scalar function* $(-DJ(u_\delta) \circ \Phi)(x)$. *In particular, when* $\delta > 0$, $\varepsilon = 0$ *is a strict local minimizer of the scalar function* $\mu_\delta^* \circ G \circ \Phi$.

Finally, we now consider the following vector optimization problem

$$C\text{-minimize}\,(G \circ \Phi)(x), \ \in [0, 1). \qquad (P(G \circ \Phi, C))$$

We note that in all the results, we are working under (2), and hence condition (10) of Lemma 2.4 is valid. We begin with the following result. We recall that Φ is said to stable at $\varepsilon = 0$, if there exists a constant $c > 0$ such that $\|\Phi(\varepsilon) - \Phi(0)\|_U \leq c\varepsilon$ for ε small enough.

Proposition 3.5 *Let* $0 \leq \delta \leq \bar{\varepsilon}$. *If* Φ *is stable at* $\varepsilon = 0$, *then*

$$\mathrm{dom}(\mu_\delta^* \circ G \circ \Phi)(0, \left(\mu_\delta^* \circ G\right)(u_0)) = \mathbb{R}_+$$

and

$$D_\uparrow \left(\mu_\delta^* \circ G \circ \Phi\right)(0, \left(\mu_\delta^* \circ G\right)(u_0))(x) \geq k \min_{v \in D_c^w \Phi(0,u_0)(x)} \|v\|_Y > 0, \ \ \textit{for every } x \neq 0. \tag{17}$$

Proof The equality $\mathrm{dom}(\mu_\delta^* \circ G \circ \Phi)(0, \left(\mu_\delta^* \circ G\right)(u_0)) = \mathbb{R}_+$ follows from the stability of $\mu_\delta^* \circ G \circ \Phi$, see for example [20, Lemma 4.4]. Reasoning as in (16), $D_c^w \Phi(0, u_0)(x) \subset D_\delta$ and we have that

$$- DJ(u_\delta)(v) \geq k \|v\|_Y, \quad \text{for every } v \in \ D_c^w \Phi(0, u_0)(x). \tag{18}$$

Moreover, as $DJ(u_\delta) = \mu_\delta^* \circ G \in U^*$ by KKT condition (1a) and linearity, we have

$$\begin{aligned} -DJ(u_\delta)\left(D_c^w \Phi(0, u_0)(x)\right) &= D_c\left((-DJ(u_\delta) \circ \Phi)(0, -DJ(u_\delta)(u_0))\right)(x) \\ &= D_c\left(\mu_\delta^* \circ G \circ \Phi\right)(0, \left(\mu_\delta^* \circ G\right)(u_0))(x) \end{aligned}$$

for every $x \geq 0$. Therefore

$$D_c\left(\mu_\delta^* \circ G \circ \Phi\right)(0, \left(\mu_\delta^* \circ G\right)(u_0))(x) = -DJ(u_\delta)\left(D_c^w \Phi(0, u_0)(x)\right),$$

and since $\mu_\delta^* \circ G \circ \Phi$ is scalar stable map, we can apply [21, Theorem 4.4] (see also [22]) to get

$$\begin{aligned} D_\uparrow\left(\mu_\delta^* \circ G \circ \Phi\right)(0, \left(\mu_\delta^* \circ G\right)(u_0))(x) &= \min D_c\left(\mu_\delta^* \circ G \circ \Phi\right)(0, \left(\mu_\delta^* \circ G\right)(u_0))(x) \\ &= \min_{v \in D_c^w \Phi(0,u_0)(x)} - DJ(u_\delta)(v) \end{aligned}$$

for every $x \geq 0$. Combining this equality with (18), we have

$$D_{\uparrow}\left(\mu_{\delta}^{*} \circ G \circ \Phi\right)(0, \left(\mu_{\delta}^{*} \circ G\right)(u_0))(x) = \min_{v \in D_c^w \Phi(0,u_0)(x)} - DJ(u_\delta)(v)$$
$$\geq k \min_{v \in D_c^w \Phi(0,u_0)(x)} \|v\|_Y .$$

For $x > 0$, we claim that $D_c^w \Phi(0, u_0)(x) \neq 0$. Indeed, if $D_c^w \Phi(0, u_0)(x) = 0$, by Lemma 2.1 we have $(x, 0) = \lambda_x(r_x, d_x)$, implying that $d_x = 0 \in \mathbb{T}$, and hence violating (2) since Φ is stable. Therefore, $D_c^w \Phi(0, u_0)(x) \neq 0$ and

$$D_{\uparrow}\left(\mu_{\delta}^{*} \circ G \circ \Phi\right)(0, \left(\mu_{\delta}^{*} \circ G\right)(u_0))(x) \geq k \min_{v \in D_c^w \Phi(0,u_0)(x)} \|v\|_Y > 0 \text{ for every } x \neq 0, \tag{19}$$

and the proof is complete. □

In particular, condition (17) corresponds with a sufficient condition of strict minimality given in [23, Corollary 3.12] for stable maps, see also [19], which confirms Corollary 3.4. In terms of the vector problem $(P(G \circ \Phi, C))$, condition (17) corresponds to a scalarization of a sufficient strict minimality condition. For this, let us consider the following condition stated in terms of the contingent derivative

$$0 \notin D_c(G \circ \Phi + C)(0, Gu_0)(x) \text{ for every } x > 0. \tag{20}$$

Condition (20), where it is implicitly assumed that $\operatorname{dom} D_c(G \circ \Phi + C)(0, Gu_0) = \mathbb{R}_+$, corresponds to a sufficient condition of strict minimality for problem $(P(G \circ \Phi, C))$ established in [24, Theorem 5.5].

Theorem 3.6 *Assume* $\operatorname{dom} D_c(G \circ \Phi + C)(0, Gu_0) = \mathbb{R}_+$. *If Φ is stable at $\varepsilon = 0$, then $\varepsilon = 0$ is a strict local minimizer of vector optimization problem* $(P(G \circ \Phi, C))$.

Proof By Proposition 3.5, condition (17) holds. Let us now prove that (20) holds. On the contrary, assume that (20) does not hold, and we have $0 \in D_c(G \circ \Phi + D)(0, Gu_0)(x)$ for some $x > 0$. By definition of τ^w-contingent derivative, since Y is reflexive and $G \circ \Phi$ is stable, by applying [25, Lemma 3.8], we have

$$0 \in D_c(G \circ \Phi + C)(0, Gu_0)(x) \subseteq D_c^w(G \circ \Phi + C)(0, Gu_0)(x)$$
$$= D_c^w(G \circ \Phi)(0, Gu_0)(x) + C.$$

Consequently, we can take an element $\bar{v} \in D_c^w(G \circ \Phi)(0, Gu_0)(x) \cap -C$ and we have $\mu_\delta^*(\bar{v}) \leq 0$ since $\mu_\delta^* \in C_\delta^* \subseteq C^*$. On the other hand, by following proof of [25, Proposition 3.9], we have

$$D_{\uparrow}\left(\mu_{\delta}^{*} \circ G \circ \Phi\right)(0, \left(\mu_{\delta}^{*} \circ G\right)(u_0))(x) = \min_{v \in D_c^w(G \circ \Phi)(0, Gu_0)(x)} \mu_\delta^*(v) \leq \mu_\delta^*(\bar{v}) \leq 0,$$

which contradicts (17). Therefore (20) holds and applying [24, Theorem 5.5] the point $\varepsilon = 0$ is a strict local minimizer of problem $(P(G \circ \Phi, D))$. □

When Y is finite dimensional, it is well known that $D_c(G \circ \Phi + C)(0, Gu_0) = \mathbb{R}_+$ provided that Φ is stable, see for example [22]. In this case, we have the following:

Corollary 3.7 *Assume Y is finite dimensional. If Φ is stable at $\varepsilon = 0$, then $\varepsilon = 0$ is a strict local minimizer of vector optimization problem $(P(G \circ \Phi, C))$.*

4 Examples

In this section, we present two illustrative examples which are taken from [6].

Example 4.1 Let $U = Y = H = \mathbb{R}^2$ be equipped with their Euclidean norm and the ordering cone $C = \mathbb{R}_+ \times \{0\}$. Given $r = (1, 1)$, we consider the following constrained optimization problem:

$$\text{minimize } J(u) = u^T u - r^T u, \quad \text{subject to } u \in C, \quad u \in U. \tag{P}$$

This example represents a problem of type (P) with unique solution given by $u_0 = \left(\frac{1}{2}, 0\right)$. In this example, the optimal regularized trajectory Φ is stable, and consequently (P) is regular and property (2) holds.

Considering the basis $\Theta = \{(1, 0)\}$ of C, the family $\{C_\varepsilon\}_{\varepsilon>0}$ of Henig dilating cones associated with C has the following analytical expression

$$C_\varepsilon = \left\{(x_1, x_2) : -\varepsilon x_1 \left(1 - \varepsilon^2\right)^{-\frac{1}{2}} \leq x_2 \leq \varepsilon x_1 \left(1 - \varepsilon^2\right)^{-\frac{1}{2}}, x_1 \geq 0\right\}.$$

The conically regularized problem is then given by

$$\text{minimize } J(u) = u^\top u - r^\top u \ \text{ subject to } u \in C_\varepsilon, \quad u \in U. \tag{P_ε}$$

Following [6, Example 1], the optimal regularized trajectory is given by

$$u_\varepsilon = \Phi(\varepsilon) = \left(\frac{\sqrt{1-\varepsilon^2}}{2}\left(\sqrt{1-\varepsilon^2} + \varepsilon\right), \ \frac{\varepsilon}{2}\left(\varepsilon + \sqrt{1-\varepsilon^2}\right)\right).$$

Furthermore,

$$J(u_\varepsilon) = -\frac{\varepsilon}{2}\sqrt{1-\varepsilon^2} - \frac{1}{4}$$

and

$$\Gamma = \left\{\left(\frac{\sqrt{1-\varepsilon^2}}{2}\left(\sqrt{1-\varepsilon^2} + \varepsilon\right), \frac{\varepsilon}{2}\left(\varepsilon + \sqrt{1-\varepsilon^2}\right)\right) : \varepsilon \in [0, 1)\right\}. \tag{21}$$

To verify Theorem 3.1, we note that we have $\|u_\varepsilon - u_0\|_U = \frac{\sqrt{2}}{2}\varepsilon$, and hence by taking $\alpha = \frac{1}{2}$, we get

$$\begin{aligned} -J(u_\varepsilon) \geq -J(u_0) + \frac{1}{2}\|u_\varepsilon - u_0\|_U \quad &\iff \quad \frac{\varepsilon}{2}\sqrt{1-\varepsilon^2} + \frac{1}{4} \geq \frac{1}{4} + \frac{\sqrt{2}}{4}\varepsilon \\ &\iff \quad \frac{1}{2} \geq \varepsilon^2, \end{aligned}$$

and this last inequality holds for every $\varepsilon \in \left[0, \frac{\sqrt{2}}{2}\right]$. Thus u_0 is a local strict minimizer of $-J$ over Γ as claimed.

On the other hand, taking $k = \frac{\sqrt{2}}{2}$, we compute the cone

$$D_0 = \{(u_1, u_2) \in U : -DJ(u_0)(u) \geq k\,\|u\|_U\} = \left\{(u_1, u_2) \in U : -\mu_0^T u \geq \frac{\sqrt{2}}{2}\,\|u\|_U\right\},$$

where $\mu_0 = (0, -1)$ is the KKT multiplier associated with problem (P). Consequently,

$$D_0 = \left\{(u_1, u_2) \in U : u_2 \geq \frac{\sqrt{2}}{2}\sqrt{u_1^2 + u_2^2}\right\}.$$

In Fig. 1b, we see that $\varepsilon = 0$ is a strong minimizer of vector optimization problem $(P(\Phi, D_0))$, that is,

$$\Phi(\varepsilon) \subseteq \Phi(0) + D_0,$$

as it was claim by Theorem 3.3. Analogously, since Φ is stable, point $\varepsilon = 0$ is a strict local minimizer of vector optimization problem $(P(G \circ \Phi, D))$, that is, there exists $\alpha > 0$ such that

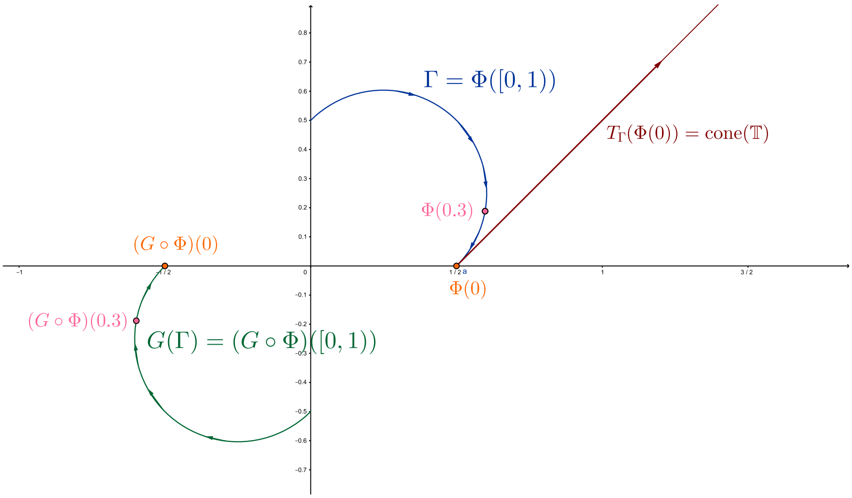
$$(G \circ \Phi)(\varepsilon) \notin (G \circ \Phi)(0) + B(0, \alpha\varepsilon) - D,$$

as claimed in Theorem 3.6 and Corollary 3.7. Indeed, taking $\alpha < \frac{1}{4}$, it holds for small ε, see Fig. 1c.

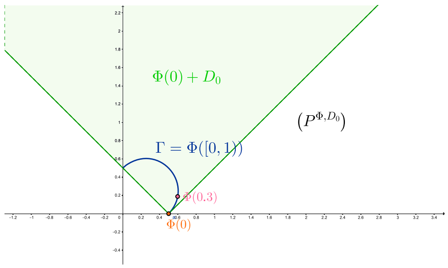
Example 4.2 Let $U = H = \mathbb{R}$, $Y = L^2[0, 1]$ be the Lebesgue space of square integrable functions, and $C = L^2_+[0, 1] = \{f \in L^2[0, 1] : f(x) \geq 0 \text{ a.e. in } [0, 1]\}$ be the associated cone of nonnegative functions. In this case $G : \mathbb{R} \to L^2[0, 1]$ is the linear bounded map defined by $G(u) = -u_F$, for every $u \in \mathbb{R}$, where u_F denotes the constant map given by $u_F(t) = u$, for every $t \in [0, 1]$. Analogously, $w \in L^2[0, 1]$ is given by $w(t) = -t$, for every $t \in [0, 1]$.

We consider the problem:

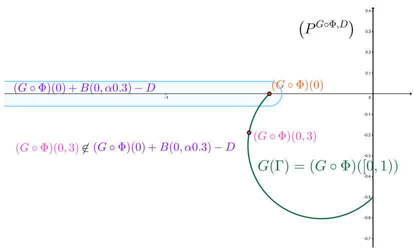
$$\text{minimize } J(u) = \frac{1}{2}u^2, \quad \text{subject to } G(u) \leq_{L^2_+[0,1]} w, \quad u \in U. \tag{P}$$



(a) Optimal regularized (state) trajectory.



(b) Illustration of Theorem 3.6.



(c) Illustration of Theorem 3.10.

Fig. 1 Example 1

Problem (P) is also a problem of type (P), where $u_0 = 1$ is the unique solution. This example corresponds with [6, Example 2], see also [26, Example 3.20]. It is an example, where (P) is not regular, Φ is not stable, but condition (2) still holds. Indeed, we have that $\mathbb{T} = \{-1\}$ and $-DJ(u_0)(-1) = 1$.

Let $\{C_\varepsilon\}_{\varepsilon>0}$ be the family of dilating cones for C, where we take the basis $\Theta = \left\{ f \in L^2_+[0,1] : \int_0^1 f(s)ds = 1 \right\}$. For each $\varepsilon > 0$, the conically regularized problem is given by

$$\text{minimize } J(u) = \frac{1}{2}u^2 \text{ subject to } G(u) \leq_{C_\varepsilon} w, \quad u \in U. \qquad (P_\varepsilon)$$

We solve numerically regularized problem (P_ε) by using a finite element discretization of the dual problem by using CVX in Matlab (see [27]). The numerical result are given in Table 1.

Table 1 Example 2. Numerical results

ε	u_ε	$\lvert u_\varepsilon - u_0\rvert$	$\frac{-J(u_\varepsilon)+J(u_0)}{\lvert u_\varepsilon-u_0\rvert}$
1e-01	0.8435	0.1565	0.9218
1e-02	0.9601	0.0399	0.9800
1e-03	0.9910	0.0090	0.9955
1e-04	0.9980	0.0020	0.9990
1e-05	0.9995	0.0005	0.9998
1e-06	0.9996	0.0004	0.9998

Fourth column in Table 1 that shows $\frac{-J(u_\varepsilon)+J(u_0)}{|u_\varepsilon-u_0|} > 0$, which agrees with the fact that u_0 is a strict local minimizer of $-J$. as predicted by Theorem 3.1. On the other hand, taking for example $k = \frac{1}{2}$, we have

$$\begin{aligned} D_\delta = \{u \in \mathbb{R} : -DJ(u_\delta)(u) \geq k\,|u|\} &= \left\{u \in \mathbb{R} : -u_\delta \frac{u}{|u|} \geq \frac{1}{2}\right\} \\ &= \left\{u \in \mathbb{R} : -u_\delta \frac{u}{|u|} \geq k\right\} = \mathbb{R}_-, \end{aligned}$$

for δ small enough. Therefore, $D_\delta = \mathbb{R}_-$ and the vector optimization problem $(P(\Phi, D_\delta)) = (P(\Phi, \mathbb{R}_-))$ corresponds to a scalar maximization problem

$$\text{maximize } \Phi(x), \quad x \in [0, 1). \tag{$P(\Phi, \mathbb{R}_-)$}$$

From Table 1 we check that $u_0 = \Phi(0) > \Phi(\varepsilon) = u_\varepsilon$ and $\varepsilon = 0$ solves $(P^{\Phi,\mathbb{R}_-})$ as predicted by Theorem 3.3.

5 Conclusions

This paper presented new qualitative and quantitative stability estimates for the conical regularization approach for linearly constrained least-squares optimization problems in Hilbert spaces. The key component of contribution is to clarify the role of property, appearing in previous works, that provides improved convergence rates. We characterized this condition as the optimality condition corresponding to different optimization problems associated with the original one least-squares optimization problems. We show that this condition corresponds to a necessary and sufficient optimality condition for the strict minimality of the objective function over the regularized trajectory. Moreover, we proved that this condition is also a necessary optimality condition for two vector optimization problems. In the first vector optimization problem, we optimize the regularized trajectory with respect to a Bishop–Phelps cone, and we proved that it is a necessary condition of proper minimality, and strong minimality under an additional assumption. In the second vector optimization problem, we optimized the regularized state trajectory with respect to the constraint cone, and we proved that it is a necessary condition for strict minimality in a vector sense for the case when the problem is regular, ensuring that in that case, it is a necessary condition for the stability of the regularized trajectory.

The given results depict a natural occurrence of vector optimization problems and the use of essential tools from the set-valued analysis in the stability analysis of linearly constrained least-squares problems in infinite-dimensional spaces. Although the importance of strict minimality is well-known (see [28–30] and references therein), in the study of stability and convergence analysis of numerical methods, to the best of our knowledge, this is the first work where tools from vector optimization have been employed to establish the stability of regularization methods.

Finally, as a future research direction, it would be interesting to apply the results of this paper to PDE-constrained optimization with state constraints. Conical regularization methods, see [3], arise in the context of such optimal control problems, where establishing an optimal regularization rate is crucial for optimal error estimates, see [31–34].

Acknowledgements The research of Akhtar Khan is supported by the National Science Foundation under Award No. 1720067. Miguel Sama is supported by Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) (Spain) and Fondo Europeo de Desarrollo Regional (FEDER) under project PGC2018-096899-B-I00 (MCIU/AEI/FEDER, UE) and Grant No. 2021-MAT11 (ETSI Industriales, UNED).

References

1. Martin, K., Ryan, C.T., Stern, M.: The Slater conundrum: duality and pricing in infinite-dimensional optimization. SIAM J. Optim. **26**(1), 111–138 (2016)
2. Jadamba, B., Khan, A.A., Sama, M., Tammer, C.: Regularization methods for scalar and vector control problems. In: Variational analysis and set optimization, pp. 296–321. CRC Press, Boca Raton, FL (2019)
3. Khan, A.A., Sama, M.: A new conical regularization for some optimization and optimal control problems: Convergence analysis and finite element discretization. Numerical Functional Analysis and Optimization **34**(8), 861–895 (2013)
4. Henig, M.I.: A cone separation theorem. J. Optim. Theory Appl. **36**(3), 451–455 (1982)
5. Henig, M.I.: Proper efficiency with respect to cones. J. Optim. Theory Appl. **36**(3), 387–407 (1982)
6. López, R., Sama, M.: Stability and sensivity analysis for conical regularization of linearly constrained least-squares problems in Hilbert spaces. J. Math. Anal. Appl. **456**(1), 476–495 (2017)
7. Huerga, L., Khan, A., Sama, M.: A henig conical regularization approach for circumventing the slater conundrum in linearly ℓ^p_+-constrained least squares problems. J. Appl. Numer. Optim. **1**, 117–129 (2019)
8. Khan, A.A., Tammer, C., Zalinescu, C.: Set-Valued Optimization. Springer, Heidelberg (2015)
9. Bishop, E., Phelps, R.R.: The support functionals of a convex set. In: Proc. Sympos. Pure Math., Vol. VII, pp. 27–35. Amer. Math. Soc., Providence, R.I. (1963)
10. Jahn, J.: Vector optimization. Springer-Verlag, Berlin (2004)
11. Borwein, J.M., Zhuang, D.: Super efficiency in vector optimization. Trans. Amer. Math. Soc. **338**(1), 105–122 (1993)
12. Aubin, J.P.: Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. In: Mathematical analysis and applications, Part A, *Adv. in Math. Suppl. Stud.*, vol. 7, pp. 159–229. Academic Press, New York-London (1981)
13. Rodríguez-Marín, L., Sama, M.: τ^w-contingent epiderivatives in reflexive spaces. Nonlinear Anal. **68**(12), 3780–3788 (2008)

14. Jahn, J., Rauh, R.: Contingent epiderivatives and set-valued optimization. Mathematical Methods of Operations Research **46**(2), 193–211 (1997)
15. Aubin, J.P., Frankowska, H.: Set-valued analysis, Systems & Control: Foundations & Applications, vol. 2. Birkhäuser Boston Inc., Boston, MA (1990)
16. Rodríguez-Marín, L., Sama, M.: Scalar Lagrange multiplier rules for set-valued problems in infinite-dimensional spaces. J. Optim. Theory Appl. **156**(3), 683–700 (2013)
17. Borwein, J.: Proper efficient points for maximizations with respect to cones. SIAM J. Control Optim. **15**(1), 57–63 (1977)
18. Jiménez, B.: Strict efficiency in vector optimization. J. Math. Anal. Appl. **265**(2), 264–284 (2002)
19. Studniarski, M.: Necessary and sufficient conditions for isolated local minima of nonsmooth functions. SIAM J. Control Optim. **24**(5), 1044–1049 (1986)
20. Rodríguez-Marín, L., Sama, M.: Variational characterization of the contingent epiderivative. J. Math. Anal. Appl. **335**(2), 1374–1382 (2007)
21. Rodríguez-Marín, L., Sama, M.: About contingent epiderivatives. J. Math. Anal. Appl. **327**(2), 745–762 (2007)
22. Tanino, T.: Sensitivity analysis in multiobjective optimization. J. Optim. Theory Appl. **56**(3), 479–499 (1988)
23. Jiménez, B., Novo, V., Sama, M.: Scalarization and optimality conditions for strict minimizers in multiobjective optimization via contingent epiderivatives. J. Math. Anal. Appl. **352**(2), 788–798 (2009)
24. Flores-Bazán, F., Jiménez, B.: Strict efficiency in set-valued optimization. SIAM J. Control Optim. **48**(2), 881–908 (2009)
25. Sama, M.: Some remarks on the existence and computation of contingent epiderivatives. Nonlinear Anal. **71**(7–8), 2997–3007 (2009)
26. Bonnans, J.F., Shapiro, A.: Perturbation analysis of optimization problems. Springer Science & Business Media (2013)
27. Grant, M., Boyd, S., Ye, Y.: Cvx: Matlab software for disciplined convex programming (2009)
28. Cromme, L.: Strong uniqueness. A far-reaching criterion for the convergence analysis of iterative procedures. Numer. Math. **29**(2), 179–193 (1977/78)
29. Auslender, A.: Stability in mathematical programming with nondifferentiable data. SIAM J. Control Optim. **22**(2), 239–254 (1984)
30. Burke, J.V., Deng, S.: Weak sharp minima revisited. I. Basic theory. pp. 439–469 (2002). Well-posedness in optimization and related topics (Warsaw, 2001)
31. Jadamba, B., Khan, A., Sama, M.: Error estimates for integral constraint regularization of state-constrained elliptic control problems. Comput. Optim. Appl. **67**(1), 39–71 (2017)
32. Hintermüller, M., Hinze, M.: Moreau-Yosida regularization in state constrained elliptic control problems: error estimates and parameter adjustment. SIAM J. Numer. Anal. **47**(3), 1666–1683 (2009)
33. Hinze, M., Meyer, C.: Variational discretization of Lavrentiev-regularized state constrained elliptic optimal control problems. Comput. Optim. Appl. **46**(3), 487–510 (2010)
34. Hinze, M., Schiela, A.: Discretization of interior point methods for state constrained elliptic optimal control problems: optimal error estimates and parameter adjustment. Comput. Optim. Appl. **48**(3), 581–600 (2011)