

A distributed algorithm for high-dimension convex quadratically constrained quadratic programs

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Received: 4 January 2021 / Accepted: 3 September 2021 / Published online: 12 October 2021 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract

We propose a Jacobi-style distributed algorithm to solve convex, quadratically constrained quadratic programs (QCQPs), which arise from a broad range of applications. While small to medium-sized convex QCQPs can be solved efficiently by interior-point algorithms, high-dimension problems pose significant challenges to traditional algorithms that are mainly designed to be implemented on a single computing unit. The exploding volume of data (and hence, the problem size), however, may overwhelm any such units. In this paper, we propose a distributed algorithm for general, non-separable, high-dimension convex QCQPs, using a novel idea of predictor–corrector primal–dual update with an adaptive step size. The algorithm enables distributed storage of data as well as parallel, distributed computing. We establish the conditions for the proposed algorithm to converge to a global optimum, and implement our algorithm on a computer cluster with multiple nodes using message passing interface. The numerical experiments are conducted on data sets of various scales from different applications, and the results show that our algorithm exhibits favorable scalability for solving high-dimension problems.

Keywords Convex QCQP \cdot Distributed algorithm \cdot Proximal method \cdot Parallel computing

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1 Introduction

In this paper, we consider the following constrained optimization problem:

$$\underset{\mathbf{x} \in \mathbb{R}^{n_1}}{\text{minimize}} \quad \frac{1}{2} \mathbf{x}^T P_0 \mathbf{x} + \mathbf{q}_0^T \mathbf{x} + r_0
\text{subject to} \quad \frac{1}{2} \mathbf{x}^T P_i \mathbf{x} + \mathbf{q}_i^T \mathbf{x} + r_i \le 0, \quad i = 1, \dots, m_1,$$
(1)

where $P_i \in \mathbb{R}^{n_1 \times n_1}$, $\mathbf{q}_i \in \mathbb{R}^{n_1}$, and $r_i \in \mathbb{R}$ for $i = 0, 1, ..., m_1$ are all given. Such a problem is referred to as a quadratically constrained quadratic program (QCQP). (Note that linear constraints are included with $P_i = \mathbf{0}$, a matrix of all 0's, for some i.) If additionally, P_0, P_1, \dots, P_m are all positive semidefinite (PSD) matrices, then the problem is convex. Convex QCQPs arise from a wide range of application areas, including multiple kernel learning [16], ranking recommendations [5], signal processing [11], radar applications [22], computer vision [1], and electric power system operation [3], to name a few. Small to medium-sized convex OCOPs can be solved efficiently by the well-established interior-point method (IPM) [19], which has polynomial running time for solving convex optimization problems. However, in order to write out the barrier function in the IPM for the feasible domain of a QCQP, decomposition of matrices $P_i = F_i^T F_i$ for $i = 1, ..., m_1$ is usually required [18], which may not be readily available through the input data. For example, in kernel-based learning applications, each quadratic constraint comprises a kernel matrix, whose components are directly defined by a kernel function: $K_{ii'} = k(\mathbf{x}_i, \mathbf{x}_{i'})$. The operations to obtain a matrix decomposition, such as through Cholesky decomposition, typically have computational complexity of $O(n^3)$, which could become very costly as the size of the matrices grows. When the dimension of the QCQPs increases dramatically due to huge amount of data, or when the data just cannot be all stored in a central location, a centralized algorithm, such as the IPM, may no longer be applicable. This directly motivates the proposed algorithm in this paper, which not only does not require any matrix decomposition, but also facilitates distributed storage of data to achieve memory efficiency and enables parallel computing even for QCQPs of non-separable constraints.

In addition to being a typical optimization problem, a convex QCQP is also a special instance of a second-order cone program (SOCP), which is in turn a special form of semi-definite program (SDP) [17]. When using commercial solvers, such as CPLEX, to solve a convex QCQP, it is usually transformed into an SOCP through preprocessing [13], and then a barrier-method-based optimizer is applied. To solve large-scale conic programs, [20] applies an operator splitting method (such as the well-known alternating direction method of multipliers, or ADMM) to the homogeneous self-dual embedding, which is an equivalent convex feasibility problem involving finding a nonzero point in the intersection of a subspace and a cone. There are also ADMM-based distributed algorithms for solving large-scale SDPs proposed in [14, 21]; but they can only be applied to a class of decomposable SDPs with special graph representations (chordal graphs, for example). To either translate a convex QCQP to a standard SOCP or use the Schur Complement to rewrite each quadratic



inequality as a linear matrix inequality (LMI) and hence translate a convex QCQP to an SDP , however, calls for matrix decomposition: $P_i = F_i^T F_i$ for $i = 1, \ldots, m_1$. As mentioned before, such operations can be very expensive for large-scale matrices. There is another ADMM-based distributed algorithm that decomposes a general QCQP with m constraints into m single-constrained QCQPs using a reformulated consensus optimization form [10]. However, even the size of the single-constrained QCQP can be very large in many applications, which may still need further decomposition, making the overall algorithm's efficiency in doubt. There is also a recent approach to transform quadratic constraints into linear constraints by sampling techniques and then to apply ADMM-based algorithms to solve the resulting high-dimension quadratic programs (QPs) [2]. This approach is studied only for QCQPs with all matrices being positive definite (PD), and all the test problems shown in [2] are of a single constraint. How would the sampling approach perform with PSD matrices in the constraints or with multiple quadratic constraints is unknown.

To overcome the above-mentioned limitations of the existing algorithms, we propose a novel first-order distributed algorithm, which decomposes a convex QCQP by a method inspired by the idea of the predictor corrector proximal multiplier method (PCPM) [6]. The advantages of our algorithm include the following: (i) non-separable, quadratic functions can become naturally separable after introducing the so-called predictor and corrector variables for both primal and dual variables, which greatly facilitates distributed computing (with Jacobi-style parallel updating, as opposed to Guass-Seidel style sequential updating); while ADMM-type algorithms cannot be directly applied to QCQPs without separable constraints; (ii) both the primal/dual predictor variables and corrector variables can be updated componentwise, making the method well-suited for massively parallel computing, and each *n*-by-*n* Hessian matrix can be stored column-wise in distributed computing units; (iii) no matrix decomposition or inversion is needed.

Convergence of our algorithm to an optimal solution will be shown, along with various numerical results. We first test the algorithm on solving standard QCQPs with randomly generated data sets of different scales, and then apply it to solve high-dimension multiple kernel learning problems. Numerical experiments are conducted on a multi-node computer cluster through message passing interface (MPI), and multiple nodes are used to highlight the benefits of distributed implementation of our algorithm. Numerical results are compared with those obtained from the commercial solver CPLEX (version 12.8.0, using the barrier optimizer). The comparison will show that our algorithm can scale to very large problems at the cost of consuming more cheap iterations to reach a higher accuracy. With a modest accuracy, our algorithm exhibits favorable scalability for solving high-dimension QCQPs when CPLEX fails to provide a solution due to memory limit or other issues.

The remainder of the paper is organized as follows. In Sect. 2, we briefly summarize the original PCPM algorithm and highlight the novel idea in our proposed algorithm. Section 3 provides convergence analyses of the algorithm, followed by discussions on how to implement the algorithm in a distributed framework in Sect. 4. Numerical performance of various testing problems is reported in Sect. 5. Finally, we conclude with some discussions in Sect. 6.



2 Algorithm description

2.1 PCPM algorithm

To present our distributed algorithm, we first briefly describe the original PCPM algorithm [6] to make this paper self-contained. For this purpose, it suffices to consider a 2-block linearly constrained convex optimization problem:

minimize
$$\mathbf{x}_1 \in \mathbb{R}^{n_1}, \mathbf{x}_2 \in \mathbb{R}^{n_2}$$
 $f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2)$ subject to $A_1\mathbf{x}_1 + A_2\mathbf{x}_2 = \mathbf{b}$, (λ)

where $f_1: \mathbb{R}^{n_1} \to (-\infty, +\infty]$ and $f_2: \mathbb{R}^{n_2} \to (-\infty, +\infty]$ are closed proper convex functions, $A_1 \in \mathbb{R}^{m \times n_1}$ and $A_2 \in \mathbb{R}^{m \times n_2}$ are full row-rank matrices, $\mathbf{b} \in \mathbb{R}^m$ is a given vector, and $\lambda \in \mathbb{R}^m$ is the corresponding Lagrangian multiplier associated with the linear equality constraint. The classic Lagrangian function $\mathcal{L}: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}$ is defined as:

$$\mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \lambda) = f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2) + \lambda^T (A_1 \mathbf{x}_1 + A_2 \mathbf{x}_2 - \mathbf{b}).$$
 (3)

It is well-known that for a convex problem of the specific form in (2) (where the linear constraint qualification automatically holds), finding an optimal solution is equivalent to finding a saddle point $(\mathbf{x}_1^*, \mathbf{x}_2^*, \lambda^*)$ such that $\mathcal{L}(\mathbf{x}_1^*, \mathbf{x}_2^*, \lambda) \leq \mathcal{L}(\mathbf{x}_1^*, \mathbf{x}_2^*, \lambda^*) \leq \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \lambda^*)$. To find such a saddle point, a simple dual decomposition algorithm can be applied to $\mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \lambda)$. More specifically, at each iteration k, given a fixed Lagrangian multiplier λ^k , the primal decision variables $(\mathbf{x}_1^{k+1}, \mathbf{x}_2^{k+1})$ can be obtained, in parallel, by minimizing $\mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \lambda^k)$. Then a dual update $\lambda^{k+1} = \lambda^k + \rho(A_1\mathbf{x}_1^{k+1} + A_2\mathbf{x}_2^{k+1} - b)$ is performed.

While the above algorithmic idea is simple, it is well-known that convergence cannot be established without more restrictive assumptions, such as strict convexity of f_1 and f_2 (e.g., Theorem 26.3 in [23]). One approach to overcome such difficulties is the proximal point algorithm, which obtains $(\mathbf{x}_1^{k+1}, \mathbf{x}_2^{k+1})$ by minimizing the proximal augmented Lagrangian function defined as $\mathcal{L}_{\rho}(\mathbf{x}_1, \mathbf{x}_2, \lambda^k) := \mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \lambda^k) + \frac{\rho}{2} \|A_1\mathbf{x}_1 + A_2\mathbf{x}_2 - \mathbf{b}\|_2^2 + \frac{1}{2\rho} \|\mathbf{x}_1 - \mathbf{x}_1^k\|_2^2 + \frac{1}{2\rho} \|\mathbf{x}_2 - \mathbf{x}_2^k\|_2^2$. The parameter ρ is given, which determines the step-size for updating both primal and dual variables in each iteration, and plays a key role in the convergence of the overall algorithm. The primal minimization step now becomes (with the Lagrangian function $\mathcal{L}(\mathbf{x}_1, \mathbf{x}_2, \lambda^k)$ explicitly written out in the form of Eq. (3)):

$$(\mathbf{x}_{1}^{k+1}, \mathbf{x}_{2}^{k+1}) = \underset{\mathbf{x}_{1} \in \mathbb{R}^{n_{1}}, \mathbf{x}_{2} \in \mathbb{R}^{n_{2}}}{\operatorname{argmin}} f_{1}(\mathbf{x}_{1}) + f_{2}(\mathbf{x}_{2}) + (\lambda^{k})^{T} (A_{1}\mathbf{x}_{1} + A_{2}\mathbf{x}_{2} - \mathbf{b})$$

$$+ \frac{\rho}{2} \|A_{1}\mathbf{x}_{1} + A_{2}\mathbf{x}_{2} - \mathbf{b}\|_{2}^{2}$$

$$+ \frac{1}{2\rho} \|\mathbf{x}_{1} - \mathbf{x}_{1}^{k}\|_{2}^{2} + \frac{1}{2\rho} \|\mathbf{x}_{2} - \mathbf{x}_{2}^{k}\|_{2}^{2}.$$

$$(4)$$



With (4), however, \mathbf{x}_1^{k+1} and \mathbf{x}_2^{k+1} can no longer be obtained in parallel due to the augmented term $\|A_1\mathbf{x}_1 + A_2\mathbf{x}_2 - \mathbf{b}\|_2^2$. To overcome this difficulty, the PCPM algorithm introduces a predictor variable $\boldsymbol{\mu}^{k+1}$:

$$\boldsymbol{\mu}^{k+1} := \lambda^k + \rho(A_1 \mathbf{x}_1^k + A_2 \mathbf{x}_2^k - \mathbf{b}). \tag{5}$$

Using the predictor variable, the optimization in (4) can be approximated as:

$$\begin{split} (\mathbf{x}_{1}^{k+1}, \mathbf{x}_{2}^{k+1}) &= \underset{\mathbf{x}_{1} \in \mathbb{R}^{n_{1}}, \mathbf{x}_{2} \in \mathbb{R}^{n_{2}}}{\operatorname{argmin}} f_{1}(\mathbf{x}_{1}) + f_{2}(\mathbf{x}_{2}) + (\boldsymbol{\mu}^{k+1})^{T} (A_{1}\mathbf{x}_{1} + A_{2}\mathbf{x}_{2} - \mathbf{b}) \\ &+ \frac{1}{2\rho} \|\mathbf{x}_{1} - \mathbf{x}_{1}^{k}\|_{2}^{2} + \frac{1}{2\rho} \|\mathbf{x}_{2} - \mathbf{x}_{2}^{k}\|_{2}^{2}, \end{split} \tag{6}$$

which allows \mathbf{x}_1^{k+1} and \mathbf{x}_2^{k+1} to be obtained in parallel again. After solving (6), the PCPM algorithm updates the dual variable as follows:

$$\lambda^{k+1} = \lambda^k + \rho (A_1 \mathbf{x}_1^{k+1} + A_2 \mathbf{x}_2^{k+1} - \mathbf{b}), \tag{7}$$

which is referred to as a corrector update.

2.2 A distributed algorithm for high-dimension convex QCQPs

Now consider a convex QCQP problem in the following form:

minimize
$$\begin{aligned} & \underset{\mathbf{x} \in \mathbb{X}, \ \mathbf{u} \in \mathbb{R}^{n_2}}{\text{minimize}} & & \frac{1}{2} \mathbf{x}^T P_0 \mathbf{x} + \mathbf{q}_0^T \mathbf{x} + \mathbf{c}_0^T \mathbf{u} + r_0 \\ & \text{subject to} & & \frac{1}{2} \mathbf{x}^T P_i \mathbf{x} + \mathbf{q}_i^T \mathbf{x} + \mathbf{c}_i^T \mathbf{u} + r_i \le 0, \quad i = 1, \dots, m_1, \\ & & & A \mathbf{x} + B \mathbf{u} = \mathbf{b}, \end{aligned}$$
 (8)

where $P_i \in \mathbb{R}^{n_1 \times n_1}$, $\mathbf{q}_i \in \mathbb{R}^{n_1}$, $r_i \in \mathbb{R}$ for $i = 0, 1, \ldots, m_1$, $A \in \mathbb{R}^{m_2 \times n_1}$, $B \in \mathbb{R}^{m_2 \times n_2}$ and $\mathbf{b} \in \mathbb{R}^{m_2}$ are all given. Note that we introduce a new variable $\mathbf{u} \in \mathbb{R}^{n_2}$ to explicitly write out the linear-only terms $\mathbf{c}_i^T \mathbf{u}$ with coefficients $\mathbf{c}_i \in \mathbb{R}^{n_2}$ for $i = 0, 1, \ldots, m_1$, and also write out a linear equality constraint $A\mathbf{x} + B\mathbf{u} = \mathbf{b}$ separately. While $\mathbb{X} = \prod_{j=1}^{n'} \mathbb{X}_j \subset \mathbb{R}^{n_1}$ and each $\mathbb{X}_j \subset \mathbb{R}^{n_j}$ can be any closed and convex set in \mathbb{R}^{n_j} with $\sum_{j=1}^{n'} n_j = n_1$, we consider specifically the one-dimension box constraint here; that is $\mathbb{X}_j = \{x_j \in \mathbb{R}^1 | 0 \le x_j \le \bar{X}_j\}$ for $j = 1, \ldots, n_1$.

The specific QCQP formulation in (8) is not more general than the standard form (1). The reason that we write out a QCQP in this specific form is to emphasize the fact that when dealing with QCQPs with linear constraints (including box constraints), our algorithm does not require the problem to be reformulated into the standard form in (1). This can be convenient from implementation perspective, as several applications, including multiple kernel learning, naturally lead to a QCQP in the form of (8).

To avoid technical difficulties, we make the blanket assumption throughout this paper that the Slater's constraint qualification (CQ) holds. Consequently, if an optimal solution exists of (8), then there always exists a corresponding Lagrangian



multiplier $(\lambda, \gamma) = (\lambda_1 \cdots \lambda_{m_1}, \gamma_1 \cdots \gamma_{m_2})^T$. To apply the PCPM algorithm to the QCQP in (8), at each iteration k, with a given primal–dual pair $(\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \gamma^k)$, we start with a dual predictor update:

• dual predictor:

$$\mu_i^{k+1} = \boldsymbol{\Pi}_{\mathbb{R}_+} \left(\lambda_i^k + \rho \left[\frac{1}{2} (\mathbf{x}^k)^T P_i \mathbf{x}^k + \mathbf{q}_i^T \mathbf{x}^k + \mathbf{c}_i^T \mathbf{u}^k + r_i \right] \right), \quad i = 1, \dots, m_1, \quad (9)$$

$$v_i^{k+1} = \gamma_i^k + \rho \left[A \mathbf{x}^k + B \mathbf{u}^k - \mathbf{b} \right]_i, \quad i = 1, \dots, m_2,$$

where $\Pi_{\mathbb{Z}}(\mathbf{z})$ denotes the projection of a vector $\mathbf{z} \in \mathbb{R}^n$ onto a closed and convex set $\mathbb{Z} \subset \mathbb{R}^n$, and \mathbb{R}_+ refers to the set of all non-negative real numbers.

After the dual predictor update step (9), we update the primal variables $(\mathbf{x}^{k+1}, \mathbf{u}^{k+1})$ by minimizing the Lagrangian function $\mathcal{L}(\mathbf{x}, \mathbf{u}, \boldsymbol{\mu}^{k+1}, \boldsymbol{\nu}^{k+1})$ evaluated at the dual predictor variable $(\boldsymbol{\mu}^{k+1}, \boldsymbol{\nu}^{k+1})$, plus the proximal terms. The primal minimization step can be written as

$$\mathbf{x}^{k+1} = \underset{\mathbf{x} \in \mathbb{X}}{\operatorname{argmin}} \frac{1}{2} \mathbf{x}^T P_0 \mathbf{x} + \mathbf{q}_0^T \mathbf{x} + \sum_{i=1}^{m_1} \mu_i^{k+1} \left(\frac{1}{2} \mathbf{x}^T P_i \mathbf{x} + \mathbf{q}_i^T \mathbf{x} \right) + (\mathbf{v}^{k+1})^T A \mathbf{x} + \frac{1}{2\rho} \|\mathbf{x} - \mathbf{x}^k\|_2^2,$$
(10a)

$$\mathbf{u}^{k+1} = \underset{\mathbf{u} \in \mathbb{R}^{n_2}}{\operatorname{argmin}} \mathbf{c}_0^T \mathbf{u} + \sum_{i=1}^{m_1} \mu_i^{k+1} \mathbf{c}_i^T \mathbf{u} + (\mathbf{v}^{k+1})^T B \mathbf{u} + \frac{1}{2\rho} \|\mathbf{u} - \mathbf{u}^k\|_2^2.$$
(10b)

Introducing the dual predictors μ and v allows parallel updating of the primal variables \mathbf{x} and \mathbf{u} , exactly as in the general PCPM algorithm. However, the primal variable $\mathbf{x} = (x_1 \dots x_j \dots x_{n_1})^T$ cannot be further decomposed into parallel updating of each component x_j , due to the coupling terms $\mathbf{x}^T P_i \mathbf{x}$, $i = 0, 1, \dots, m_1$, unless all P_i 's are diagonal matrices. To realize parallel updating of x_j 's, we propose a simple idea to use $P_i \mathbf{x}^k$ as a "predictor" for $P_i \mathbf{x}$ in the optimization (10a).

To illustrate the idea, it may be easier to consider the first-order optimality condition of (10a):

$$\frac{1}{\rho}(\mathbf{x}^{k} - \mathbf{x}^{k+1}) \in \underbrace{P_0 \mathbf{x}^{k+1}}_{(\Delta_0)} + \mathbf{q}_0 + \sum_{i=1}^{m_1} \mu_i^{k+1} \left(\underbrace{P_i \mathbf{x}^{k+1}}_{(\Delta_i)} + \mathbf{q}_i \right) + A^T \mathbf{v}^{k+1} + \mathcal{N}_{\mathbb{X}}(\mathbf{x}^{k+1}), \tag{11}$$

where $\mathcal{N}_{\mathbb{X}}(\mathbf{x}^{k+1})$ is the normal cone to the convex set $\mathbb{X} = \prod_{j=1}^{n_1} \mathbb{X}_j$ at the solution point \mathbf{x}^{k+1} . By approximating each (Δ_i) using the predictor $P_i\mathbf{x}^k$, $i = 0, 1, \dots, m_1$, the first-order optimality condition now becomes

$$\frac{1}{\rho}(\mathbf{x}^k - \mathbf{x}^{k+1}) \in P_0 \mathbf{x}^k + \mathbf{q}_0 + \sum_{i=1}^{m_1} \mu_i^{k+1} (P_i \mathbf{x}^k + \mathbf{q}_i) + A^T \mathbf{v}^{k+1} + \mathcal{N}_{\mathbb{X}}(\mathbf{x}^{k+1}). \quad (12)$$



With (12), it is easy to see that \mathbf{x}^{k+1} can be obtained through component-wise calculations. (Note that the normal cone of box constraints has explicit algebraic expressions and can also be decomposed component-wise with respect to \mathbf{x}^{k+1} .) Unfortunately, this simple idea would not work theoretically in the sense that convergence to an optimal solution cannot be established. This is mainly due to the difficulty to bound the error of $\|P_i\mathbf{x}^{k+1} - P_i\mathbf{x}^k\|$ along the iterations.

To overcome this hurdle, we propose a novel approach to split (12) into two steps by first introducing "primal predictor" variable \mathbf{y}^{k+1} for the primal decision variable \mathbf{x}^k , followed by a corrector update:

step 1 (predictor) :

$$\frac{1}{\rho}(\mathbf{x}^k - \mathbf{y}^{k+1}) \in P_0 \mathbf{x}^k + \mathbf{q}_0 + \sum_{i=1}^{m_1} \lambda_i^k \left(P_i \mathbf{x}^k + \mathbf{q}_i \right) + A^T \boldsymbol{\gamma}^k + \mathcal{N}_{\chi}(\mathbf{y}^{k+1}); \tag{13a}$$

step 2 (corrector) :

$$\frac{1}{\rho}(\mathbf{x}^{k} - \mathbf{x}^{k+1}) \in P_0 \mathbf{y}^{k+1} + \mathbf{q}_0 + \sum_{i=1}^{m_1} \mu_i^{k+1} \left(P_i \mathbf{y}^{k+1} + \mathbf{q}_i \right) + A^T \mathbf{v}^{k+1} + \mathcal{N}_{\mathbb{X}}(\mathbf{x}^{k+1}). \tag{13b}$$

By focusing on box constraints for the generic set X_j , and using the notation $[\mathbf{z}]_j$ to denote the *j*th component of a vector \mathbf{z} , we can rewrite (13a) and (13b) componentwise as follows, for each $j = 1, \dots, n_1$:

• primal predictor of x_i^k :

$$y_j^{k+1} := \Pi_{\times_j} \left(x_j^k - \rho \left[P_0 \mathbf{x}^k + \mathbf{q}_0 + \sum_{i=1}^{m_1} \lambda_i^k (P_i \mathbf{x}^k + \mathbf{q}_i) + A^T \boldsymbol{\gamma}^k \right]_j \right),$$
(14a)

• primal corrector of x_i^k :

$$x_j^{k+1} = \Pi_{\chi_j} \left(x_j^k - \rho \left[P_0 \mathbf{y}^{k+1} + \mathbf{q}_0 + \sum_{i=1}^{m_1} \mu_i^{k+1} \left(P_i \mathbf{y}^{k+1} + \mathbf{q}_i \right) + A^T \mathbf{v}^{k+1} \right]_j \right), \tag{14b}$$

where the projection onto the box constraint set X_i can be expressed as:

$$\Pi_{X_{j}}(x_{j}) := \begin{cases}
0, & \text{if } x_{j} < 0; \\
x_{j}, & \text{if } 0 \leq x_{j} \leq \bar{X}_{j}; \\
\bar{X}_{j}, & \text{if } x_{j} > \bar{X}_{j}.
\end{cases}$$
(15)

With (14a) and (14b), in addition to the apparent benefits of updating the variables component-wise, the multiplications of $P_i \mathbf{x}^k$ and $P_i \mathbf{y}^{k+1}$, $i = 0, 1, ..., m_1$ in (14a) and (14b) do not need to be carried out completely in each computing unit responsible for updating y_j^{k+1} and x_j^{k+1} . The multiplications can be divided into multiple subtasks, and each of them only needs the *j*th column of matrices P_i 's and can be



accomplished locally by each computing unit. More detailed discussions of this point are provided in Sect. 4.1.

The update of the other primal variable, \mathbf{u}^{k+1} , can be performed in a similar fashion, which is to split into two steps by first introducing a predictor variable \mathbf{v}^{k+1} for \mathbf{u}^k , followed by a corrector update:

• primal predictor of
$$u_j^k$$
:

$$v_j^{k+1} := u_j^k - \rho \left[\mathbf{c}_0 + \sum_{i=1}^{m_1} \lambda_i^k \mathbf{c}_i + B^T \boldsymbol{\gamma}^k \right]_j, \quad j = 1, \dots, n_2,$$
(16a)

• primal corrector of u_i^k :

$$u_j^{k+1} = u_j^k - \rho \left[\mathbf{c}_0 + \sum_{i=1}^{m_1} \mu_i^{k+1} \mathbf{c}_i + B^T \mathbf{v}^{k+1} \right]_j, \quad j = 1, \dots, n_2.$$
 (16b)

A dual corrector update is then performed for each Lagrangian multiplier $(\lambda^{k+1}, \gamma^{k+1})$:

· dual corrector:

$$\lambda_{i}^{k+1} = \boldsymbol{\Pi}_{\mathbb{R}_{+}} \left(\lambda_{i}^{k} + \rho \left[\frac{1}{2} (\mathbf{y}^{k+1})^{T} P_{i} \mathbf{y}^{k+1} + \mathbf{q}_{i}^{T} \mathbf{y}^{k+1} + \mathbf{c}_{i}^{T} \mathbf{v}^{k+1} + r_{i} \right] \right),$$

$$i = 1, \dots, m_{1},$$

$$\gamma_{i}^{k+1} = \gamma_{i}^{k} + \rho \left[A \mathbf{y}^{k+1} + B \mathbf{v}^{k+1} - \mathbf{b} \right]_{i}, \quad i = 1, \dots, m_{2}.$$

$$(17)$$

The overall structure of the proposed algorithm, which we name it PC²PM, to reflect the fact that two sets of predictors and correctors are utilized, is presented in Algorithm 1 below.

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Algorithm 1 PC<sup>2</sup>PM
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```
1: Initialization choose an arbitrary starting point (\mathbf{x}^0, \mathbf{u}^0, \lambda^0, \gamma^0).

2: k \leftarrow 0.

3: while termination conditions are not met do

4: • Adaptive Step-size update the step-size \rho^{k+1};

5: • Predictor Update update (\mu^{k+1}, \nu^{k+1}), \mathbf{y}^{k+1}, and \mathbf{v}^{k+1} according to (9), (14a) and (16a);

6: • Corrector Update update \mathbf{x}^{k+1}, \mathbf{u}^{k+1}, and (\lambda^{k+1}, \gamma^{k+1}) according to (14b), (16b) and (17);

7: k \leftarrow k+1

8: return (\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \gamma^k).
```

Note that the starting point of the PC^2PM algorithm can be arbitrary, and is not required to be feasible. To establish convergence of the algorithm, the specific rules to update the step-size ρ are crucial, which is the main focus of the next section. The implementation details, including distributed data storage, parallel computing through Message Passing Interface (MPI), and termination conditions, are provided in Sect. 4.



3 Convergence analysis

In this section, we establish sufficient conditions for the PC²PM algorithm to converge to an optimal solution from any starting point. First, we make a standard assumption on (8) about the existence of an optimal solution.

Assumption 1 (*Existence of an Optimal Solution*) The convex QCQP (8) is assumed to have an optimal solution, denoted by $(\mathbf{x}^*, \mathbf{u}^*)$.

With Assumption 1 and the assumption on Slater's CQ, we know that a saddle point exist for the convex QCQP (8); more specifically, $(\mathbf{x}^*, \mathbf{u}^*, \lambda^*, \gamma^*)$ is a saddle point of (8) if for any $\mathbf{x} \in \mathbb{X}$, $\mathbf{u} \in \mathbb{R}^{n_2}$, $\lambda \in \mathbb{R}^{m_1}_+$ and $\gamma \in \mathbb{R}^{m_2}$, we have that

$$\mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda, \gamma) \le \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda^*, \gamma^*) \le \mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda^*, \gamma^*), \tag{18}$$

where $\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda, \gamma)$ is the Lagrangian function of (8):

$$\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda, \gamma) := \frac{1}{2} \mathbf{x}^T P_0 \mathbf{x} + \mathbf{q}_0^T \mathbf{x} + \mathbf{c}_0^T \mathbf{u} + r_0$$

$$+ \sum_{i=1}^{m_1} \lambda_i \left(\frac{1}{2} \mathbf{x}^T P_i \mathbf{x} + \mathbf{q}_i^T \mathbf{x} + \mathbf{c}_i^T \mathbf{u} + r_i \right) + \gamma^T (A \mathbf{x} + B \mathbf{u} - \mathbf{b}).$$
(19)

The case when the convex QCQP is infeasible will be discussed in Sect. 4.4.

Next, we derive some essential lemmas for constructing the main convergence proof.

Lemma 1 (Inequality of Proximal Minimization Point) Given a closed, convex set $\mathbb{Z} \subset \mathbb{R}^n$, and a closed, convex differentiable function $F: \mathbb{Z} \to \mathbb{R}$. With a given point $\bar{\mathbf{z}} \in \mathbb{Z}$ and a positive number $\rho > 0$, if $\hat{\mathbf{z}}$ is a proximal minimization point; i.e. $\hat{\mathbf{z}} := \arg\min_{\mathbf{z} \in \mathbb{Z}} F(\mathbf{z}) + \frac{1}{2\rho} \|\mathbf{z} - \bar{\mathbf{z}}\|_2^2$, then we have that

$$2\rho[F(\widehat{\mathbf{z}}) - F(\mathbf{z})] \le \|\overline{\mathbf{z}} - \mathbf{z}\|_2^2 - \|\widehat{\mathbf{z}} - \mathbf{z}\|_2^2 - \|\widehat{\mathbf{z}} - \overline{\mathbf{z}}\|_2^2, \quad \forall \mathbf{z} \in \mathbb{Z}.$$
 (20)

Proof Denote $\Phi(\mathbf{z}) = F(\mathbf{z}) + \frac{1}{2\rho} \|\mathbf{z} - \bar{\mathbf{z}}\|_2^2$. By the definition of $\hat{\mathbf{z}}$, we have $\nabla_{\mathbf{z}}\Phi(\hat{\mathbf{z}}) = \mathbf{0}$. Since $\Phi(\mathbf{z})$ is strongly convex with modulus $\frac{1}{\rho}$, it follows that $2\rho \left[\Phi(\mathbf{z}) - \Phi(\hat{\mathbf{z}})\right] \geq \|\hat{\mathbf{z}} - \mathbf{z}\|_2^2$ for any $\mathbf{z} \in \mathbb{Z}$.

For the ease of presenting the next two lemmas, we introduce a notation for the linear approximation of the Lagrangian function (19).

Definition 1 With a given tuple $(\mathbf{x}', \lambda', \gamma') \in \mathbb{X} \times \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$, we define the following function $\mathcal{R} : \mathbb{X} \times \mathbb{R}^{n_2} \to \mathbb{R}$ as a linear approximation of the Lagrangian function $\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda, \gamma)$ evaluated at $(\mathbf{x}', \lambda', \gamma')$.



$$\mathcal{R}(\mathbf{x}, \mathbf{u}; \mathbf{x}', \lambda', \gamma') := (P_0 \mathbf{x}' + \mathbf{q}_0)^T \mathbf{x} + \mathbf{c}_0^T \mathbf{u} + r_0$$

$$+ \sum_{i=1}^{m_1} \lambda_i' [(P_i \mathbf{x}' + \mathbf{q}_i)^T \mathbf{x} + \mathbf{c}_i^T \mathbf{u} + r_i] + (\gamma')^T (A\mathbf{x} + B\mathbf{u} - \mathbf{b}),$$
(21)

for any $\mathbf{x} \in \mathbb{X}$ and $\mathbf{u} \in \mathbb{R}^{n_2}$.

Lemma 2 The update steps (9), (14a), (14a), (16a), (16b) and (17) are equivalent to obtaining proximal minimization points as follows:

$$(\boldsymbol{\mu}^{k+1}, \ \boldsymbol{v}^{k+1}) = \underset{\boldsymbol{\lambda} \in \mathbb{R}_{+}^{m_{1}}, \ \boldsymbol{\gamma} \in \mathbb{R}^{m_{2}}}{\operatorname{argmin}} - \mathcal{L}(\mathbf{x}^{k}, \mathbf{u}^{k}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) + \frac{1}{2\rho^{k+1}} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^{k}\|_{2}^{2} + \frac{1}{2\rho^{k+1}} \|\boldsymbol{\gamma} - \boldsymbol{\gamma}^{k}\|_{2}^{2};$$
(22a)

$$(\mathbf{y}^{k+1}, \ \mathbf{v}^{k+1}) = \underset{\mathbf{x} \in \mathbb{X}, \ \mathbf{u} \in \mathbb{R}^{n_2}}{\operatorname{argmin}} \mathcal{R}(\mathbf{x}, \mathbf{u}; \mathbf{x}^k, \lambda^k, \boldsymbol{\gamma}^k)$$

$$+ \frac{1}{2\rho^{k+1}} \|\mathbf{x} - \mathbf{x}^k\|_2^2 + \frac{1}{2\rho^{k+1}} \|\mathbf{u} - \mathbf{u}^k\|_2^2;$$
(22b)

$$(\mathbf{x}^{k+1}, \, \mathbf{u}^{k+1}) = \underset{\mathbf{x} \in \mathbb{X}, \, \mathbf{u} \in \mathbb{R}^{n_2}}{\operatorname{argmin}} \mathcal{R}(\mathbf{x}, \mathbf{u}; \mathbf{y}^{k+1}, \boldsymbol{\mu}^{k+1}, \boldsymbol{\nu}^{k+1}) + \frac{1}{2\rho^{k+1}} \|\mathbf{x} - \mathbf{x}^k\|_2^2 + \frac{1}{2\rho^{k+1}} \|\mathbf{u} - \mathbf{u}^k\|_2^2;$$
(22c)

$$(\lambda^{k+1}, \, \boldsymbol{\gamma}^{k+1}) = \underset{\lambda \in \mathbb{R}_{+}^{m_{1}}, \, \boldsymbol{\gamma} \in \mathbb{R}^{m_{2}}}{\operatorname{argmin}} - \mathcal{L}(\mathbf{y}^{k+1}, \mathbf{v}^{k+1}, \lambda, \boldsymbol{\gamma}) + \frac{1}{2\rho^{k+1}} \|\lambda - \lambda^{k}\|_{2}^{2} + \frac{1}{2\rho^{k+1}} \|\boldsymbol{\gamma} - \boldsymbol{\gamma}^{k}\|_{2}^{2}.$$
(22d)

Since all the four optimization in (22a)–(22d) are convex optimization problems with linear constraints, the proof follows directly from the first-order optimality conditions of each of the optimization problems, and hence is omitted.

Lemma 3 At a saddle point $(\mathbf{x}^*, \mathbf{u}^*, \lambda^*, \gamma^*)$ of the QCQP (8), the following inequality holds for any $\mathbf{x} \in \mathbb{X}$, $\mathbf{u} \in \mathbb{R}^{n_2}$, $\lambda \in \mathbb{R}^{m_1}$ and $\gamma \in \mathbb{R}^{m_2}$:

$$\mathcal{R}(\mathbf{x}^*, \mathbf{u}^*; \mathbf{x}, \lambda, \gamma) - \mathcal{R}(\mathbf{x}, \mathbf{u}; \mathbf{x}, \lambda, \gamma)$$

$$\leq \sum_{i=1}^{m_1} (\lambda_i^* - \lambda_i) \left(\frac{1}{2} \mathbf{x}^T P_i \mathbf{x} + \mathbf{q}_i^T \mathbf{x} + \mathbf{c}_i^T \mathbf{u} + r_i \right) + (\gamma^* - \gamma)^T (A\mathbf{x} + B\mathbf{u} - \mathbf{b}).$$
(23)

Proof For any $\mathbf{x} \in \mathbb{X}$, $\mathbf{u} \in \mathbb{R}^{n_2}$, $\lambda \in \mathbb{R}_+^{m_1}$ and $\gamma \in \mathbb{R}^{m_2}$, we have that $\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda^*, \gamma^*) \geq \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda, \gamma)$ by the saddle point inequality (18). We also have the inequality $\frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T P_0(\mathbf{x} - \mathbf{x}^*) + \sum_{i=1}^{m_1} \lambda_i \left[\frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T P_i(\mathbf{x} - \mathbf{x}^*)\right] \geq 0$ due to the



positive semi-definiteness of each matrix P_0, P_1, \dots, P_{m_1} . Adding the two inequalities together completes the proof.

We next establish fundamental estimates of the distance between the solution point $(\mathbf{x}^{k+1}, \mathbf{u}^{k+1}, \lambda^{k+1}, \gamma^{k+1})$ at each iteration k and the saddle point $(\mathbf{x}^*, \mathbf{u}^*, \lambda^*, \gamma^*)$.

Proposition 1 Let $(\mathbf{x}^*, \mathbf{u}^*, \lambda^*, \gamma^*)$ be a saddle point of the QCQP (8). For all $k \ge 0$, we have that

$$\|\mathbf{x}^{k+1} - \mathbf{x}^*\|_{2}^{2} + \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_{2}^{2}$$

$$\leq \|\mathbf{x}^{k} - \mathbf{x}^*\|_{2}^{2} + \|\mathbf{u}^{k} - \mathbf{u}^*\|_{2}^{2}$$

$$- (\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_{2}^{2} + \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} + \|\mathbf{v}^{k+1} - \mathbf{u}^{k}\|_{2}^{2})$$

$$+ 2\rho^{k+1} \left\{ (\mathbf{y}^{k+1} - \mathbf{x}^{k+1})^{T} P_{0}(\mathbf{y}^{k+1} - \mathbf{x}^{k}) \right.$$

$$+ \sum_{i=1}^{m_{1}} \mu_{i}^{k+1} (\mathbf{y}^{k+1} - \mathbf{x}^{k+1})^{T} P_{i}(\mathbf{y}^{k+1} - \mathbf{x}^{k})$$

$$+ \sum_{i=1}^{m_{1}} (\lambda_{i}^{*} - \mu_{i}^{k+1}) \left[\frac{1}{2} (\mathbf{y}^{k+1})^{T} P_{i} \mathbf{y}^{k+1} + \mathbf{q}_{i}^{T} \mathbf{y}^{k+1} + \mathbf{c}_{i}^{T} \mathbf{v}^{k+1} + r_{i} \right]$$

$$+ (\mathbf{y}^{*} - \mathbf{v}^{k+1})^{T} (A\mathbf{y}^{k+1} + B\mathbf{v}^{k+1} - \mathbf{b})$$

$$+ \sum_{i=1}^{m_{1}} (\mu_{i}^{k+1} - \lambda_{i}^{k}) \left[(P_{i} \mathbf{x}^{k} + \mathbf{q}_{i})^{T} (\mathbf{y}^{k+1} - \mathbf{x}^{k+1}) + \mathbf{c}_{i}^{T} (\mathbf{v}^{k+1} - \mathbf{u}^{k+1}) \right]$$

$$+ (\mathbf{v}^{k+1} - \mathbf{y}^{k})^{T} \left[A(\mathbf{y}^{k+1} - \mathbf{x}^{k+1}) + B(\mathbf{v}^{k+1} - \mathbf{u}^{k+1}) \right] \right\},$$

and

$$\|\lambda^{k+1} - \lambda^*\|_{2}^{2} + \|\gamma^{k+1} - \gamma^*\|_{2}^{2}$$

$$\leq \|\lambda^{k} - \lambda^*\|_{2}^{2} + \|\gamma^{k} - \gamma^*\|_{2}^{2}$$

$$- (\|\mu^{k+1} - \lambda^{k+1}\|_{2}^{2} + \|v^{k+1} - \gamma^{k+1}\|_{2}^{2} + \|\mu^{k+1} - \lambda^{k}\|_{2}^{2} + \|v^{k+1} - \gamma^{k}\|_{2}^{2})$$

$$+ 2\rho^{k+1} \left\{ \sum_{i=1}^{m_{1}} (\lambda_{i}^{k+1} - \lambda_{i}^{*}) \left[\frac{1}{2} (\mathbf{y}^{k+1})^{T} P_{i} \mathbf{y}^{k+1} + \mathbf{q}_{i}^{T} \mathbf{y}^{k+1} + \mathbf{c}_{i}^{T} \mathbf{v}^{k+1} + r_{i} \right] \right.$$

$$+ (\gamma^{k+1} - \gamma^*)^{T} (A\mathbf{y}^{k+1} + B\mathbf{v}^{k+1} - \mathbf{b})$$

$$+ \sum_{i=1}^{m_{1}} (\mu_{i}^{k+1} - \lambda_{i}^{k+1}) \left[\frac{1}{2} (\mathbf{x}^{k})^{T} P_{i} \mathbf{x}^{k} + \mathbf{q}_{i}^{T} \mathbf{x}^{k} + \mathbf{c}_{i}^{T} \mathbf{u}^{k} + r_{i} \right]$$

$$+ (\mathbf{v}^{k+1} - \gamma^{k+1})^{T} (A\mathbf{x}^{k} + B\mathbf{u}^{k} - \mathbf{b}) \right\}. \tag{25}$$

Proof The details of the proof are provided in "Appendix 2".



Now we are ready to present the main convergence result. A key to the proof depends on the rules to adaptively update the step-size. The rules, however, are lengthy and purely technical, and hence their details are deferred to "Appendix 1".

Theorem 1 (Global Convergence) Assume that the Slater's CQ and Assumption 1 hold. At each iteration k of Algorithm 1, let the step-size ρ^{k+1} be updated according to the update rules in "Appendix 1". Then with an arbitrary starting point $(\mathbf{x}^0, \mathbf{u}^0, \lambda^0, \gamma^0) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, the sequence $\{(\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \gamma^k)\}$ generated by Algorithm 1 converges to a saddle point $(\mathbf{x}^*, \mathbf{u}^*, \lambda^*, \gamma^*)$ of the QCQP (8).

Proof Please see "Appendix 2" for details.

A point we want to emphasize here is that the convergence result is quite strong in the sense that the entire iterative sequence, not just a subsequence, can be shown to converge to an optimization solution, with an arbitrary starting point. Such a result can help alleviate a strong assumption we made, which is to assume that a given convex QCQP has an optimal solution. While the algorithm or its convergence proof does not handle infeasible or unbounded cases, we will show in Sect. 4.4 that from a practical perspective, our algorithm can just be blindly applied to a convex QCQP, and either infeasibility or unboundedness can be inferred from observing the behavior of the residuals we use for the algorithm's stopping criteria, which are to be defined in Sect. 4.3.

4 Implementation

In this section, we discuss how to efficiently implement the PC²PM algorithm, especially within a distributed framework.

4.1 Distributed storage of data and parallel computing

As mentioned in the introduction section, one key feature of the PC²PM algorithm for solving convex QCQPs is that when implemented across multiple computing units, each computing unit does not need to store entire matrices. Instead, only each primal computing unit needs to store certain columns of the matrices (that is, the Hessian matrices in the objective function and the constraints). To illustrate this point, we use the primal predictor update (14a) as an example. Assume that ideally we have n_1 primal computing units dedicated to updating y_j , $j = 1, ..., n_1$. To ease the argument, we write out the updating rule again here:

$$y_j^{k+1} = \boldsymbol{\Pi}_{\mathbb{X}_j} \left(x_j^k - \rho \left[P_0 \mathbf{x}^k + \mathbf{q}_0 + \sum_{i=1}^{m_1} \lambda_i^k (P_i \mathbf{x}^k + \mathbf{q}_i) + A^T \boldsymbol{\gamma}^k \right]_j \right). \tag{26}$$



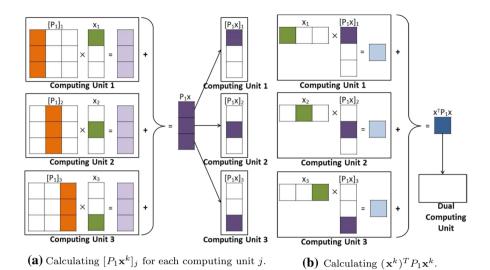


Fig. 1 Illustrations of matrix-vector multiplications using MPI functions

In each unit j, only the values of x_i^k , $[P_i]_j$, $[\mathbf{q}_i]_j$ for $i = 0, 1, \dots, m_1$ and $[A]_j$ are needed to be stored locally. To calculate $[P_i \mathbf{x}^k]_i$ for $i = 0, 1, \dots, m_1$, there is no need to store the entire P_i matrices on each computing unit. Instead, the value of $[P_i \mathbf{x}^k]_i$ can be obtained using MPI to communicate among all primal computing units, where only one column of the P_i matrices (and x_i^k) is stored locally. Here we use a simple example to illustrate the mechanism. Let $n_1 = 3$, Fig. 1a shows how $[P_1 \mathbf{x}^k]_j$, j = 1, 2, 3 are calculated in a distributed fashion through MPI. First, each computing unit j completes a subtask of multiplying $[P_1]_i$ and x_i^k using their locally stored information; then the intermediate results are summed up using the MPI_Reduce function in a root process to get the value of $P_1 \mathbf{x}^k$. Each component of the vector $P_1 \mathbf{x}^k$ is then sent back to the corresponding computing unit j using the MPI Scatter function. After obtaining the values of $[P_i \mathbf{x}^k]_i$ for $i = 0, 1, \dots, m_1$ in this way, the update step (26) can be carried out upon receiving the values of $(\lambda_1^k, \dots, \lambda_{m_k}^k)$ and γ^k from other dual computing units dedicated for updating the dual variables using MPI_Send and MPI_Recv functions, with the fact that $[A^T \gamma^k]_i = [A]_i^T \gamma^k$. Such a feature will be particularly beneficial for solving high-dimension QCQPs from real world applications, as in many such cases the number of variables $(n_1 \text{ for } \mathbf{x} \text{ and } n_2 \text{ for } \mathbf{u})$ can be enormous.

In the 3-dimension example shown in Fig. 1a, once each $[P_1\mathbf{x}^k]_j$ is received by computing unit j for j=1,2,3, a subtask of multiplying x_j^k and $[P_1\mathbf{x}^k]_j$ is needed to calculate the value of $(\mathbf{x}^k)^TP_1\mathbf{x}^k$ for dual update, such as in (9):

$$\mu_i^{k+1} = \boldsymbol{\Pi}_{\mathbb{R}_+} \left(\lambda_i^k + \rho \left[\frac{1}{2} (\mathbf{x}^k)^T P_i \mathbf{x}^k + \mathbf{q}_i^T \mathbf{x}^k + \mathbf{c}_i^T \mathbf{u}^k + r_i \right] \right). \tag{27}$$



Table 1 Values of the smallest eigenvalue d_{min} and the largest eigenvalue d_{max} for different condition numbers

Cond. num.	Smallest eigenvalue	Largest eigenvalue
$\left(\kappa = rac{d_{max}}{d_{min}} ight)$	$\left(d_{min} ight)$	(d_{max})
10 ²	0.1	10.0
10^{4}	0.003	30.0
10^{6}	0.00002	20.0

Such a process is illustrated in Fig. 1b, which shows that the locally calculated intermediate results are summed up using the MPI_Reduce function and sent to the corresponding dual computing unit. Other matrix-vector (and vector-vector) multiplications in the update steps of Algorithm 1 can all be calculated in a similar fashion. ¹

Next, we examine the speedup of using multiple compute nodes for parallel distributed computing. We run the PC²PM algorithm on a multi-node computer cluster, where each node has multiple cores (24 cores in our case). MPI is used to communicate among all parallel processes mapped to cores belonging to different nodes. For illustration purpose, we focus on a single-constraint convex QCQP:

minimize
$$\frac{1}{2} \mathbf{x}^T P_0 \mathbf{x} + \mathbf{q}_0^T \mathbf{x} + r_0$$
subject to
$$\frac{1}{2} \mathbf{x}^T P_1 \mathbf{x} + \mathbf{q}_1^T \mathbf{x} + r_1 \le 0, \qquad (\lambda_1)$$
(28)

which does not contain the block of decision variable \mathbf{u} or linear constraint $A\mathbf{x} + B\mathbf{u} = \mathbf{b}$. We test the PC²PM algorithm for solving (28) on a randomly generated data set with $P_i \in \mathbb{R}^{n_1 \times n_1}$, $\mathbf{q}_i \in \mathbb{R}^{n_1}$ and $r_i \in \mathbb{R}$ for i = 0, 1. The dimension n_1 is set at $2^{14} \approx 1.6 \times 10^4$. Each matrix P_i is randomly generated as a symmetric PSD matrix in the form of $P_i = Q^T DQ$, where $Q \in \mathbb{R}^{n_1 \times n_1}$ is a randomly generated orthogonal matrix, and $D = diag(d_1, \dots, d_{n_1})$ is a randomly generated diagonal matrix with all non-negative entries. Since d_1, \dots, d_{n_1} are also the eigenvalues of each P_i , we denote the largest eigenvalue as d_{\max} and the smallest eigenvalue as d_{\min} , and hence make the condition number of each matrix as $\kappa(P_i) = \frac{d_{\max}}{d_{\min}}$. Then, the remaining diagonal entries are randomly generated from the range $[d_{\min}, d_{\max}]$. We test different condition numbers for all matrices, increasing from 10^2 to 10^6 . The values of the smallest eigenvalue d_{\min} and the largest eigenvalue d_{\max} are listed in Table 1.

The components of each vector \mathbf{q}_i are randomly generated from the range [-1.0, 1.0], and each scalar r_i is randomly generated from the range [-1.0, 0.0] to guarantee the feasibility of the constraint sets.

Since the number of Lagrangian multipliers is 1, the number of dual computing units $n_{\text{dual-comp}}$ is also fixed as 1. The tasks of updating n_1 components of the primal decision variables \mathbf{x} and \mathbf{y} are evenly distributed among all the primal computing units with the number $n_{\text{primal-comp}}$ varying from 1 to 256 for comparison

¹ For more information, we refer the readers to our implementation codes programmed in C available online at https://github.com/BigRunTheory/A-Distributed-Algorithm-for-Large-scale-Convex-QCQPs.



Table 2 Elapsed wall-clock time used by PC²PM for solving the single-constraint convex QCQP (28) with different condition numbers (κ)

PC ² PM using multiple nodes (max. 24 cores per node)
$(\tau^{\text{PC}^2\text{PM}} = 10^{-3})$	

		Elapsed wall-clock time (h)					
n_{node}	$n_{\rm core}$	$\kappa = 10^2$	$\kappa = 10^4$	$\kappa = 10^6$			
1	1 + 1	42.08	51.72	70.45			
1	2 + 1	22.79	28.11	38.28			
1	4 + 1	14.17	17.48	23.89			
1	8 + 1	8.85	10.87	14.70			
1	16 + 1	5.60	6.91	9.62			
2	32 + 1	4.82	6.13	8.26			
3	64 + 1	4.33	5.42	7.51			
6	128 + 1	3.94	4.96	6.95			
11	256 + 1	5.51	6.84	9.32			
	Obj. val.	-420.621	-214.389	-324.428			

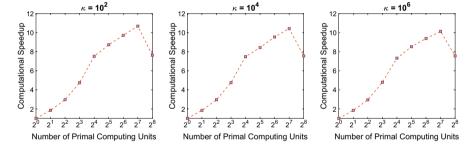


Fig. 2 Computational speedup of PC²PM for solving the single-constraint convex QCQP (28) with different condition numbers (κ)

purpose. Since each computing unit occupies a single core, the total number of cores used is $n_{\rm core} = n_{\rm primal-comp} + n_{\rm dual-comp}$. The number of nodes needed is calculated as $n_{\rm node} = \lceil n_{\rm core}/24 \rceil$ (where 24 is the number of cores per node). The elapsed wall-clock time T used by the PC²PM algorithm to converge with a tolerance $\tau^{\rm PC^2PM} = 10^{-3}$, corresponding to different condition numbers, is listed in Table 2, along with the calculated objective function values. (The specific stopping criteria are given in Sect. 4.3.)

The computational speedup S is defined as the ratio of the elapsed run time taken by a serial code to that taken by a parallel code for solving the same problem. More specifically, S is defined as

$$S := \frac{T(1+1)}{T(n_{\text{primal-comp}} + 1)}, \quad n_{\text{primal-comp}} \ge 2.$$
 (29)

The speedup of solving (28) is shown in Fig. 2.



For this specific case, parallel computing achieved linear speedup initially. However, due to communication overhead, the speedup plateaued (or even decreased) when the number of computing units is too high. As such, we suggest that a proper number of computing units needs to be carefully chosen when implementing the PC²PM algorithm to balance between computational speedup and communication overhead.

4.2 Adaptive step size with auto-learned allocation weights

In establishing the global convergence of the PC²PM algorithm, it is not specified how the values of ϵ_s , $s=1,\ldots,8$ are chosen in order to calculate the eight components $\rho_1-\rho_8$ (see the update rules in "Appendix 1"). Here we develop a practical rule to help determine the values of ϵ_s 's along the iterations. The rule may also help accelerate the algorithm's performance, based on our numerical experiments.

Generally speaking, for first-order algorithms, to which the PC²PM algorithm also belongs, the larger value a step size could take, the fewer number of iterations the algorithms would need to converge. For the step-size formula (49), it is easy to observe that the value of each ρ_s increases when the corresponding ϵ_s increases. However, the ϵ_s 's cannot be too large as their summation is bounded by $1 - \epsilon_0 \le 1$. A naive way to allocate the value of each ϵ_s is to evenly distribute the upper bound of their summation; that is, $\epsilon_s = \frac{1}{8}(1 - \epsilon_0)$ for $s = 1, \ldots, 8$, throughout the convergence. Alternatively, we introduce a weight variable $w_s > 0$ for each ϵ_s . At the beginning of the algorithm, they are all initialized as 1, indicating an even allocation of the values of the ϵ_s 's. At each iteration $k = 1, 2, \ldots$, we calculate the values of ϵ_s^k , \ldots , ϵ_s^k based on the following formulation:

$$\epsilon_s^k = \frac{w_s^k}{\sum_{s=1}^8 w_s^k} (1 - \epsilon_0), \quad s = 1, \dots, 8.$$
(30)

After the step size ρ^{k+1} is determined by $\min\{\rho_1^k,\ldots,\rho_8^k\}$ according to (49), the values of the weights $w_1^{k+1},\ldots,w_8^{k+1}$ are updated based on the ratio of ρ^{k+1} to each ρ_s^k :

$$w_s^{k+1} = \frac{\rho^{k+1}}{\rho_s^k} w_s^k, \quad s = 1, \dots, 8.$$
 (31)

The idea of the above updating rule is to make sure that all the values of the ρ_s 's will have a chance to be increased, avoiding the possibility that a particularly small ρ_s would always be chosen to determine the step size ρ^{k+1} , which would slow down the whole algorithm.

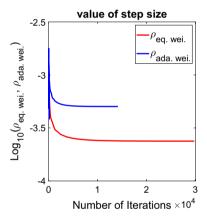
For illustration purpose, we use the PC²PM algorithm with the step-size updating rule of (49), (30) and (31) to solve the same single-constraint convex QCQP (28) in the previous subsection. We test the algorithm on a data set of matrix P_i , vector \mathbf{q}_i and scalar r_i randomly generated in the same way as in the previous subsection for i = 0, 1, but with $n_1 = 1024$ and $\kappa(P_i) = 100$. The algorithm is implemented on a single core as a serial code. (Note that the parallel computing has nothing to do with the number of iterations for the PC²PM algorithm to



Table 3 Number of iterations and elapsed wall-clock time used by PC²PM for solving a single-constraint convex QCQP (28) with different settings of $(\epsilon_0, \dots, \epsilon_5)$

	Equal weigh	t allocation	Adaptive weight allocation			
Value of ϵ_0	Num. iter.	Time (s)	Num. iter.	Time (s)		
0.5	59,496	123	28,272	62		
10^{-1}	33,055	74	15,713	36		
10^{-2}	30,050	68	14,287	32		
10^{-3}	29,780	68	14,158	32		
10^{-4}	29,753	68	14,145	32		
10^{-5}	29,750	67	14,143	31		
10^{-6}	29,750	67	14,143	31		
0	29,750	66	14,143	31		
Diminishing	30,094	70	14,378	33		

Fig. 3 Comparison of values of the step size using different weight allocations when $\epsilon_0 = 0$



converge.) Since it contains neither the decision variable **u** nor the linear equality constraint A**x** + B**u** = **b**, only $\rho_1^k, \ldots, \rho_5^k$ need to be calculated at each iteration. We compare the performance of the algorithm using equal weights versus using the adaptive weights in (31). The number of iterations and the elapsed wall-clock time used by the algorithm to converge with a tolerance $\tau^{PC^2PM} = 10^{-4}$ are listed in Table 3.

We also test using different values of ϵ_0 , including a fixed value varying from $0.5, 10^{-1}, \ldots, 10^{-6}$ to 0 and a diminishing value of $\frac{1}{\sqrt{k+1}}$. The numerical results of this specific instance suggest that by using auto-learned allocation weights, the number of iterations for the algorithm to converge is cut by more than half. Comparing each rows, we also observe that the smaller the value of ϵ_0 is, the faster the algorithm converges. Additionally, for the row of $\epsilon_0 = 0$, we plot out the comparison of values of the resulting step size ρ^k at each iteration k using different weight allocations, shown in Fig. 3.



We observe that using the adaptive weights, the step size quickly converges to a much larger value than using the equal weights, which explains the greatly reduced number of iterations.

4.3 Stopping criteria

Since we consider a convex QCQP and assume that the Slater's CQ holds, the first-order optimality conditions (aka the KKT conditions) are both necessary and sufficient. More specifically, for an optimal solution (\mathbf{x}^* , \mathbf{u}^*) of the QCQP (8) and its corresponding dual solution (λ^* , γ^*), the following conditions are satisfied:

• Stationarity:

$$-\left[P_0\mathbf{x}^* + \mathbf{q}_0 + \sum_{i=1}^{m_1} \lambda_i^* (P_i \mathbf{x}^* + \mathbf{q}_i) + A^T \boldsymbol{\gamma}^*\right]_j \in \mathcal{N}_{\mathbb{X}_j}(x_j^*), \quad j = 1, \dots, n_1$$
 (32a)

$$\mathbf{c}_0 + \sum_{i=1}^{m_1} \lambda_i^* \mathbf{c}_i + B^T \boldsymbol{\gamma}^* = \mathbf{0}$$
 (32b)

• Complementarity:

$$\lambda_i^* \left[\frac{1}{2} (\mathbf{x}^*)^T P_i \mathbf{x}^* + \mathbf{q}_i^T \mathbf{x}^* + \mathbf{c}_i^T \mathbf{u}^* + r_i \right] = 0, \quad i = 1, \dots, m_1$$
 (33)

• Primal Feasibility:

$$\frac{1}{2}(\mathbf{x}^*)^T P_i \mathbf{x}^* + \mathbf{q}_i^T \mathbf{x}^* + \mathbf{c}_i^T \mathbf{u}^* + r_i \le 0, \qquad i = 1, \dots, m_1$$
(34a)

$$A\mathbf{x}^* + B\mathbf{u}^* - \mathbf{b} = \mathbf{0} \tag{34b}$$

Dual Feasibility:

$$\lambda_i^* \ge 0, \quad i = 1, \dots, m_1 \tag{35}$$

Conversely, any primal–dual pair $(\mathbf{x}^*, \mathbf{u}^*; \lambda^*, \gamma^*)$ satisfying the above conditions is optimal to the primal and dual of the QCQP (8), respectively. Based on the optimality conditions (32a)–(35), we choose stopping criteria for our algorithm to measure stationarity, as well as complementarity and primal feasibility. (Note that dual feasibility is always maintained by the projection step in each iteration, as shown in (17).) More specifically, at each iteration k, we measure the following two residuals:

$$res_1^k = \sqrt{\frac{1}{n_1 + n_2} \left[\sum_{j=1}^{n_1} \left(res_{1-}^k x_j \right)^2 + \|\mathbf{c}_0 + \sum_{i=1}^{m_1} \lambda_i^k \mathbf{c}_i + B^T \boldsymbol{\gamma}^k \|_2^2 \right]}, \text{ and}$$
 (36)



$$res_{2}^{k} = \sqrt{\frac{1}{m_{1} + m_{2}} \left[\sum_{i=1}^{m_{1}} \left[\lambda_{i}^{k} \left| \frac{1}{2} (\mathbf{x}^{k})^{T} P_{i} \mathbf{x}^{k} + \mathbf{q}_{i}^{T} \mathbf{x}^{k} + \mathbf{c}_{i}^{T} \mathbf{u}^{k} + r_{i} \right| \right]^{2}} ,$$

$$+ \|A\mathbf{x}^{k} + B\mathbf{u}^{k} - \mathbf{b}\|_{2}^{2} \right]},$$
(37)

where $res_{1}^{k} x_{j}$ in (36) depends on the actual form of the constraint set \mathbb{X} . Again, for box constrains $0 \le x_{j} \le \bar{X}_{j}$, $j = 1, \dots, n_{1}$, we have that

$$res_{1-}^{k}x_{j} := \begin{cases} \min\{0, \left[\mathbf{grad}_{\mathbf{x}}\right]_{j}\}, & \text{if } x_{j}^{k} = 0\\ \left[\mathbf{grad}_{\mathbf{x}}\right]_{j}, & \text{if } 0 < x_{j}^{k} < \bar{X}_{j}\\ \max\{0, \left[\mathbf{grad}_{\mathbf{x}}\right]_{j}\}, & \text{if } x_{j}^{k} = \bar{X}_{j} \end{cases}$$

$$(38)$$

and $\mathbf{grad_x} = P_0 \mathbf{x}^k + \mathbf{q}_0 + \sum_{i=1}^{m_1} \lambda_i^k (P_i \mathbf{x}^k + \mathbf{q}_i) + A^T \mathbf{y}^k$. This comes from the rewriting of the optimality condition (32a) using $\mathbf{grad_x}$ as:

$$-\left[\operatorname{grad}_{\mathbf{x}}^{*}\right]_{j} \in \mathcal{N}_{\mathbb{X}_{j}}(x_{j}^{*}),\tag{39}$$

where

$$\mathcal{N}_{\mathbb{X}_{j}}(x_{j}^{*}) := \begin{cases} (-\infty, 0], & \text{if } x_{j}^{*} = 0\\ [5pt]\{0\}, & \text{if } 0 < x_{j}^{*} < \bar{X}_{j} \\ [5pt][0, +\infty), & \text{if } x_{j}^{*} = \bar{X}_{j} \end{cases}$$
(40)

We terminate our algorithm when both of the two residuals drop below a pre-specified tolerance τ . Note that the residuals defined in (36) and (37) are based on the average residuals of all the constraints. Other forms of residual metric, such as using the maximum residual of all the constraints, can also be used.

4.4 Infeasibility and unboundedness

Lastly, we examine how the PC²PM algorithm could computationally detect infeasibility or unboundedness of a convex QCQP.

First, we construct an infeasible QCQP as follows:

$$\underset{\mathbf{x} \in \mathbb{R}^{n_1}}{\text{minimize}} \quad \frac{1}{2} \mathbf{x}^T P_0 \mathbf{x} + \mathbf{q}_0^T \mathbf{x} + r_0
\text{subject to} \quad \frac{1}{2} \mathbf{x}^T P_1 \mathbf{x} + \mathbf{q}_1^T \mathbf{x} + r_1 \le 0, \qquad (\lambda_1)
\underbrace{\frac{1}{2} (\mathbf{x} + \mathbf{q}_2)^T (\mathbf{x} + \mathbf{q}_2)}_{\ge 0} + \underbrace{\Delta}_{>0} \le 0. \qquad (\lambda_2)$$

All the matrices P_i 's, vectors \mathbf{q}_i 's and scalars r_i 's in (41) are randomly generated in the same way as in Sect. 4.1, but with $n_1 = 1024$ and $\kappa(P_i) = 100$. Letting Δ denote a



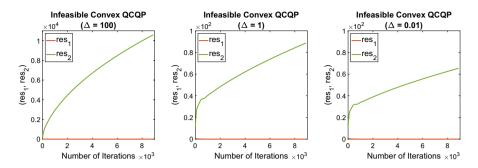


Fig. 4 Residuals of applying PC^2PM to solve an infeasible convex QCQP. The residual res_2 diverges when applying PC^2PM to solve (41)

positive scalar in the second quadratic constraint in (41) apparently makes the problem infeasible. We decrease Δ from 100 to 0.01 and apply the PC²PM algorithm to solve the resulting problems. The corresponding residuals are shown in Fig. 4.

We observe that the residual res_2 , measuring the complementarity and the primal feasibility as defined in (37), diverges for all Δ 's, while the other residual res_1 still converges. This is a strong indication that the original problem is infeasible.

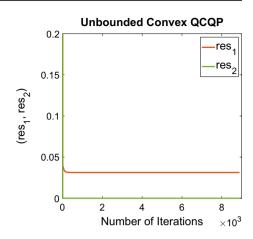
Next, we construct an unbounded convex QCQP as follows:

minimize
$$\frac{1}{2} \mathbf{x}^T D_0 \mathbf{x} + \mathbf{e}_0^T \mathbf{x} + r_0$$
subject to
$$\frac{1}{2} \mathbf{x}^T D_0 \mathbf{x} + \mathbf{e}_1^T \mathbf{x} + r_1 \le 0.$$
 (42)

The matrix $D_0 \in \mathbb{R}^{n_1 \times n_1}$ is a diagonal matrix with all but the last diagonal entry being 1, and its last diagonal entry is set as 0, hence making it a PSD matrix. The vector \mathbf{e}_0 is an n_1 -dimension vector in the form of $\mathbf{e}_0 = (0 \dots 0 \ 1)^T$. Conversely, the vector \mathbf{e}_1 is also n_1 -dimension but in the form of $\mathbf{e}_1 = (1 \dots 1 \ 0)$. The dimension n_1 is also set as 1024. All scalars r_i 's are randomly generated in the same way as in Sect. 4.1. It can be easily seen that the convex QCQP (42) is unbounded along the direction $(0, \dots, 0, -1)$. As shown in Fig. 5, when applying PC²PM to solve (42), we observe that both of the residuals converge, but the residual res_1 , measuring stationarity as defined in (36), converges to a non-zero value. This means that an optimal solution is not found. If it is known that a feasible point exists to a convex QCQP (as the example given by (42)) (and assume that a constraint qualification holds at the feasible point), then by Theorem 1, if the algorithm does not find an optimal solution, it must mean that Assumption 1 is violated, which then implies that the original problem is unbounded (as an optimal solution does not exist).



Fig. 5 Residuals of applying PC²PM to solve an unbounded convex QCQP. The residual *res*₁ converges to a non-zero value when applying PC²PM to solve (42)



5 Numerical experiments

In this section, we present more numerical results for solving high-dimension convex QCQPs using our algorithm. We first conduct numerical experiments of applying the PC²PM algorithm to solve convex QCQPs of the standard form (1), with randomly generated data sets of various sizes. We then solve convex QCQPs with explicit linear constraints as in (8), which naturally arise from multiple kernel learning applications. For both sets of experiments, we compare the performance of our algorithm with the current state-of-the-art commercial solver CPLEX 12.8.0, which uses the barrier optimizer for solving convex QCQPs. We implement PC²PM with multiple compute nodes on Purdue University's Brown cluster using MPI, called from a C program. Each node on the cluster has two 12-core Intel Xeon Gold "Sky Lake" processors (that is, 24 cores per node) and 96 GB of memory. CPLEX 12.8.0 is also called using a C program and implemented on a single compute node (with 24 cores). Note that CPLEX alone, as a centralized algorithm, cannot be run on multiple compute nodes using MPI, but it does allow multiple parallel threads that can be invoked by the barrier optimizer. More specifically, CPLEX has a parameter, CPXPARAM Threads, to call for multithread computing [12]. When CPX-PARAM Threads is set to be 1, CPLEX is single threaded; when it is set to be 0, CPLEX can use up to 32 threads, or the number of cores of the machine (with each core being a thread), whichever is smaller. In our experiments, we always set CPX-PARAM_Threads as 0, which gives CPLEX 24 threads (since each of our compute node has 24 cores).

5.1 Solving standard-form convex QCQPs

We first apply PC²PM to solve convex QCQPs of the standard form (1), without the decision variables \mathbf{u} or the explicit linear constraints $A\mathbf{x} + B\mathbf{u} - \mathbf{b} = \mathbf{0}$. The input data consist of matrix P_i , vector \mathbf{q}_i and scalar r_i for $i = 0, 1, ..., m_1$, all of which are randomly generated in the same way as in Sect. 4.1. The decision variable's



dimension n_1 is fixed as $2^{14} \approx 1.6 \times 10^4$, and the number of constraints m_1 increases from 1 to 16.

To balance between the computation speedup and communication overhead, we implement our algorithm with 128 cores allocated for primal variables' updating: (14a) (16a), (14b) and (16b), and m_1 (the number of quadratic constraints) cores for dual updating: (9) and (17). The total number of compute nodes needed is calculated as $n_{\text{node}} = \lceil \frac{n_{\text{core}}}{24} \rceil = \lceil \frac{128 + m_1}{24} \rceil$. The stopping criteria we used are defined in (36) and (37), with the tolerance $\tau^{\text{PC}^2\text{PM}}$ set to be 10^{-3} . Table 4 reports the elapsed wall-clock time used by the PC²PM algorithm, along with the amount of memory used by each compute node and the final objective function value, with respect to the increasing condition number κ . The performance of CPLEX 12.8.0 with the same convergence tolerance is also presented in Table 4 for comparison. In the first two groups of tests with $m_1 = 1$ and 2, our algorithm compares favorably to CPLEX and uses much less memory. For the rest groups of test cases, CPLEX fails to provide a solution (actually fails to complete even a single iteration) due to running out of memory; while PC²PM still converges within a reasonable amount of time. As the scale of the problem increases, our algorithm exhibits favorable scalability, due to its distributed storage of data and the capability of massively parallel computing. Another interesting observation from Table 4, though we do not know the underlying reason, is that when the number of quadratic constraints (m_1) is small, PC²PM's run time appears to be sensitive to the condition number of matrices (i.e., the Hessian matrices of the objective function and the constraints); yet when m_1 becomes larger, the effect of condition numbers on the run time appears to be subdued.

We also plot the two residuals res_1^k and res_2^k in Fig. 6, with res_1^k corresponding to the gradient of the Lagrangian function, and res_2^k corresponding to the feasibility and complementarity conditions. The three plots in a same row are with the same number of constraints m_1 , but with different condition numbers of the Hessian matrices. As seen in Fig. 6, from left to right, when m_1 is small, as the condition number κ increases, more iterations are required for the PC²PM algorithm to converge; yet when m_1 becomes larger, the number of iterations depends more on the absolute value of the objective function than the condition number. Another observation is that when the number of constraints increases (i.e., from top to bottom), the convergence of the residuals becomes more smooth.

5.2 Multiple kernel learning in support vector machine

In this subsection, we briefly introduce how the Support Vector Machine (SVM) with multiple kernel learning can be formulated as a convex QCQP, and present numerical results of applying our algorithm to solve high-dimension instances. As discussed in [8], SVM is a discriminative classifier proposed for binary classification problems. Given a set of n_{tr} pairs of independently and identically distributed training data points $\{(\mathbf{d}_j, l_j)\}_{j=1}^{n_{tr}}$, where $\mathbf{d}_j \in \mathbb{R}^{n_d}$ is the n_d -dimension input vector and $l_j \in \{-1, 1\}$ is its class label, SVM searches for a hyperplane that can best separate the points from two classes. The hyperplane is defined as $\{\mathbf{d} \in \mathbb{R}^{n_d} | f(\mathbf{d}) = \boldsymbol{\beta}^T \mathbf{d} + \beta_0 = 0\}$, where $\boldsymbol{\beta} \in \mathbb{R}^{n_d}$ is a unit vector with $\|\boldsymbol{\beta}\|_2 = 1$, and $\beta_0 \in \mathbb{R}$ is a scalar.



 $\begin{tabular}{ll} \textbf{Table 4} & Comparison of PC^2PM with $CPLEX$ 12.8.0 for solving standard-form, high-dimension convex $QCQPs$ \end{tabular}$

n_1	m_1	κ		n_{node}	n _{core}	Mem./node (GB)	Time (h)	Obj. val.
		10 ²				,	3.94	-420.621
		10^{4}	PC^2PM	6	128 + 1	1.6/node	4.96	-214.389
1.6	1	10^{6}	$(\tau^{\text{PC}^2\text{PM}} = 10^{-3})$				6.95	-324.428
$\times 10^4$		10^{2}					4.97	-420.645
		10^{4}	CPLEX 12.8.0	1	24	41.1	5.02	-214.423
		10^{6}	$(\tau^{\text{Barrier}} = 10^{-3})$				5.25	-324.465
		10^{2}					2.28	-322.232
		10^{4}	PC^2PM	6	128 + 2	1.9/node	2.41	-161.960
1.6	2	10^{6}	$(\tau^{PC^2PM} = 10^{-3})$				1.87	-244.048
$\times 10^4$		10^{2}					10.82	-322.213
		10^{4}	CPLEX 12.8.0	1	24	74.0	10.83	-161.910
		10^{6}	$(\tau^{\text{Barrier}} = 10^{-3})$				10.35	-244.035
		10^{2}					1.46	-243.154
		10^{4}	PC^2PM	6	128 + 4	2.6/node	1.23	-126.184
1.6	4	10^{6}	$(\tau^{PC^2PM} = 10^{-3})$				1.46	-189.230
$\times 10^4$		10^{2}						
		10^{4}	CPLEX 12.8.0	1	24	O.O.M.	N.A.	N.A.
		10^{6}	$(\tau^{\text{Barrier}} = 10^{-3})$			(> 96 GB)		
		10^{2}					2.13	-189.945
		10^{4}	PC^2PM	6	128 + 8	4.3/node	1.33	-97.974
1.6	8	10^{6}	$(\tau^{PC^2PM} = 10^{-3})$				1.54	-144.916
$\times 10^4$		10^{2}						
		10^{4}	CPLEX 12.8.0	1	24	O.O.M.	N.A.	N.A.
		10^{6}	$(\tau^{\text{Barrier}} = 10^{-3})$			(> 96 GB)		
		10^{2}					3.72	-147.310
		10^{4}	PC^2PM	6	128 + 16	7.7/node	2.34	-74.854
1.6	16	10^{6}	$(\tau^{\text{PC}^2\text{PM}} = 10^{-3})$				3.04	-111.490
$\times 10^4$		10^{2}						
		10^{4}	CPLEX 12.8.0	1	24	O.O.M.	N.A.	N.A.
		10^{6}	$(\tau^{\text{Barrier}} = 10^{-3})$			(> 96 GB)		

The points belonging to either class should be separated as far away from the hyperplane as possible, while still remain on the correct side. When the data points cannot be clearly separated in the original space \mathbb{R}^{n_d} , we instead search in a feature space \mathbb{R}^{n_f} , by mapping the input data space \mathbb{R}^{n_d} to the feature space through a function



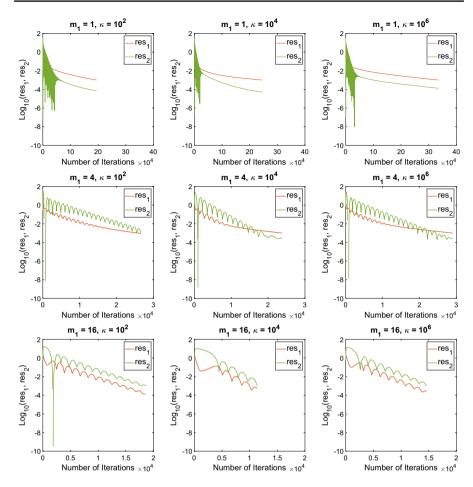


Fig. 6 Convergence of residuals

 $\Phi: \mathbb{R}^{n_d} \to \mathbb{R}^{n_f}$. For example, a 2-dimension data space can be lifted to a 3-dimension feature space. Using the function Φ , we can define a *kernel function* $k: \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \to \mathbb{R}$ as $k(\mathbf{d}, \mathbf{d}') := \langle \Phi(\mathbf{d}), \Phi(\mathbf{d}') \rangle$ for any $\mathbf{d}, \mathbf{d}' \in \mathbb{R}^{n_d}$, where \langle , \rangle denotes an inner product. The resulting discriminant function $\mathcal{G}: \mathbb{R}^{n_d} \to \{-1, 1\}$, which the SVM searches for, can be expressed as:

$$\mathcal{G}(\mathbf{d}) = \operatorname{sign}\left(\sum_{i=1}^{n_{n_r}} \alpha_j l_j k(\mathbf{d}_j, \mathbf{d}) + b\right), \quad \forall \mathbf{d} \in \mathbb{R}^{n_d},$$
(43)

where $\alpha \equiv (\alpha_1, \dots, \alpha_{n_{tr}})^T$ is the weight vector and b is the bias. The popular choices of kernel functions in the SVM literature include the linear kernel function k_{LIN} , the polynomial kernel function k_{POL} and the Gaussian kernel function k_{GAU} :



$$k_{IIN}(\mathbf{d}, \mathbf{d}') := \mathbf{d}^T \mathbf{d}', \quad \forall \mathbf{d}, \mathbf{d}' \in \mathbb{R}^{n_d}$$
 (44a)

$$k_{POL}(\mathbf{d}, \mathbf{d}') := (1 + \mathbf{d}^T \mathbf{d}')^2, \quad \forall \mathbf{d}, \mathbf{d}' \in \mathbb{R}^{n_d}$$
 (44b)

$$k_{GAU}(\mathbf{d}, \mathbf{d}')) := e^{-\frac{\|\mathbf{d} - \mathbf{d}'\|_2^2}{2\sigma^2}}, \quad \sigma > 0, \forall \mathbf{d}, \mathbf{d}' \in \mathbb{R}^{n_d}.$$

$$(44c)$$

Instead of using a single kernel function, [16] explores SVM using a kernel function that can be expressed as a non-negative combination of a pre-specified set of kernel functions $\{k_1, \ldots, k_m\}$, with the non-negative coefficients $\lambda_1, \ldots, \lambda_m$ to be allocated; that is, $k(\mathbf{d}, \mathbf{d}') = \sum_{i=1}^m \lambda_i k_i(\mathbf{d}, \mathbf{d}')$ for any $\mathbf{d}, \mathbf{d}' \in \mathbb{R}^{n_d}$ with $\lambda_1, \ldots, \lambda_m \geq 0$. The allocation process can be expressed as solving a convex QCQP, where each λ_i is the Lagrangian multiplier corresponding to each quadratic constraint. The formulation of the convex QCQP, as provided in [16], is as follows:

(i) **1-norm Soft Margin SVM** learns the coefficients through solving the following convex QCQP:

minimize
$$-\mathbf{e}^{T}\boldsymbol{\alpha} + R\boldsymbol{\alpha}_{0}$$

subject to $\frac{1}{2}\boldsymbol{\alpha}^{T} \left[\frac{1}{R_{i}} G_{i}(K_{i,tr}) \right] \boldsymbol{\alpha} - \boldsymbol{\alpha}_{0} \leq 0, \quad i = 1, \dots, m, \qquad (\lambda_{i})$
 $\sum_{j=1}^{n_{tr}} l_{j}\boldsymbol{\alpha}_{j} = 0, \qquad (\gamma)$
 $0 \leq \boldsymbol{\alpha}_{j} \leq C, \quad j = 1, \dots, n_{tr},$ (45)

(ii) 2-norm Soft Margin SVM learns the coefficients through solving the following convex QCQP:

where the vector \mathbf{e} denotes an n_{tr} -dimensional vector of all ones. Given a labeled training data set $\mathcal{S}_{tr} = \{(\mathbf{d}_j, l_j)\}_{j=1}^{n_{tr}}$ and an unlabeled test data set $\mathcal{S}_t = \{\mathbf{d}_j\}_{j=1}^{n_t}$, a matrix $K_i \in \mathbb{R}^{(n_{tr}+n_t)\times(n_{tr}+n_t)}$ can be defined on the entire data set $\mathcal{S}_{tr} \cup \mathcal{S}_t$ as

$$K_{i} := \begin{pmatrix} K_{i,tr} & K_{i,(tr,t)} \\ K_{i,(tr,t)}^{T} & K_{i,t} \end{pmatrix}. \tag{47}$$

The submatrix $K_{i,tr} \in \mathbb{R}^{n_{tr} \times n_{tr}}$ is a square symmetric matrix, whose jj'th element is directly defined by a kernel function: $[K_{i,tr}]_{jj'} := k_i(\mathbf{d}_j, \mathbf{d}_{j'})$ for any $\mathbf{d}_j, \mathbf{d}_{j'}$ in \mathcal{S}_{tr} .



The submatrices $K_{i,(tr,t)} \in \mathbb{R}^{n_{tr} \times n_t}$ and $K_{i,t} \in \mathbb{R}^{n_t \times n_t}$ are defined in the same way but with different input vectors. The matrix $G_i(K_{i,tr}) \in \mathbb{R}^{n_{tr} \times n_{tr}}$ in the quadratic constraint of (45) and (46) is a square symmetric matrix with its jj'th element being $[G_i(K_{i,tr})]_{jj'} = l_j l_{j'} [K_{i,tr}]_{jj'}$. Note that each kernel matrix $K_{i,tr}$ is a symmetric PSD matrix (see Proposition 2 in [16]), then each $G_i(K_{i,tr})$ is also a symmetric PSD matrix, since $G_i(K_{i,tr}) = LK_{i,tr}L$, where $L := diag(l_1, \ldots, l_{n_{tr}})$. Let R_i denote trace(K_i) for $i = 1, \ldots, m$, and $R = \sum_{i=1}^m \lambda_i R_i$ can be fixed as a given number. The parameter C is a fixed positive scalar from the soft margin criteria.

Once the optimal primal—dual solution $(\alpha^*; \lambda_1^*, \dots, \lambda_m^*)$ is found from either (45) or (46), combining with those pre-specified k_i 's, it can be used to label the test data set according to the following discriminant function \mathcal{G}_{MKL} : $\mathbb{R}^{n_d} \to \{-1, 1\}$:

$$\mathcal{G}_{\text{MKL}}(\mathbf{d}_{j'}) = \text{sign}\Big(\sum_{i=1}^{n_{ir}} \alpha_{j}^{*} l_{j} \Big[\sum_{i=1}^{m} \lambda_{i}^{*} k_{i}(\mathbf{d}_{j}, \mathbf{d}_{j'})\Big] + b\Big), \quad \forall \mathbf{d}_{j'} \in \mathcal{S}_{t}.$$
(48)

Compared with (43), the only difference is the replacement of a non-negative combination of k_i 's with coefficients $\lambda_1^*, \dots, \lambda_m^*$. The test set accuracy (TSA) can then be obtained by measuring the percentage of the test data points accurately labeled according to the function (48).

The formulation (45) and (46) provide instances of convex QCQPs in the form of (8), and we apply the PC²PM to solve them. The first input data set we used is the Two-norm Problem from [4], which is also used in [16]; however, our data set has a much larger size than in [16]. We first generate 8000 data points, with each data point being a 20-dimension vector, drawn from a multivariate normal distribution with a unit covariance matrix and the mean of (a, ..., a). These data points form the first class that are all labeled with 1. Another 8000 points of 20-dimension vectors are drawn from another multivariate normal distribution with also a unit covariance matrix but the mean of (-a, ..., -a). They form the second class that are all labeled with -1. The value of a is set as $\frac{2}{\sqrt{20}}$, the same as in [4]. Together, these two classes of data points form our first input data set with the size of 8000 + 8000 = 16,000. The second input data set is the *HEPMASS Data Set* from the UCI Repository.² This data set is used in high-energy physics experiments for learning particle-producing collisions from a background source. Each data point is generated from Monte Carlo simulations of collisions, and has 28 attributes. We randomly selected 16,000 data points from the original 10, 500, 000-sized data set as our inputs. We use a set of pre-specified kernel functions $\{k_1, \dots, k_5\}$ that contains all Gaussian kernel functions defined in (44c) whose σ^2 equal to 0.01, 0.1, 1, 10 and 100 respectively. Each matrix K_i is normalized and $R_i = \operatorname{trace}(K_i)$ is set to be 1.0 for i = 1, ..., 5. Then $R = \sum_{i=1}^{5} \lambda_i R_i = \sum_{i=1}^{5} \lambda_i$, is restricted to be 5.0. The value of the parameter C is

² https://archive.ics.uci.edu/ml/datasets/HEPMASS.



Table 5 Comparison of PC²PM with CPLEX 12.8.0 for solving multiple kernel learning problems using 5 Gaussian kernel functions

Two-norm problem	
(For 1-norm soft margin SVM, we let PC ² PM converge with $res_1 < 0.015$ instead of 10^{-3} , while still	
keep $res_2 < 10^{-3}$.)	

SVM		Mem./node	Time	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	TSA
Criteria		(GB)	(h)						(%)
SM1	PC ² PM	2.1/node	6.06	0.000	0.000	0.000	6.543	0.000	97.84
C = 3.0	CPLEX 12.8.0	O.O.M. (> 96)	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.
SM2	PC^2PM	2.0/node	0.72	0.000	0.000	0.000	5.005	0.000	97.83
C = 1.0	CPLEX 12.8.0	73.6	3.09	0.001	0.000	0.000	4.997	0.002	97.83

HEPMASS data set

(For 1-norm soft margin SVM, we let PC²PM converge with $res_1 < 0.02$ instead of 10^{-3} , while still keep $res_2 < 10^{-3}$.)

SVM		Mem./node	Time	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	TSA
Criteria		(GB)	(h)						(%)
SM1	PC ² PM	2.1/node	6.51	0.000	0.000	0.000	0.000	6.992	76.77
C = 5.0	CPLEX 12.8.0	O.O.M. (> 96)	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.
SM2	PC^2PM	2.0/node	0.17	0.000	0.000	0.000	0.000	5.088	78.43
C = 1.0	CPLEX 12.8.0	71.5	3.12	0.001	0.000	0.000	0.013	4.985	78.33

fixed as 1.0 for 2-norm soft margin SVMs, and is set as 3.0 for Two-norm Problem and 5.0 for HEPMASS Data Set when using 1-norm soft margin SVMs. Numerical results of both 1-norm and 2-norm soft margin SVMs using the above five kernel functions are summarized in Table 5.

Each data set of a total number of 16,000 data points is randomly partitioned into 80% for training and 20% for testing. The reported values in each row of Table 5 are averaged over five different random partitions.

We implement PC²PM using 128 cores for primal updates and 5 cores for dual updates, which amount to a total of 6 compute nodes on Purdue's Brown cluster. The average elapsed wall-clock time used by PC²PM to converge with a tolerance $\tau^{\text{PC}^2\text{PM}} = 10^{-3}$ is presented in Table 5, along with the averaged amount of memory used by each node. We also report in Table 5 the average learned non-negative coefficients $\lambda_1^*, \dots \lambda_5^*$, as well as the average TSA. The performance of CPLEX 12.8.0 with the same tolerance is also presented in Table 5 for comparison. As shown by the values of the coefficients learned, the Gaussian kernel function k_4 with $\sigma^2 = 10.0$ is selected by the models of both two soft margin SVMs for the Two-norm Problem; the HEPMASS Data Set selects the Gaussian kernel function k_5 with $\sigma^2 = 100.0$. For 2-norm soft margin SVMs, PC²PM converges much faster than CPLEX, and also uses much less memory (as expected). For TSA, both PC²PM and CPLEX obtain the same value, calculated using their own optimal solution point $(\alpha^*, \lambda_1^*, \dots, \lambda_m^*)$.



Oudoordin no.												
HEPMASS	data se	et										
PC ² PM SI	M2 (C = 1.0										
σ^2 search	m	n _{node}	n _{core}	Mem./node	Time	Non-zero λ's	TSA					
Range				(GB)	(h)		(%)					
$[10^{-4}, 10^4]$	9	6	128 + 9	2.9/node	0.43	$\lambda_7^* = 8.995 (\sigma^2 = 10^2)$	79.62					

4.4/node

Table 6 Numerical results of applying PC²PM to solve 2-norm soft margin SVMs using multiple (9, 17) Gaussian kernel functions for the HEPMASS data set

For 1-norm soft margin SVMs, CPLEX fails to provide a solution due to running out of memory, while PC²PM still solves the problem.

1.51

 $\lambda_{12}^* = 17.001 \quad (\sigma^2 = 10^{1.5})$

80.60

In Table 6, we also report the numerical results of applying PC²PM to solve 2-norm soft margin SVMs for the HEPMASS Data Set, using 9 Gaussian kernel functions with σ^2 equal to 10^{-4} , 10^{-3} , 10^{-2} , 10^{-1} , 10^0 , 10^1 , 10^2 , 10^3 and 10^4 respectively. Though the number of constraints doubles, PC²PM still converges within a reasonable amount of time, and remains memory efficient. The Gaussian kernel function k_7 with $\sigma^2 = 100.0$ is still selected by the model. We further search the range of $[10^{-4}, 10^4]$ using 17 Gaussian kernel functions with σ^2 equal to 10^{-4} , $10^{-3.5}$, 10^{-3} , ..., 10^3 , 10^3 .5, 10^4 . The Gaussian kernel function k_{12} with $\sigma^2 = 10^{1.5}$ is selected instead, and we observe a slightly increased average TSA.

While the numerical experiments so far have demonstrated the scalability of the PC^2PM algorithm due to its distributed data storage and natural decomposition to facilitate parallel computing, in the following experiments, we show the benefits of the PC^2PM algorithm for not requiring any matrix decompositions. In this test, we use three kernel functions, instead of five, to solve (45) and (46). The three kernel functions consist of k_1 —the Gaussian kernel function with $\sigma^2 = 100.0$, k_2 —a linear kernel function defined in (44a), and k_3 —a polynomial kernel function defined in (44b). The value of the parameter C is fixed as 1.0, and is only changed to 2.0 when using 1-norm soft margin SVM for HEPMASS Data Set. All the other settings remain the same as in the previous experiment (except for the value of R, which is set as 3.0). The numerical results are reported in Table 7.

For all groups of tests, CPLEX returns an error stating that the quadratic constraint containing $G_3(K_{3,tr})$ is not convex, which is theoretically impossible because each matrix $G_i(K_{i,tr})$ is at least a PSD matrix as we discussed previously; while PC²PM solves all the instances without any issues. The error returned by CPLEX is created likely by the failure of matrix decomposition of a large-scale PSD matrix due to precision limit. Once we reduce the size of the matrices in (45) and (46), CPLEX can then solve the instances without error messages. This numerical experiment illustrates that not requiring matrix decomposition in the PC²PM is not just of computational convenience; it can indeed make the algorithm more robust to solve large-scale problems without facing potential issues caused by floating point arithmetic.



17 7

128 + 17

3 kernel fu	inctions						
Two-norm	problem						
SVM		Mem./node	Time	λ_1^*	λ_2^*	λ_3^*	TSA
Criteria		(GB)	(h)				(%)
SM1	PC ² PM	1.7/node	3.16	0.000	3.029	0.000	91.29
C = 1.0	CPLEX 12.8.0	Non-con. error	N.A.	N.A.	N.A.	N.A.	N.A.
SM2	PC^2PM	1.5/node	1.69	0.000	3.054	0.000	97.85
C = 1.0	CPLEX 12.8.0	Non-con. error	N.A.	N.A.	N.A.	N.A.	N.A.
HEPMASS	S data set						
SVM		Mem./node	Time	λ_1^*	λ_2^*	λ_3^*	TSA
Criteria		(GB)	(h)				(%)
SM1	PC ² PM	1.6/node	6.43	0.000	3.021	0.000	72.81
C = 2.0	CPLEX 12.8.0	Non-con. error	N.A.	N.A.	N.A.	N.A.	N.A.
SM2	PC^2PM	1.4/node	0.83	0.000	3.019	0.000	80.51

N.A.

N.A.

N.A.

N.A.

N.A.

Non-con. error

Table 7 Comparison of PC²PM with CPLEX 12.8.0 for solving multiple kernel learning problems using 3 kernel functions

6 Conclusion and future works

CPLEX 12.8.0

C = 1.0

In this paper, we propose a novel distributed algorithm, built upon the original idea of the PCPM algorithm, that can solve non-separable convex OCOPs in a Jacobifashion (that is, parallel updating). Numerical results show that our algorithm, termed as PC²PM, exhibits much better scalability when compared to CPLEX, which uses the IPM to solve convex QCQPs. The scalability of the algorithm is attributed to the three key features of the algorithm design: first, the PC²PM algorithm can decompose primal (and dual) variables down to the scalar level and update them in parallel, even when the quadratic constraints are non-separable. Second, when implementing the algorithm, only the related columns of all the Hessian matrices need to be stored locally, instead of the entire matrices on each of computing unit in a parallel computing setting. Third, our algorithm does not need any matrix decomposition (unlike any semi-definite-programming-based approach), which can improve the algorithm's robustness, especially when solving convex QCQPs with PSD matrices, as demonstrated in our numerical experiments summarized in Table 7. The second and the third feature together make our algorithm particularly suitable to solve extreme-dimension QCQPs, which likely will cause memory issues for other algorithms.

In addition to the scalability of the PC²PM algorithm, its ability to solve non-separable, quadratically constrained problems in Jacobi-fashion should also be emphasized, as in general it is very difficult to design distributed algorithms with Jacobistyle update (as opposed to the sequential Gauss-Seidel update) to solve optimization



problems with non-separable constraints. Whether the algorithm idea from PC²PM can be extended to solve more general convex problems is certainly worth exploring. There are several other lines of research that can be done to improve the current work. First, while we proved convergence of PC2PM, we cannot prove its convergence rate as of now. Second, while the parallel updating of the primal variables is a nice property of PC²PM, it is still a synchronous algorithm in the sense that the algorithm needs to wait for all primal and dual updates to be done before it can move to the next iteration. An asynchronous implementation of the algorithm will no doubt make it even more suitable for distributed computing, and we defer it to our future work. Third, there have been increasing works on solving large-scale nonconvex QCQPs. As mentioned in the introduction section, one algorithm idea is to solve it with a sequence of convexified QCQPs, where our algorithm is then applicable. This naturally leads to an algorithm with nested loops, where the outer loop lays out sequential convexification, and the inner loop invokes our algorithm. It would be interesting to see how such a nested algorithm performs in practice, especially with high-dimension problems.

Appendix 1: Step-size update rule for ρ^{k+1}

With a given scalar $0 \le \epsilon_0 < 1$, and a series of positive scalars $\epsilon_1, \ldots, \epsilon_8 > 0$ that satisfy $\sum_{s=1}^8 \epsilon_s \le 1 - \epsilon_0$, we define the following function $\rho: \mathbb{X} \times \mathbb{R}^{n_2}_+ \times \mathbb{R}^{m_1}_+ \times \mathbb{R}^{m_2}_+ \to (0, +\infty)$ to update the adaptive step size ρ^{k+1} in Algorithm 1 at each iteration k:

$$\rho^{k+1} = \rho(\mathbf{x}^k, \mathbf{u}^k, \boldsymbol{\lambda}^k, \boldsymbol{\gamma}^k)$$

$$:= \min \left\{ \rho_1, \rho_2(\mathbf{x}^k, \mathbf{u}^k, \boldsymbol{\lambda}^k), \rho_3(\mathbf{x}^k, \boldsymbol{\lambda}^k, \boldsymbol{\gamma}^k), \rho_4, \rho_5(\mathbf{x}^k), \rho_6, \rho_7, \rho_8 \right\},$$
(49)

where

 $\begin{array}{ll} \text{(i)} & \rho_1 = \left\{\begin{array}{ll} \frac{\epsilon_1}{\|P_0\|_F}, \text{ if } \|P_0\|_F \neq 0 \\ \epsilon_1, & \text{if } \|P_0\|_F = 0, \end{array}\right. \\ \text{with } \|\cdot\|_F \text{ representing the Frobenius norm of } \\ \text{a matrix:} \end{array}$

(ii) $\rho_2(\mathbf{x}^k, \mathbf{u}^k, \lambda^k) = \min_i \{ \rho_{2i}(\mathbf{x}^k, \mathbf{u}^k, \lambda^k) \}$, where

$$\rho_{2i}(\mathbf{x}^k, \mathbf{u}^k, \boldsymbol{\lambda}^k) := \begin{cases} \frac{-b_i + \sqrt{b_i^2 + 4a_ic_i}}{2a_i}, & \text{if } a_i > 0\\ \frac{c_i}{b_i}, & \text{if } a_i = 0, b_i > 0\\ M, & \text{if } a_i = 0, b_i = 0, \end{cases}$$

for all $i = 1, ..., m_1$, with $a_i = |\frac{1}{2}(\mathbf{x}^k)^T P_i \mathbf{x}^k + \mathbf{q}_i^T \mathbf{x}^k + \mathbf{c}_i^T \mathbf{u}^k + r_i| \ge 0$, and $b_i = \lambda_i^k \ge 0$. For c_i , if $||P_i||_F \ne 0$, $c_i = \frac{\epsilon_2}{m_1 ||P_i||_F} > 0$; otherwise $c_i = \frac{\epsilon_2}{m_1} > 0$. The constant M > 0 can be any fixed, arbitrarily large scalar;



$$\begin{aligned} &(\text{iii}) \quad \rho_3(\mathbf{x}^k, \lambda^k, \pmb{\gamma}^k) = \\ &\left\{ &\min\left\{2\epsilon_3, \frac{-b + \sqrt{b^2 + 4ac}}{2a}\right\}, \text{ if } a > 0 \\ &\min\left\{2\epsilon_3, \frac{c}{b}\right\}, & \text{if } a = 0, b > 0 \\ &2\epsilon_3, & \text{if } a = 0, b = 0, \end{aligned} \right.$$

where $a = \|P_0 \mathbf{x}^k + \mathbf{q}_0 + \sum_{i=1}^{m_1} \lambda_i^k (P_i \mathbf{x}^k + \mathbf{q}_i) + A^T \mathbf{y}^k\|_2 \ge 0$, $b = 2\|\mathbf{x}^k\|_2 \ge 0$ and $c = \frac{2\epsilon_3}{\|P\|_F} > 0$ with $P \in \mathbb{R}^{m_1 n_1 \times n_1}$ denoting the stacked matrix $\begin{pmatrix} P_1 \\ \vdots \\ P_{m_s} \end{pmatrix}$;

(iv)
$$\rho_4 = \begin{cases} \frac{\epsilon_4}{\|Q\|_F}, & \text{if } \|Q\|_F \neq 0 \\ \epsilon_4, & \text{if } \|Q\|_F = 0 \end{cases}$$
, where $Q \in \mathbb{R}^{m_1 \times n_1}$ denotes matrix $\begin{pmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_{m_1}^T \end{pmatrix}$, with the \mathbf{q}_i 's being the vectors in the linear terms of \mathbf{x} in the QCQP (8);

$$(\mathbf{v}) \quad \rho_5(\mathbf{x}^k) = \left\{ \begin{array}{l} \frac{\epsilon_5}{\|\mathbf{x}^k\|_2 \|P\|_F}, \text{ if } \|\mathbf{x}^k\|_2 \neq 0 \\ \epsilon_5, & \text{if } \|\mathbf{x}^k\|_2 = 0 \end{array} \right. ;$$

(vi)
$$\rho_6 = \begin{cases} \frac{\epsilon_6}{\|C\|_F}, & \text{if } \|C\|_F \neq 0 \\ \epsilon_6, & \text{if } \|C\|_F = 0 \end{cases}$$
, where $C \in \mathbb{R}^{m_2 \times n_2}$ denotes matrix $\begin{pmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_{m_2}^T \end{pmatrix}$, with the \mathbf{c}_i 's being the vectors in the linear terms of \mathbf{u} in the QCQP (8);

(vii)
$$\rho_7 = \begin{cases} \frac{\epsilon_7}{\|A\|_F}, & \text{if } \|A\|_F \neq 0 \\ \epsilon_7, & \text{if } \|A\|_F = 0 \end{cases}$$
, where A is the matrix in the linear constraint $A\mathbf{x} + B\mathbf{u} = \mathbf{b}$ in (8);

(viii)
$$\rho_8 = \begin{cases} \frac{\epsilon_8}{\|B\|_F}, & \text{if } \|B\|_F \neq 0 \\ \epsilon_8, & \text{if } \|B\|_F = 0 \end{cases}$$
, where B is the matrix in the linear constraint $A\mathbf{x} + B\mathbf{u} = \mathbf{b}$ in (8).

While the rules to update the step-size ρ^{k+1} may appear to be very cumbersome, the calculations are actually quite straightforward. Since the Frobenius norm of all matrices can be obtained in advance, the values of ρ_1 , ρ_4 , ρ_6 , ρ_7 and ρ_8 are pre-determined. Given a current solution $(\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \boldsymbol{\gamma}^k)$, ρ_2 , ρ_3 and ρ_5 can also be easily calculated. The minimum of all the ρ_s 's then determines the value of the adaptive step size ρ^{k+1} .



Appendix 2: Proofs in Sect. 3

Proof of Proposition 1

We first prove the inequality (24). Consider the linear approximation of the Lagrangian function of a QCQP, as defined in (21), with a given point $\zeta^k \equiv (\mathbf{x}^k, \lambda^k, \gamma^k)$. Let $\hat{\mathbf{z}} = (\mathbf{y}^{k+1}, \mathbf{v}^{k+1})$, the (k+1)th iteration of the primal predictor of \mathbf{x}^k and \mathbf{u}^k in the PC²PM algorithm, as given in (14a) and (16a), respectively. By Lemma 2, we know that $\hat{\mathbf{z}}$ is the unique minimizer of the corresponding proximal minimization problem in (22b). By defining $\bar{\mathbf{z}} = (\mathbf{x}^k, \mathbf{u}^k)$ and $\mathbf{z} = (\mathbf{x}^{k+1}, \mathbf{u}^{k+1})$, and using Lemma 1, we have that

$$2\rho^{k+1}\left[\mathcal{R}(\widehat{\mathbf{z}};\zeta^k) - \mathcal{R}(\mathbf{z};\zeta^k)\right] \le \|\bar{\mathbf{z}} - \mathbf{z}\|_2^2 - \|\widehat{\mathbf{z}} - \mathbf{z}\|_2^2 - \|\widehat{\mathbf{z}} - \bar{\mathbf{z}}\|_2^2, \tag{50}$$

which leads to the following expanded inequality

$$2\rho^{k+1} \left\{ (P_{0}\mathbf{x}^{k} + \mathbf{q}_{0})^{T}\mathbf{y}^{k+1} + \mathbf{c}_{0}^{T}\mathbf{v}^{k+1} + r_{0} + \sum_{i=1}^{m_{1}} \lambda_{i}^{k} \left[(P_{i}\mathbf{x}^{k} + \mathbf{q}_{i})^{T}\mathbf{y}^{k+1} + \mathbf{c}_{i}^{T}\mathbf{v}^{k+1} + r_{i} \right] + (\boldsymbol{\gamma}^{k})^{T} (A\mathbf{y}^{k+1} + B\mathbf{v}^{k+1} - \mathbf{b}) \right\}$$

$$-2\rho^{k+1} \left\{ (P_{0}\mathbf{x}^{k} + \mathbf{q}_{0})^{T}\mathbf{x}^{k+1} + \mathbf{c}_{0}^{T}\mathbf{u}^{k+1} + r_{0} + \sum_{i=1}^{m_{1}} \lambda_{i}^{k} \left[(P_{i}\mathbf{x}^{k} + \mathbf{q}_{i})^{T}\mathbf{x}^{k+1} + \mathbf{c}_{i}^{T}\mathbf{u}^{k+1} + r_{i} \right] + (\boldsymbol{\gamma}^{k})^{T} (A\mathbf{x}^{k+1} + B\mathbf{u}^{k+1} - \mathbf{b}) \right\}$$

$$\leq \left(\|\mathbf{x}^{k} - \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{u}^{k} - \mathbf{u}^{k+1}\|_{2}^{2} \right) - \left(\|\mathbf{y}^{k+1} + \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{v}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} \right) - \left(\|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} + \|\mathbf{v}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} \right).$$

$$(51)$$

Now consider the \mathcal{R} function at a different given point $\zeta^{k+1} \equiv (\mathbf{y}^{k+1}, \mu^{k+1}, \mathbf{v}^{k+1})$. With a slight abuse of notation, we now let $\hat{\mathbf{z}} = (\mathbf{x}^{k+1}, \mathbf{u}^{k+1})$, the primal correctors at the (k+1)th iteration of the PC²PM algorithm. Also letting $\mathbf{z} = (\mathbf{x}^*, \mathbf{u}^*)$, but keeping $\bar{\mathbf{z}} = (\mathbf{x}^k, \mathbf{u}^k)$, by (22c) in Lemmas 2 and 1, we have that:



$$2\rho^{k+1}\left[\mathcal{R}(\widehat{\mathbf{z}};\boldsymbol{\zeta}^{k+1})-\mathcal{R}(\mathbf{z};\boldsymbol{\zeta}^{k+1})\right]\leq \|\bar{\mathbf{z}}-\mathbf{z}\|_2^2-\|\widehat{\mathbf{z}}-\mathbf{z}\|_2^2-\|\widehat{\mathbf{z}}-\bar{\mathbf{z}}\|_2^2,$$

which leads to the following expanded inequality

$$2\rho^{k+1} \left\{ (P_{0}\mathbf{y}^{k+1} + \mathbf{q}_{0})^{T}\mathbf{x}^{k+1} + \mathbf{c}_{0}^{T}\mathbf{u}^{k+1} + r_{0} + \sum_{i=1}^{m_{1}} \mu_{i}^{k+1} \left[(P_{i}\mathbf{y}^{k+1} + \mathbf{q}_{i})^{T}\mathbf{x}^{k+1} + \mathbf{c}_{i}^{T}\mathbf{u}^{k+1} + r_{i} \right] + (\mathbf{v}^{k+1})^{T} (A\mathbf{x}^{k+1} + B\mathbf{u}^{k+1} - \mathbf{b}) \right\}$$

$$- 2\rho^{k+1} \left\{ (P_{0}\mathbf{y}^{k+1} + \mathbf{q}_{0})^{T}\mathbf{x}^{*} + \mathbf{c}_{0}^{T}\mathbf{u}^{*} + r_{0} + \sum_{i=1}^{m_{1}} \mu_{i}^{k+1} \left[(P_{i}\mathbf{y}^{k+1} + \mathbf{q}_{i})^{T}\mathbf{x}^{*} + \mathbf{c}_{i}^{T}\mathbf{u}^{*} + r_{i} \right] + (\mathbf{v}^{k+1})^{T} (A\mathbf{x}^{*} + B\mathbf{u}^{*} - \mathbf{b}) \right\}$$

$$\leq \left(\|\mathbf{x}^{k} - \mathbf{x}^{*}\|_{2}^{2} + \|\mathbf{u}^{k} - \mathbf{u}^{*}\|_{2}^{2} \right) - \left(\|\mathbf{x}^{k+1} - \mathbf{x}^{*}\|_{2}^{2} + \|\mathbf{u}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} \right) - \left(\|\mathbf{x}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} + \|\mathbf{u}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} \right).$$

$$(52)$$

The final piece to derive inequality (24) is to utilize Lemma 3. Let $(\mathbf{x}^*, \mathbf{u}^*, \lambda^*, \gamma^*)$ be a saddle point of QCQP (8), and again, $\zeta^{k+1} = (\mathbf{y}^{k+1}, \mu^{k+1}, \mathbf{v}^{k+1})$. By Lemma 3, we have that

$$\mathcal{R}(\mathbf{x}^*, \mathbf{u}^*; \zeta^{k+1}) - \mathcal{R}(\mathbf{y}^{k+1}, \mathbf{v}^{k+1}; \zeta^{k+1}) \\
\leq \sum_{i=1}^{m} \left[(\lambda_i^* - \mu_i^{k+1}) \left(\frac{1}{2} \mathbf{y}^{k+1^T} P_i \mathbf{y}^{k+1} + \mathbf{q}_i^T \mathbf{y}^{k+1} + \mathbf{c}_i^T \mathbf{v}^{k+1} + r_i \right) \right] \\
+ (\boldsymbol{\gamma}^* - \boldsymbol{v}^{k+1})^T (A \mathbf{y}^{k+1} + B \mathbf{v}^{k+1} - \mathbf{b}). \tag{53}$$

Multiplying both sides by $2\rho^{k+1}$ and expanding the \mathcal{R} function, we have that



$$2\rho^{k+1} \left\{ (P_{0}\mathbf{y}^{k+1} + \mathbf{q}_{0})^{T}\mathbf{x}^{*} + \mathbf{c}_{0}^{T}\mathbf{u}^{*} + r_{0} + \sum_{i=1}^{m} \mu_{i}^{k+1} \left[(P_{i}\mathbf{y}^{k+1} + \mathbf{q}_{i})^{T}\mathbf{x}^{*} + \mathbf{c}_{i}^{T}\mathbf{u}^{*} + r_{i} \right] + (\mathbf{v}^{k+1})^{T} (A\mathbf{x}^{*} + B\mathbf{u}^{*} - \mathbf{b}) \right\}$$

$$-2\rho^{k+1} \left\{ (P_{0}\mathbf{y}^{k+1} + \mathbf{q}_{0})^{T}\mathbf{y}^{k+1} + \mathbf{c}_{0}^{T}\mathbf{v}^{k+1} + r_{0} + \sum_{i=1}^{m} \mu_{i}^{k+1} \left[(P_{i}\mathbf{y}^{k+1} + \mathbf{q}_{i})^{T}\mathbf{y}^{k+1} + \mathbf{c}_{i}^{T}\mathbf{v}^{k+1} + r_{i} \right] + (\mathbf{v}^{k+1})^{T} (A\mathbf{y}^{k+1} + B\mathbf{v}^{k+1} - \mathbf{b}) \right\}$$

$$\leq 2\rho^{k+1} \left\{ \sum_{i=1}^{m} (\lambda_{i}^{*} - \mu_{i}^{k+1}) \left[\frac{1}{2} (\mathbf{y}^{k+1})^{T} P_{i}\mathbf{y}^{k+1} + \mathbf{q}_{i}^{T}\mathbf{y}^{k+1} + \mathbf{c}_{i}^{T}\mathbf{v}^{k+1} + r_{i} \right] + (\gamma^{*} - \mathbf{v}^{k+1})^{T} (A\mathbf{y}^{k+1} + B\mathbf{v}^{k+1} - \mathbf{b}) \right\}.$$
(54)

Adding the three inequalities (51), (52) and (54) yields the inequality (24) in Proposition 1.

To prove the second inequality, (25), in Proposition 1, we use a similar approach as above, just replacing the linear approximation function \mathcal{R} with the original Lagrangian function \mathcal{L} . More specifically, let $\hat{\mathbf{z}} = (\boldsymbol{\mu}^{k+1}, \boldsymbol{\nu}^{k+1})$. By (22a) in Lemma 2, we know that

$$\widehat{\mathbf{z}} := (\boldsymbol{\mu}^{k+1}, \ \boldsymbol{\nu}^{k+1})$$

$$= \underset{\boldsymbol{\lambda} \in \mathbb{R}^{m_1}, \ \boldsymbol{\gamma} \in \mathbb{R}^{m_2}}{\operatorname{argmin}} - \mathcal{L}(\mathbf{x}^k, \mathbf{u}^k, \boldsymbol{\lambda}, \boldsymbol{\gamma}) + \frac{1}{2\rho^{k+1}} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^k\|_2^2 + \frac{1}{2\rho^{k+1}} \|\boldsymbol{\gamma} - \boldsymbol{\gamma}^k\|_2^2.$$
(55)

Letting $\bar{\mathbf{z}} = (\lambda^k, \gamma^k)$ and choosing a specific $\mathbf{z} = (\lambda^{k+1}, \gamma^{k+1})$, we use Lemma 1 to obtain that

$$2\rho^{k+1}\left[\left(-\mathcal{L}(\mathbf{x}^{k},\mathbf{u}^{k};\widehat{\mathbf{z}})\right)-\left(-\mathcal{L}(\mathbf{x}^{k},\mathbf{u}^{k};\mathbf{z})\right)\right] \leq \|\bar{\mathbf{z}}-\mathbf{z}\|_{2}^{2}-\|\widehat{\mathbf{z}}-\mathbf{z}\|_{2}^{2}-\|\widehat{\mathbf{z}}-\bar{\mathbf{z}}\|_{2}^{2},$$
(56)

which yields the following expanded inequality:



$$2\rho^{k+1} \left\{ \sum_{i=1}^{m_{1}} (\lambda_{i}^{k+1} - \mu_{i}^{k+1}) \left[\frac{1}{2} (\mathbf{x}^{k})^{T} P_{i} \mathbf{x}^{k} + \mathbf{q}_{0}^{T} \mathbf{x}^{k} + \mathbf{c}_{0}^{T} \mathbf{u}^{k} + r_{i} \right] + (\boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1})^{T} (A \mathbf{x}^{k} + B \mathbf{u}^{k} - \mathbf{b}) \right\}$$

$$\leq \left(\|\boldsymbol{\lambda}^{k} - \boldsymbol{\lambda}^{k+1}\|_{2}^{2} + \|\boldsymbol{\gamma}^{k} - \boldsymbol{\gamma}^{k+1}\|_{2}^{2} \right)$$

$$- \left(\|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^{k+1}\|_{2}^{2} + \|\boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^{k+1}\|_{2}^{2} \right) - \left(\|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^{k}\|_{2}^{2} + \|\boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^{k}\|_{2}^{2} \right).$$

$$(57)$$

Similarly, again with some abuse of notation, letting $\hat{\mathbf{z}} = (\lambda^{k+1}, \gamma^{k+1})$, by (22d) in Lemma 2, we have that

$$\widehat{\mathbf{z}} := (\lambda^{k+1}, \gamma^{k+1})$$

$$= \underset{\lambda \in \mathbb{R}_{+}^{m_{1}}, \ \gamma \in \mathbb{R}^{m_{2}}}{\operatorname{argmin}} - \mathcal{L}(\mathbf{y}^{k+1}, \mathbf{v}^{k+1}, \lambda, \gamma) + \frac{1}{2\rho^{k+1}} \|\lambda - \lambda^{k}\|_{2}^{2} + \frac{1}{2\rho^{k+1}} \|\gamma - \gamma^{k}\|_{2}^{2}.$$
(58)

By choosing **z** to be (λ^*, γ^*) , while keeping $\bar{\mathbf{z}}$ at (λ^k, γ^k) , we have from Lemma 1 that

$$2\rho^{k+1} \left[\left(-\mathcal{L}(\mathbf{y}^{k+1}, \mathbf{v}^{k+1}; \widehat{\mathbf{z}}) \right) - \left(-\mathcal{L}(\mathbf{y}^{k+1}, \mathbf{v}^{k+1}; \mathbf{z}) \right) \right]$$

$$\leq \|\bar{\mathbf{z}} - \mathbf{z}\|_{2}^{2} - \|\widehat{\mathbf{z}} - \mathbf{z}\|_{2}^{2} - \|\widehat{\mathbf{z}} - \bar{\mathbf{z}}\|_{2}^{2},$$

$$(59)$$

which yields the following expanded inequality:

$$2\rho^{k+1} \left\{ \sum_{i=1}^{m_{1}} (\lambda_{i}^{*} - \lambda_{i}^{k+1}) \left[\frac{1}{2} (\mathbf{y}^{k+1})^{T} P_{i} \mathbf{y}^{k+1} + \mathbf{q}_{0}^{T} \mathbf{y}^{k+1} + \mathbf{c}_{0}^{T} \mathbf{v}^{k+1} + r_{i} \right] \right.$$

$$\left. + (\boldsymbol{\gamma}^{*} - \boldsymbol{\gamma}^{k+1})^{T} (A \mathbf{y}^{k+1} + B \mathbf{v}^{k+1} - \mathbf{b}) \right\}$$

$$\leq \left(\|\boldsymbol{\lambda}^{k} - \boldsymbol{\lambda}^{*}\|_{2}^{2} + \|\boldsymbol{\gamma}^{k} - \boldsymbol{\gamma}^{*}\|_{2}^{2} \right)$$

$$\left. - \left(\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^{*}\|_{2}^{2} + \|\boldsymbol{\gamma}^{k+1} - \boldsymbol{\gamma}^{*}\|_{2}^{2} \right) - \left(\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^{k}\|_{2}^{2} + \|\boldsymbol{\gamma}^{k+1} - \boldsymbol{\gamma}^{k}\|_{2}^{2} \right).$$

$$(60)$$

Adding the two inequalities (57) and (60) leads to the second inequality, (25), in Proposition 1.

Proof of Theorem 1

By adding the two inequalities (24) and (25) in Proposition 1, we have that



$$\begin{aligned} \|\mathbf{x}^{k+1} - \mathbf{x}^*\|_2^2 + \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_2^2 + \|\lambda^{k+1} - \lambda^*\|_2^2 + \|\gamma^{k+1} - \gamma^*\|_2^2 \\ &\leq \|\mathbf{x}^k - \mathbf{x}^*\|_2^2 + \|\mathbf{u}^k - \mathbf{u}^*\|_2^2 + \|\lambda^k - \lambda^k\|_2^2 + \|\gamma^k - \gamma^*\|_2^2 \\ &- \left(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^k\|_2^2\right) \\ &- \left(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{y}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{y}^k\|_2^2\right) \\ &+ \left(2\rho^{k+1}(\mathbf{y}^{k+1} - \mathbf{x}^{k+1})^T P_0(\mathbf{y}^{k+1} - \mathbf{x}^k)\right) \\ &+ \left(2\rho^{k+1}\sum_{i=1}^{m_i} (\lambda_i^{k+1} - \mu_i^{k+1})^T P_i(\mathbf{y}^{k+1} - \mathbf{x}^k)\right) \\ &+ \left(2\rho^{k+1}\sum_{i=1}^{m_i} (\lambda_i^{k+1} - \mu_i^{k+1}) \mathbf{q}_i^T(\mathbf{y}^{k+1} - \mathbf{x}^k)\right) \\ &+ \left(2\rho^{k+1}\sum_{i=1}^{m_i} (\mu_i^{k+1} - \lambda_i^k) \mathbf{q}_i^T(\mathbf{y}^{k+1} - \mathbf{x}^k)\right) \\ &+ \left(2\rho^{k+1}\sum_{i=1}^{m_i} (\mu_i^{k+1} - \lambda_i^k) \mathbf{q}_i^T(\mathbf{y}^{k+1} - \mathbf{x}^{k+1})\right) \\ &+ \left(2\rho^{k+1}\sum_{i=1}^{m_i} (\lambda_i^{k+1} - \mu_i^{k+1}) \mathbf{c}_i^T(\mathbf{v}^{k+1} - \mathbf{x}^{k+1})\right) \\ &+ \left(2\rho^{k+1}\sum_{i=1}^{m_i} (\mu_i^{k+1} - \lambda_i^k) \mathbf{c}_i^T(\mathbf{v}^{k+1} - \mathbf{u}^k)\right) \\ &+ \left(2\rho^{k+1}\sum_{i=1}^{m_i} (\mu_i^{k+1} - \lambda_i^k) \mathbf{c}_i^T(\mathbf{v}^{k+1} - \mathbf{u}^k)\right) \\ &+ \left(2\rho^{k+1}(\mathbf{y}^{k+1} - \mathbf{v}^{k+1})^T A(\mathbf{y}^{k+1} - \mathbf{x}^k) + 2\rho^{k+1}(\mathbf{v}^{k+1} - \mathbf{y}^k)^T A(\mathbf{y}^{k+1} - \mathbf{x}^{k+1})\right) \\ &+ \left(2\rho^{k+1}(\mathbf{y}^{k+1} - \mathbf{v}^{k+1})^T B(\mathbf{v}^{k+1} - \mathbf{u}^k) + 2\rho^{k+1}(\mathbf{v}^{k+1} - \mathbf{y}^k)^T B(\mathbf{v}^{k+1} - \mathbf{u}^{k+1})\right). \end{aligned}$$



Next, we establish an upper bound for each term of the term from (a) to (l) in (61) using the adaptive step size $\rho^{k+1} = \rho(\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \gamma^k)$, as defined in (49).

(a) First, we want to show that

(a)
$$\leq \epsilon_1 \left(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 \right).$$
 (62)

To prove this (and several inequalities below), we first show an extension of the Young's inequality³ on vector products that will play a key role in the following proof. Given any two vectors $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$, we have that

$$\mathbf{z}_{1}^{T}\mathbf{z}_{2} = \sum_{j=1}^{n} z_{1j}z_{2j} = \sum_{j=1}^{n} \left(\frac{1}{\delta}z_{1j}\right) \left(\delta z_{2j}\right) \le \sum_{j=1}^{n} \left|\frac{1}{\delta}z_{1j}\right| \left|\delta z_{2j}\right|,\tag{63}$$

where δ is a non-zero real number. Applying Young's inequality on each summation term with p = q = 2, we obtain that

$$\mathbf{z}_{1}^{T}\mathbf{z}_{2} \leq \sum_{i=1}^{n} \left[\frac{1}{2} \left(\frac{1}{\delta} z_{1j} \right)^{2} + \frac{1}{2} \left(\delta z_{2j} \right)^{2} \right] = \frac{1}{2\delta^{2}} \|\mathbf{z}_{1}\|_{2}^{2} + \frac{\delta^{2}}{2} \|\mathbf{z}_{2}\|_{2}^{2}.$$
 (64)

Applying (64) on (a) yields

(a)
$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_2^2 + \frac{\delta^2}{2} \| P_0(\mathbf{y}^{k+1} - \mathbf{x}^k) \|_2^2 \right)$$

 $\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_2^2 + \frac{\delta^2}{2} \| \| P_0 \|_2^2 \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2 \right)$
 $\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_2^2 + \frac{\delta^2}{2} \| P_0 \|_F^2 \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2 \right).$ (65)

The second inequality holds due to the property that given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{z} \in \mathbb{R}^n$, $\|A\mathbf{z}\|_2 \le \|\|A\|\|_2 \|\mathbf{z}\|_2$ (see Theorem 5.6.2 in [9]), where we use the notation $\|\|\cdot\|\|_2$ to denote the matrix norm $\|\|A\|\|_2 := \sup_{\mathbf{z} \ne \mathbf{0}} \frac{\|A\mathbf{z}\|_2}{\|\mathbf{z}\|_2}$. The

last inequality holds due to the property $||A||_2 \le ||A||_F$ [7], where $||A||_F := \left(\sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^2\right)^{\frac{1}{2}}$ denotes the Frobenius norm. From (65), if $||P_0||_F \ne 0$, then letting $\delta^2 = \frac{1}{||P_0||_F}$ yields

$$(a) \le \rho^{k+1} \|P_0\|_F \Big(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 \Big). \tag{66}$$

Since $\rho^{k+1} \le \rho_1 = \frac{\epsilon_1}{\|P_0\|_F}$, we obtain (62). If, on the other hand, $\|P_0\|_F = 0$, then letting $\delta^2 = 1$ yields

³ Young's inequality states that if a and b are two non-negative real numbers, and p and q are real numbers greater than 1 such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab < \frac{a^p}{p} + \frac{b^q}{q}$.



(a)
$$\leq \rho^{k+1} \left(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} \right)$$
. (67)

Since $\rho^{k+1} \le \rho_1 = \epsilon_1$, (62) is also obtained.

(b) Here we want to show that

$$\sum_{i=1}^{m_1} (\mathbf{b})_i \le \epsilon_2 \Big(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 \Big). \tag{68}$$

Applying (64) on each term $(b)_i$ yields

$$(\mathbf{b})_{i} \leq 2\rho^{k+1}\mu_{i}^{k+1} \left(\frac{1}{2\delta_{i}^{2}} \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} + \frac{\delta_{i}^{2}}{2} \|P_{i}\|_{2}^{2} \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2}\right)$$

$$\leq 2\rho^{k+1}\mu_{i}^{k+1} \left(\frac{1}{2\delta_{i}^{2}} \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} + \frac{\delta_{i}^{2}}{2} \|P_{i}\|_{F}^{2} \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2}\right).$$

$$(69)$$

• If $||P_i||_F \neq 0$, then letting $\delta_i^2 = \frac{1}{||P_i||_F}$ yields

$$(b)_{i} \leq \rho^{k+1} \mu_{i}^{k+1} \| P_{i} \|_{F} \left(\| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_{2}^{2} + \| \mathbf{y}^{k+1} - \mathbf{x}^{k} \|_{2}^{2} \right)$$

$$\leq \rho^{k+1} \tilde{\mu}_{i}^{k+1} \| P_{i} \|_{F} \left(\| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_{2}^{2} + \| \mathbf{y}^{k+1} - \mathbf{x}^{k} \|_{2}^{2} \right),$$

$$(70)$$

where $\tilde{\mu}_i^{k+1} := \lambda_i^k + \rho^{k+1} |\frac{1}{2} (\mathbf{x}^k)^T P_i \mathbf{x}^k + \mathbf{q}_i^T \mathbf{x}^k + \mathbf{c}_i^T \mathbf{u}^k + r_i| \ge \mu_i^{k+1}$. If we can bound $\rho^{k+1} \tilde{\mu}_i^{k+1} \le \frac{\epsilon_2}{m_1 \|P_i\|_F}$, then we can achieve

$$(\mathbf{b})_{i} \le \frac{\epsilon_{2}}{m_{1}} \Big(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} \Big).$$
 (71)

By substituting $a_i = |\frac{1}{2}(\mathbf{x}^k)^T P_i \mathbf{x}^k + \mathbf{q}_i^T \mathbf{x}^k + \mathbf{c}_i^T \mathbf{u}^k + r_i| \geq 0$, $b_i = \lambda_i^k \geq 0$ and $c_i = \frac{\epsilon_2}{m_1 \|P_i\|_F} > 0$, we can rewrite $\rho^{k+1} \tilde{\mu}_i^{k+1} - \frac{\epsilon_2}{m_1 \|P_i\|_F}$ as $a_i (\rho^{k+1})^2 + b_i \rho^{k+1} - c_i$, which is simply a quadratic function of ρ^{k+1} with parameters a_i , b_i and c_i . To bound $\rho^{k+1} \tilde{\mu}_i^{k+1} \leq \frac{\epsilon_2}{m_1 \|P_i\|_F}$ is equivalent to find proper values of ρ^{k+1} that keep the quadratic function stay below zero.

- If $a_i = 0$ and $b_i = 0$, then $ρ^{k+1} ∈ (0, +∞)$.
- If $a_i = 0$ and $b_i > 0$, then $\rho^{k+1} \in (0, \frac{c_i}{b_i}]$.
- If $a_i > 0$, then $\rho^{k+1} \in (0, \frac{-b_i + \sqrt{b_i^2 + 4a_ic_i}}{2a_i}]$.

Since $\rho^{k+1} \le \rho_2(\mathbf{x}^k, \mathbf{u}^k, \lambda^k) \le \rho_{2i}(\mathbf{x}^k, \mathbf{u}^k, \lambda^k)$, it satisfies all the above three conditions, we then obtain (71), and hence (68).

• If $||P_i||_F = 0$, then letting $\delta_i^2 = 1$ yields



$$(\mathbf{b})_{i} \leq \rho^{k+1} \mu_{i}^{k+1} \left(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} \right)$$

$$\leq \rho^{k+1} \tilde{\mu}_{i}^{k+1} \left(\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} \right).$$

$$(72)$$

Similarly, if we can bound $\rho^{k+1}\tilde{\mu}_i^{k+1} \leq \frac{\epsilon_2}{m_1}$, then we can also achieve (71). By substituting $a_i = |\frac{1}{2}(\mathbf{x}^k)^T P_i \mathbf{x}^k + \mathbf{q}_i^T \mathbf{x}^k + \mathbf{c}_i^T \mathbf{u}^k + r_i| \geq 0$, $b_i = \lambda_i^k \geq 0$ and $c_i = \frac{\epsilon_2}{m_1} > 0$, we can rewrite $\rho^{k+1}\tilde{\mu}_i^{k+1} - \frac{\epsilon_2}{m_1}$ as $a_i(\rho^{k+1})^2 + b_i\rho^{k+1} - c_i$. The same analysis can be followed as discussed in the case of $\|P_i\|_F \neq 0$.

(c) Next, we want to show that

$$(c) \le \epsilon_3 \left(\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 \right).$$
By using P to denote $\begin{pmatrix} P_1 \\ \vdots \\ P_{m_1} \end{pmatrix}$, we can rewrite

(c) =
$$\rho^{k+1} \left\{ (\lambda^{k+1} - \boldsymbol{\mu}^{k+1})^T \left[I_{m_1 \times m_1} \otimes (\mathbf{x}^k + \mathbf{y}^{k+1})^T \right] P(\mathbf{y}^{k+1} - \mathbf{x}^k) \right\},$$
 (74)

where \otimes denotes the Kronecker product; that is, given a matrix $A \in \mathbb{R}^{m_1 \times n_1}$ and a matrix $B \in \mathbb{R}^{m_2 \times n_2}$, $A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1m_1}B \\ \vdots & & \vdots \\ a_{m_1}B & \cdots & a_{m_1n_1}B \end{pmatrix}. \text{ Applying (64) to (74)}$

yields

$$(c) \leq \rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \lambda^{k+1} - \boldsymbol{\mu}^{k+1} \|_{2}^{2} + \frac{\delta^{2}}{2} \| \| I_{m_{1} \times m_{1}} \otimes (\mathbf{x}^{k} + \mathbf{y}^{k+1})^{T} \| \|_{2}^{2} \| \| P \| \|_{2}^{2} \| \mathbf{y}^{k+1} - \mathbf{x}^{k} \|_{2}^{2} \right)$$

$$\leq \rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \lambda^{k+1} - \boldsymbol{\mu}^{k+1} \|_{2}^{2} + \frac{\delta^{2}}{2} \| \mathbf{x}^{k} + \mathbf{y}^{k+1} \|_{2}^{2} \| P \|_{F}^{2} \| \mathbf{y}^{k+1} - \mathbf{x}^{k} \|_{2}^{2} \right).$$

$$(75)$$

Since we have the property that $|||A \otimes B|||_2 = |||A|||_2|||B|||_2$ (see Theorem 8 in [15]), the last inequality holds due to

$$\left\| \left| I_{m_1 \times m_1} \otimes (\mathbf{x}^k + \mathbf{y}^{k+1})^T \right| \right\|_2^2 = \left\| \left| I_{m_1 \times m_1} \right| \right\|_2^2 \left\| \left| (\mathbf{x}^k + \mathbf{y}^{k+1})^T \right| \right\|_2^2, \tag{76}$$

together with $|||I_{m_1 \times m_1}|||_2 = 1$ and $|||(\mathbf{x}^k + \mathbf{y}^{k+1})^T|||_2 \le ||(\mathbf{x}^k + \mathbf{y}^{k+1})^T||_F = ||\mathbf{x}^k + \mathbf{y}^{k+1}||_2$. Note that $||P||_F \ne 0$, otherwise the QCQP is simply a QP.

• If $\|\mathbf{x}^k + \mathbf{y}^{k+1}\|_2 \neq 0$, then letting $\delta^2 = \frac{1}{\|\mathbf{x}^k + \mathbf{y}^{k+1}\|_2 \|P\|_F}$ yields

$$(c) \leq \frac{1}{2} \rho^{k+1} \| \mathbf{x}^{k} + \mathbf{y}^{k+1} \|_{2} \| P \|_{F} \left(\| \lambda^{k+1} - \boldsymbol{\mu}^{k+1} \|_{2}^{2} + \| \mathbf{y}^{k+1} - \mathbf{x}^{k} \|_{2}^{2} \right)$$

$$\leq \frac{1}{2} \rho^{k+1} \| \mathbf{x}^{k} + \tilde{\mathbf{y}}^{k+1} \|_{2} \| P \|_{F} \left(\| \lambda^{k+1} - \boldsymbol{\mu}^{k+1} \|_{2}^{2} + \| \mathbf{y}^{k+1} - \mathbf{x}^{k} \|_{2}^{2} \right),$$

$$(77)$$



where $\tilde{y}_{j}^{k+1} := x_{j}^{k} + \rho \Big| \Big[P_{0} \mathbf{x}^{k} + \mathbf{q}_{0} + \sum_{i=1}^{m_{1}} \lambda_{i}^{k} (P_{i} \mathbf{x}^{k} + \mathbf{q}_{i}) + A^{T} \boldsymbol{\gamma}^{k} \Big]_{j} \Big| \ge y_{j}^{k+1}$. If we can bound $\rho^{k+1} \| \mathbf{x}^{k} + \tilde{\mathbf{y}}^{k+1} \|_{2} \le \frac{2e_{3}}{\|P^{T}\|_{L}}$, then (73) can be obtained. We first bound

$$\rho^{k+1} \| \mathbf{x}^{k} + \tilde{\mathbf{y}}^{k+1} \|_{2} - \frac{2\epsilon_{3}}{\|P\|_{F}}$$

$$\leq \rho^{k+1} \left[2 \| \mathbf{x}^{k} \|_{2} + \rho^{k+1} \| P_{0} \mathbf{x}^{k} + \mathbf{q}_{0} + \sum_{i=1}^{m_{1}} \lambda_{i}^{k} (P_{i} \mathbf{x}^{k} + \mathbf{q}_{i}) + A^{T} \boldsymbol{\gamma}^{k} \|_{2} \right] - \frac{2\epsilon_{3}}{\|P\|_{F}}.$$

$$(78)$$

By substituting $a = \|P_0 \mathbf{x}^k + \mathbf{q}_0 + \sum_{i=1}^{m_1} \lambda_i^k (P_i \mathbf{x}^k + \mathbf{q}_i) + A^T \gamma^k\|_2 \ge 0,$ $b = 2\|\mathbf{x}^k\|_2 \ge 0$ and $c = \frac{2\epsilon_3}{\|P\|_c} > 0$, we can bound $\rho^{k+1} \|\mathbf{x}^k + \tilde{\mathbf{y}}^{k+1}\|_2 - \frac{2\epsilon_3}{\|P\|_c}$ using $a(\rho^{k+1})^2 + b\rho^{k+1} - c$, which is simply a quadratic function of ρ^{k+1} with parameters a, b and c. Bounding $\rho^{k+1} \| \mathbf{x}^k + \tilde{\mathbf{y}}^{k+1} \|_2 \le \frac{2\epsilon_3}{\|P\|_r}$ can be guaranteed by finding the proper values of ρ^{k+1} that keep the quadratic function stay below zero.

- If a = 0 and b = 0, then $\rho^{k+1} \in (0, +\infty)$.
- If a = 0 and b > 0, then $\rho^{k+1} \in (0, \frac{c}{a})$ If a > 0, then $\rho^{k+1} \in (0, \frac{-b+\sqrt{b^2+4ac^b}}{2a}]$.

Since $\rho^{k+1} \leq \rho_3(\mathbf{x}^k, \lambda^k, \boldsymbol{\gamma}^k)$, it satisfies all the above three conditions, we obtain (73).

• If $\|\mathbf{x}^k + \mathbf{y}^{k+1}\|_2 = 0$, then letting $\delta^2 = 1$ yields

$$(c) \le \frac{1}{2} \rho^{k+1} \left(\| \boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1} \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2 \right)$$
 (79)

Since $\rho^{k+1} \le \rho_3(\mathbf{x}^k, \lambda^k, \boldsymbol{\gamma}^k) \le 2\epsilon_3$, (73) is also obtained.

(d) To show that

$$(\mathbf{d}) \leq \epsilon_4 \left(\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 \right),$$
 by letting $Q = \begin{pmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_{m_1}^T \end{pmatrix}$, we can rewrite that

(d) =
$$2\rho^{k+1} \left[(\lambda^{k+1} - \boldsymbol{\mu}^{k+1})^T Q^T (\mathbf{y}^{k+1} - \mathbf{x}^k) \right].$$
 (81)

Applying (64) to (81) yields

(d)
$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \| \boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1} \|_2^2 + \frac{\delta^2}{2} \| \| Q \| \|_2^2 \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2 \right)$$

 $\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \| \boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1} \|_2^2 + \frac{\delta^2}{2} \| Q \|_F^2 \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2 \right).$ (82)

• If $||Q||_F \neq 0$, then letting $\delta^2 = \frac{1}{||Q||_F}$ yields



$$(d) \le \rho^{k+1} \|Q\|_F \Big(\|\lambda^{k+1} - \boldsymbol{\mu}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 \Big). \tag{83}$$

Since $\rho^{k+1} \le \rho_4 = \frac{\epsilon_4}{\|Q\|_F}$, we obtain (80).

• If $||Q||_F = 0$, then letting $\delta^2 = 1$ yields

$$(\mathbf{d}) \le \rho^{k+1} \Big(\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2 \Big). \tag{84}$$

Since $\rho^{k+1} \le \rho_4 = \epsilon_4$, (80) is also obtained.

(e) Similarly, to show

(e)
$$\leq \epsilon_4 \left(\| \boldsymbol{\mu}^{k+1} - \lambda^k \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_2^2 \right),$$
 (85)

we can rewrite that

(e) =
$$2\rho^{k+1} \left[(\boldsymbol{\mu}^{k+1} - \lambda^k)^T Q (\mathbf{y}^{k+1} - \mathbf{x}^{k+1}) \right].$$
 (86)

Applying (64) to (86) yields that

(e)
$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \|\boldsymbol{\mu}^{k+1} - \lambda^k\|_2^2 + \frac{\delta^2}{2} |||Q|||_2^2 \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 \right)$$

 $\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \|\boldsymbol{\mu}^{k+1} - \lambda^k\|_2^2 + \frac{\delta^2}{2} \|Q\|_F^2 \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 \right).$ (87)

• If $||Q||_F \neq 0$, then letting $\delta^2 = \frac{1}{||Q||_F}$ yields

(e)
$$\leq \rho^{k+1} \|Q\|_F (\|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^k\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2).$$
 (88)

Since $\rho^{k+1} \le \rho_4 = \frac{\epsilon_4}{\|Q\|_p}$, we obtain (85).

• If $||Q||_F = 0$, then letting $\delta^2 = 1$ yields

(e)
$$\leq \rho^{k+1} \left(\|\boldsymbol{\mu}^{k+1} - \lambda^k\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 \right).$$
 (89)

Since $\rho^{k+1} \le \rho_4 = \epsilon_4$, (85) is also obtained.

(f) To show

(f)
$$\leq \epsilon_5 (\|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^k\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2),$$
 (90)

we can rewrite

$$(\mathbf{f}) = 2\rho^{k+1} \left\{ (\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^k)^T \left[I_{m_1 \times m_1} \otimes (\mathbf{x}^k)^T \right] P(\mathbf{y}^{k+1} - \mathbf{x}^{k+1}) \right\}.$$

$$(91)$$

Applying (64), we have that



$$(f) \leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^{k} \|_{2}^{2} + \frac{\delta^{2}}{2} \| \| I_{m_{1} \times m_{1}} \otimes (\mathbf{x}^{k})^{T} \|_{2}^{2} \| \| P \|_{2}^{2} \| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_{2}^{2} \right)$$

$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^{k} \|_{2}^{2} + \frac{\delta^{2}}{2} \| \mathbf{x}^{k} \|_{2}^{2} \| P \|_{F}^{2} \| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_{2}^{2} \right).$$

$$(92)$$

Similarly, the last inequality holds due to

$$\left\| \left| \left| I_{m_1 \times m_1} \otimes (\mathbf{x}^k)^T \right| \right| \right|_2^2 = \left\| \left| I_{m_1 \times m_1} \right| \left| \left| \left| \left| \left| \mathbf{x}^k \right|^T \right| \right| \right|_2^2.$$

• If $\|\mathbf{x}^k\|_2 \neq 0$, then letting $\delta^2 = \frac{1}{\|\mathbf{x}^k\|_2 \|P\|_E}$ yields

$$(f) \le \rho^{k+1} \|\mathbf{x}^k\|_2 \|P\|_F \Big(\|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^k\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 \Big). \tag{93}$$

Since $\rho^{k+1} \le \rho_5(\mathbf{x}^k) = \frac{\epsilon_5}{\|\mathbf{x}^k\|_2 \|P\|_F}$, we obtain (90). • If $\|\mathbf{x}^k\|_2 = 0$, then letting $\delta^2 = 1$ yields

$$(f) \le \rho^{k+1} \Big(\|\boldsymbol{\mu}^{k+1} - \lambda^k\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 \Big). \tag{94}$$

Since $\rho^{k+1} \le \rho_5(\mathbf{x}^k) = \epsilon_5$, (90) is also obtained.

(g) To show

$$(\mathbf{g}) \leq \epsilon_6 \left(\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1}\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^k\|_2^2 \right),$$
By letting $C = \begin{pmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_{m_2}^T \end{pmatrix}$, we can rewrite

(g) =
$$2\rho^{k+1} \left[(\lambda^{k+1} - \boldsymbol{\mu}^{k+1})^T C(\mathbf{v}^{k+1} - \mathbf{u}^k) \right].$$
 (96)

Applying (64), we have that

$$(g) \leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \lambda^{k+1} - \boldsymbol{\mu}^{k+1} \|_{2}^{2} + \frac{\delta^{2}}{2} \| \| C \| \|_{2}^{2} \| \mathbf{v}^{k+1} - \mathbf{u}^{k} \|_{2}^{2} \right)$$

$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \lambda^{k+1} - \boldsymbol{\mu}^{k+1} \|_{2}^{2} + \frac{\delta^{2}}{2} \| C \|_{F}^{2} \| \mathbf{v}^{k+1} - \mathbf{u}^{k} \|_{2}^{2} \right).$$

$$(97)$$

• If $||C||_F \neq 0$, then letting $\delta^2 = \frac{1}{||C||_F}$ yields

$$(\mathbf{g}) \le \rho^{k+1} \|C\|_F \Big(\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1}\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^k\|_2^2 \Big). \tag{98}$$

Since $\rho^{k+1} \le \rho_6 = \frac{\epsilon_6}{\|C\|_F}$, we obtain (95).

• If $||C||_F = 0$, then letting $\delta^2 = 1$ yields

$$(g) \le \rho^{k+1} \Big(\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\mu}^{k+1}\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^k\|_2^2 \Big). \tag{99}$$



Since $\rho^{k+1} \le \rho_6 = \epsilon_6$, (95) is also obtained.

(h) Next, we want to show that

(h)
$$\leq \epsilon_6 \left(\| \boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^k \|_2^2 + \| \mathbf{v}^{k+1} - \mathbf{u}^{k+1} \|_2^2 \right).$$
 (100)

Similarly, we can rewrite

(h) =
$$2\rho^{k+1} \left[(\boldsymbol{\mu}^{k+1} - \lambda^k)^T C (\mathbf{v}^{k+1} - \mathbf{u}^{k+1}) \right].$$
 (101)

Applying (64) on the above equality leads to

(h)
$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^k\|_2^2 + \frac{\delta^2}{2} |||C|||_2^2 \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_2^2 \right)$$

 $\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^k\|_2^2 + \frac{\delta^2}{2} \|C\|_F^2 \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_2^2 \right).$ (102)

• If $||C||_F \neq 0$, then letting $\delta^2 = \frac{1}{||C||_E}$ yields

(h)
$$\leq \rho^{k+1} \|C\|_F (\|\boldsymbol{\mu}^{k+1} - \lambda^k\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_2^2).$$
 (103)

Since $\rho^{k+1} \le \rho_6 = \frac{\epsilon_6}{\|C\|_E}$, we obtain (100).

• If $||C||_F = 0$, then letting $\delta^2 = 1$ yields

$$(h) \le \rho^{k+1} \Big(\|\boldsymbol{\mu}^{k+1} - \lambda^k\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_2^2 \Big). \tag{104}$$

Since $\rho^{k+1} \le \rho_6 = \epsilon_6$, (100) is also obtained.

(i) To show

(i)
$$\leq \epsilon_7 (\| \boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1} \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2),$$
 (105)

we apply (64) on the rewriting of (i), which leads to

(i)
$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \| \boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1} \|_2^2 + \frac{\delta^2}{2} |||A|||_2^2 \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2 \right)$$

 $\leq 2\rho^{k+1} \left(\frac{1}{2\delta^2} \| \boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1} \|_2^2 + \frac{\delta^2}{2} \|A\|_F^2 \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2 \right).$ (106)

• If $||A||_F \neq 0$, then letting $\delta^2 = \frac{1}{||A||_F}$ yields

(i)
$$\leq \rho^{k+1} \|A\|_F (\|\boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1}\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^k\|_2^2).$$
 (107)

Since $\rho^{k+1} \le \rho_7 = \frac{\epsilon_7}{\|A\|_F}$, we obtain (105).

• If $||A||_F = 0$, then letting $\delta^2 = 1$ yields



(i)
$$\leq \rho^{k+1} \left(\| \boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1} \|_2^2 + \| \boldsymbol{y}^{k+1} - \boldsymbol{x}^k \|_2^2 \right).$$
 (108)

Since $\rho^{k+1} \le \rho_7 = \epsilon_7$, (105) is also obtained.

(i) Similarly, to show

$$(\mathbf{j}) \le \epsilon_7 \Big(\| \mathbf{v}^{k+1} - \mathbf{\gamma}^k \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_2^2 \Big). \tag{109}$$

we apply (64) on the rewriting of (j), which yields

$$(j) \leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^{k} \|_{2}^{2} + \frac{\delta^{2}}{2} |||\boldsymbol{A}|||_{2}^{2} \| \boldsymbol{y}^{k+1} - \boldsymbol{x}^{k+1} \|_{2}^{2} \right)$$

$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^{k} \|_{2}^{2} + \frac{\delta^{2}}{2} \| \boldsymbol{A} \|_{F}^{2} \| \boldsymbol{y}^{k+1} - \boldsymbol{x}^{k+1} \|_{2}^{2} \right).$$

$$(110)$$

• If $||A||_F \neq 0$, then letting $\delta^2 = \frac{1}{||A||_F}$ yields

$$(j) \le \rho^{k+1} \|A\|_F \left(\|\mathbf{v}^{k+1} - \mathbf{\gamma}^k\|_2^2 + \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_2^2 \right). \tag{111}$$

Since $\rho^{k+1} \le \rho_7 = \frac{\epsilon_7}{\|A\|_F}$, we obtain (109).

• If $||A||_F = 0$, then letting $\delta^2 = 1$ yields

$$(j) \le \rho^{k+1} \Big(\| \boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^k \|_2^2 + \| \boldsymbol{y}^{k+1} - \boldsymbol{x}^{k+1} \|_2^2 \Big). \tag{112}$$

Since $\rho^{k+1} \le \rho_7 = \epsilon_7$, (109) is also obtained.

(k) Next, to show

$$(\mathbf{k}) \le \epsilon_8 \Big(\| \boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1} \|_2^2 + \| \boldsymbol{v}^{k+1} - \boldsymbol{u}^k \|_2^2 \Big), \tag{113}$$

we apply (64) on the rewriting of (k):

$$(k) \leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1} \|_{2}^{2} + \frac{\delta^{2}}{2} \| \|\boldsymbol{B}\| \|_{2}^{2} \| \boldsymbol{v}^{k+1} - \boldsymbol{u}^{k} \|_{2}^{2} \right)$$

$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1} \|_{2}^{2} + \frac{\delta^{2}}{2} \| \boldsymbol{B} \|_{F}^{2} \| \boldsymbol{v}^{k+1} - \boldsymbol{u}^{k} \|_{2}^{2} \right).$$

$$(114)$$

• If $||B||_F \neq 0$, then letting $\delta^2 = \frac{1}{||B||_F}$ yields

$$(\mathbf{k}) \le \rho^{k+1} \|B\|_F \left(\|\boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1}\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^k\|_2^2 \right). \tag{115}$$

Since $\rho^{k+1} \le \rho_8 = \frac{\epsilon_8}{\|B\|_{\mathcal{E}}}$, we obtain (113).

• If $||B||_F = 0$, then letting $\delta^2 = 1$ yields

$$(k) \le \rho^{k+1} \Big(\| \boldsymbol{\gamma}^{k+1} - \boldsymbol{v}^{k+1} \|_2^2 + \| \mathbf{v}^{k+1} - \mathbf{u}^k \|_2^2 \Big). \tag{116}$$

Since $\rho^{k+1} \le \rho_8 = \epsilon_8$, (113) is also obtained.



(1) Last, to show

$$(1) \le \epsilon_8 \Big(\| \boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^k \|_2^2 + \| \boldsymbol{v}^{k+1} - \boldsymbol{u}^{k+1} \|_2^2 \Big), \tag{117}$$

we apply (64) on the rewriting of (1):

$$(1) \leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \mathbf{v}^{k+1} - \mathbf{\gamma}^{k} \|_{2}^{2} + \frac{\delta^{2}}{2} |||B|||_{2}^{2} \| \mathbf{v}^{k+1} - \mathbf{u}^{k+1} \|_{2}^{2} \right)$$

$$\leq 2\rho^{k+1} \left(\frac{1}{2\delta^{2}} \| \mathbf{v}^{k+1} - \mathbf{\gamma}^{k} \|_{2}^{2} + \frac{\delta^{2}}{2} \| B \|_{F}^{2} \| \mathbf{v}^{k+1} - \mathbf{u}^{k+1} \|_{2}^{2} \right).$$

$$(118)$$

• If $||B||_F \neq 0$, then letting $\delta^2 = \frac{1}{||B||_F}$ yields

$$(1) \le \rho^{k+1} \|B\|_F \left(\|\mathbf{v}^{k+1} - \mathbf{\gamma}^k\|_2^2 + \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_2^2 \right). \tag{119}$$

Since $\rho^{k+1} \le \rho_8 = \frac{\epsilon_8}{\|B\|_F}$, we obtain (117).

• If $||B||_F = 0$, then letting $\delta^2 = 1$ yields

$$(1) \le \rho^{k+1} \Big(\| \mathbf{v}^{k+1} - \mathbf{\gamma}^k \|_2^2 + \| \mathbf{v}^{k+1} - \mathbf{u}^{k+1} \|_2^2 \Big). \tag{120}$$

Since $\rho^{k+1} \le \rho_8 = \epsilon_8$, (117) is also obtained.

The summation of terms (a) to (l) can now be bounded as:

(a) +
$$\sum_{i=1}^{m_1} (b)_i + (c) + (d) + (e) + (f) + (g) + (h) + (i) + (j) + (k) + (1)$$

 $\leq (\epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_5 + \epsilon_7) \| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_2^2$
 $+ (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_7) \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2$
 $+ (\epsilon_6 + \epsilon_8) \| \mathbf{v}^{k+1} - \mathbf{u}^{k+1} \|_2^2 + (\epsilon_6 + \epsilon_8) \| \mathbf{v}^{k+1} - \mathbf{u}^k \|_2^2$
 $+ (\epsilon_3 + \epsilon_4 + \epsilon_6) \| \boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^{k+1} \|_2^2 + (\epsilon_4 + \epsilon_5 + \epsilon_6) \| \boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^k \|_2^2$
 $+ (\epsilon_7 + \epsilon_8) \| \mathbf{v}^{k+1} - \mathbf{y}^{k+1} \|_2^2 + (\epsilon_7 + \epsilon_8) \| \mathbf{v}^{k+1} - \mathbf{y}^k \|_2^2$
 $\leq (\sum_{s=1}^8 \epsilon_s) \left[\| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{x}^k \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{y}^k \|_2^2 \right]$
 $\leq (1 - \epsilon_0) \left[\| \mathbf{y}^{k+1} - \mathbf{x}^{k+1} \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{y}^k \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{y}^{k+1} \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{y}^k \|_2^2 - \| \mathbf{y}^{k+1} - \mathbf{y}^k \|_2^2 + \| \mathbf{y}^{k+1} - \mathbf{y}^k \|_2^2 - \| \mathbf{y}^{k+1} - \| \mathbf$



Substituting it back into (61), we have that for all $k \ge 0$,

$$\begin{aligned} \|\mathbf{x}^{k+1} - \mathbf{x}^*\|_{2}^{2} + \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_{2}^{2} + \|\lambda^{k+1} - \lambda^*\|_{2}^{2} + \|\gamma^{k+1} - \gamma^*\|_{2}^{2} \\ &\leq \|\mathbf{x}^{k} - \mathbf{x}^*\|_{2}^{2} + \|\mathbf{u}^{k} - \mathbf{u}^*\|_{2}^{2} + \|\lambda^{k} - \lambda^*\|_{2}^{2} + \|\gamma^{k} - \gamma^*\|_{2}^{2} \\ &- \epsilon_{0} \Big[\|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} + \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_{2}^{2} + \|\mathbf{v}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} \\ &+ \|\boldsymbol{\mu}^{k+1} - \lambda^{k+1}\|_{2}^{2} + \|\boldsymbol{\mu}^{k+1} - \lambda^{k}\|_{2}^{2} + \|\boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^{k+1}\|_{2}^{2} + \|\boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^{k}\|_{2}^{2} \Big], \end{aligned}$$

$$(122)$$

which implies for all $k \ge 0$:

$$0 \leq \|\mathbf{x}^{k+1} - \mathbf{x}^*\|_{2}^{2} + \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_{2}^{2} + \|\lambda^{k+1} - \lambda^*\|_{2}^{2} + \|\gamma^{k+1} - \gamma^*\|_{2}^{2}$$

$$\leq \|\mathbf{x}^{k} - \mathbf{x}^*\|_{2}^{2} + \|\mathbf{u}^{k} - \mathbf{u}^*\|_{2}^{2} + \|\lambda^{k} - \lambda^*\|_{2}^{2} + \|\gamma^{k} - \gamma^*\|_{2}^{2}$$

$$\leq \|\mathbf{x}^{k-1} - \mathbf{x}^*\|_{2}^{2} + \|\mathbf{u}^{k-1} - \mathbf{u}^*\|_{2}^{2} + \|\lambda^{k-1} - \lambda^*\|_{2}^{2} + \|\gamma^{k-1} - \gamma^*\|_{2}^{2}$$

$$\leq \cdots \leq \|\mathbf{x}^{0} - \mathbf{x}^*\|_{2}^{2} + \|\mathbf{u}^{0} - \mathbf{u}^*\|_{2}^{2} + \|\lambda^{0} - \lambda^*\|_{2}^{2} + \|\gamma^{0} - \gamma^*\|_{2}^{2}.$$

$$(123)$$

It further implies that the sequence $\{\|\mathbf{x}^k - \mathbf{x}^*\|_2^2 + \|\mathbf{u}^k - \mathbf{u}^*\|_2^2 + \|\lambda^k - \lambda^*\|_2^2 + \|\gamma^k - \gamma^*\|_2^2\}$ is monotonically decreasing and bounded below by 0; hence the sequence must be convergent to a limit, denoted by ξ :

$$\lim_{k \to +\infty} \|\mathbf{x}^k - \mathbf{x}^*\|_2^2 + \|\mathbf{u}^k - \mathbf{u}^*\|_2^2 + \|\lambda^k - \lambda^*\|_2^2 + \|\gamma^k - \gamma^*\|_2^2 = \xi.$$
 (124)

Taking the limit on both sides of (122) yields:

$$\lim_{k \to +\infty} \|\mathbf{y}^{k+1} - \mathbf{x}^{k+1}\|_{2}^{2} = 0, \qquad \lim_{k \to +\infty} \|\mathbf{y}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} = 0,
\lim_{k \to +\infty} \|\mathbf{v}^{k+1} - \mathbf{u}^{k+1}\|_{2}^{2} = 0, \qquad \lim_{k \to +\infty} \|\mathbf{v}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} = 0,
\lim_{k \to +\infty} \|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^{k+1}\|_{2}^{2} = 0, \qquad \lim_{k \to +\infty} \|\boldsymbol{\mu}^{k+1} - \boldsymbol{\lambda}^{k}\|_{2}^{2} = 0,
\lim_{k \to +\infty} \|\boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^{k+1}\|_{2}^{2} = 0, \qquad \lim_{k \to +\infty} \|\boldsymbol{v}^{k+1} - \boldsymbol{\gamma}^{k}\|_{2}^{2} = 0. \tag{125}$$

Additionally, (124) also implies that $\{(\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \gamma^k)\}$ is a bounded sequence, and there exists a sub-sequence $\{(\mathbf{x}^{k_j}, \mathbf{u}^{k_j}, \lambda^{k_j}, \gamma^{k_j})\}$ that converges to a limit point $(\mathbf{x}^{\infty}, \mathbf{u}^{\infty}, \lambda^{\infty}, \gamma^{\infty})$. We next show that the limit point is indeed a saddle point and is also the unique limit point of $\{(\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \gamma^k)\}$. Given any $\mathbf{x} \in \mathbb{X}$ and $\mathbf{u} \in \mathbb{R}^{n_2}$, we have:



$$\begin{split} &2\rho^{k+1} \left[\mathcal{L}(\mathbf{x}^{k+1}, \mathbf{u}^{k+1}, \mu^{k+1}, \nu^{k+1}) - \mathcal{L}(\mathbf{x}, \mathbf{u}, \mu^{k+1}, \nu^{k+1}) \right] \\ &= 2\rho^{k+1} \left\{ \left[\frac{1}{2} (\mathbf{x}^{k+1})^T P_0 \mathbf{x}^{k+1} - \frac{1}{2} \mathbf{x}^T P_0 \mathbf{x} \right] + \mathbf{q}_0^T (\mathbf{x}^{k+1} - \mathbf{x}) + \mathbf{c}_0^T (\mathbf{u}^{k+1} - \mathbf{u}) \right. \\ &+ \sum_{i=1}^{m_1} \mu_i^{k+1} \left[\frac{1}{2} (\mathbf{x}^{k+1})^T P_i \mathbf{x}^{k+1} - \frac{1}{2} \mathbf{x}^T P_i \mathbf{x} \right] + \sum_{i=1}^{m_1} \mu_i^{k+1} \mathbf{q}_i^T (\mathbf{x}^{k+1} - \mathbf{x}) \\ &+ \sum_{i=1}^{m_1} \mu_i^{k+1} \mathbf{c}_i^T (\mathbf{u}^{k+1} - \mathbf{u}) \\ &+ \nu^{k+1} A (\mathbf{x}^{k+1} - \mathbf{x}) + \nu^{k+1} B (\mathbf{u}^{k+1} - \mathbf{u}) \right\} \\ &= 2\rho^{k+1} \left[-\frac{1}{2} (\mathbf{x}^{k+1} - \mathbf{x})^T P_0 (\mathbf{x}^{k+1} - \mathbf{x}) - \sum_{i=1}^{m_1} \mu_i^{k+1} \frac{1}{2} (\mathbf{x}^{k+1} - \mathbf{x})^T P_i (\mathbf{x}^{k+1} - \mathbf{x}) \right] \\ &+ 2\rho^{k+1} \left[(P_0 \mathbf{x}^{k+1} + \mathbf{q}_0)^T (\mathbf{x}^{k+1} - \mathbf{x}) + \mathbf{c}_0^T (\mathbf{u}^{k+1} - \mathbf{u}) \right. \\ &+ \sum_{i=1}^{m_1} \mu_i^{k+1} (P_i \mathbf{x}^{k+1} + \mathbf{q}_i)^T (\mathbf{x}^{k+1} - \mathbf{x}) + \sum_{i=1}^{m_1} \mu_i^{k+1} \mathbf{c}_i^T (\mathbf{u}^{k+1} - \mathbf{u}) \\ &+ \nu^{k+1} A (\mathbf{x}^{k+1} - \mathbf{x}) + \nu^{k+1} B (\mathbf{u}^{k+1} - \mathbf{u}) \right] \\ &\leq 2\rho^{k+1} \left[(P_0 \mathbf{x}^{k+1} + \mathbf{q}_0)^T (\mathbf{x}^{k+1} - \mathbf{x}) + \mathbf{c}_0^T (\mathbf{u}^{k+1} - \mathbf{u}) \\ &+ \sum_{i=1}^{m_1} \mu_i^{k+1} (P_i \mathbf{x}^{k+1} + \mathbf{q}_i)^T (\mathbf{x}^{k+1} - \mathbf{x}) + \sum_{i=1}^{m_1} \mu_i^{k+1} \mathbf{c}_i^T (\mathbf{u}^{k+1} - \mathbf{u}) \\ &+ \nu^{k+1} A (\mathbf{x}^{k+1} - \mathbf{x}) + \nu^{k+1} B (\mathbf{u}^{k+1} - \mathbf{u}) \right]. \end{split}$$

The positive semi-definiteness of each P_i for all $i=0,1,\ldots,m$ guarantees the non-positiveness of (Δ) , which makes the last inequality hold. Applying Lemma 1 on (22c) with $\hat{\mathbf{z}} = (\mathbf{x}^{k+1}, \mathbf{u}^{k+1})$, $\bar{\mathbf{z}} = (\mathbf{x}^k, \mathbf{u}^k)$ and $\mathbf{z} = (\mathbf{x}, \mathbf{u})$ yields:



$$\begin{aligned} &2\rho^{k+1} \bigg\{ (P_{0}\mathbf{y}^{k+1} + \mathbf{q}_{0})^{T}\mathbf{x}^{k+1} + \mathbf{c}_{0}^{T}\mathbf{u}^{k+1} + r_{0} \\ &+ \sum_{i=1}^{m_{1}} \mu_{i}^{k+1} \Big[(P_{i}\mathbf{y}^{k+1} + \mathbf{q}_{i})^{T}\mathbf{x}^{k+1} + \mathbf{c}_{i}^{T}\mathbf{u}^{k+1} + r_{i} \Big] \\ &+ \mathbf{v}^{k+1} (A\mathbf{x}^{k+1} + B\mathbf{u}^{k+1} - \mathbf{b}) \bigg\} \\ &- 2\rho^{k+1} \bigg\{ (P_{0}\mathbf{y}^{k+1} + \mathbf{q}_{0})^{T}\mathbf{x} + \mathbf{c}_{0}^{T}\mathbf{u} + r_{0} \\ &+ \sum_{i=1}^{m} \mu_{i}^{k+1} \Big[(P_{i}\mathbf{y}^{k+1} + \mathbf{q}_{i})^{T}\mathbf{x} + \mathbf{c}_{i}^{T}\mathbf{u} + r_{i} \Big] \\ &+ \mathbf{v}^{k+1} (A\mathbf{x} + B\mathbf{u} - \mathbf{b}) \bigg\} \\ &\leq \|\mathbf{x}^{k} - \mathbf{x}\|_{2}^{2} - \|\mathbf{x}^{k+1} - \mathbf{x}\|_{2}^{2} - \|\mathbf{x}^{k+1} - \mathbf{x}^{k}\|_{2}^{2} \\ &+ \|\mathbf{u}^{k} - \mathbf{u}\|_{2}^{2} - \|\mathbf{u}^{k+1} - \mathbf{u}\|_{2}^{2} - \|\mathbf{u}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} \\ &\leq \left(\|\mathbf{x}^{k} - \mathbf{x}^{k+1}\|_{2}^{2} + \|\mathbf{x}^{k+1} - \mathbf{x}\|_{2}^{2} \right) - \|\mathbf{x}^{k+1} - \mathbf{x}\|_{2}^{2} - \|\mathbf{u}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} \\ &+ \left(\|\mathbf{u}^{k} - \mathbf{u}^{k+1}\|_{2}^{2} + \|\mathbf{u}^{k+1} - \mathbf{u}\|_{2}^{2} \right) - \|\mathbf{u}^{k+1} - \mathbf{u}\|_{2}^{2} - \|\mathbf{u}^{k+1} - \mathbf{u}^{k}\|_{2}^{2} = 0. \end{aligned} \tag{127}$$

Adding the above two inequalities yields

$$2\rho^{k+1} \left[\mathcal{L}(\mathbf{x}^{k+1}, \mathbf{u}^{k+1}, \boldsymbol{\mu}^{k+1}, \boldsymbol{\nu}^{k+1}) - \mathcal{L}(\mathbf{x}, \mathbf{u}, \boldsymbol{\mu}^{k+1}, \boldsymbol{\nu}^{k+1}) \right]$$

$$+ 2\rho^{k+1} \left\{ (\mathbf{y}^{k+1} - \mathbf{x}^{k+1})^T P_0(\mathbf{x}^{k+1} - \mathbf{x}) \right.$$

$$+ \sum_{i=1}^m \mu_i^{k+1} \left[(\mathbf{y}^{k+1} - \mathbf{x}^{k+1})^T P_i(\mathbf{x}^{k+1} - \mathbf{x}) \right] \right\} \le 0.$$
(128)

Taking the limits over an appropriate sub-sequence $\{k_j\}$ on both sides and using (125), we have:

$$\mathcal{L}(\mathbf{x}^{\infty}, \mathbf{u}^{\infty}, \lambda^{\infty}, \gamma^{\infty}) \le \mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda^{\infty}, \gamma^{\infty}), \quad \forall \mathbf{x} \in \mathbb{X}, \forall \mathbf{u} \in \mathbb{R}^{n_2}.$$
 (129)

Similarly, given any $\lambda \in \mathbb{R}^{m_1}_+$ and $\gamma \in \mathbb{R}^{m_2}_+$, applying Lemma 1 on (22d) with $\hat{\mathbf{z}} = (\lambda^{k+1}, \gamma^{k+1}), \bar{\mathbf{z}} = (\lambda^k, \gamma^k)$ and $\mathbf{z} = (\lambda, \gamma)$ yields

$$\begin{split} &2\rho^{k+1} \left[\mathcal{L}(\mathbf{y}^{k+1}, \mathbf{v}^{k+1}, \lambda, \gamma) - \mathcal{L}(\mathbf{y}^{k+1}, \mathbf{v}^{k+1}, \lambda^{k+1}, \gamma^{k+1}) \right] \\ &\leq \|\lambda^{k} - \lambda\|_{2}^{2} - \|\lambda^{k+1} - \lambda\|_{2}^{2} - \|\lambda^{k+1} - \lambda^{k}\|_{2}^{2} \\ &+ \|\gamma^{k} - \gamma\|_{2}^{2} - \|\gamma^{k+1} - \gamma\|_{2}^{2} - \|\gamma^{k+1} - \gamma^{k}\|_{2}^{2} \\ &\leq \left(\|\lambda^{k} - \lambda^{k+1}\|_{2}^{2} - \|\lambda^{k+1} + \lambda\|_{2}^{2} \right) - \|\lambda^{k+1} - \lambda\|_{2}^{2} - \|\lambda^{k+1} - \lambda^{k}\|_{2}^{2} \\ &+ \left(\|\gamma^{k} - \gamma^{k+1}\|_{2}^{2} - \|\gamma^{k+1} + \gamma\|_{2}^{2} \right) - \|\gamma^{k+1} - \gamma\|_{2}^{2} - \|\gamma^{k+1} - \gamma^{k}\|_{2}^{2} = 0. \end{split}$$

$$\tag{130}$$

Taking the limits over an appropriate sub-sequence $\{k_j\}$ on both sides and using (125), we have:



$$\mathcal{L}(\mathbf{x}^{\infty}, \mathbf{u}^{\infty}, \lambda, \gamma) \le \mathcal{L}(\mathbf{x}^{\infty}, \mathbf{u}^{\infty}, \lambda^{\infty}, \gamma^{\infty}), \quad \forall \lambda \in \mathbb{R}^{m_1}_+, \forall \gamma \in \mathbb{R}^{m_2}.$$
 (131)

Therefore, we show that $(\mathbf{x}^{\infty}, \mathbf{u}^{\infty}, \lambda^{\infty}, \gamma^{\infty})$ is indeed a saddle point of the Lagrangian function $\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda, \gamma)$. Then (124) implies that

$$\lim_{k \to +\infty} \|\mathbf{x}^k - \mathbf{x}^{\infty}\|_2^2 + \|\mathbf{u}^k - \mathbf{u}^{\infty}\|_2^2 + \|\lambda^k - \lambda^{\infty}\|_2^2 + \|\gamma^k - \gamma^{\infty}\|_2^2 = \xi.$$
 (132)

Since we have argued (after Eq. (125)) that there exists a bounded sequence of $\{(\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \boldsymbol{\gamma}^k)\}$ that converges to $(\mathbf{x}^\infty, \mathbf{u}^\infty, \lambda^\infty, \boldsymbol{\gamma}^\infty)$; that is, there exists $\{k_j\}$ such that $\lim_{k_j \to +\infty} \|\mathbf{x}^{k_j} - \mathbf{x}^\infty\|_2^2 + \|\mathbf{u}^{k_j} - \mathbf{u}^\infty\|_2^2 + \|\lambda^{k_j} - \lambda^\infty\|_2^2 + \|\boldsymbol{\gamma}^{k_j} - \boldsymbol{\gamma}^\infty\|_2^2 = 0$, which then implies that $\xi = 0$. Therefore, we show that $\{(\mathbf{x}^k, \mathbf{u}^k, \lambda^k, \boldsymbol{\gamma}^k)\}$ converges globally to a saddle point $(\mathbf{x}^\infty, \mathbf{u}^\infty, \lambda^\infty, \boldsymbol{\gamma}^\infty)$.

Acknowledgements The authors would like to acknowledge the support of National Science Foundation Grant CMMI-1832688 and the Emerging Frontiers grant from the School of Industrial Engineering at Purdue University. Specially, we wish to thank Professor Jong-Shi Pang of University of Southern California for the helpful comments and discussions. In addition, we would like to thank Purdue Rosen Center for Advanced Computing for providing the computing resources and technical support.

References

- Aholt, C., Agarwal, S., Thomas, R.: A QCQP approach to triangulation. In: European Conference on Computer Vision. Springer, pp. 654–667 (2012)
- Basu, K., Saha, A., Chatterjee, S.: Large-scale quadratically constrained quadratic program via lowdiscrepancy sequences. In: Advances in Neural Information Processing Systems, pp. 2297–2307 (2017)
- Bose, S., Gayme, D.F., Chandy, K.M., Low, S.H.: Quadratically constrained quadratic programs on acyclic graphs with application to power flow. IEEE Trans. Control Netw. Syst. 2(3), 278–287 (2015)
- 4. Breiman, L., et al.: Arcing classifier. Ann. Stat. **26**(3), 801–849 (1998)
- Chatterjee, S., Saha, A., Basu, K.: Constrained multi-slot optimization for ranking recommendations (2018). US Patent App. 15/400738
- Chen, G., Teboulle, M.: A proximal-based decomposition method for convex minimization problems. Math. Program. 64(1–3), 81–101 (1994)
- Golub, G.H., Van Loan, C.F.: Matrix Computations. Johns Hopkins University Press, Baltimore (2013)
- 8. Hastie, T., Tibshirani, R., Friedman, J.: The Elements of Statistical Learning: Data Mining, Inference, and Prediction. Springer, Berlin (2009)
- 9. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (2012)
- Huang, K., Sidiropoulos, N.D.: Consensus-ADMM for general quadratically constrained quadratic programming. IEEE Trans. Signal Process. 64(20), 5297–5310 (2016)
- Huang, Y., Palomar, D.P.: Randomized algorithms for optimal solutions of double-sided QCQP with applications in signal processing. IEEE Trans. Signal Process. 62(5), 1093–1108 (2014)
- IBM ILOG CPLEX optimization studio CPLEX Parameters Reference, Version 12 Release 8 (1987–2017)
- 13. IBM ILOG CPLEX optimization studio CPLEX User's Manual, Version 12 Release 7 (1987–2017)
- Kalbat, A., Lavaei, J.: A fast distributed algorithm for decomposable semidefinite programs. In: 54th IEEE Conference on Decision and Control, pp. 1742–1749 (2015)
- Lancaster, P., Farahat, H.K.: Norms on direct sums and tensor products. Math. Comput. 26(118), 401–414 (1972)



16. Lanckriet, G.R., Cristianini, N., Bartlett, P., Ghaoui, L.E., Jordan, M.I.: Learning the kernel matrix with semidefinite programming. J. Mach. Learn. Res. 5(Jan), 27–72 (2004)

- 17. Lobo, M.S., Vandenberghe, L., Boyd, S., Lebret, H.: Applications of second-order cone programming. Linear Algebra Appl. **284**(1–3), 193–228 (1998)
- Nemirovski, A.: Interior point polynomial time methods in convex programming. Lecture Notes (2004)
- Nesterov, Y., Nemirovskii, A.: Interior-Point Polynomial Algorithms in Convex Programming. SIAM, Philadelphia (1994)
- O'donoghue, B., Chu, E., Parikh, N., Boyd, S.: Conic optimization via operator splitting and homogeneous self-dual embedding. J. Optim. Theory Appl. 169(3), 1042–1068 (2016)
- Pakazad, S.K., Hansson, A., Andersen, M.S., Rantzer, A.: Distributed semidefinite programming with application to large-scale system analysis. IEEE Trans. Autom. Control 63(4), 1045–1058 (2018)
- Rabaste, O., Savy, L.: Mismatched filter optimization for radar applications using quadratically constrained quadratic programs. IEEE Trans. Aerosp. Electron. Syst. 51(4), 3107–3122 (2015)
- 23. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (2015)

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