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NONLINEAR QUASI-HEMIVARIATIONAL INEQUALITIES: EXISTENCE AND OPTIMAL CONTROL*

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Abstract. In this paper, we investigate a generalized nonlinear quasi-hemivariational inequality (QHI) involving a multivalued map in a Banach space. Under general assumptions, by using a fixed point theorem combined with the theory of nonsmooth analysis and the Minty technique, we prove that the set of solutions for the hemivariational inequality associated to the QHI problem is nonempty, bounded, closed, and convex. Then, we prove the existence of a solution to QHI. Furthermore, an optimal control problem governed by QVI is introduced, and a solvability result for the optimal control problem is established. Finally, an approximation of an elastic contact problem with the constitutive law involving a convex subdifferential inclusion is studied as an illustrative application, in which approximate contact boundary conditions are described by a multivalued version of the normal compliance contact condition with frictionless effect and a frictional contact law with the slip dependent coefficient of friction.

Key words. quasi-hemivariational inequality, existence, optimal control, Kuratowski limit, elastic approximate contact problem

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1. Introduction. In numerous complicated physical processes and engineering applications, mathematical models often lead to inequalities instead of the more commonly seen equations. In this context, two classes of inequality problems have been widely studied, namely, variational inequalities and hemivariational inequalities. Variational inequalities emerge from applied models with an underlying convex structure and have been studied extensively since the early sixties. Some representative references include [4, 6, 8, 9, 16, 33, 44] on mathematical theories and [17, 29] on numerical treatment. On the other hand, hemivariational inequalities, introduced

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by Panagiotopoulos in the early 1980s, stem from engineering problems involving non-smooth, nonmonotone, and possibly multivalued relations describing deformable bodies; see [46]. Compared to the theory of variational inequalities, the analysis of hemivariational inequalities is more involved and makes use of the properties of Clarke's subgradient, defined for locally Lipschitz, possibly nonconvex functions. Substantial advances have been made on modeling, analysis, numerical approximation, and computer simulations of hemivariational inequalities. Comprehensive references in the area include [7, 45, 47] in the earlier years and [32, 38, 40, 43] more recently.

In the study of classical variational and hemivariational inequalities, the constraint set, if present, remains independent of the sought solution. However, in many critical situations arising in engineering and economic models, such as Nash equilibrium problems with shared constraints and transport optimization feedback control problems, the constraint set depends explicitly on the unknown solution. This dependence leads naturally to the notion of quasi-variational and quasi-hemivariational inequalities. Recently, numerous authors have contributed to strengthening the theory and applicability of quasi-variational and quasi-hemivariational inequalities. In the following, we provide a brief review of some of the related developments. Gwinner et al. [20] investigated inverse problems of identifying variable parameters in variational and quasi-variational inequalities by using an abstract nonsmooth regularization approach. They discretized the inverse problem and gave the convergence analysis for the discrete problems. In the same vein, by employing Mosco-type continuity properties and Kluge's fixed point theorem for the multivalued map, Khan and Motreanu [25] presented existence results for elliptic and evolutionary variational and quasi-variational inequalities. Liu, Motreanu, and Zeng [35] have examined a notion of well-posedness for differential mixed quasi-variational inequalities in Hilbert spaces. Khan and Sama [27] have proved existence results for an optimal control problem for a quasi-variational inequality with multivalued pseudomonotone maps, and when some noise contaminates the data for the underlying quasi-variational inequality, they provided a convergence analysis of the control. Khan, Tammer, and Zalinescu [28] have employed an elliptic regularization technique to study an ill-posed quasi-variational inequality with contaminated data and showed that a sequence of bounded regularized solutions converges strongly to a solution of the original quasi-variational inequality. For more details on this topic, the reader is referred to Alleche and Rădulescu [1], Aussel, Sultana, and Vetrivel [3], Khan, Migórski, and Sama [24], Migórski, Khan, and Zeng [39], Aussel, Gupta, and Mehra [2], Gwinner [19], Liu and Zeng [36], Khan and Motreanu [26], and the cited references therein.

Before any advancement, let us first introduce the problem that will play the central role in this study. Let V be a real reflexive Banach space with the norm $\|\cdot\|_V$, V^* be the dual space of V, and X and Y be two Banach spaces. Let C be a nonempty, closed, and convex subset of V, $K\colon C \to 2^C$ and $T\colon V \to 2^{V^*}$ be two multivalued maps, $\varphi\colon V \times V \to \mathbb{R}$ be a function, $J\colon X \to \mathbb{R}$ be a locally Lipschitz functional, $\gamma\colon V \to X$ and $\pi\colon V \to Y$ be two operators, and $f \in Y^*$.

PROBLEM 1.1. *Given $f \in Y^*$, we consider the generalized nonlinear quasi-hemi variational inequality of finding $u \in C$ and $u^* \in T(u)$ such that $u \in K(u)$ and*

$$\langle u^*, v-u\rangle + \varphi(v,u) + J^0(\gamma u; \gamma(v-u)) \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} \text{ for all } v \in K(u). \tag{1.1}$$

The main contribution of this research is threefold. First, we aim to deliver solvability conditions for the above quasi-hemivariational inequality. Second, we investigate an optimal control problem associated to the above quasi-hemivariational inequality. Finally, we provide novel applications of our abstract results to an approximate elastic contact model. We recall that the constitutive law in the model is described by a convex subdifferential inclusion, while the contact boundary conditions are considered with the following two perspectives:

(i) a multivalued version of the normal compliance contact condition with frictionless effect,
(ii) a frictional contact law with the slip dependent coefficient of friction.

The adopted methodology to achieve the main goals is as follows. For the solvability of the optimal control problem, Problem 4.1, we first verify that the solution map with respect to f is bounded, weakly upper semicontinuous, and weakly closed. Then, by applying these results, we prove the existence of the optimal control problem by employing a Weierstrass type theorem and the concept of the Kuratowski upper limit.

The outline of the paper is as follows. Section 2 collects the necessary notation and preliminary results. In section 3, we prove an existence result for a generalized nonlinear quasi-hemivariational inequality by applying the Kluge fixed point principle. In section 4, we formulate an optimal control problem governed by the quasi-hemivariational inequality and provide an existence theorem for the optimal control problem. Section 5 provides an application of these results to an approximate elastic contact problem.

2. Preliminaries.

In this section, we briefly review basic notation and results which are needed in the paper. For more details, we refer to the monographs [7, 12, 13, 58].

Throughout the paper, we denote by $\langle \cdot, \cdot \rangle_{Y^* \times Y}$ the duality pairing between a Banach space Y and its dual Y^*. The symbols "$\rightharpoonup$" and "$\to$" stand for the weak and the strong convergence in various spaces. The norm in a normed space Y is denoted by $\| \cdot \|_Y$. Given a subset D of Y, we write $\|D\|_Y := \sup\{\|v\|_Y \mid v \in Y\}$. If no confusion arises, we often drop the subscripts. Furthermore, we denote by $\mathcal{L}(Y_1, Y_2)$ the space of linear and bounded operators from a normed space Y_1 to a normed space Y_2 endowed with the usual norm $\| \cdot \|_{\mathcal{L}(Y_1,Y_2)}$.

We first recall definitions and properties of semicontinuous multivalued maps.

Definition 2.1. *Let X and Y be topological spaces, and let $F\colon X \to 2^Y$ be a multivalued map. The map F is called*

(i) *upper semicontinuous at $x \in X$ if for every open set $O \subset Y$ with $F(x) \subset O$, there exists a neighborhood $N(x)$ of x such that $F(N(x)) := \cup_{y \in N(x)} F(y) \subset O$, and if this holds for every $x \in X$, then F is called upper semicontinuous;*
(ii) *closed at $x_0 \in X$ if for every sequence $\{(x_n, y_n)\} \subset \mathrm{Gr}(F)$ such that $(x_n, y_n) \to (x_0, y_0)$ in $X \times Y$, we have $(x_0, y_0) \in \mathrm{Gr}(F)$, where $\mathrm{Gr}(F)$ is the graph of F defined by $\mathrm{Gr}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$. We say that F is closed (or F has a closed graph) if it is closed at every $x_0 \in X$.*

Proposition 2.2. *Let $F\colon X \to 2^Y$ with X and Y topological spaces. The following statements are equivalent:*

(i) *F is upper semicontinuous.*
(ii) *For each closed set $C \subset Y$, $F^-(C) := \{x \in X \mid F(x) \cap C \neq \emptyset\}$ is closed in X.*
(iii) *For each open set $O \subset Y$, $F^+(O) := \{x \in X \mid F(x) \subset O\}$ is open in X.*

Theorem 2.3. *Let X, Y be topological spaces and $F\colon X \to 2^Y$ be an upper semicontinuous multivalued map such that for each $x \in X$ the set $F(x)$ is compact in Y. If $\{x_\alpha\}$ is a net in X with $x_\alpha \to x_0$ and $y_\alpha \in F(x_\alpha)$ for each α, then there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Let $(X, \|\cdot\|_X)$ be a Banach space. A function $J\colon X \to \mathbb{R}$ is called locally Lipschitz continuous at $u \in X$ if there exists a neighborhood $N(u)$ of u and a constant $L_u > 0$ such that

$$|J(w) - J(v)| \le L_u \|w - v\|_X \quad \text{for all} \quad w, v \in N(u).$$

Definition 2.4. *Given a locally Lipschitz function $J\colon X \to \mathbb{R}$, we denote by $J^0(u; v)$ the generalized (Clarke) directional derivative of J at the point $u \in X$ in the direction $v \in X$ defined by*

$$J^0(u; v) = \limsup_{\lambda \to 0^+,\, w \to u} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

The generalized gradient of $J\colon X \to \mathbb{R}$ at $u \in X$ is given by

$$\partial J(u) = \{\, \xi \in X^* \mid J^0(u; v) \ge \langle \xi, v\rangle_{X^* \times X} \quad \text{for all} \quad v \in X \,\}.$$

The generalized gradient and generalized directional derivative of a locally Lipschitz function enjoy nice properties and rich calculus. Here we collect some basic results (see [40, Proposition 3.23]).

Proposition 2.5. *Let $J\colon X \to \mathbb{R}$ be a locally Lipschitz function. Then, the following holds:*

(i) *For every $x \in X$, the function $X \ni v \mapsto J^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $J^0(x; \lambda v) = \lambda J^0(x; v)$ for all $\lambda \ge 0$, $v \in X$, and $J^0(x; v_1 + v_2) \le J^0(x; v_1) + J^0(x; v_2)$ for all v_1, $v_2 \in X$, respectively.*

(ii) *For every $v \in X$, it holds that $J^0(x; v) = \max\{\, \langle \xi, v\rangle_{X^* \times X} \mid \xi \in \partial J(x) \,\}$.*

(iii) *The function $X \times X \ni (u, v) \mapsto J^0(u; v) \in \mathbb{R}$ is upper semicontinuous.*

We conclude this section with the following two fixed point theorems for multivalued maps which will play a critical role in the existence results for the inequality problems.

Theorem 2.6 (see [57]). *Let $K \ne \emptyset$ be a convex subset of a Hausdorff topological vector space E. Let $G\colon K \to 2^K$ be a multivalued map such that*

(i) *for each $u \in K$, the set $G(u)$ is a nonempty convex subset of K,*

(ii) *for each $v \in K$, $G^{-1}(v) = \{u \in K \mid v \in G(u)\}$ contains a relatively open subset O_v (O_v may be empty for some v),*

(iii) $\bigcup_{v \in K} O_v = K$,

(iv) *there exists a nonempty set K_0 contained in a compact convex subset K_1 of K such that $D = \bigcap_{v \in K_0} O_v^{\complement}$ is either empty or compact, where $O_v^{\complement}$ denotes the complement of O_v.*

Then, there exists $u_0 \in K$ such that $u_0 \in G(u_0)$.

Theorem 2.7 (see [30]). *Let Z be a reflexive Banach space and $C \subset Z$ be nonempty, closed, and convex. Assume that $\Psi\colon C \to 2^C$ is a multivalued map such that for every $u \in C$, the set $\Psi(u)$ is nonempty, closed, and convex, and the graph of Ψ is sequentially weakly closed. If either C is bounded or $\Psi(C)$ is bounded, then the map Ψ has at least one fixed point in C.*

3. Existence results.

In this section, we are interested in giving existence results for the generalized quasi-hemivariational inequality. For this, we impose the following hypotheses on the data:

(H_C) C is a nonempty, closed, and convex subset of V.
(H_J) $J\colon X \to \mathbb{R}$ is a locally Lipschitz function.
(H_γ) $\gamma\colon V \to X$ is a linear, bounded operator with its adjoint $\gamma^*\colon X^* \to V^*$.
(H_0) $\pi\colon V \to Y$ is a linear, bounded operator with its adjoint $\pi^*\colon Y^* \to V^*$, and $f \in Y^*$.
(H_T) $T\colon C \to 2^{V^*}$ is a multivalued map such that
(i) $T\colon C \to 2^{V^*}$ is upper semicontinuous and compact and convex valued;
(ii) the multivalued map $C \ni u \mapsto T(u) + \gamma^*\partial J(\gamma u) \in V^*$ is (φ, h)-stably pseudomonotone with respect to $\{\pi^* f\}$, i.e., for all u, $v \in C$ if there exist $u^* \in T(u)$ and $\eta_u \in \partial J(\gamma u)$ such that

$$\langle u^* + \gamma^*\eta_u - \pi^* f, v - u\rangle + \varphi(v, u) \geq 0,$$

then

$$\langle v^* + \gamma^*\eta_v - \pi^* f, v - u\rangle + \varphi(v, u) \geq h(v - u)$$

for all $v^* \in T(v)$ and all $\eta_v \in \partial J(\gamma v)$, where $h\colon V \to \mathbb{R}$ is such that

$$\limsup_{t\to 0^+} \frac{h(tu)}{t} \geq 0 \quad \text{for all} \quad u \in V,$$

and, for any sequence $\{v_n\} \subset V$ with $v_n \rightharpoonup v$ in V, satisfies

$$h(v) \leq \limsup_{n\to\infty} h(v_n). \tag{3.1}$$

(H_φ) $\varphi\colon V \times V \to \mathbb{R}$ is such that
(i) for each $u \in V$, the function $V \ni v \mapsto \varphi(v, u)$ is convex and lower semicontinuous;
(ii) for each $v \in V$, the function $V \ni u \mapsto \varphi(v, u)$ is concave and upper semicontinuous;
(iii) for all $v \in V$, we have $\varphi(v, v) = 0$.
(H_K) $K\colon C \to 2^C$ is such that for all $u \in C$, the set $K(u) \subseteq C$ is nonempty, closed, convex, and
(i) for any sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x$, and for any $y \in K(x)$, there exists a sequence $\{y_n\} \subset C$ such that $y_n \in K(x_n)$ and $y_n \to y$, as $n \to \infty$;
(ii) for all sequences $\{x_n\}$ and $\{y_n\}$ in C with $y_n \in K(x_n)$, if $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, then $y \in K(x)$.
(H_{C_0}) There is a bounded subset C_0 of V with $K(u) \cap C_0 \neq \emptyset$ for each $u \in C$. Further, if C is unbounded, then for all $v_0 \in C_0$, it holds that

$$\lim_{u\in C,\, \|u\|_V\to\infty} \frac{\inf_{u^*\in T(u)}\langle u^*, u - v_0\rangle + \inf_{\eta_u\in\partial J(\gamma u)}\langle \eta_u, u - v_0\rangle_{X^*\times X} - \varphi(v_0, u)}{\|u\|_V} = +\infty. \tag{3.2}$$

Remark 3.1. The notion of (φ, h)-stable pseudomonotonicity used in hypothesis (H_T)(ii) has been considered in several works; see [37, 56, 59]. If $u \mapsto T(u) + \gamma^*\partial J(\gamma u)$ is monotone, then it is (φ, h)-stably pseudomonotone with $h = 0$. On the other hand, when φ is defined by $\varphi(v, u) = \phi(v) - \phi(u)$ with a convex and lower semicontinuous function $\phi\colon V \to \mathbb{R}$, then hypothesis (H_φ) holds automatically.

An interesting example of the function φ is given in the following.

Example 3.2. Let $V = L^2(\Omega)$ and $K = \{u \in V \mid u(x) \geq 0 \text{ for a.e. } x \in \Omega\}$, where Ω is a bounded domain in $\mathbb{R}^N$. Then, the function $\varphi\colon K \times K \to \mathbb{R}$ defined by

$$\varphi(v, u) = \int_\Omega v(x)u(x)\, dx - \|u\|^2_{L^2(\Omega)} \quad \text{for} \quad u, v \in K$$

satisfies assumption (H_φ). Moreover, $\varphi(v, u) + \varphi(u, v) \leq 0$ for all u, $v \in K$.

For the solvability of Problem 1.1, we first consider the following generalized hemivariational inequality.

PROBLEM 3.3. *For a given $f \in Y^*$, find $u \in C$ such that there exists $u^* \in T(u)$ and the following inequality holds:*

$$\langle u^*, v - u\rangle + \varphi(v, u) + J^0(\gamma u; \gamma(v - u)) \geq \langle f, \pi(v - u)\rangle_{Y^*\times Y} \quad \textit{for all } v \in C. \tag{3.3}$$

In what follows, the solution set of Problem 3.3 will be denoted by $\mathrm{SOL}(C; T, J, \varphi, f)$. The following result of Minty type provides the nonemptiness, convexity, and closedness of $\mathrm{SOL}(C; T, J, \varphi, f)$.

THEOREM 3.4. *Assume that (H_C), (H_J), (H_γ), (H_0), (H_T), (H_φ), and* (3.2) *hold. Then,*

(i) *an element $u \in C$ is a solution to Problem* 3.3 *if and only if it solves the following inequality: find $u \in C$ such that for all $v^* \in T(v)$, $\eta_v \in \partial J(\gamma v)$ and all $v \in C$, we have*

$$\langle v^*, v-u\rangle + \varphi(v, u) + \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u); \tag{3.4}$$

(ii) *the solution set $\mathrm{SOL}(C; T, J, \varphi, f)$ is nonempty, bounded, and weakly closed in V;*

(iii) *if, in addition, h is a convex function, then $\mathrm{SOL}(C; T, J, \varphi, f)$ is convex in V.*

Proof. (i) Let $u \in C$ be a solution to Problem 3.3. Then, there exists $u^* \in T(u)$ such that inequality (3.3) holds. Let $v \in C$. It follows from Proposition 2.5(ii) that one can find an element $\xi_u \in \partial J(\gamma u)$ such that

$$J^0(\gamma u; \gamma(v - u)) = \langle \xi_u, \gamma(v - u)\rangle_{X^*\times X}.$$

This equality combined with (3.3) implies

$$0 \leq \langle u^* - \pi^* f, v-u\rangle + \varphi(v, u) + J^0(\gamma u; \gamma(v-u)) = \langle u^* - \pi^* f, v-u\rangle + \varphi(v, u) + \langle \xi_u, \gamma(v-u)\rangle_{X^*\times X}.$$

The latter, due to the (φ, h)-stable pseudomonotonicity of $T(\cdot) + \gamma^*\partial J(\gamma\cdot)$, entails

$$\langle v^*, v - u\rangle + \varphi(v, u) + \langle \eta_v, \gamma(v - u)\rangle_{X^*\times X} \geq \langle f, \pi(v - u)\rangle_{Y^*\times Y} + h(v - u)$$

for all $v^* \in Tv$, $\eta_v \in \partial J(\gamma v)$ and all $v \in C$. Hence, u solves problem (3.4).

Conversely, assume that $u \in C$ is a solution to problem (3.4). Let $w \in C$ and $t \in (0, 1)$ be arbitrary. We now insert $v = v_t := tw + (1 - t)u$ into (3.4) to get

$$\begin{aligned} &t\langle f, \pi(w - u)\rangle_{Y^*\times Y} + h(t(w - u)) \\ &\quad\leq t\langle v_t^*, w - u\rangle + \varphi(v_t, u) + t\langle \eta_{v_t}, \gamma(w - u)\rangle_{X^*\times X} \\ &\qquad (\text{for all } \eta_{v_t} \in \partial J(\gamma v_t)) \\ &\quad\leq t\langle v_t^*, w - u\rangle + t\varphi(w, u) + (1 - t)\varphi(u, u) + tJ^0(\gamma v_t; \gamma(w - u)) \\ &\quad= t\langle v_t^*, w - u\rangle + t\varphi(w, u) + tJ^0(\gamma v_t; \gamma(w - u)) \end{aligned}$$

for all $v_t^* \in T(v_t)$, where we used (H_φ)(i), (iii) and the definition of the Clarke subgradient. Therefore,

$$\langle v_t^*, w-u\rangle + \varphi(w, u) + J^0(\gamma v_t; \gamma(w-u)) \geq \langle f, \pi(w-u)\rangle_{Y^*\times Y} + \frac{h(t(w - u))}{t} \text{ for all } v_t^* \in T(v_t). \tag{3.5}$$

Fix a sequence $\{v_t^*\}$ with $v_t^* \in T(v_t)$ for $t \in (0,1)$. Recalling that T is upper semicontinuous with compact values (see hypothesis (H_T)(i)), it follows from Theorem 2.3 that there exists a subnet of $\{v_t^*\}$ which converges to a point of Tu, as $t \to 0^+$. Without any loss of generality, we may suppose that

$$v_t^* \to u^*(w) \quad \text{for some} \quad u^*(w) \in T(u), \quad \text{as} \quad t \to 0^+. \tag{3.6}$$

Passing to the upper limit, as $t \to 0^+$, in inequality (3.5) and taking into account (3.6), Proposition 2.5(iii), and the fact that $\limsup_{t\to 0^+} \frac{h(tu)}{t} \geq 0$ for all $u \in V$, we have

$$\begin{aligned}
&\langle u^*(w), w-u\rangle + \varphi(w,u) + J^0(\gamma u; \gamma(w-u)) \\
&\quad \geq \limsup_{t\to 0^+} \langle v_t^*, w-u\rangle + \varphi(w,u) + \limsup_{t\to 0^+} J^0(\gamma v_t; \gamma(w-u)) \\
&\quad \geq \limsup_{t\to 0^+} \left[\langle v_t^*, w-u\rangle + \varphi(w,u) + J^0(\gamma v_t; \gamma(w-u))\right] \\
&\quad \geq \langle f, \pi(w-u)\rangle_{Y^*\times Y} + \limsup_{t\to 0^+} \frac{h(t(w-u))}{t} \\
&\quad \geq \langle f, \pi(w-u)\rangle_{Y^*\times Y}.
\end{aligned}$$

Summing up, we have shown that for each $w \in C$, there exists an element $u^*(w) \in T(u)$ such that

$$\langle u^*(w), w-u\rangle + \varphi(w,u) + J^0(\gamma u; \gamma(w-u)) \geq \langle f, \pi(w-u)\rangle_{Y^*\times Y}. \tag{3.7}$$

Next, we assume that $u \in C$ is not a solution to Problem 3.3. Then for each $u^* \in T(u)$ there is $v \in C$ such that

$$\langle u^*, v-u\rangle + \varphi(v,u) + J^0(\gamma u; \gamma(v-u)) < \langle f, \pi(v-u)\rangle_{Y^*\times Y}.$$

We denote $R(u) = T(u) + \gamma^*\partial J(\gamma u)$. From the geometric version of the Hahn–Banach theorem (see, e.g., [5, Theorem 1.7]), we infer that for each $v^* \in R(u)$, there exists $v \in C$ such that

$$\langle v^*, v-u\rangle < -\varphi(v,u) + \langle f, \pi(v-u)\rangle_{Y^*\times Y}. \tag{3.8}$$

For any $v \in C$, we now consider the set $S_v \subset R(u)$ defined by

$$S_v := \{v^* \in R(u) \mid \langle v^*, v-u\rangle < -\varphi(v,u) + \langle f, \pi(v-u)\rangle_{Y^*\times Y}\}.$$

Moreover, it is not difficult to prove that for every $v \in C$, the set S_v is weakly open in V^*. Besides, we observe that $\{S_v\}_{v\in C}$ is an open covering of $R(u)$. The latter combined with the facts that V is reflexive and $R(u)$ is weakly compact and convex in V^* ensures that $R(u)$ has a finite subcovering in $\{S_v\}_{v\in C}$. Let $\{S_{v_1}, S_{v_2}, \ldots, S_{v_n}\}$ be the finite subcovering indicated by the points $\{v_1, v_2, \ldots, v_n\}$. Let κ_1, $\kappa_2, \ldots, \kappa_n$ be a partition of unity for $R(u)$, where for each $i = 1, 2, \ldots, n$, $\kappa_i \colon R(u) \to [0,1]$ is a weakly continuous function such that $\sum_{i=1}^n \kappa_i(v^*) = 1$ for all $v^* \in R(u)$ (see [18, Lemma 7.3]). Subsequently, we introduce a function $\mathcal{N}\colon R(u) \to V$ defined by

$$\mathcal{N}(v^*) = \sum_{i=1}^n \kappa_i(v^*)(v_i) \quad \text{for all} \quad v^* \in R(u).$$

Obviously, the function $\mathcal{N}$ is also weakly continuous due to the weak continuity of κ_i for $i = 1, 2, \ldots, n$. For any $v^* \in R(u)$, the estimate

$$\langle v^*, \mathcal{N}(v^*) - u\rangle = \langle v^*, \sum_{i=1}^{n} \kappa_i(v^*)v_i - u\rangle \le \sum_{i=1}^{n} \kappa_i(v^*)\langle v^*, v_i - u\rangle$$
$$< -\sum_{i=1}^{n} \kappa_i(v^*)\varphi(v_i, u) + \sum_{i=1}^{n} \kappa_i(v^*)\langle f, \pi(v_i - u)\rangle_{Y^*\times Y}$$

and the convexity of $v \mapsto \varphi(v, u)$ imply

$$\langle v^*, \mathcal{N}(v^*) - u\rangle < -\varphi\left(\sum_{i=1}^{n} \kappa_i(v^*)v_i, u\right) + \left\langle f, \pi\left(\sum_{i=1}^{n} \kappa_i(v^*)v_i - u\right)\right\rangle_{Y^*\times Y}$$
$$= -\varphi(\mathcal{N}(v^*), u) + \langle f, \pi(\mathcal{N}(v^*) - u)\rangle_{Y^*\times Y} \tag{3.9}$$

for all $v^* \in R(u)$. Let us also define two functions $\Phi\colon C \to 2^{R(u)}$ and $\Psi\colon R(u) \to 2^{R(u)}$ by

$$\Phi(v) := \left\{v^* \in R(u) \mid \langle v^*, v - u\rangle \ge -\varphi(v, u) + \langle f, \pi(v - u)\rangle_{Y^*\times Y}\right\} \text{ for all } v \in C$$

and

$$\Psi(v^*) := \Phi(\mathcal{N}(v^*)) \text{ for all } v^* \in R(u).$$

We observe that (3.7) ensures that Φ has nonempty, weakly compact, and convex values. Also, we assert that Φ is upper semicontinuous from V to the weak topology of V^*. From Proposition 2.2(ii), it is enough to verify that for each weakly closed set D in V^*, the set

$$\Phi^-(D) := \left\{v \in C \mid \Phi(v) \cap D \ne \emptyset\right\}$$

is closed in V. Let $\{v_n\} \subset \Phi^-(D)$ be a sequence such that $v_n \to v$, as $n \to \infty$. Then, for each $n \in \mathbb{N}$, we are able to find $v_n^* \in R(u)$ satisfying

$$\langle v_n^*, v_n - u\rangle \ge -\varphi(v_n, u) + \langle f, \pi(v_n - u)\rangle_{Y^*\times Y}. \tag{3.10}$$

From the weak compactness of $R(u)$, without any loss of generality, we may suppose that $v_n^* \rightharpoonup v^*$ in V^*, as $n \to \infty$, for some $v^* \in R(u)$. Using the continuity of the function $v \mapsto \varphi(v, u)$ (see, e.g., [15, Corollary 2.5]) and passing to the upper limit, as $n \to \infty$, in (3.10), we have

$$\langle v^*, v - u\rangle \ge -\varphi(v, u) + \langle f, \pi(v - u)\rangle_{Y^*\times Y},$$

i.e., $v^* \in \Phi(v)$. The weak closedness of D implies $v^* \in D$. Therefore, we find that $v^* \in \Phi(v) \cap D$, namely, $v \in \Phi^-(D)$. Applying Proposition 2.2(ii) we obtain that Φ is weakly upper semicontinuous.

On the other hand, the continuity of the function $\mathcal{N}$ and [23, Theorem 1.2.8] imply that Ψ is also weakly upper semicontinuous. For function Ψ, we are now in a position to employ the Tychonov fixed point principle (see [18, Theorem 8.6]) to conclude that there exists $v^* \in R(u)$ such that

$$\langle v^*, \mathcal{N}(v^*) - u\rangle \ge -\varphi(\mathcal{N}(v^*), u) + \langle f, \mathcal{N}(v^*) - u\rangle.$$

This leads to a contraction with (3.9). Consequently, we conclude that $u \in C$ solves Problem 3.3 as well, i.e., there exists $u^* \in T(u)$ such that (3.3) holds. This completes the proof of part (i).

(ii) In this part of the proof, we shall distinguish two cases: C is bounded and C is unbounded.

First, we consider the situation that C is bounded. We argue by contradiction and suppose that Problem 3.3 has no solution, i.e., for each $u \in C$, we can find $v \in C$ such that

$$\sup_{u^* \in T(u)} \langle u^*, v-u\rangle + \varphi(v,u) + J^0(\gamma u; \gamma(v-u)) < \langle f, \pi(v-u)\rangle_{Y^*\times Y}.$$

Based on this fact, we introduce a multivalued map $G\colon C \to 2^C$ defined by

$$G(u) := \left\{ v \in C \mid \sup_{u^* \in T(u)} \langle u^*, v-u\rangle + \varphi(v,u) + J^0(\gamma u; \gamma(v-u)) < \langle f, \pi(v-u)\rangle_{Y^*\times Y} \right\}$$

for all $u \in C$. Under the assumption that Problem 3.3 has no solution, we get

(a) for each $u \in C$, the set $G(u)$ is nonempty,

(b) the multivalued map G has no fixed point in C, i.e., $u \notin G(u)$ for all $u \in C$.

It is obvious that if we trigger the contradiction that the multivalued map G admits a fixed point in C, then we can conclude that Problem 3.3 has at least a solution in C. To do so, we shall apply the fixed point theorem, Theorem 2.6, to verify that the set of fixed points of G is nonempty. Indeed, we have the following four claims.

Claim 1. For each $u \in C$, the set $G(u)$ is convex in C.

For $u \in C$ fixed, let $v_1,\ v_2 \in G(u)$ and $t \in (0,1)$ be arbitrary, and denote $v_t = tv_1 + (1-t)v_2$. Then, for $i = 1,\ 2$, we have

$$\langle u^*, v_i-u\rangle + \varphi(v_i,u) + J^0(\gamma u; \gamma(v_i-u)) < \langle f, \pi(v_i-u)\rangle_{Y^*\times Y} \quad \text{for all} \quad u^* \in T(u).$$

The convexity of $v \mapsto \varphi(v,u)$ together with positive homogeneity and subadditivity of $w \mapsto J^0(u;w)$ (see Proposition 2.5(i)) implies

$$\begin{aligned}
&\langle u^*, v_t-u\rangle + \varphi(v_t,u) + J^0(\gamma u; \gamma(v_t-u)) \\
&\quad \le t\big[\langle u^*, v_1-u\rangle + \varphi(v_1,u) + J^0(\gamma u; \gamma(v_1-u))\big] \\
&\qquad + (1-t)\big[\langle u^*, v_2-u\rangle + \varphi(v_2,u) + J^0(\gamma u; \gamma(v_2-u))\big] \\
&\quad < t\langle f, \pi(v_1-u)\rangle_{Y^*\times Y} + (1-t)\langle f, \pi(v_2-u)\rangle_{Y^*\times Y} \\
&\quad = \langle f, \pi(v_t-u)\rangle_{Y^*\times Y}
\end{aligned}$$

for all $u^* \in T(u)$, which means that

$$\sup_{u^* \in T(u)} \langle u^*, v_t-u\rangle + \varphi(v_t,u) + J^0(\gamma u; \gamma(v_t-u)) < \langle f, \pi(v_t-u)\rangle_{Y^*\times Y}.$$

Hence, the set $G(u)$ is convex.

Further, for each $v \in C$ fixed, we introduce the set O_v defined by

$$\begin{aligned}
O_v := \Bigg\{ u \in C \mid &\inf_{v^* \in T(v)} \langle v^*, v-u\rangle + \varphi(v,u) + \inf_{\eta_v \in \partial J(\gamma v)} \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} \\
&< \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u) \Bigg\}.
\end{aligned}$$

Claim 2. For each $v \in C$, the set O_v is weakly open, and $G^{-1}(v)$ contains O_v, where $G^{-1}(v)$ is defined by

$$G^{-1}(v) = \{u \in C \mid v \in G(u)\}.$$

Evidently, the claim is equivalent to the statement that the complement set $O_v^{\complement}$ is weakly closed and $\left[G^{-1}(v)\right]^{\complement} \subset O_v^{\complement}$. Let $u \in \left[G^{-1}(v)\right]^{\complement}$ be arbitrary, videlicet,

$$\sup_{u^* \in T(u)} \langle u^*, v-u\rangle + \varphi(v,u) + J^0(\gamma u; \gamma(v-u)) \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y}.$$

Recalling that T has compact values, we can use Proposition 2.5(ii) again to find

$$\langle \widetilde{u}^* + \gamma^*\xi, v-u\rangle + \varphi(v,u) \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y}, \tag{3.11}$$

where $\widetilde{u}^* \in T(u)$ and $\xi \in \partial J(\gamma u)$ are such that

$$\sup_{u^* \in T(u)} \langle u^*, v-u\rangle = \langle \widetilde{u}^*, v-u\rangle \quad \text{and} \quad J^0(\gamma u; \gamma(v-u)) = \langle \xi, \gamma(v-u)\rangle_{X^*\times X}.$$

By (3.11) and the (φ, h)-stable pseudomonotonicity of $T(\cdot) + \gamma^*\partial J(\gamma\cdot)$, we have

$$\langle v^*, v-u\rangle + \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} + \varphi(v,u) \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u)$$
$$\text{for all } v^* \in T(v), \text{ for all } \eta_v \in \partial J(\gamma v).$$

Taking the infimum in the above inequality with $v^* \in T(v)$ and $\eta_v \in \partial J(\gamma v)$, respectively, we have

$$\inf_{v^* \in T(v)} \langle v^*, v-u\rangle + \inf_{\eta_v \in \partial J(\gamma v)} \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} + \varphi(v,u) \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u),$$

which means that $u \in O_v^{\complement}$, and hence inclusion $\left[G^{-1}(v)\right]^{\complement} \subset O_v^{\complement}$ is valid.

To demonstrate the weak closedness of $O_v^{\complement}$, let $\{u_n\} \subset O_v^{\complement}$ be a weakly convergent sequence, i.e., $u_n \rightharpoonup u$ in V, as $n \to \infty$, for some $u \in C$. Then, for $n \in \mathbb{N}$, we have

$$\langle v^*, v-u_n\rangle + \langle \eta_v, \gamma(v-u_n)\rangle_{X^*\times X} + \varphi(v,u_n) \geq \langle f, \pi(v-u_n)\rangle_{Y^*\times Y} + h(v-u_n) \tag{3.12}$$

for all $v^* \in T(v)$ and all $\eta_v \in \partial J(\gamma v)$. Since $u \mapsto \varphi(v,u)$ is concave and upper semicontinuous, it is weakly upper semicontinuous. Passing to the upper limits, as $n \to \infty$, in inequality (3.12) and using (3.1), we get

$$\begin{aligned}
&\langle v^*, v-u\rangle + \varphi(v,u) + \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X}\\
&\quad \geq \limsup_{n\to\infty} \langle v^*, v-u_n\rangle + \limsup_{n\to\infty} \varphi(v,u_n)\\
&\qquad + \limsup_{n\to\infty} \langle \eta_v, \gamma(v-u_n)\rangle_{X^*\times X}\\
&\quad \geq \lim_{n\to\infty} \langle f, \pi(v-u_n)\rangle_{Y^*\times Y} + \limsup_{n\to\infty} h(v-u_n)\\
&\quad \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u)
\end{aligned}$$

for all $v^* \in T(v)$, $\eta_v \in \partial J(\gamma v)$ and all $v \in C$. Hence

$$\inf_{v^* \in T(v)} \langle v^*, v-u\rangle + \inf_{\eta_v \in \partial J(\gamma v)} \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} + \varphi(v,u) \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u)$$

for all $v \in C$. Hence $u \in O_v^{\complement}$, confirming that $O_v^{\complement}$ is weakly closed in V, or O_v is weakly open for each $v \in C$.

Claim 3. It holds that $\bigcup_{v\in C} O_v = C$.

It is easy to see that $\bigcup_{v\in C} O_v \subset C$. It remains to show that $\bigcup_{v\in C} O_v \supset C$. Let $u \in C$ be fixed. Note that Problem 3.3 has no solution, so invoking assertion (i), we conclude that there exists $v \in C$ such that

$$\inf_{v^*\in T(v)} \langle v^*, v-u\rangle + \inf_{\eta_v\in\partial J(\gamma v)} \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} + \varphi(v,u) < \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u).$$

This, in view of the definition of O_v, ensures that $u \in O_v$, and hence $\bigcup_{v\in C} O_v \supset C$ holds.

Claim 4. The set $\bigcap_{v\in C} O_v^{\complement}$ is weakly compact if it is nonempty.

Suppose that $\bigcap_{v\in C} O_v^{\complement}$ is nonempty. Claim 2 indicates that $\bigcap_{v\in C} O_v^{\complement}$ is weakly closed, thanks to the weak closedness of $O_v^{\complement}$. On the other hand, since C is bounded, closed, and convex, and V is reflexive, so C is weakly compact in V. Therefore, we infer that the set $\bigcap_{v\in C} O_v^{\complement}$ is weakly compact, which proves the claim.

To summarize, we have verified all conditions of Theorem 2.6. Therefore, we deduce that G admits a fixed point in C. This leads to a contradiction. Thus, Problem 3.3 has at least one solution in C.

Next, we consider the case that C is unbounded. Let $n \in \mathbb{N}$. We are able to choose $n \in \mathbb{N}$ large enough such that $C_n = \{u \in C \mid \|u\|_V \le n\} \neq \emptyset$. Consider the following inequality problem: find $u_n \in C_n$ and $u_n^* \in T(u_n)$ such that

$$\langle u_n^*, v-u_n\rangle + \varphi(v,u_n) + J^0(\gamma u_n; \gamma(v-u_n)) \ge \langle f, \pi(v-u_n)\rangle_{Y^*\times Y} \text{ for all } v \in C_n. \tag{3.13}$$

Note that the first part of the proof provides $u_n \in C_n$, which solves (3.13). We now make the following claim.

Claim 5. There exists $N_0 \in \mathbb{N}$ such that

$$\|u_{N_0}\|_V < N_0. \tag{3.14}$$

Suppose that for each $n \in \mathbb{N}$, we have $\|u_n\|_V = n$. Let $v_0 \in C_0$ be fixed. Observe that

$$0 \ge \langle u_n^*, u_n - v_0\rangle - \varphi(v_0,u_n) + \langle \xi_{u_n}, \gamma(u_n-v_0)\rangle_{X^*\times X} + \langle f, \pi(v_0-u_n)\rangle_{Y^*\times Y} \tag{3.15}$$

with $u_n^* \in T(u_n)$, $\xi_{u_n} \in \partial J(\gamma u_n)$, and $\langle \xi_{u_n}, \gamma(v_0-u_n)\rangle_{X^*\times X} = J^0(\gamma u_n; \gamma(v_0-u_n))$. We now apply (3.2) and (3.15) to find a function $r\colon \mathbb{R}_+ \to \mathbb{R}$ with $r(s) \to +\infty$, as $s \to +\infty$, such that

$$\begin{aligned} 0 &\ge \langle u_n^*, u_n - v_0\rangle - \varphi(v_0,u_n) + \langle \xi_{u_n}, \gamma(u_n-v_0)\rangle_{X^*\times X} + \langle f, \pi(v_0-u_n)\rangle_{Y^*\times Y} \\ &\ge \inf_{u_n^*\in T(u_n)} \langle u_n^*, u_n - v_0\rangle - \varphi(v_0,u_n) + \inf_{\xi\in\partial J(\gamma u_n)} \langle \xi, \gamma(u_n-v_0)\rangle_{X^*\times X} \\ &\quad - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V,Y)}\|v_0-u_n\|_V \\ &\ge r(\|u_n\|_V)\|u_n\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V,Y)}\|u_n\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V,Y)}\|u_0\|_V. \end{aligned} \tag{3.16}$$

Recall that $\|u_n\|_V \to \infty$, as $n \to \infty$, and $r(s) \to +\infty$, as $s \to +\infty$. It is easy to find $N_1 \in \mathbb{N}$ large enough such that $r(\|u_{N_1}\|_V)\|u_{N_1}\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V,Y)}\|u_{N_1}\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V,Y)}\|u_0\|_V < 0$. This leads to a contradiction with (3.16). So, we infer that (3.14) is satisfied.

Let $N_0 \in \mathbb{N}$ and $u_{N_0} \in C_{N_0}$ be such that (3.14) holds. For any $w \in C$, (3.14) allows us to choose $t \in (0,1)$ small enough so that $v_t = tw + (1-t)u_{N_0} \in C_{N_0}$. Inserting $v = v_t$ in (3.13) for $n = N_0$, we get

$$\begin{aligned}
t\big[\langle u^*_{N_0}, w-u_{N_0}\rangle &+ \varphi(w,u_{N_0}) + J^0(\gamma u_{N_0};\gamma(w-u_{N_0}))\big] \\
&= t\langle u^*_{N_0}, w-u_{N_0}\rangle + t\varphi(w,u_{N_0}) \\
&\quad + (1-t)\varphi(u_{N_0},u_{N_0}) + tJ^0(\gamma u_{N_0};\gamma(w-u_{N_0})) \\
&\geq t\langle u^*_{N_0}, w-u_{N_0}\rangle + \varphi(v_t,u_{N_0}) + J^0(\gamma u_{N_0};\gamma(v_t-u_{N_0})) \\
&\geq t\langle f, \pi(w-u_{N_0})\rangle_{Y^*\times Y}.
\end{aligned}$$

Here, we have used hypotheses (H_φ)(i), (iii) and positive homogeneity of $v\mapsto J^0(u;v)$. We obtain

$$\langle u^*_{N_0}, w-u_{N_0}\rangle + \varphi(w,u_{N_0}) + J^0(\gamma u_{N_0};\gamma(w-u_{N_0})) \geq \langle f, \pi(w-u_{N_0})\rangle_{Y^*\times Y}.$$

Because $w\in C$ is arbitrary and $u^*_{N_0}\in T(u_{N_0})$, we find that $u_{N_0}\in C$ is a solution to Problem 3.3.

It remains to show that $\mathrm{SOL}(C;T,J,\varphi,f)$ is bounded and weakly closed. For the boundedness of the solution set, let $u\in \mathrm{SOL}(C;T,J,\varphi,f)$. Then, a simple calculation based on the coercivity condition (3.2) (see (3.16)) gives

$$r(\|u\|_V)\|u\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V,Y)}\|u\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V,Y)}\|u_0\|_V \leq 0.$$

Hence, it is clear that u stays in a bounded set in V. So, the solution set to Problem 3.3, $\mathrm{SOL}(C;T,J,\varphi,f)$, is bounded.

For the weak closedness of the solution set, let $\{u_n\}\subset \mathrm{SOL}(C;T,J,\varphi,f)$ be a sequence converging weakly to some $u\in C$. By assertion (i) of the theorem, we have

$$\langle v^*, v-u_n\rangle + \varphi(v,u_n) + \langle \eta_v, \gamma(v-u_n)\rangle_{X^*\times X} \geq \langle f,\pi(v-u_n)\rangle_{Y^*\times Y} + h(v-u_n)$$

for all $v^*\in T(v)$, $\eta_v\in\partial J(\gamma v)$ and all $v\in C$. Passing to the upper limit as $n\to\infty$ in the above inequality, and using hypotheses (H_φ)(ii) and (3.1), we obtain

$$\begin{aligned}
\langle v^*, v-u\rangle + \varphi(v,u) + \langle \eta_v,\gamma(v-u)\rangle_{X^*\times X} &\geq \limsup_{n\to\infty}\langle v^*, v-u_n\rangle + \limsup_{n\to\infty}\varphi(v,u_n) \\
&\quad + \limsup_{n\to\infty}\langle \eta_v,\gamma(v-u_n)\rangle_{X^*\times X} \\
&\geq \lim_{n\to\infty}\langle f,\pi(v-u_n)\rangle_{Y^*\times Y} + \limsup_{n\to\infty} h(v-u_n) \\
&\geq \langle f,\pi(v-u)\rangle_{Y^*\times Y} + h(v-u)
\end{aligned}$$

for all $v^*\in T(v)$, $\eta_v\in\partial J(\gamma v)$ and all $v\in C$. Invoking assertion (i) again, we see that $u\in \mathrm{SOL}(C;T,J,\varphi,f)$, and thus $\mathrm{SOL}(C;T,J,\varphi,f)$ is weakly closed.

(iii) Suppose, in addition, that h is convex. Let u_1, $u_2\in \mathrm{SOL}(C;T,J,\varphi,f)$ and $t\in(0,1)$ be arbitrary, and denote $u_t = tu_1 + (1-t)u_2$. Then, from assertion (i), for $i=1$, 2, we have

$$\langle v^*, v-u_i\rangle + \varphi(v,u_i) + \langle \eta_v,\gamma(v-u_i)\rangle_{X^*\times X} \geq \langle f,\pi(v-u_i)\rangle_{Y^*\times Y} + h(v-u_i)$$

for all $v^*\in T(v)$, $\eta_v\in\partial J(\gamma v)$ and all $v\in C$. By the concavity of function $u\mapsto\varphi(v,u)$ (see (H_φ) (ii)), we get

$$\begin{aligned}
\langle v^*, v-u_t\rangle &+ \varphi(v,u_t) + \langle \eta_v,\gamma(v-u_t)\rangle_{X^*\times X} \\
&\geq t\big[\langle v^*, v-u_1\rangle + \varphi(v,u_1) + \langle \eta_v,\gamma(v-u_1)\rangle_{X^*\times X}\big] \\
&\quad + (1-t)\big[\langle v^*, v-u_2\rangle + \varphi(v,u_2) + \langle \eta_v,\gamma(v-u_2)\rangle_{X^*\times X}\big] \\
&\geq t[\langle f,\pi(v-u_1)\rangle_{Y^*\times Y} + h(v-u_1)] \\
&\quad + (1-t)\big[\langle f,\pi(v-u_2)\rangle_{Y^*\times Y} + h(v-u_2)\big] \\
&\geq \langle f,\pi(v-u_t)\rangle_{Y^*\times Y} + h(v-u_t)
\end{aligned}$$

for all $v^* \in T(v)$, $\eta_v \in \partial J(\gamma v)$ and all $v \in C$. Again, exploiting assertion (i), we conclude that $u_t \in \mathrm{SOL}(C; T, J, \varphi, f)$, hence $\mathrm{SOL}(C; T, J, \varphi, f)$ is a convex set. ◻

A careful reading of the proof of Theorem 3.4 allows one to state the following result.

PROPOSITION 3.5. *Besides hypotheses* (H_C), (H_J), (H_γ), (H_0), (H_φ), *and* (3.2), *assume that* $T\colon C \to V^*$ *satisfies the following:*
(i) T *is hemicontinuous.*
(ii) *The multivalued map* $u \mapsto T(u) + \gamma^* \partial J(\gamma u)$ *is* (φ, h)*-stably pseudomonotone with respect to* $\{\pi^* f\}$, *where* $h\colon V \to \mathbb{R}$ *enjoys the same conditions as in* (H_T)(ii).
Then, the following conclusions hold:
(i) $u \in C$ *is a solution to the following hemivariational inequality: find* $u \in C$ *such that*

$$\langle T(u), v-u\rangle + \varphi(v,u) + J^0(\gamma u; \gamma(v-u)) \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} \tag{3.17}$$

for all $v \in C$ *if and only if it solves the inequality*

$$\langle T(v), v-u\rangle + \varphi(v,u) + \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u)$$

for all $\eta_v \in \partial J(\gamma v)$ *and all* $v \in C$.
(ii) *The set of solutions to problem* (3.17) *is nonempty, bounded, and weakly closed in* V.
(iii) *If, in addition,* h *is convex, then the set of solutions to problem* (3.17) *is convex in* V.

Remark 3.6. We note that Theorem 3.4 extends recent results given in [34, Lemma 3.3 and Theorem 3.4]. In fact, for Problem 3.3, we do not require that the operator $\gamma\colon V \to X$ is compact, and we consider a more general function φ than in [34]. This extends the scope of applications of hemivariational inequality. Moreover, the key idea of the proof of Theorem 3.4 is entirely different from the one used in [34], where instead of the Fan–Knaster–Kuratowski–Mazurkiewicz theorem, we employed a fixed point principle.

In particular, if φ is specialized to $\varphi(v,u) = \phi(v) - \phi(u)$, where $\phi\colon V \to \mathbb{R}$ is a convex and lower semicontinuous function, then we obtain the following.

COROLLARY 3.7. *Let* $\phi\colon C \to \mathbb{R}$ *be a convex and lower semicontinuous function. Assume that* (H_C), (H_γ), (H_J), (H_0), (H_T), *and* (3.2) *with* $\varphi(v,u) = \phi(v) - \phi(u)$ *for all* v, $u \in C$, *are fulfilled. Then,*
(i) $u \in C$ *is a solution to the following hemivariational inequality: find* $u \in C$ *such that there* $u^* \in T(u)$ *which satisfies*

$$\langle u^*, v-u\rangle + \phi(v) - \phi(u) + J^0(\gamma u; \gamma(v-u)) \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} \tag{3.18}$$

for all $v \in C$ *if and only if it solves the inequality*

$$\langle v^*, v-u\rangle + \phi(v) - \phi(u) + \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} \geq \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u)$$

for all $v^* \in T(v)$, $\eta_v \in \partial J(\gamma v)$ *and all* $v \in C$;
(ii) *the solution set of problem* (3.18) *is nonempty, bounded, and weakly closed in* V;
(iii) *if, in addition,* h *is convex, then the solution set of problem* (3.18) *is convex in* V.

We are now in a position to develop an existence result to the generalized quasi-hemivariational inequality, Problem 1.1. In what follows, we denote by $\Gamma(f)$ the set of solutions to Problem 1.1 for $f \in Y^*$.

THEOREM 3.8. *Assume that* (H_C), (H_J), (H_γ), (H_0), (H_T) *with* h *being convex,* (H_φ), (H_K), *and* (H_{C_0}) *hold. Assume, in addition, that* γ *is compact, and for any sequences* $\{v_n\}$, $\{u_n\} \subset C$ *such that* $v_n \to v$ *and* $u_n \rightharpoonup u$ *in* V *for some* $u, v \in C$, *we have*

$$\limsup_{n\to\infty} \varphi(v_n, u_n) \le \varphi(v, u). \tag{3.19}$$

Then, the following conclusion holds:

(i) *An element* $u \in C$ *is a solution to Problem* 1.1 *if and only if it solves the following inequality problem: find* $u \in C$ *such that* $u \in K(u)$ *and for all* $v^* \in T(v)$, $\eta_v \in \partial J(\gamma v)$ *and all* $v \in K(u)$, *we have*

$$\langle v^*, v-u\rangle + \varphi(v,u) + \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} \ge \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u). \tag{3.20}$$

(ii) *The solution set of Problem* 1.1, $\Gamma(f)$, *is nonempty, bounded, and weakly closed.*

Proof. (i) It follows by arguments similar to those used in the proof of Theorem 3.4(i).

(ii) We will first show that $\Gamma(f) \neq \emptyset$ for all $f \in Y^*$. Let $f \in Y^*$ be fixed. Given an element $w \in C$, we now consider the generalized hemivariational inequality: find $u \in K(w)$ such that there exists $u^* \in T(u)$ and

$$\langle u^*, v-u\rangle + \varphi(v,u) + J^0(\gamma u; \gamma(v-u)) \ge \langle f, \pi(v-u)\rangle_{Y^*\times Y} \quad \text{for all } \ v \in K(w). \tag{3.21}$$

Next, we define the so-called variational selection $S\colon C \to 2^C$, which associates to any $w \in C$ the set of solutions to inequality (3.21), that is,

$$S(w) := \big\{u \in K(w) \mid u \text{ solves the problem (3.21)}\big\}.$$

It is not difficult to see that any fixed point of multivalued map S is a solution to quasi-hemivariational inequality, Problem 1.1. Based on this fact, we will verify that $\Gamma(f) \neq \emptyset$ for all $f \in Y^*$ via showing that the variational selection S satisfies the assumptions imposed on the map Φ in Theorem 2.7.

For each $w \in C$, it follows from Theorem 3.4 that the set $S(w)$ is nonempty, bounded, closed, and convex. We claim that the graph of variational selection S is sequentially weakly closed. To this end, let $\{w_n\}$, $\{u_n\}$ be sequences such that $u_n \in S(w_n)$ with $w_n \rightharpoonup w$ and $u_n \rightharpoonup u$ in V, as $n \to \infty$, for some $w, u \in C$. From assertion (i), we can see that $u_n \in K(w_n)$ and for all $v^* \in T(v)$, $\eta_v \in \partial J(\gamma v)$ and all $v \in K(w_n)$, we have

$$\langle v^*, v-u_n\rangle + \varphi(v,u_n) + \langle \eta_v, \gamma(v-u_n)\rangle_{X^*\times X} \ge \langle f, \pi(v-u_n)\rangle_{Y^*\times Y} + h(v-u_n). \tag{3.22}$$

Recalling that $(w_n, u_n) \in \mathrm{Gr}(K)$ with $(w_n, u_n) \rightharpoonup (w, u)$ in $C \times C$, by hypothesis (H_K)(ii), we have $u \in K(w)$. On the other hand, for any $z \in K(w)$, condition (H_K)(i) allows us to find a sequence $\{v_n\} \subset C$ with $v_n \in K(w_n)$ for all $n \in \mathbb{N}$ such that $v_n \to z$, as $n \to \infty$. Let us fix a sequence $\{\widetilde{v}_n^*\}$ in V^* with $\widetilde{v}_n^* \in T(v_n)$. Keeping in mind that T is upper semicontinuous with compact values and $v_n \to z$ in V, as $n \to \infty$, from Theorem 2.3, we are able to find $z^* \in T(z)$ and a subsequence of $\{v_n^*\}$, still denoted by $\{v_n^*\}$, such that $v_n^* \to z^*$ in V^*, as $n \to \infty$. Inserting $v = v_n$ and $v^* = v_n^*$ into (3.22), and passing to the upper limit, as $n \to \infty$, using (3.1), (3.19), and the upper semicontinuity of the function $(u, v) \mapsto J^0(u; v)$, we get

$$
\begin{aligned}
&\langle z^*, z-u\rangle + \varphi(z,u) + J^0(\gamma z;\gamma(v-u))\\
&\quad\geq \lim_{n\to\infty}\langle v_n^*, v_n-u_n\rangle + \limsup_{n\to\infty}\varphi(v_n,u_n) + \limsup_{n\to\infty} J^0(\gamma v_n;\gamma(v_n-u_n))\\
&\quad\geq \lim_{n\to\infty}\langle v_n^*, v_n-u_n\rangle + \limsup_{n\to\infty}\varphi(v_n,u_n) + \limsup_{n\to\infty}\langle \eta_{v_n}, \gamma(v_n-u_n)\rangle_{X^*\times X}\\
&\quad\geq \limsup_{n\to\infty}\left[\langle v_n^*, v_n-u_n\rangle + \varphi(v_n,u_n) + \langle \eta_{v_n}, \gamma(v_n-u_n)\rangle_{X^*\times X}\right]\\
&\quad\geq \lim_{n\to\infty}\langle f, \pi(v_n-u_n)\rangle_{Y^*\times Y} + \limsup_{n\to\infty} h(v_n-u_n)\\
&\quad\geq \langle f, \pi(z-u)\rangle_{Y^*\times Y} + h(z-u)
\end{aligned}
$$

for all $\eta_{v_n}\in\partial J(\gamma v_n)$. This implies that for each $z\in K(w)$, there exists $z^*\in T(z)$ such that

$$\langle z^*, z-u\rangle + \varphi(z,u) + J^0(\gamma z;\gamma(z-u)) \geq \langle f,\pi(z-u)\rangle_{Y^*\times Y} + h(z-u).$$

Let $y\in K(w)$ and $t\in(0,1)$ be arbitrary. Taking $z = z_t := ty+(1-t)u\in K(w)$ into the above inequality, we obtain

$$\langle z_t^*, y-u\rangle + \varphi(y,u) + J^0(\gamma z_t;\gamma(y-u)) \geq \langle f,\pi(y-u)\rangle_{Y^*\times Y} + \frac{h(t(y-u))}{t}. \tag{3.23}$$

Moreover, Theorem 2.3 ensures, by passing to a subsequence if necessary, that there is an element $u^*\in T(u)$ such that $z_t^*\to u^*$ in V^*, as $t\to 0^+$. Further, passing to the upper limit, as $t\to 0^+$, in (3.23), we have

$$
\begin{aligned}
&\langle u^*, y-u\rangle + \varphi(y,u) + J^0(\gamma u;\gamma(y-u))\\
&\quad\geq \lim_{t\to 0^+}\langle z_t^*, y-u\rangle + \varphi(y,u) + \limsup_{t\to 0^+} J^0(\gamma z_t;\gamma(y-u))\\
&\quad\geq \langle f,\pi(y-u)\rangle_{Y^*\times Y} + \limsup_{t\to 0^+}\frac{h(t(y-u))}{t}\\
&\quad\geq \langle f,\pi(y-u)\rangle_{Y^*\times Y}
\end{aligned} \tag{3.24}
$$

for all $y\in K(w)$ with $u^*\in T(u)$. Hence, $u\in S(w)$, proving that the graph of S is sequentially weakly closed.

Furthermore, we claim that the set $S(C)$ is bounded. We will apply 3.2, similarly as in (3.16). Arguing by contradiction, assume that $S(C)$ is unbounded, and then there are sequences $\{w_n\}$ and $\{u_n\}$ such that $u_n\in S(w_n)$ and $\|u_n\|_V\to\infty$, as $n\to\infty$. By definition of S, for each $n\in\mathbb{N}$, there exists $u_n^*\in T(u_n)$ such that $u_n\in K(w_n)$ and

$$\langle u_n^*, v-u_n\rangle + \varphi(v,u_n) + \langle \eta_{u_n}, \gamma(v-u_n)\rangle_{X^*\times X} \geq \langle f,\pi(v-u_n)\rangle_{Y^*\times Y} \quad \text{for all } v\in K(u_n), \tag{3.25}$$

where $\eta_{u_n}\in\partial J(\gamma u_n)$ is such that $\langle \eta_{u_n}, \gamma(v-u_n)\rangle_{X^*\times X} = J^0(\gamma u_n;\gamma(v-u_n))$. Next, it follows from (3.25) and assumption $H(C_0)$ that there exist a function $r\colon\mathbb{R}_+\to\mathbb{R}$ with $r(s)\to+\infty$, as $s\to\infty$, and a sequence $\{v_n\}\subset C_0$ with $v_n\in C_0\cap K(w_n)$ such that

$$
\begin{aligned}
0 &\geq \inf_{u_n^*\in T(u_n)}\langle u_n^*, u_n-v_n\rangle + \inf_{\eta_{u_n}\in\partial J(\gamma u_n)}\langle \eta_{u_n}, u_n-v_n\rangle_{X^*\times X} - \varphi(v_n,u_n)\\
&\quad + \langle f,\pi(v_n-u_n)\rangle_{Y^*\times Y}\\
&\geq r(\|u_n\|_V)\|u_n\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V;Y)}\|v_n-u_n\|_V\\
&\geq r(\|u_n\|_V)\|u_n\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V;Y)}\big(\|v_n\|_V+\|u_n\|_V\big)\\
&\geq r(\|u_n\|_V)\|u_n\|_V - \|f\|_{Y^*}\|\pi\|_{\mathcal{L}(V;Y)}\big(M_{C_0}+\|u_n\|_V\big),
\end{aligned}
$$

where $M_{C_0} > 0$ is such that $\|v\|_V \le M_{C_0}$ for all $v \in C_0$. The latter combined with the facts that $r(s) \to +\infty$, as $s \to \infty$, and $\|u_n\|_V \to \infty$, as $n \to \infty$, leads to a contradiction. We deduce that the set $S(C)$ is bounded.

Having verified all hypotheses of Theorem 2.7 for the map S, we deduce that S has a fixed point. Consequently, for each $f \in Y^*$, we have $\Gamma(f) \neq \emptyset$. Since $\Gamma(f) \subset S(C)$, the set $\Gamma(f)$ is bounded as well.

Finally, it remains to prove that for each $f \in Y^*$, the set $\Gamma(f)$ is weakly closed. Let $\{u_n\} \subset \Gamma(f)$ be such that $u_n \rightharpoonup u$ in C, as $n \to \infty$. Then, for each $n \in \mathbb{N}$, by assertion (i), one has $u_n \in K(u_n)$ and

$$\langle v^*, v-u_n\rangle + \varphi(v,u_n) + \langle \eta_v, \gamma(v-u_n)\rangle_{X^*\times X} \ge \langle f, \pi(v-u_n)\rangle_{Y^*\times Y} + h(v-u_n) \tag{3.26}$$

for all $v^* \in T(v)$, $\eta_v \in \partial J(\gamma v)$ and all $v \in K(u_n)$. Moreover, hypothesis (H_K)(ii) and convergence $u_n \rightharpoonup u$ in V with $u_n \in K(u_n)$ imply $u \in K(u)$. On the other hand, for any $z \in K(u)$, using condition (H_K)(i), there exists a sequence $\{v_n\} \subset C$ such that $v_n \in K(u_n)$ and $v_n \to z$, as $n \to \infty$. Fix a sequence $\{v_n^*\} \subset V^*$ with $v_n^* \in T(v_n)$. The convergence $v_n \to z$, as $n \to \infty$, and Theorem 2.3 show that there exist an element $z^* \in T(z)$ and a subsequence of $\{v_n^*\}$, still denoted by $\{v_n^*\}$, such that $v_n^* \to z^*$, as $n \to \infty$. Inserting $v = v_n$ and $v^* = v_n^*$ into (3.26), and passing to the upper limit, as $n \to \infty$, we conclude

$$\langle z^*, z-u\rangle + \varphi(z,u) + J^0(\gamma z; \gamma(v-u)) \ge \langle f, \pi(z-u)\rangle_{Y^*\times Y} + h(z-u)$$

with $z^* \in T(z)$ and for all $z \in K(u)$. The same reasoning as in (3.23) and (3.24) guarantees that we can find an element $u^* \in T(u)$ such that

$$\langle u^*, y-u\rangle + \varphi(y,u) + J^0(\gamma u; \gamma(y-u)) \ge \langle f, \pi(y-u)\rangle_{Y^*\times Y}$$

for all $y \in K(u)$. Hence $u \in \Gamma(f)$, which completes the proof that $\Gamma(f)$ is weakly closed. ∎

In a particular case, if T is a single-valued map , then by Proposition 3.5 and Theorem 3.8, we obtain the following.

COROLLARY 3.9. *Besides* (H_C), (H_J), (H_γ), (H_0), (H_φ), (H_K), (H_{C_0}), *and* (3.19), *assume that* γ *is compact and* $T\colon C \to V^*$ *satisfies the following conditions:*
(i) T *is continuous,*
(ii) *the multivalued map* $u \mapsto T(u) + \gamma^*\partial J(\gamma u)$ *is* (φ, h)*-stably pseudomonotone with respect to* $\{\pi^* f\}$, *where* $h\colon V \to \mathbb{R}$ *is a convex function such that all conditions in* $H(T)$(ii) *are fulfilled.*

Then, the following conclusions hold:
(i) $u \in C$ *is a solution to the following quasi-hemivariational inequality: find* $u \in C$ *such that* $u \in K(u)$ *and*

$$\langle Tu, v-u\rangle + \varphi(v,u) + J^0(\gamma u; \gamma(v-u)) \ge \langle f, \pi(v-u)\rangle_{Y^*\times Y} \tag{3.27}$$

for all $v \in K(u)$ *if and only if it solves the following inequality problem: find* $u \in C$ *such that* $u \in K(u)$ *and*

$$\langle Tv, v-u\rangle + \varphi(v,u) + \langle \eta_v, \gamma(v-u)\rangle_{X^*\times X} \ge \langle f, \pi(v-u)\rangle_{Y^*\times Y} + h(v-u)$$

for all $\eta_v \in \partial J(\gamma v)$ *and all* $v \in K(u)$.
(ii) *The solution set of problem* (3.27) *is nonempty, bounded, and weakly closed.*

Remark 3.10. Note that if $J(w) = 0$ for all $w \in X$, π is the identity operator of V (i.e., $\pi u = u$ for all $u \in V$), $h(u) = 0$ for all $u \in V$, and $\varphi(v,u) = \phi(v) - \phi(u)$, where $\phi\colon K \to \mathbb{R}$ is a convex, and lower semicontinuous function, then Corollary 3.9 reduces to the recent result in [39, Theorem 2.3].

4. Optimal control in quasi-hemivariational inequalities. In this section, we study the optimal control problem governed by a generalized quasi-hemivariational inequality.

Given a function $g\colon V\times Y^*\to\mathbb{R}$, we consider the following optimal control problem.

PROBLEM 4.1. *Find $f\in Y^*$ such that*

$$f\in\arg\min_{f\in Y^*}F(f)\quad\text{with}\quad F(f):=\inf_{u\in\Gamma(f)}g(u,f), \tag{4.1}$$

where $\Gamma(f)\subset V$ stands for the set of solutions to Problem 1.1 *corresponding to $f\in Y^*$.*

We assume that g in the above definition of the cost function F satisfies the following condition:

(H_g) $g\colon V\times Y^*\to\mathbb{R}$ is weakly lower semicontinuous on $V\times Y^*$, and for all $(v,f)\in V\times Y^*$, we have

$$g(v,f)\ge c_0+c_1\|f\|_{Y^*}\quad\text{with}\quad c_0\in\mathbb{R},\ c_1\ge 0.$$

We begin with the following.

LEMMA 4.2. *Under assumptions of Theorem* 3.8*, the following conclusions hold:*

(i) *For each bounded set B in Y^*, the set $\Gamma(B)$ is bounded in V as well.*

(ii) *If $\pi\colon V\to Y$ is compact and $\{f_n\}$ is a sequence weakly convergent to f in Y^*, then*

$$w\text{–}\limsup_{n\to\infty}\Gamma(f_n)\subset\Gamma(f),$$

where $w\text{–}\limsup_{n\to\infty}\Gamma(f_n)$ stands for the sequential Kuratowski upper limit of $\{\Gamma(f_n)\}$ with respect to the weak topology of V, namely,

$$w\text{–}\limsup_{n\to\infty}\Gamma(f_n):=\big\{u\in C\,|\,\exists\,\{u_{n_k}\}\subset\{u_n\},\ u_{n_k}\in\Gamma(f_{n_k}),\ u_{n_k}\rightharpoonup u\ \textit{in}\ V,\ \text{as}\ k\to\infty\big\}.$$

Proof. (i) Let B be a bounded set in Y^*. If the set $\Gamma(B)$ is unbounded, then we are able to find sequences $\{f_n\}\subset B$ and $\{u_n\}\subset V$ with $u_n\in\Gamma(f_n)$ for all $n\in\mathbb{N}$ such that $\|u_n\|_V\to\infty$, as $n\to\infty$. Therefore, for each $n\in\mathbb{N}$, there exists $u_n^*\in T(u_n)$ such that $u_n\in K(u_n)$ and

$$\langle u_n^*,v-u_n\rangle+\varphi(v,u_n)+J^0(\gamma u_n;\gamma(v-u_n))\ge\langle f_n,\pi(v-u_n)\rangle_{Y^*\times Y}\quad\text{for all } v\in K(u_n).$$

Arguing as in (3.16), from hypothesis $H(C_0)$, we can pick up a sequence $\{v_n\}\subset C_0$ with $v_n\in C_0\cap K(u_n)$ and find a function $r\colon\mathbb{R}_+\to\mathbb{R}$ with $r(s)\to+\infty$, as $s\to\infty$, such that

$$\begin{aligned}0&\ge\langle u_n^*,u_n-v_n\rangle-J^0(\gamma u_n;\gamma(v_n-u_n))-\varphi(v_n,u_n)+\langle f_n,\pi(v_n-u_n)\rangle_{Y^*\times Y}\\&\ge\inf_{\widetilde{u}_n^*\in T(u_n)}\langle\widetilde{u}_n^*,u_n-v_n\rangle+\inf_{\eta_{u_n}\in\partial J(\gamma u_n)}\langle\eta_{u_n},\gamma(u_n-v_n)\rangle_{X^*\times X}\\&\quad-\varphi(v_n,u_n)+\langle f_n,\pi(v_n-u_n)\rangle_{Y^*\times Y}\\&\ge r(\|u_n\|_V)\|u_n\|_V-\|f_n\|_{Y^*}\|\pi\|_{\mathcal{L}(V;Y)}\|v_n-u_n\|_V\\&\ge r(\|u_n\|_V)\|u_n\|_V-M_B\|\pi\|_{\mathcal{L}(V;Y)}\big(M_{C_0}+\|u_n\|_V\big),\end{aligned}\tag{4.2}$$

where M_B, $M_{C_0}>0$ are such that $\|v\|_V\le M_{C_0}$ for all $v\in C_0$ and $\|f\|_{Y^*}\le M_B$ for all $f\in B$. Taking into account the facts $r(s)\to+\infty$, as $s\to+\infty$, $\|u_n\|_V\to\infty$, as

$n \to \infty$, and inequality (4.2), we are lead to a contradiction. Consequently, for each bounded set B in Y^*, the set $\Gamma(B)$ is bounded in V.

(ii) Let $\{f_n\} \subset Y^*$ be such that $f_n \rightharpoonup f$ in Y^*, as $n \to \infty$, and

$$u \in w\text{–}\limsup_{n\to\infty} \Gamma(f_n).$$

By virtue of definition of the Kuratowski upper limit, there is a sequence $\{u_{n_k}\} \subset C$ with $u_{n_k} \in \Gamma(f_{n_k})$ such that

$$u_{n_k} \rightharpoonup u \quad \text{in } V, \quad \text{as } k \to \infty. \tag{4.3}$$

Hence, for each $k \in \mathbb{N}$, Theorem 3.8(i) provides $u_{n_k} \in K(u_{n_k})$ such that

$$\langle v^*, v - u_{n_k}\rangle + \varphi(v, u_{n_k}) + \langle \eta_v, \gamma(v - u_{n_k})\rangle_{X^*\times X} \ge \langle f_{n_k}, \pi(v - u_{n_k})\rangle_{Y^*\times Y} + h(v - u_{n_k}) \tag{4.4}$$

for all $v^* \in T(v)$, $\eta_v \in \partial J(\gamma v)$ and all $v \in K(u_{n_k})$. Combining (H_K)(i), convergence (4.3), and $u_{n_k} \in K(u_{n_k})$, we have $u \in K(u)$. Moreover, let us fix $z \in K(u)$. Hypothesis (H_K)(ii) allows one to get a sequence $\{v_k\} \subset V$ with $v_k \in K(u_{n_k})$ such that $v_k \to z$, as $k \to \infty$. Next, fix a sequence $\{v_k^*\} \subset V^*$ such that $v_k^* \in T(v_k)$. From condition $H(T)$(i) and Theorem 2.3, we deduce, along a relabeled subsequence if necessary, that $v_k^* \to z^*$ as $k \to \infty$ for some $z^* \in T(z)$. We insert $v = v_n$ and $v^* = v_n^*$ into (4.4). Applying the compactness of operator π and hypotheses (3.1), (3.19), and passing to the upper limit, as $n \to \infty$, we conclude

$$\langle z^*, z - u\rangle + \varphi(z, u) + J^0(\gamma z; \gamma(z - u)) \ge \langle f, \pi(z - u)\rangle_Y + h(z - u)$$

for all $v \in K(u)$. An argument exploited in the proof of Theorem 3.8(ii) provides $u^* \in T(u)$ such that

$$\langle u^*, v - u\rangle + \varphi(v, u) + J^0(\gamma u; \gamma(v - u)) \ge \langle f, \pi(v - u)\rangle_{Y^*\times Y}$$

for all $v \in K(u)$. This means that $u \in \Gamma(f)$, so the desired conclusion is proved. ∎

The following result is a direct consequence of Lemma 4.2.

LEMMA 4.3. *Under assumptions of Theorem* 3.8*, if, in addition,* $\pi\colon V \to Y$ *is compact, then the multivalued map* $Y^* \ni f \mapsto \Gamma(f) \subset C$ *is weakly closed and weakly upper semicontinuous.*

Proof. Let $\{f_n\}$ be a sequence weakly convergent to f in Y^* and $u_n \in \Gamma(f_n) \subset C$ be such that $u_n \rightharpoonup u$ in V, as $n \to \infty$. By Lemma 4.2(ii), it follows that $u \in \Gamma(f)$, which means that the graph of the multivalued map $Y^* \ni f \mapsto \Gamma(f) \subset C$ is closed in $Y^* \times V$ equipped with weak topologies.

Since, by Theorem 3.8(ii), for any $f \in Y^*$, $\Gamma(f)$ is a bounded set in a reflexive Banach space V, it is weakly compact. Applying [12, Proposition 4.1.16], we deduce that the multivalued map is also weakly upper semicontinuous. ∎

We have the following result on the optimal control problem.

THEOREM 4.4. *Besides the hypotheses of Theorem* 3.8*, assume that* (H_g) *holds. Then the set of solutions to Problem* 4.1 *is nonempty and weakly compact.*

Proof. We will use a Weierstrass type theorem. First, we will show that function $F\colon Y^* \to \mathbb{R}$ given in Problem 4.1 is weakly lower semicontinuous. We need to show that for each $\lambda \in \mathbb{R}$, the set $S_\lambda = \{f \in Y^* \mid F(f) \le \lambda\}$ is weakly closed. Let $\{f_n\} \subset S_\lambda$ and assume $f_n \rightharpoonup f$ in Y^*, as $n \to \infty$. By the definition of F, we know that we can find $u_n \in \Gamma(f_n)$ such that

$$F(f_n) = g(u_n, f_n) \quad \text{and} \quad F(f_n) \le \lambda.$$

It follows from Theorem 3.8(ii) that the set $\Gamma(f)$ is bounded and weakly closed in V. Hence and from the reflexivity of V, one can assume that $u_n \rightharpoonup u$ in V for some $u \in V$. Moreover, by Lemma 4.2(ii), it follows that $u \in \Gamma(f)$. Therefore, by $H(g)$, we have

$$F(f) \leq g(u, f) \leq \liminf_{n\to\infty} g(u_n, f_n) = \lim_{n\to\infty} F(f_n) \leq \lambda,$$

that is, $F(f) \leq \lambda$, which implies $f \in S_\lambda$. So S_λ is weakly closed in Y^*, which shows that $F\colon Y^* \to \mathbb{R}$ is weakly lower semicontinuous as claimed.

Now, we consider Problem 4.1. We set $\varrho := \inf_{f\in Y^*} F(f) \in \mathbb{R}$ and let $\{f_n\} \subset Y^*$ be a minimizing sequence for F, namely,

$$\varrho = \lim_{n\to\infty} F(f_n). \tag{4.5}$$

We use hypothesis $H(g)$ to deduce that $\{f_n\}$ is bounded in Y^*. Therefore, from the reflexivity of Y^*, there exists $f^* \in Y^*$ such that, passing to a subsequence still denoted in the same way, one has

$$f_n \rightharpoonup f^* \quad \text{in } Y^*, \quad \text{as} \quad n \to \infty. \tag{4.6}$$

Since $F\colon Y^* \to \mathbb{R}$ is weakly lower semicontinuous, we get

$$\varrho \leq F(f^*) \leq \liminf_{n\to\infty} F(f_n) = \varrho,$$

which proves that $f^* \in Y^*$ is a solution to the optimal control problem, Problem 4.1.

It remains to verify that the set of solutions to Problem 4.1 is weakly compact. Let $\{f_n\}$ be a sequence of solutions to Problem 4.1, i.e., $F(f_n) \leq F(k)$ for all $k \in Y^*$. The hypothesis $H(g)$ implies that the sequence $\{f_n\}$ is bounded in Y^*. Therefore, we may find $f^* \in Y$ such that, by passing to a subsequence still denoted as $\{f_n\}$, convergence (4.6) holds. By the weak lower semicontinuity of F, it follows $F(f^*) \leq \liminf_{n\to\infty} F(f_n) \leq F(k)$ for all $k \in Y^*$. We conclude that f^* is a solution to Problem 4.1. This shows that the set of solutions to Problem 4.1 is weakly compact. The proof is complete. ∎

5. An application to approximate elastic contact problems.

To illustrate the applicability of the theoretical results derived in sections 3 and 4, in this section, we study an approximation of a complicated static elastic contact model involving a multivalued version of the normal compliance contact condition with frictionless effect, a frictional contact law with slip dependent coefficient of friction, and an additional constraint on the displacement. This model is an excellent approximation of real contact models. An optimal control problem for the aforementioned approximate contact model is also considered.

The physical setting of the model is as follows. We suppose that the nonlinear elastic body occupies a bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary $\Gamma := \partial\Omega$, where the boundary is given by $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_{C_1} \cup \Gamma_{C_2}$, where Γ_D, Γ_N, Γ_{C_1}, and Γ_{C_2} are pairwise disjoint and measurable sets with $\mathrm{m}(\Gamma_D) > 0$.

The classical formulation of the approximated contact model reads as follows.

PROBLEM 5.1. *Find a displacement field* $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$ *such that*

(5.1)
$$\boldsymbol{\sigma} \in \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})) + \partial\phi_B(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega,$$

(5.2)
$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega,$$

(5.3)
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D,$$

(5.4)
$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{on} \quad \Gamma_N,$$

(5.5)
$$\begin{cases} -\sigma_\nu \in \partial j_\nu(\boldsymbol{x}, u_\nu) \\ \boldsymbol{\sigma}_\tau = \boldsymbol{0} \end{cases} \quad \text{on} \quad \Gamma_{C_1},$$

(5.6)
$$\begin{cases} -\sigma_\nu = S \\ \|\boldsymbol{\sigma}_\tau\|_{\mathbb{R}^d} \le \mu(\|\boldsymbol{u}_\tau\|_{\mathbb{R}^d})|S|, \quad -\boldsymbol{\sigma}_\tau = \mu(\|\boldsymbol{u}_\tau\|_{\mathbb{R}^d})|S|\dfrac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|_{\mathbb{R}^d}} \quad \text{if} \quad \boldsymbol{u}_\tau \ne \boldsymbol{0} \end{cases} \quad \text{on} \quad \Gamma_{C_2},$$

(5.7)
$$\int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{\mathbb{S}^d}\, d\boldsymbol{x} \le U^2(\boldsymbol{u}).$$

In Problem 5.1, we adopt the standard notation; see [40, 52, 54]. The unit outward normal vector on the boundary and the position vector in the body are denoted by ν and $\boldsymbol{x} \in \overline{\Omega} = \Omega \cup \partial\Omega$, respectively. Let $(\mathbb{S}^d, \|\cdot\|_{\mathbb{S}^d})$ be the space of second-order symmetric tensors on $\mathbb{R}^d$. The standard inner products and norms in $\mathbb{R}^d$ and $\mathbb{S}^d$ are denoted by $\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i$, $\|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}}$ for $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in \mathbb{R}^d$, and $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $\|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{\frac{1}{2}}$ for $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d$, accordingly. All indices i, j, k, l run from 1 to d, and the summation convention over repeated indices is used. For a vector $\boldsymbol{\xi}$ on the boundary, its normal and tangential components are defined by $\xi_\nu = \boldsymbol{\xi}\cdot\boldsymbol{\nu}$ and $\boldsymbol{\xi}_\tau = \boldsymbol{\xi} - \xi_\nu\boldsymbol{\nu}$. Analogously, for the tensor $\boldsymbol{\sigma}$, its normal and tangential components on the boundary are given by $\sigma_\nu = (\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$. We denote the displacement vector, the stress tensor, and the linearized strain tensor by $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, and

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\big(u_{i,j} + u_{j,i}\big), \quad i, j = 1, \ldots, d.$$

For simplicity, we often will not indicate explicitly the dependence on the variable $\boldsymbol{x}$.

We briefly provide a description of equations and conditions in Problem 5.1. The inclusion (5.1) represents an elastic constitutive law for the locking material in which $\mathcal{E}$ is a nonlinear elasticity operator, ϕ_B stands for the indicator function of a constraint set $B \subset \mathbb{S}^d$, i.e.,

$$\phi_B(\boldsymbol{\varepsilon}) = \begin{cases} 0 & \text{if } \boldsymbol{\varepsilon} \in B \\ +\infty & \text{otherwise} \end{cases} \quad \text{for} \quad \boldsymbol{\varepsilon} \in \mathbb{S}^d,$$

and $\partial\phi_B$ denotes the convex subdifferential operator of function ϕ_B. Note that locking materials belong to a class of hyperelastic bodies for which the strain tensor is constrained to stay in a convex set B. The theory of locking materials was initiated by Prager; see [48, 49, 50]. The set B describes the properties of the material. Several forms of B are met in the literature for models of an ideal-locking effect, of materials with limited compressibility, and of the behaviour of rubber and some types of plastic materials. A model of torsion of a cylindrical bar made of a locking material was studied in [11]. The variational-hemivariational inequalities for locking materials have been studied only recently in [42, 52].

The relation (5.2) is the equilibrium equation, where "Div" represents the divergence operator given by

$$\mathrm{Div}\boldsymbol{\sigma} = (\sigma_{ij,j}) = \Big(\frac{\partial \sigma_{ij}}{\partial x_j}\Big)$$

and $\boldsymbol{f}_0$ stands for the density of volume forces. The boundary conditions (5.3) and (5.4) characterize the physical phenomena that the elastic body is clamped on Γ_D and it is subjected to surface tractions of density $\boldsymbol{f}_N$ on Γ_N.

The approximate contact condition (5.5) represents a subdifferential condition supplemented with the frictionless condition. The latter means that the tangential part of the stress (called the friction force) vanishes on Γ_{C_1}, i.e., $\boldsymbol{\sigma}_\tau = \boldsymbol{0}$. The frictionless condition is a sufficiently good approximation of the reality in some situations; it was used in several publications (see [21, 40, 51, 54] and the references therein). The condition in the normal direction is called an approximate contact law in which ∂j_ν represents the generalized subgradient of the prescribed function $j_\nu\colon \Gamma_{C_1} \times \mathbb{R} \to \mathbb{R}$ in the second variable. Note that many contact laws of classical elasticity are particular cases of the condition (5.5), and in most of these cases the function $j_\nu(\boldsymbol{x}, \cdot)$ is convex, hence leading to monotone boundary conditions; see, for example, the Winkler boundary condition obtained for $j_\nu(\boldsymbol{x}, r) = \frac{k}{2} r^2$, $k > 0$. The nonmonotone approximate contact law (5.5) can be single-valued and multivalued; see [41, Examples 16 and 17]. In particular, if $p_\nu\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, then we can choose $j_\nu(r) = \int_0^r p_\nu(s)\, ds$ for $r \in \mathbb{R}$ to obtain $\partial j_\nu(r) = p_\nu(r)$ for $r \in \mathbb{R}$. Therefore, the general normal compliance contact condition of the form $-\sigma_\nu = p_\nu(u_\nu)$ on Γ_{C_1} can be equivalently written as in (5.5). Other examples of nonmonotone contact laws can be found in [47, section 2.4], [45, section 1.4], [55, section 8.1], and [40, section 6.3].

The frictional contact law (5.6) has been considered in [53]. Condition $-\sigma_\nu = S$ on Γ_{C_2} states the normal stress is prescribed on this part of the boundary, where $S\colon \Gamma_{C_2} \to \mathbb{R}$ is the given function. The associated frictional condition represents a relation between the tangential displacement $\boldsymbol{u}_\tau$ and the tangential stress $\boldsymbol{\sigma}_\tau$, and $\mu\colon \Gamma_{C_2} \times \mathbb{R} \to [0, +\infty)$ is the coefficient of friction which depends on the slip $\|\boldsymbol{u}_\tau\|_{\mathbb{R}^d}$. This nonstandard frictional law is a mathematical model suitable for proportional loadings. It is also a first approximation of a more realistic model, based on a friction law involving the time derivative of $\boldsymbol{u}_\tau$; see, for instance, [10]. The nonmonotone slip dependent friction laws can be also found in [42]. Moreover, in the model, we consider constraint (5.7) in which the quantity

$$\int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))\|^2_{\mathbb{S}^d}\, d\boldsymbol{x}$$

represents the deformation of the elastic body corresponding to the displacement $\boldsymbol{u}$, and $U\colon L^2(\Omega; \mathbb{R}^d) \to \mathbb{R}$ is a given constraint function.

To establish the variational formulation of Problem 5.1, let us introduce the function spaces:

$$V = \{ \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0} \text{ on } \Gamma_D\}, \quad Q = L^2(\Omega;\mathbb{S}^d).$$

Since $\mathrm{m}(\Gamma_D) > 0$, it is not difficult to verify that V is a Hilbert space endowed with the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) : \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\, d\boldsymbol{x} \quad \text{for } \boldsymbol{u},\, \boldsymbol{v} \in V$$

and the associated norm $\|\cdot\|_V$. Further, Q is a Hilbert space equipped with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau}\rangle_Q = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\tau}(\boldsymbol{x})\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{\sigma},\, \boldsymbol{\tau} \in Q$$

and the associated norm $\|\cdot\|_Q$. In what follows, we denote by $\gamma_1 \colon V \to L^2(\Gamma_{C_1};\mathbb{R}^d)$, $\gamma_2 \colon V \to L^2(\Gamma_{C_2};\mathbb{R}^d)$, and $\gamma_3 \colon V \to L^2(\Gamma_N;\mathbb{R}^d)$ the trace operators. From the Sobolev trace theorem, we have $\|\boldsymbol{v}\|_{L^2(\Gamma_{C_i};\mathbb{R}^d)} \le \|\gamma_i\|\|\boldsymbol{u}\|_V$ for all $\boldsymbol{v} \in V$, $i = 1$, 2. In addition, we introduce the space $Y = L^2(\Omega;\mathbb{R}^d) \times L^2(\Gamma_N;\mathbb{R}^d)$ equipped with the canonical product topology and consider the map $\pi \colon V \to Y$ defined

$$\pi\boldsymbol{v} = (\boldsymbol{v}, \gamma_3\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V. \tag{5.8}$$

Next, we introduce the set $C := \{\boldsymbol{v} \in V \mid \boldsymbol{\varepsilon}(\boldsymbol{v}) \in B \text{ for a.e. } \boldsymbol{x} \in \Omega\}$ and the multivalued map $K \colon C \to 2^C$ defined by

$$K(\boldsymbol{u}) := \{\boldsymbol{v} \in C \mid \|\boldsymbol{v}\|_V \le U(\boldsymbol{u})\} \quad \text{for all } \boldsymbol{u} \in C. \tag{5.9}$$

The elasticity operator $\mathcal{E}$ and the set B are supposed to satisfy the following conditions:

$$\begin{cases} \mathcal{E} \colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\ \text{(a) } \mathcal{E}(\cdot, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \quad \text{the function } \boldsymbol{x} \mapsto \mathcal{E}(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^d}) \text{ belongs to } L^2(\Omega;\mathbb{S}^d), \\ \text{(b) there exists a constant } L_{\mathcal{E}} > 0 \text{ such that} \\ \qquad \|\mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d} \le L_{\mathcal{E}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ and a.e. } \boldsymbol{x} \in \Omega, \\ \text{(c) there exists a constant } m_{\mathcal{E}} > 0 \text{ such that} \\ \qquad \big(\mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\big) : \big(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\big) \ge m_{\mathcal{E}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d} \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ and for a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{5.10}$$

$$\begin{cases} B \text{ is a closed convex subset of } \mathbb{S}^d \text{ such that } \boldsymbol{0}_{\mathbb{S}^d} \in B, \\ S \in L^2(\Gamma_{C_2}), \;\; S(\boldsymbol{x}) \ge 0 \quad \text{for a.e. } \boldsymbol{x} \in \Gamma_{C_2}. \end{cases} \tag{5.11}$$

The potential function j_ν fulfills the following assumption:

$$(5.12)\qquad \begin{cases} j_\nu\colon \Gamma_{C_1}\times\mathbb{R}\to\mathbb{R} \text{ is such that}\\ \text{(a) } j_\nu(\cdot,r) \text{ is measurable on } \Gamma_{C_1} \text{ for all } r\in\mathbb{R},\\ \quad\text{the function } \boldsymbol{x}\mapsto j_\nu(\boldsymbol{x},0) \text{ belongs to } L^1(\Gamma_{C_1}),\\ \text{(b) } j_\nu(\boldsymbol{x},\cdot) \text{ is locally Lipschitz continuous, a.e. } \boldsymbol{x}\in\Gamma_{C_1},\\ \text{(c) there exists a constant } m_{j_\nu}\ge 0 \text{ such that}\\ \qquad (\zeta_1-\zeta_2)(r_1-r_2)\ge -m_{j_\nu}|r_1-r_2|^2\\ \quad\text{for all } r_i\in\mathbb{R},\ \zeta_i\in\partial j_\nu(\boldsymbol{x},r_i),\ i=1,2 \text{ and a.e. } \boldsymbol{x}\in\Gamma_{C_1},\\ \text{(d) either } j_\nu(\boldsymbol{x},\cdot) \text{ or } -j_\nu(\boldsymbol{x},\cdot) \text{ is regular for a.e. } \boldsymbol{x}\in\Gamma_{C_1},\\ \text{(e) there exist } \theta\in[1,2),\ \alpha_{j_\nu}\ge 0 \text{ and } \beta_{j_\nu}>0 \text{ such that}\\ \qquad j_\nu^0(\boldsymbol{x},r;-r)\le \alpha_{j_\nu}+\beta_{j_\nu}|r|^\theta\\ \quad\text{for all } r\in\mathbb{R} \text{ and a.e. } \boldsymbol{x}\in\Gamma_{C_1}. \end{cases}$$

The coefficient of friction μ satisfies the condition

$$(5.13)\qquad \begin{cases} \mu\colon \Gamma_{C_2}\times[0,+\infty)\to[0,+\infty) \text{ is such that}\\ \text{(a) } \mu(\cdot,r) \text{ is measurable on } \Gamma_{C_2} \text{ for all } r\ge 0,\\ \text{(b) } r\mapsto\mu(\boldsymbol{x},r)r \text{ is convex on } [0,+\infty) \text{ a.e. } \boldsymbol{x}\in\Gamma_{C_2},\\ \text{(c) } r\mapsto\mu(\boldsymbol{x},r) \text{ is continuous and nondescreasing on } [0,+\infty), \text{ a.e. } \boldsymbol{x}\in\Gamma_{C_2}. \end{cases}$$

An example of the function μ which satisfies condition (5.13) is given by $\mu(r) = a_0+\sqrt{r}$ for $r\ge 0$ with $a_0\ge 0$. The constraint function U satisfies the following conditions:

$$(5.14)\qquad U\colon L^2(\Omega;\mathbb{R}^d)\to\mathbb{R} \text{ is continuous, } U(\boldsymbol{w})\ge\rho \text{ for all } \boldsymbol{w}\in L^2(\Omega;\mathbb{R}^d) \text{ with some } \rho>0.$$

Finally, we suppose that the densities of volume forces and surface tractions have the following regularity:

$$(5.15)\qquad \boldsymbol{f}_0\in L^2(\Omega;\mathbb{R}^d) \quad\text{and}\quad \boldsymbol{f}_N\in L^2(\Gamma_N;\mathbb{R}^d).$$

Under the above functional framework, we now assume that there exists a pair of functions $(\boldsymbol{u},\boldsymbol{\sigma})$, which is smooth enough such that (5.1)–(5.7) hold. Let $\boldsymbol{v}\in K(\boldsymbol{u})$. We multiply the equation of equilibrium (5.2) by $\boldsymbol{v}-\boldsymbol{u}$, and then employing the Green formula (see, e.g., [40, Theorem 2.25]), one has

$$\langle\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q = \langle\boldsymbol{f}_0,\boldsymbol{v}-\boldsymbol{u}\rangle_{L^2(\Omega;\mathbb{R}^d)} + \int_{\partial\Omega}\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}\cdot(\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x}))\,d\Gamma.$$

Keeping in mind the identity

$$\begin{aligned}\int_{\partial\Omega}&\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}\cdot(\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x}))\,d\Gamma\\ &=\int_{\Gamma_D}\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}\cdot(\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x}))\,d\Gamma+\int_{\Gamma_N}\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}\cdot(\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x}))\,d\Gamma\\ &\quad+\int_{\Gamma_{C_1}}\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}\cdot(\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x}))\,d\Gamma+\int_{\Gamma_{C_2}}\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}\cdot(\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x}))\,d\Gamma,\end{aligned}$$

we use boundary conditions (5.3) and (5.4) to show that

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q = \langle \boldsymbol{f}_0, \boldsymbol{v} - \boldsymbol{u}\rangle_{L^2(\Omega;\mathbb{R}^d)} + \int_{\Gamma_N} \boldsymbol{f}_N(\boldsymbol{x}) \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma$$
$$+ \int_{\Gamma_{C_1}} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu} \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma + \int_{\Gamma_{C_2}} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu} \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma.$$

Hence, we have

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q = \langle \boldsymbol{f}, \pi(\boldsymbol{v} - \boldsymbol{u})\rangle_{Y^*\times Y} + \int_{\Gamma_{C_1}} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu} \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma$$
$$+ \int_{\Gamma_{C_2}} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu} \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma, \tag{5.16}$$

where $\boldsymbol{f} = (\boldsymbol{f}_0, \boldsymbol{f}_N) \in Y^*$. Taking into account boundary condition (5.5) and the equality

$$\int_{\Gamma_{C_1}} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}\cdot(\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma = \int_{\Gamma_{C_1}} \sigma_\nu(\boldsymbol{x})(v_\nu(\boldsymbol{x}) - u_\nu(\boldsymbol{x}))\, d\Gamma + \int_{\Gamma_{C_1}} \boldsymbol{\sigma}_\tau(\boldsymbol{x})\cdot(\boldsymbol{v}_\tau(\boldsymbol{x}) - \boldsymbol{u}_\tau(\boldsymbol{x}))\, d\Gamma,$$

we obtain

$$-\int_{\Gamma_{C_1}} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu} \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma = -\int_{\Gamma_{C_1}} \sigma_\nu(\boldsymbol{x})(v_\nu(\boldsymbol{x}) - u_\nu(\boldsymbol{x}))\, d\Gamma$$
$$\le \int_{\Gamma_{C_1}} j_\nu^0(\boldsymbol{x}, u_\nu(\boldsymbol{x}); v_\nu(\boldsymbol{x}) - u_\nu(\boldsymbol{x}))\, d\Gamma. \tag{5.17}$$

On the other hand, relation (5.6) yields

(5.18)

$$\int_{\Gamma_{C_2}} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu} \cdot (\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma \ge \int_{\Gamma_{C_2}} S(\boldsymbol{x})(u_\nu(\boldsymbol{x}) - v_\nu(\boldsymbol{x}))\, d\Gamma$$
$$+ \int_{\Gamma_{C_2}} \mu(\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d})|S(\boldsymbol{x})|\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d}\, d\Gamma - \int_{\Gamma_{C_2}} \mu(\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d})|S(\boldsymbol{x})|\|\boldsymbol{v}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d}\, d\Gamma.$$

Next, using inclusion (5.1), it is clear that there exists $\boldsymbol{\xi}(\boldsymbol{x}) \in \partial\phi_B(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))$ such that

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) + \boldsymbol{\xi}(\boldsymbol{x})$$

for a.e. $\boldsymbol{x} \in \Omega$. Hence, we have

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q = \langle \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})) + \boldsymbol{\xi}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q$$
$$\le \langle \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q + \int_\Omega \phi_B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))\, d\boldsymbol{x} - \int_\Omega \phi_B(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))\, d\boldsymbol{x}.$$

Recall that $\boldsymbol{u} \in C$ and $\boldsymbol{v} \in K(\boldsymbol{u})$, so the above inequality can be rewritten as

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q \le \langle \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q. \tag{5.19}$$

Combining (5.16)–(5.19), we obtain the following variational formulation of Problem 5.1.

PROBLEM 5.2. *Find a displacement field* $\boldsymbol{u} \in C$ *such that* $\boldsymbol{u} \in K(\boldsymbol{u})$ *and for all* $\boldsymbol{v} \in K(\boldsymbol{u})$*, we have*

(5.20)

$$\langle \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q$$
$$+ \int_{\Gamma_{C_1}} j_\nu^0(\boldsymbol{x}, u_\nu(\boldsymbol{x}); v_\nu(\boldsymbol{x}) - u_\nu(\boldsymbol{x}))\, d\Gamma + \int_{\Gamma_{C_2}} S(\boldsymbol{x})(v_\nu(\boldsymbol{x}) - u_\nu(\boldsymbol{x}))\, d\Gamma$$
$$+ \int_{\Gamma_{C_2}} \mu(\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d})|S(\boldsymbol{x})|(\|\boldsymbol{v}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d} - \|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d})\, d\Gamma \ge \langle \boldsymbol{f}, \pi(\boldsymbol{v} - \boldsymbol{u})\rangle_{Y^*\times Y}.$$

THEOREM 5.3. *Under assumptions* (5.10)–(5.15), *if* $m_{\mathcal{E}} \geq m_{j_\nu}\|\gamma_1\|^2$ *holds, then the solution set of Problem* 5.2 *is nonempty, bounded, and weakly closed in* V.

Proof. We will apply Corollary 3.9. To this end, in what follows, we will verify all its hypotheses. Let $X = L^2(\Gamma;\mathbb{R}^d)$ and denote $C_0 = \{\mathbf{0}_V\}$, and $\gamma = \gamma_1$. It follows from assumptions $\mathbf{0} \in B$ and (5.14) that $C_0 \subset K(\boldsymbol{u})$ for all $\boldsymbol{u} \in V$. We now introduce the operator $T\colon V \to V^*$ and functions $\varphi\colon V \times V \to \mathbb{R}$ and $J\colon X \to \mathbb{R}$ defined by

$$\langle T\boldsymbol{u},\boldsymbol{v}\rangle_{V^*\times V} = \langle \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_Q, \tag{5.21}$$

$$\varphi(\boldsymbol{v},\boldsymbol{u}) = \int_{\Gamma_{C_2}} S(\boldsymbol{x})(v_\nu(\boldsymbol{x}) - u_\nu(\boldsymbol{x}))\,d\Gamma + \int_{\Gamma_{C_2}} \mu(\boldsymbol{x},\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d})|S(\boldsymbol{x})|\big(\|\boldsymbol{v}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d} - \|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d}\big)\,d\Gamma, \tag{5.22}$$

$$J(\boldsymbol{w}) = \int_{\Gamma_{C_1}} j_\nu(\boldsymbol{x},w_\nu(\boldsymbol{x}))\,d\Gamma, \tag{5.23}$$

respectively, for $\boldsymbol{u},\boldsymbol{v} \in V$ and $\boldsymbol{w} \in X$. Next, let $\boldsymbol{u}$, $\boldsymbol{v} \in V$. Hypothesis (5.10) and Hölder's inequality imply

$$\begin{aligned}\|T\boldsymbol{u} - T\boldsymbol{v}\|_{V^*} &= \sup_{\boldsymbol{w}\in V,\,\|\boldsymbol{w}\|_V=1} |\langle T\boldsymbol{u} - T\boldsymbol{v},\boldsymbol{w}\rangle_V| \\ &= \sup_{\boldsymbol{w}\in V,\,\|\boldsymbol{w}\|_V=1} \int_\Omega \big|\big(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))\big) : \boldsymbol{\varepsilon}(\boldsymbol{w}(\boldsymbol{x}))\big|\,d\boldsymbol{x} \\ &\leq \left(\int_\Omega \big\|\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))\big\|^2_{\mathbb{S}^d}\,d\boldsymbol{x}\right)^{\frac{1}{2}} \leq L_{\mathcal{E}}\|\boldsymbol{u} - \boldsymbol{v}\|_V\end{aligned}$$

for all $\boldsymbol{u},\boldsymbol{v} \in V$, and

$$\begin{aligned}&\langle T\boldsymbol{u} - T\boldsymbol{v},\boldsymbol{u} - \boldsymbol{v}\rangle_{V^*\times V} \\ &\quad= \int_\Omega \big(\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))\big) : \big(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) - \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\big)\,d\boldsymbol{x} \geq m_{\mathcal{E}}\|\boldsymbol{u} - \boldsymbol{v}\|^2_V.\end{aligned}$$

Hence, T is Lipschitz continuous and strongly monotone with constants $L_{\mathcal{E}}$ and $m_{\mathcal{E}}$, respectively.

From assumption (5.12) and [40, Theorem 3.47], we conclude that $J\colon X \to \mathbb{R}$ is locally Lipschitz continuous and

$$\begin{cases} J^0(\boldsymbol{u};\boldsymbol{v}) = \displaystyle\int_{\Gamma_{C_1}} j^0_\nu(\boldsymbol{x},u_\nu(\boldsymbol{x});v_\nu(\boldsymbol{x}))\,d\Gamma \ \text{ for all } \ \boldsymbol{u},\boldsymbol{v} \in X, \\ \partial J(\boldsymbol{u}) = \displaystyle\int_{\Gamma_{C_1}} \partial j_\nu(\boldsymbol{x},u_\nu(\boldsymbol{x}))\,d\Gamma \ \text{ for all } \ \boldsymbol{u} \in X, \\ \boldsymbol{u} \mapsto \gamma^*\partial J(\gamma\boldsymbol{u}) \text{ is relaxed monotone with constant } m_J = m_{j_\nu}\|\gamma\|^2. \end{cases} \tag{5.24}$$

Moreover, for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$, $\boldsymbol{\xi}_u \in \partial J(\gamma\boldsymbol{u})$, and $\boldsymbol{\xi}_v \in \partial J(\gamma\boldsymbol{v})$, the estimate

$$\langle T\boldsymbol{u} - T\boldsymbol{v},\boldsymbol{u} - \boldsymbol{v}\rangle + \langle \boldsymbol{\xi}_u - \boldsymbol{\xi}_v,\gamma(\boldsymbol{u} - \boldsymbol{v})\rangle_{X^*\times X} \geq (m_{\mathcal{E}} - m_{j_\nu}\|\gamma\|^2)\|\boldsymbol{u} - \boldsymbol{v}\|^2_V \geq 0$$

shows that the multivalued map $\boldsymbol{u} \mapsto T\boldsymbol{u} + \gamma^*\partial J(\gamma\boldsymbol{u})$ is monotone. Note that $C_0 = \{\mathbf{0}_V\}$, and therefore, for all $\boldsymbol{u} \in V$ and $\boldsymbol{\xi} \in \partial J(\gamma\boldsymbol{u})$, we have

$$
\begin{aligned}
\langle T\boldsymbol{u},\boldsymbol{u}\rangle + \langle \boldsymbol{\xi},\gamma\boldsymbol{u}\rangle_{X^*\times X} - \varphi(\boldsymbol{0}_V,\boldsymbol{u}) &= \langle T\boldsymbol{u}-T\boldsymbol{0}_V,\boldsymbol{u}\rangle + \langle T\boldsymbol{0}_V,\boldsymbol{u}\rangle - \langle \boldsymbol{\xi},-\gamma\boldsymbol{u}\rangle_{X^*\times X} - \varphi(\boldsymbol{0}_V,\boldsymbol{u})\\
&\geq m_{\mathcal{E}}\|\boldsymbol{u}\|_V^2 - \|T\boldsymbol{0}_V\|_{V^*}\|\boldsymbol{u}\|_V - J^0(\gamma\boldsymbol{u};-\gamma\boldsymbol{u})\\
&\quad + \int_{\Gamma_{C_2}} |S(\boldsymbol{x})|\big(u_\nu(\boldsymbol{x}) + \mu(\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d})\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d}\big)\,d\Gamma\\
&\geq m_{\mathcal{E}}\|\boldsymbol{u}\|_V^2 - \|T\boldsymbol{0}_V\|_{V^*}\|\boldsymbol{u}\|_V - J^0(\gamma\boldsymbol{u};-\gamma\boldsymbol{u}) + \int_{\Gamma_{C_2}} |S(\boldsymbol{x})|u_\nu(\boldsymbol{x})\,d\Gamma\\
&\geq m_{\mathcal{E}}\|\boldsymbol{u}\|_V^2 - \|T\boldsymbol{0}_V\|_{V^*}\|\boldsymbol{u}\|_V - \|S\|_{L^2(\Gamma_{C_2})}\|\gamma_2\|\|\boldsymbol{u}\|_V - \alpha_{j_\nu}\,\mathrm{m}(\Gamma_{C_1})\\
&\quad - \beta_{j_\nu}\|\gamma\|\,\mathrm{m}(\Gamma_{C_1})^{\frac{2-\theta}{2}}\|\boldsymbol{u}\|_V^\theta,
\end{aligned}
$$

where we have used condition (5.12)(e) and properties (5.24). Hence,

$$
\lim_{\boldsymbol{u}\in V,\,\|\boldsymbol{u}\|_V\to\infty} \frac{\langle T\boldsymbol{u},\boldsymbol{u}\rangle_V + \inf_{\boldsymbol{\xi}\in\partial J(\gamma\boldsymbol{u})}\langle\boldsymbol{\xi},\gamma\boldsymbol{u}\rangle_{X^*\times X} - \varphi(\boldsymbol{0}_V,\boldsymbol{u})}{\|\boldsymbol{u}\|_V} = +\infty,
$$

due to $\theta < 2$.

On the other hand, from definition (5.22) of φ, it is not difficult to observe that for each $u \in V$, the function $v \mapsto \varphi(v,u)$ is convex and lower semicontinuous, and for all $v \in V$, we have $\varphi(v,v) = 0$, that is, $H(\varphi)$(i) and (iii) hold. To verify condition $H(\varphi)$(ii), we need to show that the function

$$
\boldsymbol{u} \mapsto \int_{\Gamma_{C_2}} \mu(\boldsymbol{x},\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d})\|\boldsymbol{u}_\tau(\boldsymbol{x})\|_{\mathbb{R}^d}\,d\Gamma \quad \text{for} \quad \boldsymbol{u}\in V \tag{5.25}
$$

is convex. The latter follows directly from hypothesis (5.13)(b). The continuity of the function $r \mapsto \mu(\boldsymbol{x},r)$ for a.e. $\boldsymbol{x}\in\Gamma_{C_2}$ implies that the function (5.25) is continuous. In conclusion, we deduce that φ enjoys all conditions in hypotheses (H_φ). Additionally, by using the Sobolev embedding theorems, we can see that $\pi\colon V\to Y$ and $\gamma\colon V\to X$ are both linear, bounded, and compact operators.

Since $C_0 \subset K(\boldsymbol{u})$ for all $\boldsymbol{u}\in V$, the convexity and continuity of $\boldsymbol{u}\mapsto\|\boldsymbol{u}\|_V$ imply that for each $\boldsymbol{u}\in V$ the set $K(\boldsymbol{u})$ is nonempty, convex, and closed. Further, we shall show that the multivalued map K satisfies condition (H_K). Let $\{\boldsymbol{u}_n\}\subset C$ be a sequence such that $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in V. For any $\boldsymbol{v}\in K(\boldsymbol{u})$, we define the sequence $\{\boldsymbol{v}_n\}\subset C$ by

$$
\boldsymbol{v}_n := \frac{U(\boldsymbol{u}_n)}{U(\boldsymbol{u})}\,\boldsymbol{v}.
$$

Recall that $U(\boldsymbol{w}) \geq \rho > 0$ for all $\boldsymbol{w}\in V$, the embedding of V to $L^2(\Omega;\mathbb{R}^d)$ is compact, and $U\colon L^2(\Omega;\mathbb{R}^d)\to\mathbb{R}$ is continuous. Hence, the sequence $\{\boldsymbol{v}_n\}$ converges to $\boldsymbol{v}$ strongly in V. Also, a simple calculation shows that

$$
\|\boldsymbol{v}_n\|_V = \frac{U(\boldsymbol{u}_n)}{U(\boldsymbol{u})}\|\boldsymbol{v}\|_V \leq U(\boldsymbol{u}_n),
$$

where the last inequality follows from the fact that $\boldsymbol{v}\in K(\boldsymbol{u})$, i.e., $\|\boldsymbol{v}\|_V \leq U(\boldsymbol{u})$. This means that for each $n\in\mathbb{N}$, $\boldsymbol{v}_n\in K(\boldsymbol{u}_n)$ and $\boldsymbol{v}_n\to\boldsymbol{v}$ in V. Thus, condition (H_K)(i) is valid.

Subsequently, let $\{\boldsymbol{u}_n\}$, $\{\boldsymbol{v}_n\}\subset C$ be two sequences satisfying

$$
\boldsymbol{v}_n\in K(\boldsymbol{u}_n),\quad \boldsymbol{u}_n\rightharpoonup\boldsymbol{u} \quad \text{and} \quad \boldsymbol{v}_n\rightharpoonup\boldsymbol{v} \quad \text{in } V, \text{ as } n\to\infty.
$$

The definition of K and the weak lower semicontinuity of the norm $\|\cdot\|_V$ imply

$$\|\boldsymbol{v}\|_V \le \liminf_{n\to\infty} \|\boldsymbol{v}_n\|_V \le \liminf_{n\to\infty} U(\boldsymbol{u}_n) = U(\boldsymbol{u}).$$

Hence, we have $\boldsymbol{v} \in K(\boldsymbol{u})$, and condition (H_K)(ii) follows.

Consequently, all hypotheses of Corollary 3.9 have been verified. Invoking this corollary, we deduce that the solution set of Problem 5.2 is nonempty, bounded, and weakly closed in V. This completes the proof. ∎

We move our attention to an optimal control problem associated with Problem 5.1. We denote $\boldsymbol{f} = (\boldsymbol{f}_0, \boldsymbol{f}_N) \in Y^*$ and consider the following cost functional $\mathcal{L}\colon Y^* \to \mathbb{R}$ defined by

$$\mathcal{L}(\boldsymbol{f}) := \inf_{\boldsymbol{u}\in\Sigma(\boldsymbol{f})} \widetilde{g}(\boldsymbol{u}) + m(\boldsymbol{f}). \tag{5.26}$$

Here, $\Sigma(\boldsymbol{f})$ stands for the solution set of Problem 5.2 corresponding to $\boldsymbol{f} \in Y^*$, and $\widetilde{g}\colon V \to \mathbb{R}$ and $m\colon Y \to \mathbb{R}$ are defined by

$$\widetilde{g}(\boldsymbol{u}) := \rho_1 \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))\|^2_{\mathbb{S}^d}\, d\boldsymbol{x} + \rho_2 \int_{\Gamma_{C_2}} |u_\nu(\boldsymbol{x}) - \psi(\boldsymbol{x})|^2\, d\Gamma, \tag{5.27}$$

$$m(\boldsymbol{f}) = \rho_3 \|\boldsymbol{f}_0\|^2_{L^2(\Omega;\mathbb{R}^d)} + \rho_4 \|\boldsymbol{f}_N\|^2_{L^2(\Gamma_N;\mathbb{R}^d)} \tag{5.28}$$

for some positive parameters (weights) ρ_i, $i = 1, \ldots, 4$, and $\psi \in L^2(\Gamma_{C_2})$ is a given function (desired target). The first term in function $\widetilde{g}$ represents the deformation in the body corresponding to the displacement $\boldsymbol{u}$, while the second term allows one to minimize the corresponding normal displacement u_ν to be as close as possible to the "desired displacement" ψ.

The optimal control problem we are interested in is formulated as follows.

Problem 5.4. *Find $\boldsymbol{f} \in Y^*$ such that*

$$\boldsymbol{f} \in \arg\min_{\boldsymbol{f}\in Y^*} \mathcal{L}(\boldsymbol{f}). \tag{5.29}$$

Theorem 5.5. *Assume that hypotheses of Theorem* 5.3 *hold. Then, the set of solutions of Problem* 5.4 *is nonempty and weakly compact in Y^*.*

Proof. It follows from the Sobolev embedding theorem that the function $\widetilde{g}$ is weakly lower semicontinuous and $\widetilde{g}(\boldsymbol{u}) \ge 0$ for all $\boldsymbol{u} \in V$. Moreover, the function m is convex and continuous, so the function $g(\boldsymbol{u}, \boldsymbol{f}) := \widetilde{g}(\boldsymbol{u}) + m(\boldsymbol{f})$ satisfies (H_g). Therefore, the desired conclusion is a consequence of Theorems 4.4 and 5.3. ∎

Finally, it would be of interest to focus future investigations on construction of suitable and reliable numerical techniques to find optimal state and control. To this end, it is important to derive optimality conditions for the optimal control problem (4.1). For the related differentiability of the cost function we refer to [31], and to [22] for numerical strategies in multivalued contact problems.

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