Chapter 15

An Equation Error Approach for Identifying a Random Parameter in a Stochastic Partial Differential Equation

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15.1 Introduction

Stochastic partial differential equations (SPDEs) involving random variables parameters appear in abundance in applied models. The typical examples include the stochastic diffusion coefficient in the diffusion equation, the flexural rigidity coefficient in the Cauchy-Euler beam and plate models, the stochastic Láme parameters in the linear elasticity system equations, and the stochastic viscosity

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in the Stokes's equations. The significance of stochastic equations, differential and integral, was evident long ago; see the fundamental contributions by Engl and Nashed [21, 22], Nashed and Engl [40], and Nashed and Salehi [41], and the monograph by Bharucha-Reid [12]. Earlier research on SPDEs, besides establishing useful theoretical results, also developed some computational schemes. However, it is only in recent years when concentrated research efforts on the systematic numerical treatment of SPDEs have been made. See the basic results developed by Ghanem and Spanos [25], Deb, Babu**v**ska, and Oden [20], Babu**v**ska and Chatzipantelidis [5], Babu**v**ska, Tempone, and Zouraris [7, 8], Babu**v**ska, Nobile, and Tempone [6], and the cited references.

Assume that $(\Omega, \mathcal{F}, \mu)$ is a probability space, and $D \subset \mathbb{R}^m$ is a sufficiently smooth bounded domain and ∂D is its boundary. Given random fields $a : \Omega \times D \to \mathbb{R}$ and $f : \Omega \times D \to \mathbb{R}$, the direct problem in this work seeks a random field $u : \Omega \times D \to \mathbb{R}$ that almost surely satisfies the following boundary value problem (BVP) with random data:

$$-\nabla \cdot (a(\omega,x)\nabla u(\omega,x)) = f(\omega,x), \text{ in } D, \tag{15.1a}$$

$$u(\omega,x) = 0, \text{ on } \partial D. \tag{15.1b}$$

SPDE (15.1) appears in many applied models and has been extensively studied. SPDE (15.1) constitutes the direct problems in this study, and there are two associated inverse problems. The first inverse problem seeks, from some statistical information of $u(x,\omega)$, the statistical information of the random coefficient $a(x,\omega)$. This inverse problem is commonly referred to as the parameter identification problem. The second inverse problem seeks the right-hand side f, again from some information concerning the solution u. This inverse problem is the source identification problem. The latter problem is closely connected to the optimal control problem of finding the control $\theta(\omega,x)$ when the source term has the form $f(\omega,x) \equiv f(\omega,x) + \theta(\omega,x)$. In this work, we will focus on the inverse problem of parameter identification.

The problem of estimating stochastic or deterministic parameters in stochastic PDEs from a measurement of the solution of the SPDE has been widely studied in the last few years. We note that a common approach for inverse problems with data uncertainty is the Bayesian approach (see [23, 24, 45, 43]). The so-called variational approach attracted quite a bit of attention in recent years due to some of the shortcomings of the Bayesian approach. The variational approach is suitable for identifying distributed and spatially correlated parameters in SPDEs. It involves posing an optimization problem whose solution can provide information on statistical quantities associated with an unknown parameter. The key advantages of the variational approach include access to an array of efficient and reliable optimization algorithms, a theoretical framework for convergence analysis, and the ability to embed the parameter structure into the inversion framework. For some of the recent developments in stochastic control problems and stochastic inverse problems, we refer the reader

to [1, 4, 10, 13, 14, 15, 16, 17, 18, 32, 34, 35, 36, 37, 39, 42, 44, 46, 47, 48, 49, 50] and the cited references.

The most commonly used optimization formulation for the stochastic inverse problem of parameter identification is the following output least-squares:

$$J_0(a) := \frac{1}{2}\mathbb{E}\left[\|u_a(\omega,x) - z(\omega,x)\|^2\right], \tag{15.2}$$

where $u_a(\omega,x)$ is the solution of (15.1) for $a(\omega,x)$, $z(\omega,x) \in L^2(\Omega;L^2(D))$ is the measured data, and $\|\cdot\|$ is a suitable norm, For example, $L_2(D)$-norm was considered in [1], whereas Wyk [48] employed $H_1(D)$-norm; $H_1(D)$-seminorm is another possibility. Here $\mathbb{E}[\cdot]$ is the expectation.

Recently, in the context of (15.1), the following modified output least-squares (MOLS) objective functional was introduced in [33]:

$$J(a) = \frac{1}{2}\mathbb{E}\left[\int_D a(\omega,x)\nabla(u_a(\omega,x) - z(\omega,x)) \cdot \nabla(u_a(\omega,x) - z(\omega,x))dx\right],$$

where $u_a(\omega,x)$ is the weak solution of (15.1) for $a(\omega,x)$ and $z(\omega,x)$ is the measured data.

In [33], it was shown that the MOLS functional given above is convex. Similar results for the inverse problem of identifying a stochastic flexural rigidity coefficient in a fourth-order plate model are available in [30].

The primary objective of this work is to propose an equation error approach for solving the inverse problem of parameter identification in stochastic PDE (15.1). Besides advocating for the usefulness of the equation error approach for inverse problems, the developed framework also pitches it as a reliable and efficient fast alternative to other optimization formulations.

We divide the contents of this chapter into seven sections. In Section 2, we collect preliminary results related to the stochastic PDE and discuss the solvability of the associated variational form. Section 3 presents an overview of some of the most commonly used techniques for solving stochastic PDEs. Section 4 is devoted to the study of the new equation error formulation for the considered stochastic inverse problem. We give the unique solvability of the regularized stochastic optimization problem emerging from the equation error approach. We provide discrete formulas in Section 5. Section 6 gives preliminary numerical results. The chapter concludes with some general remarks.

15.2 Solvability of the Direct Problem

We begin with an overview of SPDEs and common computational techniques, noting that a majority of the available literature focused on (15.1). Before discussing the solvability of (15.1) and the associated inverse problem, we first recall some function spaces. Given the domain D, for $1 \leq p < \infty$, by $L^p(D)$, we

represent the space of pth Lebesgue integrable functions, that is

$$L^p(D) = \left\{ y : D \mapsto \mathbb{R} \text{ is measurable, and } \int_D |y|^p \, dx < +\infty \right\}.$$

The space $L^\infty(D)$ consists of measurable functions that are bounded almost everywhere (a.e.) on D. We also recall that the Sobolev spaces are given by

$$H^1(D) = \left\{ y \in L^2(D),\ \ \partial_{x_i} y \in L^2(D),\ i = 1, \ldots, n \right\},$$
$$H_0^1(D) = \left\{ y \in H^1(D),\ \ y|_{\partial D} = 0 \right\},$$

and $H^{-1}(D) = (H_0^1(D))^*$ is the topological dual of $H_0^1(D)$. For $m \in \mathbb{N}$, higher-order Sobolev spaces $H^m(D)$ consist of $L^2(D)$ functions with all partial derivatives up to order m reside in $L^2(D)$.

Moreover, we recall that given a real Banach space X, a probability space $(\Omega, \mathcal{F}, \mu)$, and an integer $p \in [1, \infty)$, the Bochner space $L^p(\Omega; X)$ consists of Bochner integrable functions $u : \Omega \to X$ with finite p-th moment, that is,

$$\|u\|_{L^p(\Omega;X)} := \left(\int_\Omega \|u(\omega)\|_X^p d\mu(\omega) \right)^{1/p} = \mathbb{E}\left[\|u(\omega)\|_X^p \right]^{1/p} < \infty,$$

where $\mathbb{E}[\cdot]$ is the expectation. For details and properties of these spaces, see [38].

If $p = \infty$, then $L^\infty(\Omega; X)$ is the space of Bochner measurable functions $u : \Omega \to X$ such that

$$\operatorname{ess\,sup}_{\omega \in \Omega} \|u(\omega)\|_X < \infty.$$

Many useful features of $L^p(D)$ spaces of Lebesgue integrable functions translate naturally to Bochner spaces $L^p(\Omega; X)$. Moreover, it is known that $L^\infty(\Omega; L^\infty(D)) \subset L^\infty(\Omega \times D)$, but $L^\infty(\Omega; L^\infty(D)) \neq L^\infty(\Omega \times D)$, in general. Furthermore, the space $L^p(\Omega; L^q(D))$, for $p, q \in [1, \infty)$, is isomorphic to

$$\left\{ v : \Omega \times D \to \mathbb{R}^n \middle| \int_\Omega \left(\int_D |v(\omega, x)|^q dx \right)^{p/q} d\mu(\omega) < \infty \right\}.$$

A critical part of the study of SPDEs is the **finite-dimensional noise** representation of the random fields by a finite number of mutually independent random variables (see [7, 38]): Given M random variables $\xi_k : \Omega \mapsto \Gamma_k$, for $k = 1, \ldots, M$, a function $v \in L^2(\Omega; L^2(D))$ of the form $v(\xi(\omega), x)$ for $x \in D$ and $\omega \in \Omega$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_M) : \Omega \mapsto \Gamma \subset \mathbb{R}^M$ and $\Gamma := \Gamma_1 \times \Gamma_2 \cdots \times \Gamma_M$, is called a finite-dimensional noise.

If a random field $v(\omega, x)$ is finite-dimensional noise, a change of variables can be performed for computing expectations. To be specific, denoting by σ, the joint density of ξ, we have

$$\|v\|_{L^2(\Omega;L^2(D))}^2 = \mathbb{E}\left[\|v\|_{L^2(D)}^2 \right] = \int_\Gamma \sigma(y) \|v(y, \cdot)\|_{L^2(D)}^2 dy.$$

Therefore, by defining $y_k := \xi_k(\omega)$ and setting $y = (y_1, y_2, \ldots, y_M)$, we associate a random field $v(\omega, x)$ with a finite-dimensional noise by a function $v(x, y)$ in the weighted L^2 space:

$$L^2_\sigma(\Gamma; L^2(D)) := \left\{ v : \Gamma \times D \to \mathbb{R} : \int_\Gamma \sigma(y) \|v(\cdot, y)\|^2_{L^2(D)} dy < \infty \right\}.$$

Assume that $a(\omega, x)$ and $f(\omega, x)$ are finite-dimensional noises and given by

$$a(\omega, x) = a_0(x) + \sum_{k=1}^{P} a_k(x)\xi_k(\omega), \tag{15.3}$$

$$f(\omega, x) = f_0(x) + \sum_{k=1}^{L} f_k(x)\xi_k(\omega), \tag{15.4}$$

where the real-valued functions a_k and f_k are uniformly bounded. Then, as a consequence of the celebrated Doob-Dynkin lemma, a solution of (15.6) is finite-dimensional noise and u is a function of ξ where $\xi = (\xi_1, \xi_2, \ldots, \xi_M) : \Omega \mapsto \Gamma$ and $M := \max\{P, L\}$, see [38].

For the solvability of the variational problem (15.6), in the following we assume that there are constants k_0 and k_1 with

$$0 < k_0 \leq a(\omega, x) \leq k_1 < \infty, \text{ almost everywhere in } \Omega \times D. \tag{15.5}$$

For the variational form of BVP (15.1), we will use $\widehat{V} := L^2(\Omega; H^1(D))$ which is a Hilbert space with the inner product defined by

$$\langle u, v \rangle = \int_\Omega \langle u(\omega, x), v(\omega, x) \rangle_{H^1(\Omega)} \, d\mu(\omega).$$

To impose the boundary conditions, we will use $V = L^2(\Omega; H^1_0(\Omega)) \subset \widehat{V}$.

To derive the variational formulation, we take $u \in L^2(\Omega; H^2(D))$ and multiply (15.1) by a test function $v \in V$ and by integrating the product on both sides, invoking the Green's identity, and using the boundary conditions, we obtain

$$\int_\Omega \int_D a(\omega, x) \nabla u(\omega, x) \cdot \nabla v(\omega, x) dx d\mu(\omega) \tag{15.6}$$
$$= \int_\Omega \int_D f(\omega, x) v(\omega, x) \, dx d\mu(\omega), \text{ for every } v \in V.$$

Therefore, we are looking for elements $u \in V$ such that (15.6) holds for all $v \in V$.

For the solvability of (15.6), we introduce the following notation

$$s(u, v) = \int_\Omega \int_D a(\omega, x) \nabla u(\omega, x) \nabla v(\omega, x) \, dx d\mu(\omega), \tag{15.7}$$

$$m(v) = \int_\Omega \int_D f(\omega, x) v(\omega, x) \, dx d\mu(\omega), \tag{15.8}$$

and, for a fixed $a(\omega,x)$, write (15.6) as the problem of finding $u \in V$ with

$$s(u,v) = m(v), \quad \text{for every } v \in V. \tag{15.9}$$

Since $a(\omega,x) \in L^\infty(\Omega \times D)$ and $V \subset L^2(\Omega;L^2(D)) \cong L^2(\Omega \times D)$, we can show that

$$|s(u,v)| \leq \|a(\omega,x)\|_{L^\infty(\Omega\times D)} \|u(\omega,x)\|_V \|v(\omega,x)\|_V,$$

which proves that the bilinear form s is continuous.

Furthermore, the bilinear form $s(\cdot,\cdot)$ is coercive as well because

$$s(u,v) = \mathbb{E}\left[\int_D a(\omega,x)\nabla v(\omega,x)\cdot\nabla v(\omega,x)dx\right] \geq \alpha\|v(\omega,x)\|_V^2,$$

where α is a positive constant involving the Poincare's constant.

For the given $f \in L^2(\Omega;H^1(D)^*)$ and for any $v \in V$, for the functional $m(\cdot)$, we have

$$|m(v)| = \left|\mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\,dx\right]\right| \leq \|f(\omega,x)\|_{H^1(\Omega;H^1(D)^*)}\|v(\omega,x)\|_V,$$

which proves the continuity of m. Hence, the unique solvability of (15.6) ensues from the Lax-Milgram lemma.

For the inverse problem of parameter identification, analytic properties of the parameter-to-solution map $a \mapsto u_a(\omega,x)$, that assigns to a, the unique solution $u_a(\omega,x)$ of (15.6), are crucial. For this, we define $K \subset B := L^\infty(\Omega;L^\infty(D))$ to be the set of feasible parameters with a nonempty interior. We emphasize that the feasible parameters in K must satisfy (15.5). We recall the following result that gives the Lipschitz continuity of the parameter-to-solution map.

Proposition 15.2.1 [33] *The map $K \ni a(\omega,x) \mapsto u_a(\omega,x)$ is Lipschitz continuous.*

We next recall the following result that gives a derivative characterization of the parameter-to-solution map:

Theorem 15.2.2 [33] *Let $a(\omega,x)$ be in the interior of K. Then, the derivative $\delta u_a(\omega,x) := Du_a(\delta a(\omega,x))$ of $u_a(\omega,x)$ in the direction $\delta a(\omega,x)$ is the unique solution of the stochastic variational problem: Find $\delta u_a(\omega,x) \in V$ such that*

$$\begin{aligned}\int_\Omega\int_D a(\omega,x)\nabla\delta u_a(\omega,x)\cdot\nabla v(\omega,x)dx\,d\mu(\omega) \\ = -\int_\Omega\int_D \delta a\nabla u_a(\omega,x)\cdot\nabla v(\omega,x)\,dx d\mu(\omega), \ \textit{for every } v \in V.\end{aligned} \tag{15.10}$$

The above derivative characterization plays a central role in developing a gradient-based optimization framework for solving the inverse problem and is instrumental in obtaining discrete formulas for the gradient of various objective functionals.

We note that under (15.3) and (15.4), the variational problem (15.6) reduces to the following parametric variational problem, which is commonly used for numerical simulations: Find $u(y,x) \in V_\sigma := L^2_\sigma(\Gamma; H^1_0(D))$ such that

$$\int_\Gamma \sigma(y) \int_D a(y,x) \nabla u(y,x) \cdot \nabla v(y,x) dx\, dy = \int_\Gamma \sigma(y) \int_D f(y,x) v(y,x)\, dx\, dy, \quad \text{for all } v(y,x) \in V_\sigma. \tag{15.11}$$

15.3 Numerical Techniques for Stochastic PDEs

In the following, we briefly describe three of the commonly used numerical methods for solving SPDEs.

15.3.1 Monte Carlo Finite Element Type Methods

A heavily used approach for the numerical treatment of (15.11) is the sampling-based Monte Carlo Finite Element Method (MC-FE) approach, where for the deterministic component, a finite element discretization scheme is used. For the MC-FE method, s realizations of the random variable $y_j := (y_j^1, \ldots, y_j^m)$, $j = 1, \ldots, s$, are generated, and for each realization y_j, a solution of $u_j = u(y_j, x)$ of (15.11) is obtained involving the realizations of $a(y,x)$ and $f(y,x)$. From the solution samples, the desired statistics ensues. For instance, the ℓ-th moment of the solution $u(y,x)$ is obtained by

$$\mathbb{E}\left[u(\cdot,x)^\ell\right] \approx \frac{1}{s} \sum_{j=1}^{s} u(y_j,x)^\ell.$$

The MC-FE approach is robust, easily implementable, uses existing finite element solvers, and does not impose regularity restrictions on the data. However, a glaring pitfall is a slow convergence, with asymptotic order $\frac{1}{\sqrt{s}}$, requiring a large sample size for an acceptable approximation.

One of the many generalization of the MC-FE method, the **Multilevel Monte Carlo Finite Element** (MLMC-FE) method (see Giles [27], Barth et al. [11]), significantly reduces its computational cost. The key idea of the MLMC-FE is to employ hierarchical finite element spaces, acquiring sample solution on varying mesh-sizes and gradually decreasing the sample size on each finer mesh, finding a fine compromise between the convergence and discretization error.

15.3.2 The Stochastic Collocation Method

The stochastic collocation method (see [7]) is based on combining collocation on Γ and finite element discretization of D. It is similar to the MC-FE method in that it samples the random data, but without choosing the sampling point randomly. Given collocation points, $\{y_j\}_{j=1}^s$ in Γ, the collocation scheme defines an approximate solution as the Lagrangian interpolant

$$u_P(y,x) = \sum_{k=1}^{s} u(y_k,x)L_k(y),$$

where $L_k : \Gamma \mapsto \mathbb{R}$ is the Lagrange polynomial satisfying $L_k(y_s) = \delta_{ks}$, and $u(y_k,x)$ is the solution of the parametric PDE (15.11) at y_k.

Employing the Galerkin finite element discretization for the spatial component and denoting the finite element basis by $\{\phi_1,\ldots,\phi_N\}$, the complete approximate solution is given by

$$u_{hP}(y,x) = \sum_{k=1}^{s}\sum_{l=1}^{N} u_l^k \phi_l(x)L_k(y).$$

The commonly used collocation points are those generated by Smolyak's algorithm and lie on a sparse grid. For high stochastic dimension, Smolyak sparse grids have fewer points than the full tensor product, but only a slightly slow order of convergence. There are many variants of the stochastic collocation method. Bäck et al. [9] presented a comprehensive comparison of the stochastic Galerkin and stochastic collocation methods.

15.3.3 The Stochastic Galerkin Method

The stochastic Galerkin method (see [38]) builds around the parametric variational problem (15.11). Let V_h be an N-dimensional subspace of $H_0^1(D)$ and S_k be a Q-dimensional subspace of $L_\sigma^2(\Gamma)$ with

$$V_h = \text{span}\{\phi_1,\phi_2,\ldots,\phi_N\},$$
$$S_k = \text{span}\{\psi_1,\psi_2,\ldots,\psi_Q\},$$

where we assume that the basis $\{\psi_1,\psi_2,\ldots,\psi_Q\}$ is orthonormal with respect to σ. In stochastic Galerkin, the NQ-dimensional subspace V_σ can be constructed by tensorising the basis functions ϕ_i and ψ_j:

$$V_{hk} := V_h \otimes S_k := \text{span}\{\phi_i\psi_j |\ i = 1,\ldots,N,\ j = 1,\ldots,Q\}.$$

The **stochastic Galerkin solution** $u_{hk} \in V_{hk}$ then satisfies for all $v \in V_{hk}$:

$$\int_\Gamma \sigma(y)\int_D a(y,x)\nabla u_{hk}(y,x)\cdot\nabla v(y,x)dxdy = \int_\Gamma \sigma(y)\int_D f(y,x)v(y,x)dxdy.$$

In the above variational equations, taking the representation

$$u_{hk} = \sum_{k=1}^{N} \sum_{m=1}^{Q} U_{km} \phi_k(x) \psi_m(y),$$

a sparse linear system emerges which gives an approximate solution.

15.4 An Equation Error Approach

For inverse problems of identifying deterministic parameters, the extensively studied equation error approach is considered a compromise between an optimization-based iterative scheme and a direct solution method for finding the unknown parameter. Although the equation error approach is an optimization formulation, it bears no connection to the underlying PDE while solving the optimization problem. The OLS-based or MOLS-based optimization problems need to solve the underlying PDEs at every update, a strategy that makes these methods computationally demanding. In contrast, the equation error approach results in a speedy process for inverse problems. Moreover, being an optimization problem, the equation error approach allows for the natural inclusion of the regularization for stability. However, a drawback of not being related to the PDEs is that it heavily relies on the supplied data quality and performs poorly under high data contamination. For details on the equation error approach, see [2, 3, 19, 26, 28, 29].

For the stochastic inverse problem of estimating $a(\omega,x)$ from the data $z(\omega,x) \in L^2(\Omega;H_0^1(D))$, we propose the following equation error formulation:

$$J_{\mathcal{E}}(a(\omega,x)) := \frac{1}{2}\mathbb{E}\left[\|e(a,z)\|^2_{H^1(D)}\right] = \frac{1}{2}\int_{\Omega} \|e(a,z)\|^2_{H^1(D)}\, d\mu(\omega), \qquad (15.12)$$

where $e(\cdot,\cdot) \in L^2(\Omega;H_0^1(D))$ is such that for every $v \in L^2(\Omega;H_0^1(D))$, we have

$$\begin{aligned} &\langle e(a,u), v\rangle_{L^2(\Omega;H_0^1(D))} \\ &\quad = \int_{\Omega}\int_{D} \left[a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x) - f(\omega,x)v(\omega,x)\right] dx\, d\mu(\omega). \end{aligned} \qquad (15.13)$$

For regularizing the equation error objective, we define the following admissible set

$$A := \left\{a \in H := L^2(\Omega;H(D)) :\ 0 < k_0 \le a(\omega,x) \le k_1 \text{ a.s. } \Omega\times D\right\},$$

where the regularization space H, a separable Hilbert space, is compactly embedded into $B := L^\infty(\Omega;L^\infty(D))$, and $H(D)$ is continuously embedded in $L^\infty(\Omega)$.

We now propose the following regularized equation error problem:

$$\min_{a\in A} J^{\kappa}_{\mathcal{E}}(a) := \frac{1}{2}\mathbb{E}\left[\|e(a,z)\|^2_{H^1(D)}\right] + \frac{\kappa}{2}\|a(\omega,x)\|^2_H, \qquad (15.14)$$

where $z(\omega,x) \in L^2(\Omega;H_0^1(D))$ is the measured data, $\kappa > 0$ is a fixed regularization parameter, and $\|\cdot\|_H^2$ is the quadratic regularizer.

We have the following existence result:

Theorem 15.4.1 *For each $\kappa > 0$, the equation error based optimization problem (15.14) has a unique solution.*

Proof. Since $J_{\mathcal{E}}^{\kappa}(a) \geq 0$ for all $a \in A$, there exists a minimizing sequence $\{a_n\}$ in A, such that $\lim_{n\to\infty} J_{\mathcal{E}}^{\kappa}(a_n) = \inf_{a\in A} J_{\mathcal{E}}^{\kappa}(a)$. By the definition of the regularized equation error functional, the sequence $\{a_n\}$ is bounded in the H-norm. This, however, implies the existence of a subsequence, still denoted by $\{a_n\}$, that converges to some $\bar{a} \in A$ in the B-norm. In view of the definition of $e(\cdot,\cdot)$, for every $v \in L^2(\Omega;H_0^1(D))$, we have

$$\langle e(a_n,z),v\rangle_{L^2(\Omega;H_0^1(D))} = \mathbb{E}\left[\int_D [a_n \nabla z(\omega,x)\cdot\nabla v(\omega,x) - f(\omega,x)v(\omega,x)]\,dx\right],$$
$$\langle e(\bar{a},z),v\rangle_{L^2(\Omega;H_0^1(D))} = \mathbb{E}\left[\int_D [\bar{a} \nabla z(\omega,x)\cdot\nabla v(\omega,x) - f(\omega,x)v(\omega,x)]\,dx\right].$$

Subtracting the above two equations and setting $v = e(a_n,z) - e(\bar{a},z)$, we get

$$\begin{aligned}\|e(a_n,z) - e(\bar{a},z)\|^2_{L^2(\Omega;H_0^1(D))} &= \mathbb{E}\left[\int_D (a_n-\bar{a})\nabla z\cdot\nabla(e(a_n,z)-e(\bar{a},z))\,dx\right]\\ &\leq \|a_n-\bar{a}\|_{L^\infty(\Omega;L^\infty(D))}\|e(a_n,z)-e(\bar{a},z)\|_{L^2(\Omega;H_0^1(D))}\|z\|_{L^2(\Omega;H_0^1(D))},\end{aligned}$$

which confirms that $\|e(a_n,z) - e(\bar{a},z)\|^2_{L^2(\Omega;H_0^1(D))} \to 0$.

Consequently,

$$\begin{aligned}J_{\mathcal{E}}^{\kappa}(\bar{a}) &= \frac{1}{2}\mathbb{E}\left[\|e(\bar{a},z)\|^2_{H^1(D)}\right] + \frac{\kappa}{2}\|\bar{a}(\omega,x)\|_H^2\\ &\leq \lim_{n\to\infty}\frac{1}{2}\mathbb{E}\left[\|e(a_n,z)\|^2_{H^1(D)}\right] + \liminf_{n\to\infty}\frac{\kappa}{2}\|a_n(\omega,x)\|_H^2\\ &\leq \liminf_{n\to\infty}\left\{\frac{1}{2}\mathbb{E}\left[\|e(\bar{a},z)\|^2_{H^1(D)}\right] + \frac{\kappa}{2}\|a_n(\omega,x)\|_H^2\right\}\\ &= \inf\{J_{\mathcal{E}}^{\kappa}(a) \mid a(\omega,x)\in A\},\end{aligned}$$

confirming that $\bar{a}(\omega,x)$ is a solution of (15.14). The uniqueness follows from the strong convexity of the regularizer and convexity of the equation error objective. The proof is complete. □

15.5 Discrete Formulae

We will now give some discrete formulas for numerical simulations. Recall that the parametric variational problem seeks $u \in V_\sigma := L^2_\sigma(\Gamma, H^1_0(D))$ such that

$$\int_\Gamma \sigma(y) \int_D a(y,x)\nabla u(y,x)\cdot\nabla v(y,x)dx\,dy = \int_\Gamma \sigma(y)\int_D f(y,x)v(y,x)dx\,dy,$$

for all $v \in V_\sigma$.

Given a finite-dimensional subspace V_{hk} of V_σ, an element $u_{hk} \in V_{hk}$ is the stochastic Galerkin solution if

$$\int_\Gamma \sigma(y) \int_D a(y,x)\nabla u_{hk}(y,x)\cdot\nabla v(y,x)dx\,dy = \int_\Gamma \sigma(y)\int_D f(y,x)v(y,x)dx\,dy, \text{ for all } v \in V_{hk}.$$

Assume that V_h is an N-dimensional subspace of $H^1_0(D)$ and S_k is a Q-dimensional subspace of $L^2_\sigma(\Gamma)$ such that

$$V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\},$$
$$S_k = \text{span}\{\psi_1, \psi_2, \ldots, \psi_Q\}.$$

We assume that the basis $\{\psi_1, \psi_2, \ldots, \psi_Q\}$ is orthonormal with respect to σ, that is,

$$\int_\Gamma \sigma(y)\psi_n(y)\psi_m(y)dy = \delta_{nm},$$

where δ_{nm} is the Kronecker delta: $\delta_{nm} = 1$ for $n = m$, $\delta_{nm} = 0$ for $n \neq m$. To obtain a finite-dimensional subspace of V_σ, we use the commonly used process of tensorising the basic functions ϕ_i and ψ_j. That is, we define the following NQ-dimensional subspace for solving the discrete variational problem:

$$V_{hk} = V_h \otimes S_k := \text{span}\{\phi_i\psi_j |\ i = 1,\ldots,N,\ j = 1,\ldots,Q\}.$$

Therefore, any $v \in V_h \otimes S_k$ has the representation

$$v(y,x) = \sum_{i=1}^{N}\sum_{j=1}^{Q} V_{ij}\phi_i(x)\psi_j(y) = \sum_{j=1}^{Q}\left[\sum_{i=1}^{N} V_{ij}\phi_i(x)\right]\psi_j(y) = \sum_{j=1}^{Q} V_j(x)\psi_j(y),$$

where

$$V_j(x) \equiv \sum_{i=1}^{N} V_{ij}\phi_i(x).$$

Setting, $V_j := \left[V_{1j}, \cdots, V_{Nj}\right]^\top \in \mathbb{R}^N$, we introduce the following vectorized notation:

$$\begin{aligned} V &= \mathrm{vec}(V_{ij}) \\ &= [V_{11}, \cdots, V_{N1}, V_{12}, \cdots, V_{N2}, \cdots, V_{1Q}, \cdots, V_{NQ}]^\top \\ &= [V_1, V_2, \cdots, V_Q]^\top \in \mathbb{R}^{QN\times 1}. \end{aligned}$$

Inspired by the use of the KL expansion (see [38]), we assume that the unknown random field $a(\cdot,\cdot)$ admits a finite linear expansion:

$$a(y,x) = a_0(x) + \sum_{s=1}^{M} y_s a_s(x) = \sum_{s=0}^{M} y_s a_s(x), \tag{15.15}$$

where, by convention, we take $y_0 = 1$. We discretize the spatial components a_s by using a P-dimensional space

$$A_h = \mathrm{span}\{\varphi_1, ..., \varphi_P\}.$$

Using the vector notation once again, we have

$$a(y,x) = \sum_{i=1}^{P} A_{i0}\varphi_i(x) + \sum_{s=1}^{M}\left(\sum_{i=1}^{P} A_{is}\varphi_i(x)\right) y_s = \sum_{s=0}^{M} A_s y_s \tag{15.16}$$

where the vectors $A_s(x) \equiv (A_{is}) \in \mathbb{R}^P$ for $s = 0\ldots, M$, and

$$A = [A_0, A_1, \cdots, A_M]^\top \in \mathbb{R}^{P(M+1)\times 1}.$$

Recall that the discrete variational problem seeks $u_{hk}(y,x) \in V_h \otimes S_Q$ such that

$$\begin{aligned} &\int_\Gamma \sigma(y)\psi_n(y)\left(\int_D a(y,x)\nabla u_{hk}(y,x)\nabla\phi_i(x)dx\right)dy \\ &\qquad = \int_\Gamma \sigma(y)\psi_n(y)\left(\int_D f(y,x)\phi_i(x)dx\right)dy, \end{aligned}$$

for every $i = 1,\ldots,N,\ n = 1,\ldots,Q$.

Then using in the above variational problem the representation

$$u_{hk} = \sum_{k=1}^{N}\sum_{m=1}^{Q} U_{km}\phi_k(x)\psi_m(y),$$

we obtain

$$\begin{aligned} &\int_\Gamma \sigma(y)\psi_n(y)\left(\int_D a(y,x)\nabla\left(\sum_{k=1}^{N}\sum_{m=1}^{Q} U_{km}\phi_k(x)\psi_m(y)\right)\nabla\phi_i(x)dx\right)dy \\ &\qquad = \int_\Gamma \sigma(y)\psi_n(y)\left(\int_D f(y,x)\phi_i(x)dx\right)dy. \end{aligned}$$

By substituting the expansion (15.16) in the above identity, and performing a simple calculation, we obtain, for every $i = 1,\dots,N,\ n = 1,\dots,Q$:

$$\sum_{k=1}^{N}\sum_{m=1}^{Q} U_{km}\int_{\Gamma}\sigma(y)\psi_n(y)\psi_m(y)\left(\int_D a(y,x)\nabla\phi_k(x)\nabla\phi_i(x)dx\right)dy$$
$$= \left(K(A_0)+\sum_{s=1}^{M} g^s_{nn}K(A_s)\right)U_n + \sum_{N\neq N}\sum_{s=1}^{M} g^s_{nm}K(A_s)U_m$$

where for every $s \in \{0,\dots,M\}$, we define $K(A_s) \in \mathbb{R}^{n\times n}$ and $g^s_{nm} \in \mathbb{R}$ by

$$K(A_s)_{i,k} = \int_D A_s(x)\nabla\phi_k(x)\nabla\phi_i(x)dx,$$
$$g^s_{nm} = \int_{\Gamma}\sigma(y)\psi_n(y)\psi_m(y)y_s dy.$$

Now, for $s \in \{0,\dots,M\}$, we set

$$G^s = (g^s_{nm}) \in \mathbb{R}^{Q\times Q},$$

where, the case $s = 0$, by orthonormality, corresponds to the unit matrix as follows

$$G^0 = \left(\int_{\Gamma}\sigma(y)\psi_n(y)\psi_m(y)dy\right) = I.$$

On the other hand, we discretize the right-hand side as follows

$$(F_n)_i = \int_{\Gamma}\sigma(y)\psi_n(y)\int_D f(y,x)\phi_i(x)dx\,dy, \quad \text{for every } \ n \in \{1,...,Q\}.$$

Summarizing, the discrete variational problem reads

$$\left(K(A_0)+\sum_{s=1}^{M} g^s_{nn}K(A_s)\right)U_n + \sum_{m\neq n}\sum_{s=1}^{M} g^s_{nm}K(A_s)U_m = F_n, \text{ for every } n = 1,\dots,Q,$$

which corresponds to solving the linear system $K(A)U = F$ for $U = [U_1,U_2,\cdots,U_Q]^\top$, where $F = [F_1,F_2,\cdots,F_Q]^\top$ and the matrix $K(A)$ is given by:

$$K(A) := \left[\sum_{s=0}^{M} G^s \otimes K(A_s)\right],$$

where $\otimes$ is the Kronecker product.

The parametric analogue of (15.13) defining $e_{hk}(\cdot,\cdot) \in V_\sigma$ satisfies, for every $v \in V_\sigma$,

$$\langle e_{hk}(a,u),v\rangle_{V_\sigma} = \int_{\Gamma}\int_D \left[a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x) - f(\omega,x)v(\omega,x)\right]dx\,d\mu(\omega). \tag{15.17}$$

For a discrete form of the regularized equation error functional, we first obtain a discrete form of three inner products in V_σ. For $v_{hk}, w_{hk} \in V_\sigma$ with the following representations

$$v_{hk} = \sum_{k=1}^{N}\sum_{m=1}^{Q} V_{km}\phi_k(x)\psi_m(y),$$
$$w_{hk} = \sum_{k=1}^{N}\sum_{m=1}^{Q} W_{km}\phi_k(x)\psi_m(y),$$

we have

$$\begin{aligned}
\langle v_{hk}, w_{hk}\rangle_0 &:= \int_\Gamma \sigma(y)\int_D v_{hk}w_{hk}dxdy \\
&= \int_\Gamma\int_D \left(\sum_{i_1=1}^{N}\sum_{j_1=1}^{Q} V_{i_1j_1}\phi_{i_1}(x)\psi_{j_1}(y)\right)\left(\sum_{i_2=1}^{N}\sum_{j_2=1}^{Q} W_{i_2j_2}\phi_{i_2}(x)\psi_{j_2}(y)\right)\sigma(y)dxdy \\
&= \sum_{i_1,i_2=1}^{N}\sum_{j_1,j_2=1}^{Q} V_{i_1j_1}W_{i_2j_2}\left(\int_D \phi_{i_1}(x)\phi_{i_2}(x)dx\right)\left(\int_\Gamma \psi_{j_1}(y)\psi_{j_2}(y)\sigma(y)dy\right) \\
&= \sum_{j_1,j_2=1}^{Q} \widehat{V}_{j_1}^\top M_u \widehat{W}_{j_2}\left(\int_\Gamma \psi_{j_1}(y)\psi_{j_2}\sigma(y)dy\right) \\
&= V^\top (I\otimes M_u)W, \qquad (15.18)
\end{aligned}$$

where $\widehat{V}_{j_1} = [V_{1j_1},\ldots,V_{N,j_1}]^\top \in \mathbb{R}^N$, $V = [\widehat{V}_1,\ldots,\widehat{V}_Q]^\top \in \mathbb{R}^{NQ}$, $I \in \mathbb{R}^{Q\times Q}$ is the identity matrix, and $M_u \in \mathbb{R}^{N\times N}$ are given by

$$(M_u)_{i_1i_2} = \int_D \phi_{i_1}(x)\phi_{i_2}(x)dx.$$

Analogously, for computing semi-norms, we have

$$\begin{aligned}
\langle v_{hk}, w_{hk}\rangle_1 &:= \int_\Gamma\int_D \nabla v_{hk}\cdot\nabla w_{hk}\sigma(y)dxdy \\
&= \int_\Gamma\int_D \nabla\left(\sum_{i_1=1}^{N}\sum_{j_1=1}^{Q} V_{i_1j_1}\phi_{i_1}(x)\psi_{j_1}(y)\right)\cdot\nabla\left(\sum_{i_2=1}^{N}\sum_{j_2=1}^{Q} W_{i_2j_2}\phi_{i_2}(x)\psi_{j_2}(y)\right)\sigma(y)dxdy \\
&= \sum_{i_1,i_2=1}^{N}\sum_{j_1,j_2=1}^{Q} V_{i_1j_1}W_{i_2j_2}\left(\int_D \nabla\phi_{i_1}(x)\cdot\nabla\phi_{i_2}(x)dx\right)\left(\int_\Gamma \psi_{j_1}(y)\psi_{j_2}(y)\sigma(y)dy\right) \\
&= \sum_{j_1,j_2=1}^{Q} \widehat{V}_{j_1}^\top K\widehat{W}_{j_2}\left(\int_\Gamma \psi_{j_1}(y)\psi_{j_2}\sigma(y)dy\right) \\
&= V^\top (I\otimes K_u)W, \qquad (15.19)
\end{aligned}$$

where $K_u \in \mathbb{R}^{N\times N}$ is given by

$$(K_u)_{i_1 i_2} = \int_D \nabla\phi_{i_1}(x)\cdot\nabla\phi_{i_2}(x)dx.$$

Finally, combining the above two estimates, we have the following form:

$$\begin{aligned}\langle v_{hk}, w_{hk}\rangle_{V_\sigma} &= \int_\Gamma \sigma(y)\int_D [v_{hk}w_{hk} + \nabla v_{hk}\cdot\nabla w_{hk}]\,dxdy \\ &= V^\top (I\otimes(K_u+M_u))\,W.\end{aligned}$$

Assume that the discrete $e_{hk}(a,z)$ and the data z_{hk} have the representations:

$$\begin{aligned}e_{hk} &= \sum_{k=1}^{N}\sum_{m=1}^{Q} E_{km}(A,Z)\phi_k(x)\psi_m(y),\\ z_{hk} &= \sum_{k=1}^{N}\sum_{m=1}^{Q} Z_{km}\phi_k(x)\psi_m(y),\end{aligned}$$

where $E(A,Z), Z \in \mathbb{R}^{NQ}$.

Then, in view of the above calculations, we obtain from (15.17) that

$$(I\otimes(K_u+M_u))\,E(A,Z) = K(A)Z - F = \left[\sum_{s=0}^{M} G^s\otimes K(A_s)\right]Z - F,$$

or equivalently,

$$\begin{aligned}E(A,Z) &= (I\otimes(K_u+M_u))^{-1}\left[\left[\sum_{s=0}^{M} G^s\otimes K(A_s)\right]Z - F\right]\\ &= I\otimes(K_u+M_u)^{-1}\left[\left[\sum_{s=0}^{M} G^s\otimes K(A_s)\right]Z - F\right].\end{aligned} \tag{15.20}$$

The above expression at once gives the discrete formula for the equation error formulation (without the regularization term):

$$J_{\mathcal{E}}(A) = \frac{1}{2}\|E(A,Z)\|^2 = \left\langle K(A)Z - F, (I\otimes(K_u+M_u))^{-1}(K(A)Z-F)\right\rangle,$$

where $K(A) = \left[\sum_{s=0}^{M} G^s\otimes K(A_s)\right]$.

A formula for the gradient of $J_{\mathcal{E}}$ can be computed using the linearity of E with respect to A. Moreover, a discrete formula for a norm regularizer can also be easily obtained by the formulas given above for various inner products.

15.6 Computational Experiments

In this section, we present results of some numerical experiments for one-dimensional problems where we use the equation error method with stochastic Galerkin. We assume that the desired parameter is in form of a finite linear combination:

$$a(\omega,x) = a_0(x) + \sum_{\ell=1}^{M} a_\ell(x)\, Y_\ell(\omega),$$

and that the joint distribution of $(Y_1, Y_2, \ldots, Y_M)$ is known a priori. Piecewise linear finite elements are used and we use the H^1 semi-norm regularization for the objective functional. Relative errors in the mean and variance of a and u are computed by the error functional defined below. For the mean and variance of the coefficient a, we use

$$\varepsilon^{SG}_{\text{mean}}(a) = \frac{\sqrt{\int_D \left(\mathbb{E}[\bar{a}(\cdot,x)] - \mathbb{E}[a^{SG}(\cdot,x)]\right)^2 dx}}{\sqrt{\int_D \mathbb{E}[\bar{a}(\cdot,x)]^2 dx}},$$

$$\varepsilon^{SG}_{\text{var}}(a) = \frac{\sqrt{\int_D \left(\mathbb{V}\text{ar}[\bar{a}(\cdot,x)] - \mathbb{V}\text{ar}[a^{SG}(\cdot,x)]\right)^2 dx}}{\sqrt{\int_D \mathbb{V}\text{ar}[\bar{a}(\cdot,x)]^2 dx}}.$$

Similarly, the relative error functional for the mean and variance of u are defined by

$$\varepsilon^{SG}_{\text{mean}}(u) = \frac{\sqrt{\int_D \left(\mathbb{E}[\bar{u}(\cdot,x)] - \mathbb{E}[u(a^{SG})(\cdot,x)]\right)^2 dx}}{\sqrt{\int_D \mathbb{E}[\bar{u}(\cdot,x)]^2 dx}},$$

$$\varepsilon^{SG}_{\text{var}}(u) = \frac{\sqrt{\int_D \left(\mathbb{V}\text{ar}[\bar{a}(\cdot,x)] - \mathbb{V}\text{ar}[u(a^{SG})(\cdot,x)]\right)^2 dx}}{\sqrt{\int_D \mathbb{V}\text{ar}[\bar{u}(\cdot,x)]^2 dx}}.$$

Example 15.6.1 This example has a single random variable Y_1 and we assume that $Y_1 \sim U[0,1]$ (i.e., Y_1 is uniformly distributed on $[0,1]$). Moreover, SPDE (15.1) is satisfied by the following data set on the domain $D = (0,1)$:

$$\begin{aligned}
\bar{a}(\omega,x) &= 1 + Y_1(\omega),\\
\bar{u}(\omega,x) &= x(1-x) + Y_1(\omega)\sin(\pi x),\\
f(\omega,x) &= (1+Y_1(\omega))\left(2+\pi^2 Y_1(\omega)\sin(\pi x)\right).
\end{aligned}$$

Figure 15.1 shows realizations of the exact and the estimated coefficients $a(\omega,x)$, and corresponding point-wise errors in the coefficient in the top row. The second row figures are realizations of the simulated solution $u = u(a)$ (each curve shows

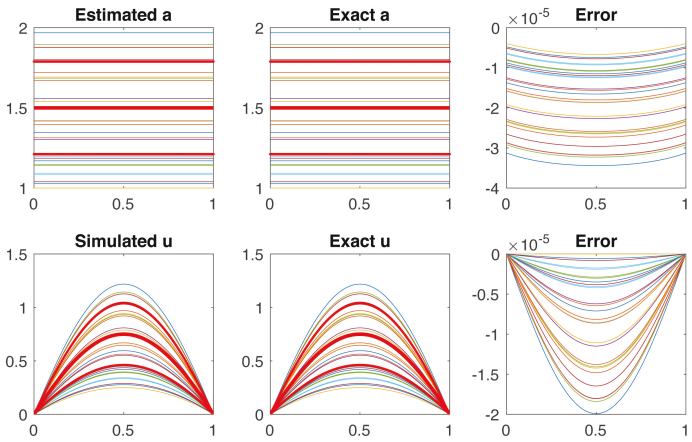

Figure 15.1: A comparison of 30 realizations of the coefficient a and the solution u in the inverse problem of identifying a in Example 15.6.1. Mesh size is $h = 1/50$ and the regularization parameter is $\kappa = 10^{-6}$. The thick red line represents the mean of the random fields a and u, and the red dotted lines represent ± 1 standard deviation from the mean.

a solution of the direct problem using the estimated coefficient a for a fixed value of ω) along with the exact solution u and corresponding point-wise errors. Relative errors for the mean and variance of a and u for various values of the $R = \dim(V_h)$ (mesh size is $h = 1/(R+1)$) are shown in the Table 15.1.

Table 15.1: Relative errors for the equation error method with stochastic Galerkin for Example 15.6.1. The numbers correspond to the case where the regularization parameter κ is fixed at 10^{-6}.

$\dim(V_h)$	$\varepsilon^{SG}_{\text{mean}}(a)$	$\varepsilon^{SG}_{\text{var}}(a)$	$\varepsilon^{SG}_{\text{mean}}(u)$	$\varepsilon^{SG}_{\text{var}}(u)$
50	2.1425e−4	9.1465e−4	7.7522e−5	6.5774e−4
100	5.3570e−5	2.2874e−4	1.9385e−5	1.6448e−4
150	2.3810e−5	1.0167e−4	8.6157e−6	7.3105e−5
200	1.3393e−5	5.7189e−5	4.8464e−6	4.1122e−5

Example 15.6.2 We take $D = (0,1)$, and assume that the two random variables involved are independent from each other and $Y_1, Y_2 \sim U[0,1]$. Moreover, SPDE (15.1) is satisfied by the following data set:

$$\bar{a}(\omega,x) = 3 + x^2 + Y_1(\omega)\cos(\pi x) + Y_2(\omega)\sin(2\pi x),$$
$$\bar{u}(\omega,x) = x(1-x)Y_1(\omega).$$

Realizations of the exact coefficient, the identified coefficient, the exact solution, and the simulated solution are shown in Figure 15.2 along with the corresponding

errors. Relative errors for the mean and variance of a and u for various values of the $Q = \dim(V_h)$ are shown in the Table 15.2.

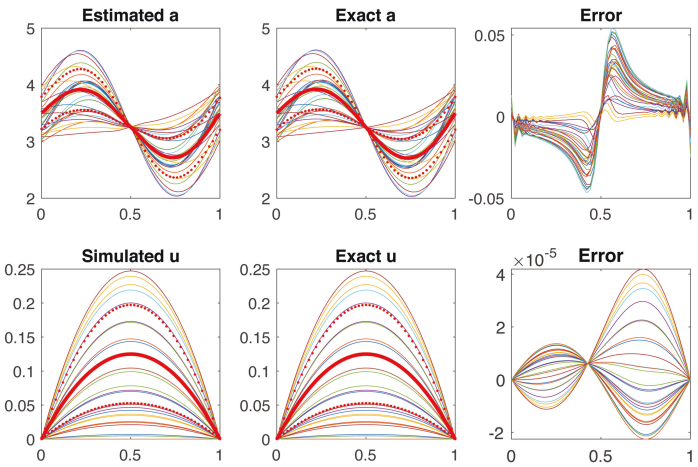

Figure 15.2: A comparison of 30 realizations of a and u in the inverse problem of identifying a in Example 15.6.2. Mesh size is $h = 1/200$ and the regularization parameter is $\kappa = 10^{-5}$. The thick red line represents the mean of the random fields a and u, and the red dotted lines represent ± 1 standard deviation from the mean.

Table 15.2: Relative errors for the equation error method with stochastic Galerkin for Example 15.6.2. The numbers correspond to the case where the regularization parameter κ is 10^{-6}.

$\dim(V_h)$	$\varepsilon^{SG}_{\text{mean}}(a)$	$\varepsilon^{SG}_{\text{var}}(a)$	$\varepsilon^{SG}_{\text{mean}}(u)$	$\varepsilon^{SG}_{\text{var}}(u)$
50	0.0036	0.0096	1.7321e−4	3.4020e−4
100	0.0034	0.0098	4.3343e−5	8.5117e−5
150	0.0033	0.0097	1.9267e−5	3.7835e−5
200	0.0033	0.0095	1.0838e−5	2.1283e−5

Example 15.6.3 We consider an example from [31]. In the example, we have $D = (0,1)$ and the coefficient and the solution have the following expansions:

$$\bar{u}(\omega,x) = x\left(1-x^2\right) + \sum_{n=1}^{M} \sin(2n\pi x)\, Y_n,$$

$$\bar{a}(\omega,x) = \left(1+x^3\right) + \sum_{n=1}^{M} \cos\left(\frac{n\pi x}{2M}\right) Y_n.$$

Here, $Y_1, Y_2, \ldots, Y_M$ are independent and uniformly distributed on $[0,1]$.

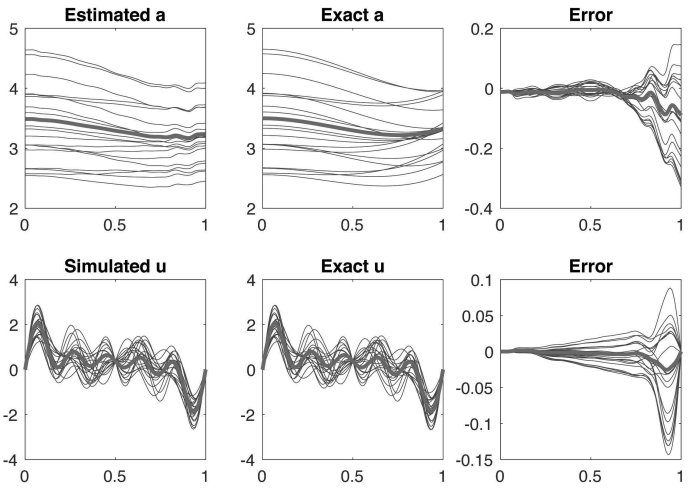

Figure 15.3: A comparison of 20 realizations of a and u in the inverse problem of identifying a in Example 15.6.3. The thick bold lines represent the means of the random fields a and u.

Results of the numerical experiments with 5 random variables ($M = 5$ case) are shown in Figure 15.3. We note that the computational time is dependent on the number of random variables present in the problem (for stochastic Galerkin method, stiffness matrices are assembled for each term a_ℓ in the finite linear expansion of the coefficient a). Equation error method is very efficient in solving the inverse problem compared to other traditional methods such as output least squares (OLS) since the direct problem is not solved at every optimization iteration. For Example 15.6.2 (two random variables), the method takes less than a minute in MATLAB for any reasonable mesh size h that gives a good resolution of both the solution u and the coefficient a (for example, $h = 1/100$ or $1/200$). Computational efforts are scaled proportionally for problems with more random variables (see Example 15.6.3) in one-dimensional case.

15.7 Concluding Remarks

We presented an overview of the recent developments in a stochastic inverse problem and proposed a new approach using equation error method in this work. Preliminary numerical results we obtained demonstrate that the equation error approach is very efficient and delivers good quality identifications for the coefficient. However, a thorough comparison with other methods that use optimization formulations (especially for problems in two- or three-dimensions) is necessary. Sensitivity of the method for data with noise should be also studied carefully. Convergent behavior of the method is observed numerically, and we note that

any advance towards developing a rigorous error estimate for the identified coefficient would be of particular importance.

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