ANALYSIS & PDE

Volume 13 No. 6 2020

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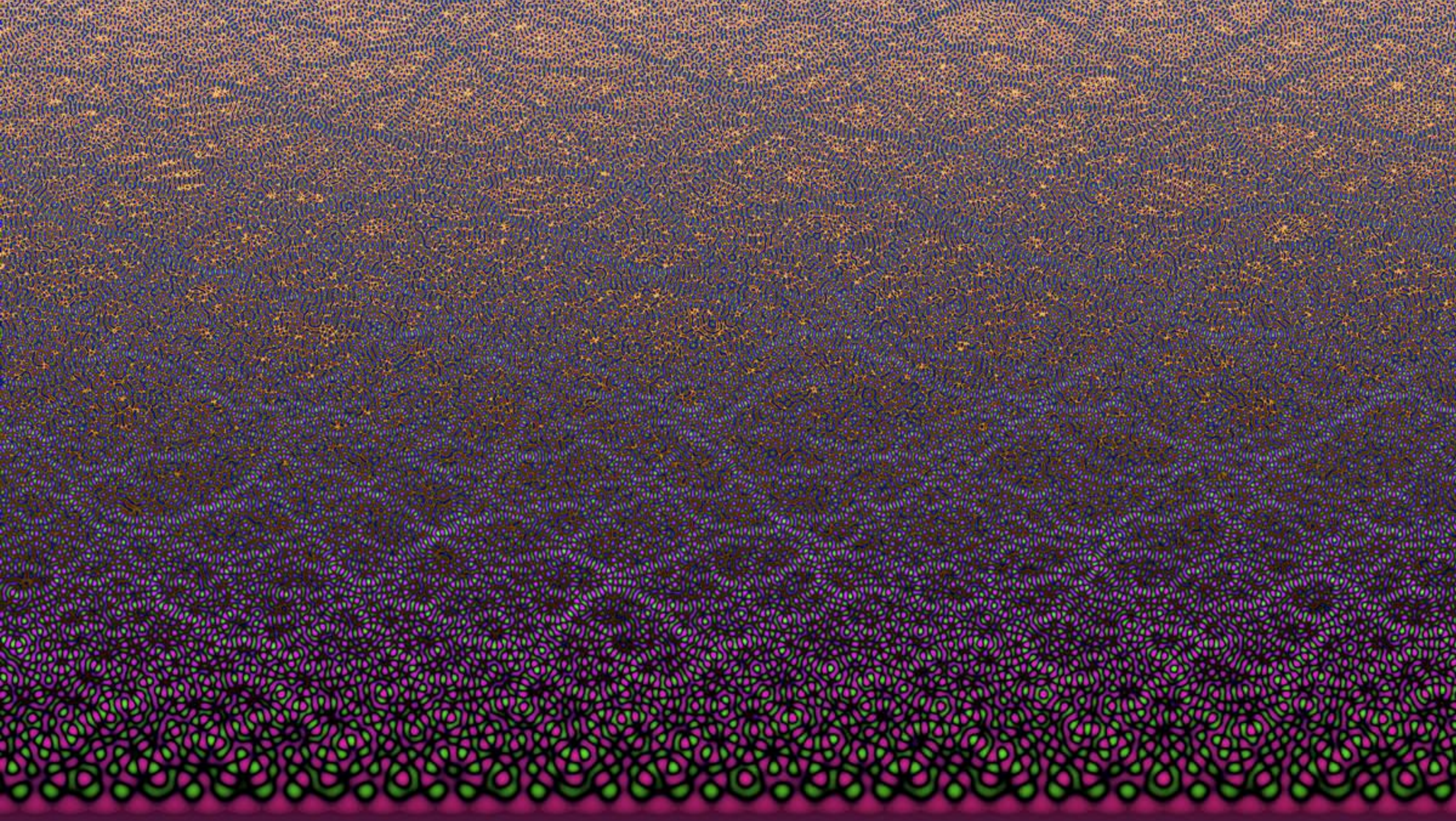

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ANALYSIS AND PDE
Vol. 13, No. 6, 2020
https://doi.org/10.2140/apde.2020.13.1633

EIGENVALUE BOUNDS FOR NON-SELF-ADJOINT SCHRÖDINGER OPERATORS WITH NONTRAPPING METRICS

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We study eigenvalues of non-self-adjoint Schrödinger operators on nontrapping asymptotically conic manifolds of dimension $n \geq 3$. Specifically, we are concerned with the following two types of estimates. The first one deals with Keller-type bounds on individual eigenvalues of the Schrödinger operator with a complex potential in terms of the L^p-norm of the potential, while the second one is a Lieb–Thirring-type bound controlling sums of powers of eigenvalues in terms of the L^p-norm of the potential. We extend the results of Frank (2011), Frank and Sabin (2017), and Frank and Simon (2017) on the Keller- and Lieb–Thirring-type bounds from the case of Euclidean spaces to that of nontrapping asymptotically conic manifolds. In particular, our results are valid for the operator $\Delta_g + V$ on $\mathbb{R}^n$ with g being a nontrapping compactly supported (or suitably short-range) perturbation of the Euclidean metric and $V \in L^p$ complex-valued.

1. Introduction and statement of results

The purpose of this paper is to establish bounds of Keller- and Lieb–Thirring-type for eigenvalues of non-self-adjoint Schrödinger operators on nontrapping asymptotically conic manifolds. Before stating our results, let us proceed to describe these two types of bounds in the more familiar Euclidean setting, motivating the significance of extending them to the case of asymptotically conic manifolds.

1A. ***Keller- and Lieb–Thirring-type bounds in the Euclidean case.*** Recently there have been numerous works devoted to the study of eigenvalues of the Schrödinger operator $\mathcal{P} = \Delta + V$ in $L^2(\mathbb{R}^n)$, with Δ being the nonnegative Laplace operator and V being a complex-valued potential. Of particular interest here is the problem of obtaining quantitative information concerning the localization and distribution of the eigenvalues of $\mathcal{P}$ under the sole assumption that $V \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$. Here we may note that the spectrum of $\mathcal{P}$ in $\mathbb{C} \setminus [0, \infty)$ consists then of isolated eigenvalues of finite algebraic multiplicity; see [Frank 2018, Proposition B.2].

The following two types of results are of particular interest for this problem. The first one deals with Keller-type bounds [1961] on the individual eigenvalues of $\mathcal{P}$ in terms of the L^p-norm of the potential. If V is real-valued, so that $\mathcal{P}$ admits a natural self-adjoint realization, then the eigenvalues of $\mathcal{P}$ in $\mathbb{C} \setminus [0, \infty)$

MSC2010: 35P15, 42B37, 58J40, 58J50.
Keywords: non-self-adjoint Schrödinger operators, eigenvalue bounds, asymptotically conic manifolds.

are negative and by the variational principle and Sobolev's inequalities, for any eigenvalue $\lambda < 0$ of $\mathcal{P}$, we have the scale-invariant bounds

$$|\lambda|^\gamma \le C_{\gamma,n} \int_{\mathbb{R}^n} |V(x)|^{\gamma+\frac{n}{2}}\, dx \tag{1-1}$$

for every $\gamma \ge \frac{1}{2}$ if $n = 1$ and every $\gamma > 0$ if $n \ge 2$. Here the constant $C_{\gamma,n} > 0$ depends on γ and n only; see [Frank and Simon 2017; Keller 1961; Lieb and Thirring 1976].

If the potential V is complex-valued, the problem is more involved due to the lack of variational techniques and the absence of a spectral resolution theorem. In dimension $n = 1$ the bound (1-1) with $\gamma = \frac{1}{2}$ was proved by Abramov, Aslanyan, and Davies [Abramov et al. 2001]. In dimensions $n \ge 2$, Frank [2011] established the bound (1-1) for all eigenvalues $\lambda \in \mathbb{C} \setminus [0,\infty)$ and for all $0 < \gamma \le \frac{1}{2}$; see also [Frank and Simon 2017]. The work [Frank 2018] gives a replacement of the bound (1-1) for all $\gamma > \frac{1}{2}$. We refer to [Cuenin 2017; Cuenin and Kenig 2017; Enblom 2016; Laptev and Safronov 2009; Mizutani 2016] for some other recent works on bounds on the individual eigenvalues for non-self-adjoint operators of Schrödinger type.

The second type of result is concerned with bounds on sums of powers of absolute values of eigenvalues of $\mathcal{P}$, generalizing the classical Lieb–Thirring bounds [1976] to the non-self-adjoint case. If V is real-valued then the Lieb–Thirring inequality has the form

$$\sum |\lambda|^\gamma \le C_{\gamma,n} \int_{\mathbb{R}^n} V_-(x)^{\gamma+\frac{n}{2}}\, dx, \tag{1-2}$$

where $V_- = \max(-V, 0)$, $\gamma \ge \frac{1}{2}$ if $n = 1$, $\gamma > 0$ if $n = 2$, and $\gamma \ge 0$ if $n \ge 3$. The summation in the left-hand side in (1-2) extends over all negative eigenvalues of $\mathcal{P}$, counted with their multiplicities. The situation in the non-self-adjoint case is less clear. In particular, Bögli [2017] established that for any $p > n$, there exists a nonreal potential $V \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that the Schrödinger operator $\mathcal{P}$ has infinitely many nonreal eigenvalues accumulating at every point of the essential spectrum $[0,\infty)$, thus showing that inequalities like (1-2) cannot hold in the non-self-adjoint case for $p > n$. A possible modification of the Lieb–Thirring inequality (1-2) to the non-self-adjoint case was suggested in [Demuth et al. 2013b]:

$$\sum \frac{d(\lambda)^{\gamma+\frac{n}{2}}}{|\lambda|^{\frac{n}{2}}} \le C_{\gamma,n} \int_{\mathbb{R}^n} |V(x)|^{\gamma+\frac{n}{2}}\, dx, \tag{1-3}$$

where

$$d(\lambda) = \operatorname{dist}(\lambda, [0,\infty)). \tag{1-4}$$

We refer to [Demuth et al. 2009; 2013a; Frank et al. 2006; Frank and Sabin 2017; Sambou 2014] for some of the important contributions to generalizations of the Lieb–Thirring inequality (1-2) to the setting of complex potentials.

A crucial idea of Frank [2011] in establishing bounds (1-1) on the individual eigenvalues of the Schrödinger operator $\mathcal{P}$ with a complex-valued potential was to make use of the uniform L^p resolvent estimates for Δ of Kenig, Ruiz, and Sogge [Kenig et al. 1987]. Recently, this approach was extended to the case of non-self-adjoint Schrödinger operators with inverse-square potentials in [Mizutani 2019], to the case of magnetic Schrödinger and Pauli operators with complex electromagnetic potentials in [Cuenin and Kenig 2017], and to the case of the Dirac and fractional Schrödinger operators with complex potentials in [Cuenin 2017].

Developing the idea of [Frank 2011] further, Frank and Sabin [2017] obtained some very interesting uniform weighted bounds for the resolvent of Δ in suitable Schatten classes, and applied these bounds to derive uniform estimates on the sums of eigenvalues of non-self-adjoint Schrödinger operators, thus obtaining some results towards proving the conjectured Lieb–Thirring inequality (1-3) in the case of complex potentials. Recently, this approach was extended in [Cuenin 2017] to the case of the Dirac and fractional Schrödinger operators with complex potentials.

1B. ***Asymptotically conic manifolds.*** Notice that in all the works described above the principal part of the operators considered has constant coefficients. It is nevertheless of significant interest to extend both types of results to the case of complex potential perturbations of the Laplace–Beltrami operator Δ_g considered on $\mathbb{R}^n$ or more generally, on a class of complete noncompact Riemannian manifolds.

The class we consider here is the class of *asymptotically conic manifolds*, whose Riemannian metric outside a compact set is asymptotic to the end of a metric cone. Metric cones are Riemannian manifolds of the form $N \times (0,\infty)_r$ with metric $dr^2 + r^2 G$ for some metric G on N. They were studied in [Cheeger 1983; Cheeger and Taylor 1982] but have a long history going back to [Sommerfeld 1896]. As defined by Melrose [1994] (who used the term "scattering metric"), (M, g) is *asymptotically conic* if M is the interior of a smooth compact manifold with boundary $\overline{M}$ and g is a smooth metric on M satisfying the following property: there exists a smooth boundary-defining function[1] x on $\overline{M}$ such that (M, g) is isometric outside a compact set to a collar $(0,\varepsilon)_x \times \partial\overline{M}$ equipped with the metric of the form

$$\frac{dx^2}{x^4} + \frac{h(x)}{x^2} = \frac{dx^2}{x^4} + \frac{\sum_{j,k} h_{jk}(x,y)\, dy_j\, dy_k}{x^2} \tag{1-5}$$

for some smooth one-parameter family of metrics h on the boundary $\partial\overline{M}$. If $y = (y_1, \dots, y_{n-1})$ stands for local coordinates on $\partial\overline{M}$ and (x, y) are the corresponding local coordinates on M near $\partial\overline{M}$, the function $r = \frac{1}{x}$ near $x = 0$ can be thought of as a "radial" variable near infinity and $y = (y_1, \dots, y_{n-1})$ can be regarded as $n-1$ "angular" variables. Rewriting (1-5) in the (r, y)-coordinates, we have

$$g = dr^2 + r^2 h(r^{-1}) = dr^2 + r^2 \sum h_{jk}(r^{-1}, y)\, dy^j\, dy^k, \tag{1-6}$$

and we observe that the metric g is asymptotic to an exact conic metric $dr^2 + r^2 h(0)$ on $(r_0,\infty)_r \times \partial\overline{M}$ as $r \to \infty$. The most important example of an asymptotically conic manifold is Euclidean space $M = \mathbb{R}^n$ equipped with a short-range perturbation of the Euclidean metric (δ_{ij}), which is of the form

$$g_{ij} = \delta_{ij} + |z|^{-2} k_{ij}\left(\frac{z}{|z|}, \frac{1}{|z|}\right), \quad |z| \to \infty, \tag{1-7}$$

where k_{ij} are smooth on $\mathbb{S}^{n-1} \times [0, 1)$; see [Melrose and Zworski 1996].

[1] A boundary-defining function is a nonnegative smooth function x such that $\partial\overline{M} = x^{-1}(\{0\})$ and $dx|_{\partial M}$ does not vanish on $\partial\overline{M}$.

Let $z = (z_1, \dots, z_n)$ be local coordinates away from $\partial \overline{M}$. We say that M is *nontrapping* if every geodesic $z(s)$ in M reaches $\partial \overline{M}$ as $s \to \pm\infty$. This places restrictions on the compactification $\overline{M}$. For example, a compact perturbation of the Euclidean metric is nontrapping provided that it is sufficiently small in C^2; see [Hassell et al. 2006]. However, a nontrapping asymptotically conic metric g may be far from asymptotically Euclidean. Indeed, there is such a nontrapping metric g on $\mathbb{R}^n$ for every limiting metric $h(0)$ on the sphere $\mathbb{S}^{n-1}$, identified with $\partial \overline{M}$ in this case.

In terms of the Weyl calculus, the symbol of the Laplacian for an asymptotically conic metric on $\mathbb{R}^n$ is in the calculus corresponding to the metric on $T^*\mathbb{R}^n$

$$\frac{dz^2}{\langle z \rangle^2} + \frac{d\xi^2}{\langle \xi \rangle^2}.$$

This class of symbols was studied by Parenti [1972], Cordes [1976], Schrohe [1987], Hörmander [1985, Equation (19.3.11) and Theorem 19.3.1′] and others. Melrose [1994] adopted a different point of view, working from the outset on the compactification $\overline{M}$ (which can be any manifold with boundary) and introducing the *scattering calculus* as the natural class of pseudodifferential operators associated with the *scattering Lie algebra* of vector fields on $\overline{M}$. He seems to have been the first to exploit the fact that in this calculus one has propagation of singularities at spatial infinity *at all finite frequencies*. Using the scattering calculus, the second author in collaboration with Vasy, Wunsch, the first author, and Sikora, worked out detailed properties of the spectral measure; see [Guillarmou et al. 2013a; Hassell and Vasy 2001; Hassell and Wunsch 2008].

Let us remark on why we elect to work with the class of nontrapping asymptotically conic manifolds. On the one hand, it is a sufficiently *general* class which includes compactly supported or suitable short-range perturbations of Euclidean space as well as geometrically interesting examples such as metrics with strictly negative curvature, which are not present in the class of asymptotically Euclidean manifolds. On the other hand, it is sufficiently *restricted* to allow us to obtain detailed results on the resolvent and spectral measure, analogous in some sense to that for flat Euclidean space.

1C. *Main results.* Throughout the paper, we let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Since g is complete, the Laplacian Δ_g associated with the metric g is nonnegative self-adjoint on $L^2(M)$ with domain $H^2(M)$. The spectrum of Δ_g is purely absolutely continuous and is given by $\mathrm{Spec}(\Delta_g) = [0, \infty)$: the absence of singular continuous spectrum follows for example from [Froese and Hislop 1989] using a Mourre estimate, and the absence of embedded L^2-eigenvalues follows from adapting [Hörmander 1985, Theorem 17.2.8] as in [Melrose 1994, Section 10].

Our starting point is the following uniform L^p resolvent estimate of Kenig–Ruiz–Sogge-type for the Laplace operator Δ_g on an asymptotically conic nontrapping manifold, established in [Guillarmou and Hassell 2014].

Theorem 1. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Then for all $p \in \left[\frac{2n}{n+2}, \frac{2(n+1)}{n+3}\right]$, there is a constant $C > 0$ such that for all $z \in \mathbb{C}$ and for all $f \in L^p(M)$, we have*

$$\|(\Delta_g - z)^{-1} f\|_{L^{p'}(M)} \leq C|z|^{n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \|f\|_{L^p(M)}. \tag{1-8}$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$.

As explained in [Guillarmou and Hassell 2014], when $z \in (0, +\infty)$, the operator in (1-8) may be taken to be either the outgoing or incoming resolvent $(\Delta_g - (z \pm i0))^{-1}$, defined by

$$(\Delta_g - (z \pm i0))^{-1} = \lim_{\delta \to 0^+} (\Delta_g - (z \pm i\delta))^{-1}$$

as a map $x^{1/2+\varepsilon} L^2(M) \to x^{-1/2-\varepsilon} L^2(M)$ for all $\varepsilon > 0$, where x is the boundary-defining function, thanks to the limiting absorption principle; see [Hassell and Vasy 2001; Melrose 1994] for details.

The main technical contribution of the present paper is the following weighted uniform Schatten class estimate for the resolvent of Δ_g, generalizing [Frank and Sabin 2017, Theorem 12], obtained in the Euclidean setting. This result is the key ingredient which allows us to extend the Lieb–Thirring-type bounds of [Frank and Sabin 2017; Frank 2018] to our setting. Below, $\mathcal{C}_q(L^2(M))$ denotes the Schatten space of order q (see Section 2A for definition).

Theorem 2. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Let $p \in \left[\frac{n}{2}, \frac{n+1}{2}\right]$. Then there exists $C > 0$ such that for all $z \in \mathbb{C} \setminus \{0\}$ and all $W_1, W_2 \in L^{2p}(M)$, we have $W_1(\Delta_g - z)^{-1} W_2 \in \mathcal{C}_q(L^2(M))$, $q = \frac{p(n-1)}{n-p} \in [n-1, n+1]$, and*

$$\|W_1(\Delta_g - z)^{-1} W_2\|_{\mathcal{C}_q(L^2(M))} \leq C|z|^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{1-9}$$

Remark 1.1. When $z \in (0, +\infty)$, the operator in (1-9) may be taken to be either the outgoing or incoming resolvent $(\Delta_g - (z \pm i0))^{-1}$.

In what follows we shall write $E_{\sqrt{\Delta_g}}(\lambda) = 1_{(-\infty,\lambda)}(\sqrt{\Delta_g})$, $\lambda > 0$, for the spectral projection of $\sqrt{\Delta_g}$, and remark that the spectral measures $d(E_{\sqrt{\Delta_g}}(\lambda)u, u)_{L^2(M)}$ are absolutely continuous with respect to the Lebesgue measure for any $u \in L^2(M)$. Let us write

$$dE_{\sqrt{\Delta_g}}(\lambda) := \frac{d}{d\lambda} E_{\sqrt{\Delta_g}}(\lambda).$$

The proof of Theorem 2 is based on the following weighted Schatten norm estimates on the spectral measure $dE_{\sqrt{\Delta_g}}(\lambda)$ of $\sqrt{\Delta_g}$, which extend the corresponding estimates of [Frank and Sabin 2017, Theorem 2], obtained in the Euclidean setting. We believe that these estimates may be of some independent interest.

Theorem 3. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Let $p \in \left[1, \frac{n+1}{2}\right]$. Then there exists $C > 0$ such that for all $\lambda > 0$ and all $W_1, W_2 \in L^{2p}(M)$, we have $W_1 dE_{\sqrt{\Delta_g}}(\lambda) W_2 \in \mathcal{C}_q(L^2(M))$, $q = \frac{p(n-1)}{n-p} \in [1, n+1]$, and*

$$\|W_1 dE_{\sqrt{\Delta_g}}(\lambda) W_2\|_{\mathcal{C}_q(L^2(M))} \leq C\lambda^{-1+\frac{n}{p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{1-10}$$

Remark 1.2. If the nontrapping assumption is dropped, the estimates in Theorem 3, and therefore also Theorem 2, may fail. Instead, the estimates will hold for all $\lambda \leq \lambda_0$ for a constant C which depends on λ_0. A "metric bottle" example illustrating this, for which the best $C(\lambda_0)$ grows exponentially in λ_0, is given in [Guillarmou et al. 2013b, Remark 8.8].

Let us now consider the Schrödinger operator $\Delta_g + V$ with a complex-valued potential $V \in L^p(M)$, $\frac{n}{2} \leq p < \infty$. As explained in Section 6, this operator has a natural m-sectorial realization on $L^2(M)$, and the spectrum of $\Delta_g + V$ in $\mathbb{C} \setminus [0, \infty)$ consists of isolated eigenvalues of finite algebraic multiplicity.

As an application of Theorem 1, we have the following generalization of the results of [Frank 2011; 2018; Frank and Simon 2017] concerning Keller-type bounds on the individual eigenvalues of non-self-adjoint Schrödinger operators in the Euclidean setting to that of an asymptotically conic nontrapping manifold; see also [Fanelli et al. 2018].

Theorem 4. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$:*

(i) *Let $V \in L^{\gamma+n/2}(M)$ for some $0 < \gamma \leq \frac{1}{2}$. Then any eigenvalue $\lambda \in \mathbb{C}$ of the operator $\Delta_g + V$ satisfies*

$$|\lambda|^{\gamma} \leq C_{\gamma,n} \|V\|^{\gamma+\frac{n}{2}}_{L^{\gamma+n/2}(M)}, \tag{1-11}$$

where the constant $C_{\gamma,n} > 0$ depends on γ and n only.

(ii) *If $V \in L^{n/2}(M)$ is such that $\|V\|_{L^{n/2}(M)}$ is sufficiently small, then the operator $\Delta_g + V$ has no eigenvalues.*

(iii) *Let $V \in L^{\gamma+n/2}(M)$ for some $\gamma > \frac{1}{2}$. Then any eigenvalue $\lambda \in \mathbb{C}$ of the operator $\Delta_g + V$ satisfies*

$$d(\lambda)^{\gamma-\frac{1}{2}} |\lambda|^{\frac{1}{2}} \leq C_{\gamma,n} \|V\|^{\gamma+\frac{n}{2}}_{L^{\gamma+n/2}(M)}, \tag{1-12}$$

where $d(\lambda)$ is given by (1-4) *and the constant $C_{\gamma,n} > 0$ depends on γ and n only.*

Remark 1.3. Parts (i) and (ii) of Theorem 4 have been established in [Guillarmou and Hassell 2014, Proposition 7.2] without specifying the radius of the disk containing the eigenvalues of $\Delta_g + V$ in part (i).

As a consequence of Theorem 2, we obtain the following analogue of [Frank and Sabin 2017, Theorem 16], concerning Lieb–Thirring-type inequalities for the sums of eigenvalues of $\Delta_g + V$ in the case of a short-range potential $V \in L^p(M)$, $p = \frac{n}{2} + \gamma$, where $0 \leq \gamma \leq \frac{1}{2}$.

Theorem 5. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$, and let $V \in L^p(M)$ with p such that*

$$\tfrac{n}{2} \leq p \leq \tfrac{n+1}{2}.$$

Let us denote by λ_j the eigenvalues of $\Delta_g + V$ in $\mathbb{C} \setminus [0, \infty)$, repeated according to their algebraic multiplicities. The following estimates then hold:

(i) *If $p = \frac{n}{2}$, we have*

$$\sum_j \frac{\operatorname{Im} \sqrt{\lambda_j}}{1 + |\lambda_j|} < \infty, \tag{1-13}$$

where the branch of the square root is chosen to have positive imaginary part.

(ii) *If $\frac{n}{2} < p \leq \frac{n+1}{2}$, then*

$$\sum_j \frac{d(\lambda_j)}{|\lambda_j|^{\frac{1-\varepsilon}{2}}} \leq C_{\varepsilon,p,n} \|V\|^{\frac{(1+\varepsilon)p}{2p-n}}_{L^p(M)} \tag{1-14}$$

for all ε satisfying

$$\begin{cases} \varepsilon \geq 0, & \frac{n}{2} < p < \frac{n^2}{2n-1}, \\ \varepsilon > \frac{p(2n-1)-n^2}{n-p} \geq 0, & \frac{n^2}{2n-1} \leq p \leq \frac{n+1}{2}. \end{cases}$$

Remark 1.4. If $\frac{n}{2} < p \le \frac{n+1}{2}$, then by Theorem 4 we know that the eigenvalues of $\Delta_g + V$ are confined to an open disk centered at the origin. Furthermore, it follows from (1-14) that if a sequence of eigenvalues $\lambda_{j_k} \in \mathbb{C} \setminus [0,\infty)$ converges to $E > 0$ then $\operatorname{Im} \lambda_{j_k} \in \ell^1$. In the case $p = \frac{n}{2}$ the bound (1-13) controls a possible accumulation rate of eigenvalues in $\mathbb{C} \setminus [0,\infty)$ at infinity, and it implies in particular, with the help of

$$\operatorname{Im}(\sqrt{\lambda}) = \frac{|\operatorname{Im} \lambda|}{\sqrt{2(|\lambda| + \operatorname{Re} \lambda)}},$$

that if a sequence of eigenvalues $\lambda_{j_k} \in \mathbb{C} \setminus [0,\infty)$ converges to $E > 0$ then $\operatorname{Im} \lambda_{j_k} \in \ell^1$.

As another application of the Schatten class estimates for the resolvent of Δ_g given in Theorem 2, we get the following generalization of [Frank 2018, Theorem 1.2], concerning Lieb–Thirring-type inequalities for the sums of eigenvalues $\Delta_g + V$ in the case of a long-range potential $V \in L^p(M)$, $p = \gamma + \frac{n}{2}$, $\gamma > \frac{1}{2}$.

Theorem 6. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \ge 3$, and let $V \in L^p(M)$ with $p = \gamma + \frac{n}{2}$, $\gamma > \frac{1}{2}$. Then the eigenvalues $\lambda_j \in \mathbb{C} \setminus [0,\infty)$ of $\Delta_g + V$, repeated according to their algebraic multiplicities, satisfy the following bounds: for any $\varepsilon > 0$,*

$$\Bigg(\sum_{|\lambda_j|^\gamma \le C_{\gamma,n} \int_M |V|^{\gamma+n/2}\,dx} d(\lambda_j)^{2\gamma+\varepsilon} \Bigg)^{\frac{\gamma}{2\gamma+\varepsilon}} \le L_{\varepsilon,\gamma,n} \int_M |V|^{\gamma+\frac{n}{2}}\,dx,$$

and for any $\varepsilon > 0$, $0 < \varepsilon' < \frac{\gamma}{\gamma+n/2}$, and $\mu \ge 1$,

$$\Bigg(\sum_{|\lambda_j|^\gamma \ge \mu C_{\gamma,n} \int_M |V|^{\gamma+n/2}\,dx} \frac{d(\lambda_j)^{2\gamma+\varepsilon}}{|\lambda_j|^{2\gamma - \frac{\gamma}{\gamma+n/2} + \varepsilon + \varepsilon'}} \Bigg)^{\frac{\gamma(\gamma+n/2)}{\gamma - \varepsilon'(\gamma+n/2)}} \le L_{\varepsilon,\varepsilon',\gamma,n} \mu^{-\frac{\varepsilon'(\gamma+n/2)}{\gamma-\varepsilon'(\gamma+n/2)}} \int_M |V|^{\gamma+\frac{n}{2}}\,dx.$$

Remark 1.5. As observed in [Frank 2018], Theorem 6 has the following consequence: let $\gamma > \frac{1}{2}$ and $V \in L^{\gamma+n/2}(M)$. If $(\lambda_j)_{j=1}^{\infty}$ is a sequence of eigenvalues of $\Delta_g + V$ with $\lambda_j \to \lambda_0 \in [0,\infty)$ then $\operatorname{Im} \lambda_j \in l^p$ for any $p > 2\gamma$.

Remark 1.6. Let us emphasize once more that all our results, Theorems 2–6, are valid for the metric Schrödinger operator in the Euclidean space $\mathbb{R}^n$, with a metric that is a nontrapping short-range perturbation of the Euclidean one, in the sense of (1-7). In particular, the results hold true for the metric Schrödinger operator in the Euclidean space $\mathbb{R}^n$, with a metric that is a sufficiently small compactly supported perturbation of the Euclidean one.

1D. ***Outline of the paper.*** The plan of the paper is as follows. In Section 2 we present our strategy for proving Theorem 2, which is the main result of the paper. Section 3 is devoted to the proof of Theorem 3, giving Schatten norm estimates on the spectral measure. In Section 4 we derive some Schatten norm estimates on the resolvent of the Laplacian, as a direct consequence of the Schatten norm estimates on the spectral measure, and give their analogues at the endpoint case $p = \frac{n}{2}$, needed in the proof of Theorem 2. The principal step in the proof of Theorem 2, corresponding to the estimates on the spectrum, is carried out in Section 5. Section 6 contains the proof of Theorem 4, which follows the arguments of [Frank 2018; Frank and Simon 2017] closely, relying on Theorem 1, with some small adjustments due to the fact that we are no longer in the Euclidean setting. Finally, we observe in Section 7 that Theorems 5 and 6 are direct consequences of Theorem 2 combined with the arguments of [Frank and Sabin 2017, Theorem 16] and [Frank 2018, Theorem 1.2]. Appendix A contains the proof of Lemma 5.5, needed in the main text. Appendix B is concerned with the analysis of the microlocal structure of the spectrally localized outgoing and incoming resolvent, used in the proof of Theorem 2.

2. Strategy of the proof of Theorem 2

2A. ***Schatten norm estimates.*** We first recall the definition of the Schatten spaces of operators on $L^2(M)$; see [Simon 1979]. Let A be a compact operator on $L^2(M)$, and let $\mu_j(A)$ be the singular values of A, given by $\mu_j(A) = \lambda_j((A^*A)^{1/2})$. Here $\lambda_j(B)$ denotes the eigenvalues of a positive self-adjoint compact operator B, arranged in decreasing order. The Schatten norm of A of order $1 \le q < \infty$ is defined as

$$\|A\|^q_{\mathcal{C}_q(L^2(M))} = \sum_{j=1}^{\infty} \mu_j(A)^q = \operatorname{tr}((A^*A)^{\frac{q}{2}}).$$

The basic mechanism for proving the Schatten norm estimates of Theorems 2 and 3 comes from the fact that the Schatten spaces are complex interpolation spaces, see [Simon 1979, Theorem 2.9; 2015, p. 154], and from [Frank and Sabin 2017, Proposition 1].

Proposition 2.1. *Let T_s be an analytic family of operators, defined on the strip $\{s \in \mathbb{C} \mid -\lambda_0 \le \operatorname{Re} s \le 0\}$ for some $\lambda_0 > 1$, acting on functions on M. Assume that we have operator norm bounds*

$$\|T_{ir}\|_{L^2(M)\to L^2(M)} \le M_0 e^{a|r|}, \quad \|T_{-\lambda_0+ir}\|_{L^1(M)\to L^\infty(M)} \le M_1 e^{a|r|} \quad \textit{for all } r \in \mathbb{R},$$

for some $a \ge 0$ and $M_0, M_1 > 0$. Then for any $W_1, W_2 \in L^{2\lambda_0}(M)$, the operator $W_1 T_{-1} W_2$ belongs to the Schatten class $\mathcal{C}_{2\lambda_0}(L^2(M))$ and we have the estimate

$$\|W_1 T_{-1} W_2\|_{\mathcal{C}_{2\lambda_0}} \le M_0^{1-\frac{1}{\lambda_0}} M_1^{\frac{1}{\lambda_0}} \|W_1\|_{L^{2\lambda_0}(M)} \|W_2\|_{L^{2\lambda_0}(M)}.$$

Let us recall briefly the proof of Proposition 2.1. The result is established by considering the analytic family of operators $S_s = |W_1|^{-1-s} W_1 T_s W_2 |W_2|^{-1-s}$. This family has the property that $S_{-1} = W_1 T_{-1} W_2$ and it satisfies the following estimates on the boundary of the strip. For $s = ir$, r real, we have

$$\|S_{ir}\|_{L^2(M)\to L^2(M)} \le \|T_{ir}\|_{L^2(M)\to L^2(M)} \le M_0 e^{a|r|},$$

and for $s = -\lambda_0 + ir$, we note that T_s has its Schwartz kernel bounded pointwise by $M_1 e^{a|r|}$ (due to the $L^1 \to L^\infty$ bound on T_s) and $|W_1|^{-s}$, $|W_2|^{-s}$ are L^2 functions; hence S_s is a Hilbert–Schmidt operator with the Hilbert–Schmidt norm bounded by $M_1 e^{a|r|} \|W_1\|^{\lambda_0}_{L^{2\lambda_0}(M)} \|W_2\|^{\lambda_0}_{L^{2\lambda_0}(M)}$. Interpolating between the operator norm and the Hilbert–Schmidt norm gives us a bound on the Schatten norms, in particular at $s = -1$, where we obtain the Schatten norm at exponent $2\lambda_0$.

2B. ***Strategy.*** The principal idea of the proof of the Euclidean analogue of Theorem 2, which is due to Frank and Sabin [2017, Theorem 12], is to establish the following pointwise bound for the Schwartz kernel of the powers of the resolvent $(\Delta - z)^{-\alpha}$:

$$|(\Delta - z)^{-\alpha}(x, y)| \le C e^{C(\mathrm{Im}(\alpha))^2} |z|^{\frac{n-1}{4} - \frac{\mathrm{Re}(\alpha)}{2}} |x - y|^{\mathrm{Re}(\alpha) - \frac{n+1}{2}}, \quad x, y \in \mathbb{R}^n. \tag{2-1}$$

Here $z \in \mathbb{C} \setminus [0, \infty)$, $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \in [\frac{n-1}{2}, \frac{n+1}{2}]$. The desired Schatten bound (1-9) in the Euclidean case is therefore a consequence of (2-1) combined with the Hölder and Hardy–Littlewood–Sobolev inequalities as well as an interpolation argument.

Unfortunately, the natural analogue of the pointwise bound (2-1) does not hold in general, for z close to the spectrum of Δ_g, for asymptotically conic manifolds, essentially because there can be conjugate points for the geodesic flow, and to prove the bound (1-9) we have to proceed differently.

Our strategy of the proof of Theorem 2 is to establish the Schatten norm estimate (1-9) for $W_1(\Delta_g - z)^{-1} W_2$ for z on the negative real axis, and for z just above and below the spectrum, that is, for $W_1(\Delta_g - (z \pm i0))^{-1} W_2$ for $z > 0$. We then use the Phragmén–Lindelöf theorem to obtain the result on the whole of the complex plane, excluding the origin.

Let us give the proof of Theorem 2, assuming that it has been established for $z < 0$ and for $z \pm i0$, $z > 0$. Let $W_1, W_2 \in L^{2p}(M)$ with $p \in \left[\frac{n}{2}, \frac{n+1}{2}\right]$, and let us consider the following bilinear form for $z \in \mathbb{C} \setminus [0, \infty)$:

$$B_z(W_1, W_2) := W_1(\Delta_g - z)^{-1} W_2. \tag{2-2}$$

When $z \in (0, \infty)$, we extend the definition of B_z by taking the outgoing resolvent $(\Delta_g - (z + i0))^{-1}$ in (2-2). Thus, we know that for $z \in \mathbb{R} \setminus \{0\}$, B_z is a bounded bilinear form

$$B_z : L^{2p}(M) \times L^{2p}(M) \to \mathcal{C}_q(L^2(M)), \quad p \in \left[\tfrac{n}{2}, \tfrac{n+1}{2}\right], \quad q = \tfrac{p(n-1)}{n-p},$$

such that

$$\|B_z(W_1, W_2)\|_{\mathcal{C}_q} \le C |z|^{-1 + \frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{2-3}$$

We now complete the proof of Theorem 2 by a Phragmén–Lindelöf argument. In doing so, let $W_1, W_2 \in C_0^\infty(M)$. We claim that the function $H(z) := B_z(W_1, W_2)$ is holomorphic in $\mathrm{Im}\, z > 0$ with values in $\mathcal{C}_q(L^2(M))$ such that

$$\|H(z)\|_{\mathcal{C}_q} \le C(|z|^{-\frac{1}{2}} + |z|^{\frac{1}{2}}).$$

Indeed, for $\mathrm{Im}\, z > 0$, the operator $W_1(\Delta_g - z)^{-1} W_2 : L^2(M) \to H^2(M) \cap \mathcal{E}'(K)$ is bounded, where K is a compact set containing the support of W_1. Furthermore, it depends holomorphically on z with $\mathrm{Im}\, z > 0$ and satisfies the bound

$$\|W_1(\Delta_g - z)^{-1} W_2\|_{\mathcal{L}(L^2(M), H^2(M))} \le C(|z|^{-\frac{1}{2}} + |z|^{\frac{1}{2}}), \quad \mathrm{Im}\, z \ge 0,\ z \ne 0;$$

see [Melrose 1994] for intermediate values of z, [Vasy and Zworski 2000] for $|z| \to \infty$ and [Rodnianski and Tao 2015, Proposition 1.26] for $|z| \to 0$. Now the embedding $H^2(M) \cap \mathcal{E}'(K) \to L^2(M)$ is an operator in $\mathcal{C}_{n/2+\varepsilon}$ for all $\varepsilon > 0$ in view of the Weyl law for the Laplacian on a compact manifold. Since $q > \frac{n}{2}$, we deduce the claim.

The function $H(z)$ is continuous for $\operatorname{Im} z \geq 0,\ z \neq 0$, with values in $\mathcal{C}_q(L^2(M))$, and to avoid the problem at $z=0$, we consider the map

$$F(z) := \langle H(e^z), T\rangle e^{(1-\frac{n}{2p})z}$$

for a fixed $T \in \mathcal{C}_{q'}(L^2(M))$ with norm $\|T\|_{\mathcal{C}_{q'}} = 1$. Here $\frac{1}{q'}+\frac{1}{q}=1$ and the product is the duality pairing between the Banach space $\mathcal{C}_q$ and its dual $\mathcal{C}_{q'}$. Then $F(z)$ is holomorphic in $\operatorname{Im} z \in (0,\pi)$, continuous on the closure, and enjoys the bounds

$$|F(z)| \leq C e^{C|z|} \quad \text{for } 0 \leq \operatorname{Im} z \leq \pi,$$
$$|F(z)| \leq C\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)} \quad \text{for } \operatorname{Im} z \in \{0,\pi\}$$

in view of (2-3). Applying the Phragmén–Lindelöf principle, we deduce that

$$|F(z)| \leq C\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}$$

for all $z \in \mathbb{C}$ such that $0 \leq \operatorname{Im} z \leq \pi$, and therefore

$$\|H(z)\|_{\mathcal{C}_q} \leq C|z|^{-1+\frac{n}{2p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}, \quad \operatorname{Im} z \geq 0,\ z \neq 0.$$

By a density argument, we obtain the bound (1-9) for $\operatorname{Im} z \geq 0,\ z \neq 0$. By considering the adjoint of the operator B_z, we complete the proof of Theorem 2.

This argument reduces the problem to proving estimate (1-9) for $z \in \mathbb{R}\setminus\{0\}$. We find it convenient to first prove the corresponding estimate for the spectral measure given in Theorem 3. The proof of Theorem 3 relies crucially on the TT^* structure of the spectral measure.

When $z \in (-\infty,0)$ and $p \in \left(\frac{n}{2},\frac{n+1}{2}\right]$, the Schatten norm estimate (1-9) is a direct consequence of Theorem 3, and at the endpoint case $p=\frac{n}{2}$, the Schatten norm estimate (1-9) follows from the heat kernel estimates due to [Grigoryan 1997; Varopoulos 1985].

Establishing the Schatten norm estimate (1-9) for $W_1(\Delta_g-(z\pm i0))^{-1}W_2$ with $z>0$ represents the main difficulty in the proof of Theorem 2. When doing so, following [Guillarmou and Hassell 2014; Guillarmou et al. 2013b; Hassell and Zhang 2016], we use a microlocal partition of the identity

$$\sum_{i=1}^{N} Q_i(\eta) = \mathrm{Id},$$

where $Q_i(\eta)$are pseudodifferential operators depending on the energy parameter $0<\eta\sim|z|^{1/2}$, constructed in [Guillarmou et al. 2013b]. Splitting up the operator $W_1(\Delta_g-(z\pm i0))^{-1}W_2$ by means of the partition of the identity, we are led to estimate the individual terms $W_1Q_i(\eta)^*(\Delta_g-(z\pm i0))^{-1}Q_j(\eta)W_2$, and here the most interesting contributions arise when $i=j$. When handling those, we proceed by establishing pointwise bounds for the Schwartz kernel of the operator

$$Q_i(\eta)^*\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g-(z\pm i0))^{-s}Q_j(\eta), \quad \operatorname{Re} s \in \left[\tfrac{n-1}{2},\tfrac{n+1}{2}\right],$$

analogous to the Euclidean estimates (2-1). Here ϕ is a cut-off near 1.

3. Schatten norm estimates on the spectral measure: proof of Theorem 3

Our starting point for the proof is the operator partition of unity, $\mathrm{Id} = \sum_{i=1}^N Q_i(\eta)$, depending on $\eta > 0$, constructed in [Guillarmou et al. 2013b]. This partition of unity enjoys the following estimates in particular: there exists $\delta > 0$ sufficiently small but fixed such that, for all $k = 0, 1, 2, \dots$, there is $C_k > 0$ such that, for all $m, m' \in M$, we have

$$\begin{aligned}\big|\partial_\lambda^k\big(Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta)\big)(m, m')\big| \\ \le C_k \lambda^{n-1-k}(1+\lambda d(m,m'))^{-\frac{(n-1)}{2}+k}, \quad \lambda \in [(1-\delta)\eta, (1+\delta)\eta], \quad (3\text{-}1)\end{aligned}$$

with $d(\cdot\,,\cdot)$ being the Riemannian distance on M. We say more about this partition of the identity in Section 5A below; here, we can use results of [Chen 2018; Guillarmou et al. 2013b] as a "black box". Then for all $\lambda \in \big[\big(1-\frac{\delta}{2}\big)\eta, \big(1+\frac{\delta}{2}\big)\eta\big]$, we use the partition of unity to decompose the spectral measure sandwiched between two L^{2p} functions:

$$W_1 dE_{\sqrt{\Delta_g}}(\lambda) W_2 = \sum_{i,j=1}^N W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2. \quad (3\text{-}2)$$

Let $p \in \big[1, \frac{n+1}{2}\big]$ and $q = \frac{p(n-1)}{n-p} \in [1, n+1]$. In the first step, we shall prove microlocalized estimates of the form

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)} \quad (3\text{-}3)$$

for the diagonal $(i=j)$ terms of the decomposition (3-2). In doing so, we shall follow [Frank and Sabin 2017, proof of Theorem 2] and start by showing (3-3) at the endpoints $p = \frac{n+1}{2}$ and $p = 1$; i.e.,

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2\|_{\mathcal{C}_{n+1}} \le C\lambda^{\frac{n-1}{n+1}} \|W_1\|_{L^{n+1}(M)} \|W_2\|_{L^{n+1}(M)}, \quad (3\text{-}4)$$

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2\|_{\mathcal{C}_1} \le C\lambda^{n-1} \|W_1\|_{L^2(M)} \|W_2\|_{L^2(M)}, \quad (3\text{-}5)$$

respectively. Once the estimates (3-4) and (3-5) have been established, the bound (3-3) follows by a complex interpolation argument applied to the analytic family of operators

$$\zeta \mapsto W_1^{\frac{2}{n+1}+\zeta\frac{n-1}{n+1}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2^{\frac{2}{n+1}+\zeta\frac{n-1}{n+1}}$$

in the strip $0 \le \operatorname{Re}\zeta \le 1$, with $W_j \ge 0$ being simple functions such that $\|W_j\|_{L^2(M)} = 1$, $j = 1, 2$; see [Simon 1979, Theorem 2.9].

Now to prove the estimate (3-4), we shall consider the family of operators

$$T_s := Q_i(\eta)^* \phi\bigg(\frac{\sqrt{\Delta_g}}{\lambda}\bigg) \chi_+^s(\lambda - \sqrt{\Delta_g}) Q_i(\eta), \quad -\frac{(n+1)}{2} \le \operatorname{Re} s \le 0,$$

introduced in [Chen 2018; Guillarmou et al. 2013b, Definition 3.2]. Here $\phi \in C_0^\infty\big(\big(1-\frac{\delta}{4}, 1+\frac{\delta}{4}\big)\big)$ is such that $\phi(t) = 1$ in a neighborhood of $t = 1$, and χ_+^s is the family of distributions on $\mathbb{R}$, entire analytic in $s \in \mathbb{C}$ and such that

$$\chi_+^s(\lambda) = \frac{\lambda_+^s}{\Gamma(s+1)}, \quad \operatorname{Re} s > -1,$$

where $\lambda_+ = \max(\lambda, 0)$; see [Hörmander 1990, Section 3.2]. Note that, at least formally, we have

$$\chi_+^0(\lambda - \sqrt{\Delta_g}) = E_{\sqrt{\Delta_g}}(\lambda), \quad \chi_+^{-k}(\lambda - \sqrt{\Delta_g}) = \left(\frac{d}{d\lambda}\right)^{k-1} dE_{\sqrt{\Delta_g}}(\lambda), \quad k = 1, 2, \dots.$$

Recall from [Guillarmou et al. 2013b, Definition 3.2] that T_s is the operator whose Schwartz kernel is given by

$$\begin{aligned}\left(Q_i(\eta)^* \phi\left(\frac{\sqrt{\Delta_g}}{\lambda}\right)\chi_+^s(\lambda - \sqrt{\Delta_g}) Q_i(\eta)\right)(m, m')\\ = \int \chi_+^{k+s}(\lambda - \mu)\partial_\mu^k \left(Q_i(\eta)^* \phi\left(\frac{\mu}{\lambda}\right) dE_{\sqrt{\Delta_g}}(\mu) Q_i(\eta)\right)(m, m')\, d\mu,\end{aligned} \tag{3-6}$$

where $k \in \mathbb{N}$ is such that $\operatorname{Re} s + k > -1$. As $\mu \in [\eta(1-\delta), \eta(1+\delta)]$ for $\lambda \in \left[\left(1 - \frac{\delta}{2}\right)\eta, \left(1 + \frac{\delta}{2}\right)\eta\right]$ and $\frac{\mu}{\lambda} \in \operatorname{supp}(\phi)$, thanks to the estimates (3-1) the integral in (3-6) is well-defined.

As explained in [Guillarmou et al. 2013b], the family of operators T_s is analytic in the sense of Stein in the strip $-\frac{(n+1)}{2} \le \operatorname{Re} s \le 0$. When $\operatorname{Re} s = 0$ we have

$$\|T_s\|_{L^2(M) \to L^2(M)} \le C e^{\frac{\pi |s|}{2}},$$

and relying on the estimates (3-1) it was shown in [Chen 2018; Guillarmou et al. 2013b] that when $\operatorname{Re} s = -\frac{(n+1)}{2}$ we have

$$\|T_s\|_{L^1(M) \to L^\infty(M)} \le C(1 + |r|) e^{\frac{\pi |r|}{2}} \lambda^{\frac{n-1}{2}}, \quad s = -\frac{(n+1)}{2} + ir, \quad r \in \mathbb{R}.$$

Applying Proposition 2.1, we get, for any two complex-valued functions $W_1, W_2 \in L^{n+1}(M)$,

$$\begin{aligned} W_1 T_{-1} W_2 &= W_1 Q_i(\eta)^* \phi\left(\frac{\sqrt{\Delta_g}}{\lambda}\right)\chi_+^{-1}(\lambda - \sqrt{\Delta_g}) Q_i(\eta) W_2 \\ &= W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2 \end{aligned}$$

is in the Schatten $\mathcal{C}_{n+1}$ class and (3-4) holds.

To show (3-5), we recall from [Guillarmou et al. 2013b] that we have a pointwise kernel bound on the (microlocalized) spectral measure,

$$\|Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta)\|_{L^1(M) \to L^\infty(M)} \le C \lambda^{n-1}. \tag{3-7}$$

Also, we have

$$dE_{\sqrt{\Delta_g}}(\lambda) = (2\pi)^{-1} P(\lambda) P^*(\lambda), \tag{3-8}$$

where $P(\lambda) : L^2(\partial M) \to L^r(M)$, $r \in \left[\frac{2(n+1)}{n-1}, \infty\right]$, is the Poisson operator; see [Guillarmou et al. 2013b]. Using the T^*T trick, it follows from (3-7) and (3-8) that

$$\|Q_i(\eta)^* P(\lambda)\|_{L^2(\partial M) \to L^\infty(M)} \le C \lambda^{\frac{n-1}{2}}.$$

The Schwartz kernel $Q_i(\eta)^* P(\lambda)(m, m')$ of the operator $Q_i(\eta)^* P(\lambda)$ satisfies therefore

$$\|Q_i(\eta)^* P(\lambda)(m, \cdot)\|_{L^2(\partial M)} \le C \lambda^{\frac{n-1}{2}}$$

for almost all $m \in M$. Thus, for any $W_1 \in L^2(M)$, the operator $W_1 Q_i(\eta)^* P(\lambda) : L^2(\partial M) \to L^2(M)$ is Hilbert–Schmidt with the norm bounded by $C\lambda^{(n-1)/2}\|W_1\|_{L^2(M)}$. Taking adjoints, we find that $P(\lambda)^* Q_i(\eta) W_2$ is a Hilbert–Schmidt operator with norm bounded by $C\lambda^{(n-1)/2}\|W_2\|_{L^2(M)}$. Therefore, $(2\pi)^{-1}$ times the composition of these two operators, which is precisely $W_1 Q_i(\eta)^* dE_{\sqrt{\Delta}}(\lambda) Q_i(\eta) W_2$, is of trace class and (3-5) follows.

In the second step, we shall bound the Schatten norm of the off-diagonal ($i \neq j$) terms in the decomposition (3-2); i.e., we shall prove the estimate

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{3-9}$$

As above, we shall exploit the T^*T structure of the spectral measure.

Let $T : L^2(M) \to L^2(\partial M)$ be a compact operator and $q \ge 1$. Then $T^*T \in \mathcal{C}_q(L^2(M))$ if and only if $T \in \mathcal{C}_{2q}(L^2(M), L^2(\partial M))$, and moreover, $\|T^*T\|_{\mathcal{C}_q} = \|T\|^2_{\mathcal{C}_{2q}}$. This is a consequence of the following equality for the singular values:

$$\mu_k(T^*T) = \mu_k(T)^2. \tag{3-10}$$

Moreover, if T_1, T_2 are in $\mathcal{C}_{2q}(L^2(M), L^2(\partial M))$, then $T_1^* T_2$ is in $\mathcal{C}_q(L^2(M))$, and

$$\|T_1^* T_2\|^q_{\mathcal{C}_q} \le \|T_1^* T_1\|^q_{\mathcal{C}_q} + \|T_2^* T_2\|^q_{\mathcal{C}_q}; \tag{3-11}$$

see for example [McCarthy 1967]. Using (3-8), we write

$$W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2 = (2\pi)^{-1} T_1^* T_2, \tag{3-12}$$

where $T_1 = P(\lambda)^* Q_i(\eta) \overline{W}_1$, and $T_2 = P(\lambda)^* Q_j(\eta) W_2$. Now it follows from (3-3) that $T_1^* T_1 \in \mathcal{C}_q(L^2(M))$, $T_2^* T_2 \in \mathcal{C}_q(L^2(M))$, and we have

$$\|T_1^* T_1\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}} \|W_1\|^2_{L^{2p}(M)}, \quad \|T_2^* T_2\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}} \|W_2\|^2_{L^{2p}(M)}.$$

By the discussion above, this is equivalent to the fact that $T_1 \in C_{2q}(L^2(M), L^2(\partial M))$ and $T_2 \in C_{2q}(L^2(M), L^2(\partial M))$. It follows from (3-12) and discussion above that $W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2 \in \mathcal{C}_q(L^2(M))$, and using (3-11), we get that

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}} (\|W_1\|^2_{L^{2p}(M)} + \|W_2\|^2_{L^{2p}(M)}).$$

Thus, (3-9) follows by bilinearity in W_1, W_2. This completes the proof of Theorem 3.

4. Consequences of the spectral measure estimates for $p \in \left(\frac{n}{2}, \frac{n+1}{2}\right]$ and their analogues at the endpoint $p = \frac{n}{2}$

4A. ***Consequences of the spectral measure Schatten norm estimate.*** Using Theorem 3 and Minkowski's integral inequality, we can deduce some Schatten estimates on the resolvent. In this subsection, we only treat the case $p > \frac{n}{2}$.

The first result applies for z in any sector excluding the positive real axis.

Proposition 4.1. *Let $p \in (\frac{n}{2}, \frac{n+1}{2}]$, and suppose $W_1, W_2 \in L^{2p}(M)$. Let $\varepsilon > 0$ be arbitrary. Then for $z \in \mathbb{C}$ such that $z \neq 0$, $\arg z \in [\varepsilon, 2\pi - \varepsilon]$, we have the sandwiched resolvent $W_1(\Delta_g - z)^{-1} W_2$ is in the Schatten class $\mathcal{C}_q(L^2(M))$ with $q = \frac{p(n-1)}{n-p} \in (n-1, n+1]$ and*

$$\|W_1(\Delta_g - z)^{-1} W_2\|_{\mathcal{C}_q} \le C|z|^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)},$$

where C depends on p, ε and (M, g), but not z.

Proof. We express the operator $W_1(\Delta_g - z)^{-1} W_2$ as

$$W_1(\Delta_g - z)^{-1} W_2 = \int_0^\infty (\lambda^2 - z)^{-1} W_1 \, dE_{\sqrt{\Delta_g}}(\lambda) W_2 \, d\lambda.$$

The result follows by estimating the Schatten norm of $W_1 \, dE_{\sqrt{\Delta_g}}(\lambda) W_2$ using Theorem 3 and noting that, provided $p > \frac{n}{2}$, we have

$$\int_0^\infty |\lambda^2 - z|^{-1} \lambda^{-1+\frac{n}{p}} \, d\lambda \le C|z|^{-1+\frac{n}{2p}},$$

where C depends on p and ε but does not depend on z in the given sector. □

In a similar manner we obtain "elliptic" estimates on the resolvent, where we remove the singularity in the spectral multiplier. In this way we can obtain estimates on the positive real axis. To state these, we fix a function $\phi : [0, \infty) \to [0, 1]$ such that $\phi(t) = 1$ for t in a neighborhood of $t = 1$ and has support in a slightly bigger neighborhood of $t = 1$.

Proposition 4.2. *Let $p \in \left(\frac{n}{2}, \frac{n+1}{2}\right]$, and suppose $W_1, W_2 \in L^{2p}(M)$. Then for $z \in \mathbb{C} \setminus \{0\}$, the operator $W_1(1-\phi)(\Delta_g/|z|)(\Delta_g - z)^{-1} W_2$ is in the Schatten class $\mathcal{C}_q(L^2(M))$ with $q = \frac{p(n-1)}{n-p} \in (n-1, n+1]$, and we have*

$$\left\| W_1(1-\phi)\left(\frac{\Delta_g}{|z|}\right)(\Delta_g - z)^{-1} W_2 \right\|_{\mathcal{C}_q} \le C|z|^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)},$$

where C depends on p and on (M, g), but not z.

Proof. Again we express the operator using an integral over the spectral measure, and estimate the Schatten norm of the spectral measure using Theorem 3 and Minkowski's integral inequality. This time we obtain the integral

$$\int_0^\infty |\lambda^2 - z|^{-1} (1-\phi)\left(\frac{\lambda^2}{|z|}\right) \lambda^{-1+\frac{n}{p}} \, d\lambda$$

and it is straightforward to check that this is bounded by $C|z|^{-1+n/(2p)}$ uniformly in z. □

4B. ***Analogues at the endpoint $p = \frac{n}{2}$.*** In the case $p = \frac{n}{2}$, the arguments used in the proofs of Propositions 4.1 and 4.2 are no longer valid and need to be replaced. In view of the Phragmén–Lindelöf argument, explained in Section 2B, we only need to do this for z negative in the case of Proposition 4.1 and z positive in the case of Proposition 4.2. To this end we prove the following two results.

Proposition 4.3. *Let $p = \frac{n}{2}$. There is $C > 0$ such that for all $z < 0$ and for all $W_1, W_2 \in L^n(M)$, the operator $W_1(\Delta_g - z)^{-1} W_2$ is in $\mathcal{C}_{n-1}(L^2(M))$ and we have*

$$\| W_1(\Delta_g - z)^{-1} W_2 \|_{\mathcal{C}_{n-1}} \le C \| W_1 \|_{L^n(M)} \| W_2 \|_{L^n(M)}. \tag{4-1}$$

Proof. Here we use a slight variation of Proposition 2.1. Let W_1, W_2 be nonnegative simple functions and consider the analytic family of operators

$$S_s = W_1^{-s}(\Delta_g - z)^s W_2^{-s}, \quad -\tfrac{(n-1)}{2} \le \operatorname{Re} s \le 0.$$

Clearly, when $\operatorname{Re} s = 0$, we have

$$\| S_s \|_{L^2(M) \to L^2(M)} \le C. \tag{4-2}$$

Next, we will show that, when $\operatorname{Re} s = -\frac{(n-1)}{2}$, then S_s is Hilbert–Schmidt and we have

$$\| S_s \|_{\mathcal{C}_2} \le C e^{C|\operatorname{Im} s|} \| W_1 \|_{L^n(M)}^{\frac{n-1}{2}} \| W_2 \|_{L^n(M)}^{\frac{n-1}{2}}. \tag{4-3}$$

This allows us to run the interpolation argument in the proof of Proposition 2.1.

To prove (4-3), on the line $\operatorname{Re} s = -\frac{(n-1)}{2}$, we express $(\Delta_g - z)^s$ in terms of the heat kernel:

$$\Gamma(-s)(\Delta_g - z)^s(m, m') = \int_0^\infty t^{-s-1} e^{tz} e^{-t\Delta_g}(m, m')\, dt. \tag{4-4}$$

We now use heat kernel estimates. Due to [Varopoulos 1985], we have the estimate $\| e^{-t\Delta_g} \|_{L^1 \to L^\infty} \le C t^{-n/2}$ and by a result of [Grigoryan 1997], this implies a pointwise upper Gaussian estimate on the heat kernel

$$|e^{-t\Delta_g}(m, m')| \le C t^{-\frac{n}{2}} e^{-\frac{c d(m,m')^2}{t}}, \quad t > 0, \tag{4-5}$$

for some $c > 0$. The integral in (4-4) is convergent for all $m \ne m'$ due to (4-5). We thus get for all $m \ne m'$ and $z \in (-\infty, 0)$, and uniformly for all s such that $\operatorname{Re} s = -\frac{(n-1)}{2}$

$$\begin{aligned} |\Gamma(-s)(\Delta_g - z)^s(m, m')| &\le C \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{c d(m,m')^2}{t} + zt}\, dt \\ &\le C d(m, m')^{-1} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{c}{t} + z d(m,m')^2 t}\, dt \\ &\le C d(m, m')^{-1}. \end{aligned} \tag{4-6}$$

Using Hölder's inequality, the generalized Hardy–Littlewood–Sobolev inequality of [García-Cuerva and Gatto 2004] and (4-6), we obtain for $\operatorname{Re} s = -\frac{(n-1)}{2}$,

$$\begin{aligned} \| W_1^{-s}(\Delta_g - z)^s W_2^{-s} \|_{\mathcal{C}_2(M)}^2 &\le C |\Gamma(-s)|^{-1} \int_{M \times M} W_1(m)^{n-1} d(m, m')^{-2} W_2(m')^{n-1}\, dV_g(m)\, dV_g(m') \\ &\le C |\Gamma(-s)|^{-1} \| W_1^{n-1} \|_{L^{n/(n-1)}(M)} \| W_2^{n-1} \|_{L^{n/(n-1)}(M)} \\ &\le C e^{C|\operatorname{Im} s|} \| W_1 \|_{L^n(M)}^{n-1} \| W_2 \|_{L^n(M)}^{n-1}, \end{aligned}$$

where the factor $e^{C|\operatorname{Im} s|}$ is contributed by the Gamma function. This shows (4-3).

We now interpolate using the family S_s between (4-2) and (4-3), as in the proof of Proposition 2.1, and we obtain at $s = -1$

$$\|W_1(\Delta_g - z)^{-1} W_2\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}, \tag{4-7}$$

which completes the proof for W_1 and W_2 nonnegative and simple. The extension to general $W_1, W_2 \in L^n(M)$ is standard. ☐

We now prove an analogue of Proposition 4.2.

Proposition 4.4. *Let $p = \frac{n}{2}$ and suppose $W_1, W_2 \in L^n(M)$, and let ϕ be as in Proposition 4.2. Then for $z > 0$, the operator $W_1(1-\phi)(\Delta_g/z)(\Delta_g - z)^{-1} W_2$ is in the Schatten class $\mathcal{C}_{n-1}(L^2(M))$ and*

$$\left\| W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g - z)^{-1} W_2 \right\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}$$

uniformly in z.

Proof. We first note that for $z > 0$, the operator

$$W_1 \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1} W_2$$

is in the Schatten class $\mathcal{C}_{n-1}(L^2(M))$, and

$$\left\| W_1 \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1} W_2 \right\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}$$

uniformly in z. This follows from the spectral measure estimate (1-10), since

$$\int_0^\infty \lambda \phi\left(\frac{\lambda^2}{z}\right)(\lambda^2 + z)^{-1}\, d\lambda$$

is bounded uniformly in z. Combining this with Proposition 4.3, we see that $W_1(1-\phi)(\Delta_g/z)(\Delta_g+z)^{-1} W_2$ is in $\mathcal{C}_{n-1}(L^2(M))$ and we have

$$\left\| W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1} W_2 \right\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)} \tag{4-8}$$

uniformly in z.

Now we write

$$W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g - z)^{-1} W_2 = W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1} W_2 + 2z W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1}(\Delta_g - z)^{-1} W_2. \tag{4-9}$$

The first term in the right-hand side of (4-9) has already been shown to lie in $\mathcal{C}_{n-1}$ with the bound (4-8). We write the second term on the right-hand side of (4-9) in terms of the spectral measure and apply

Minkowski's integral inequality together with the spectral measure estimate (1-10), and find that the norm in $\mathcal{C}_{n-1}$ is bounded by

$$C\left(z\int_0^\infty (1-\phi)\left(\frac{\lambda^2}{z}\right)(\lambda^2+z)^{-1}(\lambda^2-z)^{-1}\lambda\,d\lambda\right)\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)}$$

and a change of variable shows that this integral is convergent and independent of z, completing the proof. □

5. Resolvent estimates on the spectrum: completion of the proof of Theorem 2

The key difficulty in proving Theorem 2 is to obtain estimates on the limiting resolvent at the spectrum $(\Delta_g-(z+i0))^{-1}$ for $z>0$. Given Propositions 4.2 and 4.4, we only need to do this localized near the singularity at z of the spectral multiplier $(\lambda^2-z)^{-1}$. In doing so, following [Guillarmou and Hassell 2014; Guillarmou et al. 2013b; Hassell and Zhang 2016], we shall use a microlocal partition of unity.

5A. ***Operator partition of unity.*** We begin by recalling some results of [Guillarmou and Hassell 2014; Hassell and Zhang 2016] on high- and low-frequency microlocal estimates on the spectral measure and resolvents of Δ_g.

Proposition 5.1. ***High-frequency microlocal estimates.*** *For all high energies $\eta\geq\frac{1}{2}$, there exists a family of bounded operators $Q_i(\eta):L^2(M)\to L^2(M)$, $i=1,\dots,N_h$, with N_h independent of η and with the norm satisfying*

$$\|Q_i(\eta)\|_{L^2(M)\to L^2(M)}\leq C \quad \textit{for some } C \textit{ independent of } \eta, \tag{5-1}$$

so that the following properties hold:

(1) *The operators $Q_i(\eta)$ form an operator partition of unity*:

$$\sum_{i=1}^{N_h} Q_i(\eta)=\mathrm{Id}. \tag{5-2}$$

(2) *Let $\eta\geq\frac{1}{2}$ and $(i,j)\in\{1,\dots,N_h\}^2$. There exists $\delta>0$ small such that for all $z>0$ such that $\sqrt{z}\in[(1-\delta)\eta,(1+\delta)\eta]$, one of the following three alternatives holds*:

(2.i) *One has for the* ***outgoing*** *resolvent*

$$\big(Q_i(\eta)^*(\Delta_g-(z+i0))^{-1}Q_j(\eta)\big)(m,m')\in x(m)^\infty x(m')^\infty z^{-\infty}C^\infty(\overline{M}\times\overline{M}) \tag{5-3}$$

for all $m,m'\in M$, where the $C^\infty(\overline{M}\times\overline{M})$-part depends also on z and is uniformly bounded in z in the smooth topology.

(2.ii) *One has for the* ***incoming*** *resolvent*

$$\big(Q_i(\eta)^*(\Delta_g-(z-i0))^{-1}Q_j(\eta)\big)(m,m')\in x(m)^\infty x(m')^\infty z^{-\infty}C^\infty(\overline{M}\times\overline{M}) \tag{5-4}$$

for all $m,m'\in M$.

(2.iii) *The spectral measure satisfies, for* $\lambda = \sqrt{z} \in [(1-\delta)\eta, (1+\delta)\eta]$, *the following bounds: for all* $k = 0, 1, 2, \dots$, *there is* $C_k > 0$ *such that for all* $m, m' \in M$

$$\left|\partial_\lambda^k\big(Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta)\big)(m,m')\right| \le C_k \lambda^{n-1-k}(1+\lambda d(m,m'))^{-\frac{(n-1)}{2}+k}, \tag{5-5}$$

$$\big(Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta)\big)(m,m') = \lambda^{n-1}\bigg(\sum_{\pm} e^{\pm i\lambda d(m,m')} a_\pm(\lambda,m,m') + b(\lambda,m,m')\bigg), \tag{5-6}$$

with $a_\pm, b$ *satisfying the estimates, for all* $k = 0, 1, 2, \dots$,

$$|\partial_\lambda^k a_\pm(\lambda,m,m')| \le C_k \lambda^{-k}(1+\lambda d(m,m'))^{-\frac{(n-1)}{2}}, \tag{5-7}$$

$$|\partial_\lambda^k b(\lambda,m,m')| \le C_k \lambda^{-k}(1+\lambda d(m,m'))^{-K} \quad \text{for all } K > 1. \tag{5-8}$$

Moreover the alternative (2.iii) *always holds if* $i = j$.

Low-frequency microlocal estimates. *Similarly, for all low energies* $\eta \le 2$, *there exists a family of bounded operators* $Q_i(\eta) : L^2(M) \to L^2(M)$, $i = 0, *, 1, \dots, N_l$, *with* N_l *independent of* η *satisfying* (5-1) *and* (5-2) (*with the sum in this case ranging over* $i = 0, *, 1, \dots, N_l$), *satisfying the following*:

(3) *Let* $0 < \eta \le 2$ *and* i, j *range independently in* $\{0, *, 1, \dots, N_l\}$. *There exists* $\delta > 0$ *small such that, for all* $z > 0$ *satisfying* $\lambda := \sqrt{z} \in [(1-\delta)\eta, (1+\delta)\eta]$, *one of the following three alternatives holds*:

(3.i) *One has the pointwise kernel bound for the* ***outgoing*** *resolvent* (*for all* $N \in \mathbb{N}$)

$$\left|\big(Q_i(\eta)^*(\Delta_g-(z+i0))^{-1}Q_j(\eta)\big)(m,m')\right| \le C_N\left(\frac{x}{x+\lambda}\right)^N\left(\frac{x'}{x'+\lambda}\right)^N \frac{(xx')^{\frac{n-1}{2}}\big(\chi(\frac{x}{\lambda})+\chi(\frac{x'}{\lambda})\big)}{x+x'+\lambda}, \tag{5-9}$$

where $x = x(m)$, $x' = x(m')$, *and* $\chi \in C_0^\infty((-\varepsilon,\varepsilon),[0,\infty))$ *is such that* $\chi = 1$ *in* $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$. *Here* $\varepsilon > 0$ *is small enough.*

(3.ii) *One has the pointwise kernel bound for the* ***incoming*** *resolvent* (*for all* $N \in \mathbb{N}$)

$$\left|\big(Q_i(\eta)^*(\Delta_g-(z-i0))^{-1}Q_j(\eta)\big)(m,m')\right| \le C_N\left(\frac{x}{x+\lambda}\right)^N\left(\frac{x'}{x'+\lambda}\right)^N \frac{(xx')^{\frac{n-1}{2}}\big(\chi(\frac{x}{\lambda})+\chi(\frac{x'}{\lambda})\big)}{x+x'+\lambda}. \tag{5-10}$$

(3.iii) *For all* $k = 0, 1, 2, \dots$, *there is* $C_k > 0$ *such that* (5-5), (5-6), (5-7) *and* (5-8) *hold.*

Moreover if $i = j$, *the alternative* (3.iii) *holds.*

Remark 5.2. The two partitions of the identity do not quite match up in the intermediate energy regime, $\frac{1}{2} \le \eta \le 2$. Because of this, it would be more notationally accurate to label the partitions Q_i^{high} and Q_j^{low}; to avoid cumbersome notation, we do not do this. We emphasize that in this intermediate regime either partition can be used.

Remark 5.3. In the low-energy case, $\eta \le 2$, let us first point out the meaning of the right-hand side of (5-9) and (5-10). In [Guillarmou et al. 2013a] it was shown that the Schwartz kernel of the resolvent $(\Delta_g - (\lambda^2 \pm i0))^{-1}$ for $\lambda \in [0, \lambda_0]$ has some polyhomogeneous structure on the "low-energy space", which is a blowup of $\overline{M} \times \overline{M} \times [0, \lambda_0]$. Ignoring the artificial boundary at $\lambda = \lambda_0$, this blown-up space has seven boundary hypersurfaces corresponding to seven different types of asymptotics displayed by the resolvent kernel. These are the left boundary lb, the right boundary rb, which arise from $\partial\overline{M}\times\overline{M}\times[0,\lambda_0]$ and $\overline{M}\times\partial\overline{M}\times[0,\lambda_0]$; the b-face bf, which arises from blowing up $\partial\overline{M}\times\partial\overline{M}\times[0,\lambda_0]$; the "zero face" zf, arising from $\overline{M}\times\overline{M}\times\{0\}$; and three faces at $\lambda=0$ produced by blowing up. These are bf_0, arising from blowing up $\partial\overline{M}\times\partial\overline{M}\times\{0\}$; the face lb_0, arising from blowing up $\partial\overline{M}\times\overline{M}\times\{0\}$; and lastly rb_0, arising from blowing up $\overline{M}\times\partial\overline{M}\times\{0\}$. See Figure 1 of [Guillarmou et al. 2013a].

The resolvent (microlocally away from the conormal bundle of the diagonal) was shown in [Guillarmou et al. 2013a] to be polyhomogeneous and vanish to order $n-2$ at the boundary hypersurfaces lb_0, rb_0, bf_0, and to vanish to order $\frac{n-1}{2}$ at lb and rb. Cases (3.i) and (3.ii) apply when the microlocalizing operators Q_i and Q_j remove the wavefront set at lb, rb and bf, meaning there is infinite-order vanishing there. Moreover, the cutoff factor $\chi\big(\frac{x}{\lambda}\big)+\chi\big(\frac{x'}{\lambda}\big)$ vanishes in a neighborhood of zf. Now notice that x vanishes to first order at lb, lb_0 and bf_0, while x' vanishes to first order at rb, rb_0 and bf_0 and $x+x'+\lambda$ vanishes to first order at bf_0. So the product on the right-hand side of (5-9) and (5-10) precisely encodes the order of vanishing at these remaining boundary hypersurfaces.

Proof. This is a combination of several results from [Guillarmou and Hassell 2014; Guillarmou et al. 2013b]. In the high-energy case, $\eta\ge\frac{1}{2}$, Lemma 5.3 of [Guillarmou and Hassell 2014] tells us that the pairs (i,j) split into four cases. In the first two cases, $Q_i(\eta)^*$ is either not-incoming or not-outgoing related to $Q_j(\eta)$, and then Proposition 6.7 of [Guillarmou and Hassell 2014] applies; note that the estimates in (2.i) and (2.ii) above appear in the proof, rather than the statement, of Proposition 6.7. In the third and fourth cases, Theorem 1.12 of [Guillarmou et al. 2013b] applies and shows that estimates (5-5) hold; see also Proposition 6.4 of [Guillarmou and Hassell 2014]. Also in the third and fourth cases, Proposition 1.5 of [Hassell and Zhang 2016] holds and gives the estimates (5-6), (5-7) and (5-8). Note that [Hassell and Zhang 2016, Proposition 1.5] is written in the case when $i=j$ but the proof of that proposition shows that it remains valid more generally when $i\ne j$ but the microsupports are close enough.

In the low-energy case, as shown in Section 6 of [Guillarmou and Hassell 2014], case (3.iii) applies to the pairs $(0,0)$, $(*,*)$, and (i,j) where $i,j\ge1$ and $|i-j|\le1$. Moreover, case (3.iii) also applies to any pair where either $i=*$ or $j=*$. That is because in these cases, the operator $Q_*(\eta)$ annihilates the wavefront set of the spectral measure at bf, with the consequence that the spectral measure estimates

$$\big|\partial_\lambda^k\big(Q_i(\eta)^*dE_{\sqrt{\Delta_g}}(\lambda)Q_j(\eta)\big)(m,m')\big|\le C_k\lambda^{n-1-k}(1+\lambda d(m,m'))^{-\frac{(n-1)}{2}+k} \tag{5-11}$$

hold if either $i=*$ or $j=*$, and this leads to estimates (5-5) as in the high-energy case. For (3.iii) with $i,j\ge1$, the estimates (5-6), (5-7) and (5-8) are proven in [Hassell and Zhang 2016, Proposition 1.5] in the case when $i=j$ but the proof shows that it remains valid more generally when $i\ne j$ but the microsupports are close enough. The case $i,j\in\{0,*\}$ in (3.iii) is also shown in [Hassell and Zhang 2016, Proposition 1.5].

The cases $i=0$ and $j\ge1$, and $i\ge1$ and $j=0$, fit any one of the cases (3.i), (3.ii), (3.iii) above. This is because here the wavefront set at bf is wiped out by $Q_0(\eta)$, while the wavefront set at fiber-infinity is wiped out by $Q_j(\eta)$ for $j\ge1$.

The final case remaining, where $i, j \geq 1$ and $|i-j| \geq 2$, fits into cases (3.i) or (3.ii) according to whether $Q_i(\eta)^*$ is not incoming-related or not outgoing-related to $Q_j(\eta)$, as shown in Proposition 6.9 of [Guillarmou and Hassell 2014]. □

Cases (3.i) and (3.ii) will be treated using the following lemma.

Lemma 5.4. *Let (M, g) be an asymptotically conic manifold of dimension $n \geq 3$. Then if an integral operator K has kernel $K(m, m')$ bounded pointwise by*

$$C\frac{(xx')^{\frac{n-1}{2}}\big(\chi\big(\frac{x}{\lambda}\big)+\chi\big(\frac{x'}{\lambda}\big)\big)}{x+x'+\lambda}, \quad 0<\lambda\leq 3,$$

then for $W_1, W_2 \in L^{2p}(M)$, $p \in \big[\frac{n}{2}, \frac{n+1}{2}\big]$, the operator $W_1 K W_2$ is Hilbert–Schmidt and we have

$$\|W_1 K W_2\|_{\mathcal{C}_2} \leq C\lambda^{-2+\frac{n}{p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}. \tag{5-12}$$

Proof. Using Hölder's inequality with $\frac{1}{p'}+\frac{1}{p}=1$ and $p' \in \big[\frac{n+1}{n-1}, \frac{n}{n-2}\big]$, we get

$$\|W_1 K W_2\|_{\mathcal{C}_2} \leq \|W_1\|_{L^{2p}}\|W_2\|_{L^{2p}}\bigg(\int_{M\times M}\frac{(x(m)x(m'))^{(n-1)p'}\big(\chi\big(\frac{x(m)}{\lambda}\big)+\chi\big(\frac{x(m')}{\lambda}\big)\big)^{2p'}}{(x(m)+x(m')+\lambda)^{2p'}}\,dV_g(m)\,dV_g(m')\bigg)^{\frac{1}{2p'}}.$$

We use the coordinates $m=(x,y)$, $m'=(x',y')$ near the boundary, where the measure $dV_g(m)$ is comparable to $dx\,dy/x^{n+1}$. Let us introduce the polar coordinates $(x,x')=(R\sin(\theta), R\cos(\theta))$ with $\theta\in\big[0,\frac{\pi}{2}\big]$, near $x=x'=0$. Using that $(n-1)p'-(n+1)\geq 0$ and $x+x'\sim R$, we get

$$\begin{aligned}
\bigg(\int_{M\times M}&\frac{(xx')^{(n-1)p'}\chi\big(\frac{x}{\lambda}\big)}{(x+x'+\lambda)^{2p'}}\,dV_g\,dV_{g'}\bigg)^{\frac{1}{2p'}}\\
&\leq C\bigg(\int_{0<x<2\lambda}\frac{(xx')^{(n-1)p'-(n+1)}}{(x+x'+\lambda)^{2p'}}\,dx\,dx'\bigg)^{\frac{1}{2p'}}\\
&\leq C\bigg(\int_0^\infty\int_{0<\sin\theta<\frac{2\lambda}{R}}\frac{R^{2(n-1)p'-2n-1}}{(R+\lambda)^{2p'}}\,dR\,d\theta\bigg)^{\frac{1}{2p'}}\\
&\leq C\frac{1}{\lambda}\bigg(\int_0^{2\lambda}R^{2(n-1)p'-2n-1}\,dR\bigg)^{\frac{1}{2p'}}+C\bigg(\int_{2\lambda}^\infty\int_{0<\theta\leq\frac{\tilde{C}\lambda}{R}}R^{2(n-1)p'-2p'-2n-1}\,dR\,d\theta\bigg)^{\frac{1}{2p'}}\\
&\leq C\lambda^{\frac{n}{p}-2}+C\lambda^{\frac{1}{2p'}}\bigg(\int_{2\lambda}^\infty R^{2(n-2)p'-2n-2}\,dR\bigg)^{\frac{1}{2p'}}\leq C\lambda^{\frac{n}{p}-2}.
\end{aligned}$$

Here we used that $(n-1)p'>n$ and $2(n-2)p'-2n-1<0$. The same argument works with the term involving $\chi\big(\frac{x'}{\lambda}\big)$ and the estimate (5-12) follows. □

5B. ***Analytic family of operators.*** In this section we closely follow Section 4 of [Guillarmou and Hassell 2014], especially Remark 4.2 (which is essentially due to Adam Sikora). Let $\phi\in C_0^\infty\big(\big(\big(1-\frac{\delta}{4}\big)^2, \big(1+\frac{\delta}{4}\big)^2\big)\big)$ be such that $\phi(t)=1$ in a neighborhood of $t=1$, where $\delta>0$ is small, and consider the analytic family of operators in $\mathrm{Re}(s) \le 0$

$$H_{s,z,\varepsilon}(\Delta_g) = \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z + i\varepsilon))^s, \quad z > 0,\ \varepsilon > 0.$$

By the spectral theorem, we have

$$H_{s,z,\varepsilon}(\Delta_g) = z^{s+\frac{1}{2}} \int_0^\infty \left(\lambda - \left(1 + i\frac{\varepsilon}{z}\right)\right)^s \frac{\phi(\lambda)}{2\sqrt{\lambda}} dE_{\sqrt{\Delta_g}}(z^{\frac{1}{2}}\lambda^{\frac{1}{2}})\, d\lambda. \tag{5-13}$$

Let $\eta > 0$ be such that $z^{1/2} \in \left[\left(1 - \frac{\delta}{2}\right)\eta, \left(1 + \frac{\delta}{2}\right)\eta\right]$ and let $Q_i(\eta)$ and $Q_j(\eta)$ be such that the condition (2.iii) or (3.iii) of Proposition 5.1 holds, in the high-energy, respectively, low-energy case. Then using (5-13), we have on the level of Schwartz kernels, for $m, m' \in M$,

$$\left(Q_i(\eta)^* H_{s,z,\varepsilon}(\Delta_g) Q_j(\eta)\right)(m, m') = z^{s+\frac{1}{2}} \int_0^\infty \left(\lambda - \left(1 + i\frac{\varepsilon}{z}\right)\right)^s \psi(\lambda)\, d\lambda, \tag{5-14}$$

where

$$\psi(\lambda) = \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}) Q_j(\eta)(m, m').$$

Here, as $\delta > 0$ is small, we have $z^{1/2}\lambda^{1/2} \in [(1-\delta)\eta, (1+\delta)\eta]$ when $z^{1/2} \in \left[\left(1 - \frac{\delta}{2}\right)\eta, \left(1 + \frac{\delta}{2}\right)\eta\right]$ and $\lambda \in \mathrm{supp}(\phi)$, and therefore, in view of (5-5), we have $\psi(\lambda) \in C_0^\infty(\mathbb{R})$.

Letting $\varepsilon \to 0$ in (5-14), we define $Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)$ when $z^{1/2} \in \left[\left(1-\frac{\delta}{2}\right)\eta, \left(1+\frac{\delta}{2}\right)\eta\right]$ as operators whose Schwartz kernels are given by

$$\begin{aligned}\left(Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)\right)(m, m') &= z^{s+\frac{1}{2}} \int_0^\infty (\lambda - (1 + i0))^s \psi(\lambda)\, d\lambda \\ &= z^{s+\frac{1}{2}}((\lambda - i0)^s * \psi(\lambda))(1).\end{aligned} \tag{5-15}$$

We are interested in pointwise estimates for the kernel of $Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)$ and to this end we shall need the following result of [Guillarmou and Hassell 2014, Remark 4.2]. Even though the proof is almost the same as that of [Guillarmou et al. 2013b, Lemma 3.3], for completeness we provide a proof in Appendix A.

Lemma 5.5. *Let $a < b < c \le 0$ and let us write $b = \theta a + (1-\theta)c$, $0 < \theta < 1$. Then there is $C > 0$ such that, for all $f \in C_0^\infty(\mathbb{R})$, all $t \in \mathbb{R}$, and all $0 < \varepsilon \ll 1$, we have*

$$\|(\lambda \pm i\varepsilon)^{b+it} * f\|_{L^\infty_\lambda} \le C(1 + |t|)e^{\frac{3\pi|t|}{2}} \|\chi_+^a * f\|^\theta_{L^\infty_\lambda} \|\chi_+^c * f\|^{1-\theta}_{L^\infty_\lambda}. \tag{5-16}$$

We have the following result.

Proposition 5.6. *Suppose that (i, j) are such that the condition (2.iii) or (3.iii) holds in the high-energy, respectively, low-energy case. Then there is $C > 0$ such that the kernel of the operator $Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)$ with $z > 0$ and $z^{1/2} \in \left[\left(1-\frac{\delta}{2}\right)\eta, \left(1+\frac{\delta}{2}\right)\eta\right]$ has the following pointwise estimates,*

(i) *For $\mathrm{Re}(s) = -\frac{(n+1)}{2}$, we have*

$$|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m, m')| \le Ce^{C|\mathrm{Im}(s)|} z^{-\frac{1}{2}} \tag{5-17}$$

for all $m, m' \in M$, uniformly in z and η.

(ii) *For* $\mathrm{Re}(s) = -\frac{(n-1)}{2}$, *we have*

$$|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m,m')| \le C e^{C|\mathrm{Im}(s)|} d(m,m')^{-1} \tag{5-18}$$

for all $m, m' \in M$, *uniformly in* z *and* η.

Proof. Estimate (5-17) is proved in [Guillarmou and Hassell 2014, Remark 4.2]. Estimate (5-18) is proved in the same way, except for the case $n = 3$, relying on the estimates (5-5) only. Indeed, in the case $n \ge 5$ is odd, we take $a = -\frac{(n+1)}{2}$ and $c = -\frac{(n-3)}{2}$ in Lemma 5.5 and using that

$$\chi_+^{-k} = \delta_0^{(k-1)}, \quad k = 1, 2, \dots,$$

we get

$$\begin{aligned}|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m,m')| &\le C z^{\frac{2-n}{2}} (1 + |\mathrm{Im}(s)|) e^{\frac{3\pi|\mathrm{Im}(s)|}{2}} \\ &\times \left\| \partial_\lambda^{\frac{n-1}{2}} \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty}^{\frac12} \\ &\times \left\| \partial_\lambda^{\frac{n-5}{2}} \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty}^{\frac12},\end{aligned}$$

and therefore, using (5-5), we obtain

$$\begin{aligned}|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m,m')| &\le C e^{C|\mathrm{Im}(s)|} z^{\frac12} (1 + z^{\frac12} d(m,m'))^{-1} \\ &\le C e^{C|\mathrm{Im}(s)|} d(m,m')^{-1}.\end{aligned} \tag{5-19}$$

For $n \ge 4$ even, taking $a = -\frac{n}{2}$, $c = -\frac{(n-2)}{2}$ in Lemma 5.5 and using (5-5), we also get (5-19). We have therefore established (5-18) for all $n \ge 4$.

When $n = 3$, using Lemma 5.5 with $a = -2$ and $c = 0$, and the fact that $\chi_+^0(\lambda) = H(\lambda)$ is the Heaviside function, we obtain

$$\begin{aligned}|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m,m')| &\le C z^{-\frac12} (1+|\mathrm{Im}(s)|) e^{\frac{3\pi|\mathrm{Im}(s)|}{2}} \\ &\times \left\| \partial_\lambda \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty}^{\frac12} \\ &\times \left\| H * \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty}^{\frac12}.\end{aligned} \tag{5-20}$$

By (5-5), we get

$$\left\| \partial_\lambda \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty} \le C z. \tag{5-21}$$

Now if we show that

$$\left\| H * \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty} \le C d(m,m')^{-2}, \tag{5-22}$$

then the estimate (5-18) will follow from (5-20), (5-21) and (5-22). To prove (5-22), using (5-6), we write

$$
\begin{aligned}
H * \Big(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m')\Big)(\lambda)
&= \int_0^{\lambda^{1/2}} \phi(\mu^2) Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\mu) Q_j(\eta)(m,m')\, d\mu \\
&= \int_0^{\lambda^{1/2}} \phi(\mu^2) z\mu^2 \Big[\sum_{\pm} e^{\pm i z^{1/2}\mu d(m,m')} a_\pm(z^{\frac12}\mu, m, m') + b(z^{\frac12}\mu, m, m')\Big]\, d\mu. \qquad (5\text{-}23)
\end{aligned}
$$

The terms involving $a_\pm$ in (5-23) can be treated similarly and in what follows we shall only consider the term involving a_+ and drop the sign $+$. To estimate this term, we integrate by parts and get

$$
\begin{aligned}
&\int_0^{\lambda^{1/2}} \phi(\mu^2) z\mu^2 e^{i z^{1/2}\mu d(m,m')} a(z^{\frac12}\mu, m, m')\, d\mu \\
&\quad = \frac{1}{i z^{\frac12} d(m,m')} \Big[\phi(\mu^2) z\mu^2 e^{i z^{1/2}\mu d(m,m')} a(z^{\frac12}\mu, m, m')\big|_{\mu=0}^{\mu=\lambda^{1/2}} \\
&\qquad\qquad - \int_0^{\lambda^{1/2}} \partial_\mu\big(\phi(\mu^2) z\mu^2 a(z^{\frac12}\mu, m, m')\big) e^{i z^{1/2}\mu d(m,m')}\, d\mu \Big]. \qquad (5\text{-}24)
\end{aligned}
$$

Estimating the terms in the left-hand side of (5-24) with the help of (5-7), we obtain that

$$
\left| \int_0^{\lambda^{1/2}} \phi(\mu^2) z\mu^2 e^{i z^{\frac12}\mu d(m,m')} a(z^{\frac12}\mu, m, m')\, d\mu \right| \le C\lambda^{\frac12} d(m,m')^{-2}, \qquad (5\text{-}25)
$$

uniformly in z. To estimate the term involving the remainder b in (5-23), we use (5-8) with $K = 2$ and get

$$
\begin{aligned}
\int_0^{\lambda^{1/2}} \phi(\mu^2) z\mu^2 |b(z^{\frac12}\mu, m, m')|\, d\mu &\le C \int_0^{\lambda^{1/2}} \phi(\mu^2) z\mu^2 (1 + z^{\frac12}\mu d(m,m'))^{-2}\, d\mu \\
&\le C d(m,m')^{-2}. \qquad (5\text{-}26)
\end{aligned}
$$

Now (5-22) follows from (5-23), (5-25) and (5-26). This completes the proof of estimate (5-18). □

When proving the Schatten bound on the resolvent on the spectrum in Section 5C below, the cases (2.iii) and (3.iii) of Proposition 5.1 will be treated using the following result.

Proposition 5.7. *Suppose that (i, j) are such that the condition (2.iii) or (3.iii) holds in the high-energy, respectively low-energy case. Let $p \in [\frac{n}{2}, \frac{n+1}{2}]$. Then there is $C > 0$ such that, for all $z \in (0,\infty)$, $z^{1/2} \in \big[\big(1-\frac{\delta}{2}\big)\eta, \big(1+\frac{\delta}{2}\big)\eta\big]$ and all $W_1, W_2 \in L^{2p}(M)$, we have $W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2 \in \mathcal{C}_q(L^2(M))$, $q = \frac{p(n-1)}{n-p}$, and*

$$
\|W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2\|_{\mathcal{C}_q} \le C z^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \qquad (5\text{-}27)
$$

Proof. First thanks to Proposition 5.6, case (i), we know that for $\operatorname{Re} s = -\frac{(n+1)}{2}$,

$$
\|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)\|_{L^1(M)\to L^\infty(M)} \le C e^{C|\operatorname{Im}(s)|} z^{-\frac12}.
$$

By the spectral theorem, we also know that for $\operatorname{Re} s = 0$

$$\|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)\|_{L^2(M)\to L^2(M)} \le C e^{\pi|\operatorname{Im}(s)|}.$$

Hence, Proposition 2.1 implies that $W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2 \in \mathcal{C}_{n+1}(L^2(M))$ and, moreover,

$$\|W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2\|_{\mathcal{C}_{n+1}} \le C z^{-\frac{1}{n+1}} \|W_1\|_{L^{n+1}(M)} \|W_2\|_{L^{n+1}(M)}. \tag{5-28}$$

Now when $\operatorname{Re} s = -\frac{(n-1)}{2}$, thanks to Proposition 5.6(ii), the kernel of the operator $Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)$ has the bound (5-18), which is the same as the bound (4-6) in the proof of Proposition 4.3. Proceeding exactly as in the proof of Proposition 4.3, we get

$$\|W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}. \tag{5-29}$$

In view of (5-28) and (5-29), the bound (5-27) follows by a complex interpolation argument applied to the analytic family of operators

$$\zeta \mapsto W_1^{\frac{2}{n+1}+\zeta\frac{2}{n(n+1)}} Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2^{\frac{2}{n+1}+\zeta\frac{2}{n(n+1)}}$$

in the strip $0 \le \operatorname{Re} \zeta \le 1$, with $W_j \ge 0$ being simple functions such that $\|W_j\|_{L^2(M)} = 1$, $j = 1, 2$; see [Simon 2015, p. 154]. □

5C. *Resolvent estimates on the spectrum.* The final ingredient in the proof of Theorem 2 is the following result.

Proposition 5.8. *Let $\phi \in C_0^\infty\big(\big((1-\frac{\delta}{4})^2, (1+\frac{\delta}{4})^2\big)\big)$ be such that $\phi(t) = 1$ in a neighborhood of $t = 1$, where $\delta > 0$ is small, and let $p \in \big[\frac{n}{2}, \frac{n+1}{2}\big]$. Then there is $C > 0$ such that for all $z \in (0,\infty)$ and all $W_1, W_2 \in L^{2p}(M)$, for $q = \frac{p(n-1)}{n-p}$ we have $W_1 \phi(\Delta_g/z)(\Delta_g - (z+i0))^{-1} W_2 \in \mathcal{C}_q(L^2(M))$ and*

$$\left\| W_1 \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} W_2 \right\|_{\mathcal{C}_q} \le C z^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{5-30}$$

Proof. Let us first take the high-energy case $z \ge 1$ and let $\eta \ge 1$ be such that $\sqrt{z} \in \big[(1-\frac{\delta}{2})\eta, (1+\frac{\delta}{2})\eta\big]$. We decompose the spectrally localized outgoing resolvent $\phi(\Delta_g/z)(\Delta_g - (z+i0))^{-1}$ into microlocalized pieces

$$W_1 \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} W_2 = \sum_{i,j=1}^{N_h} W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2.$$

The bound (5-30) will follow if we show that for all (i, j) we have

$$\left\| W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2 \right\|_{\mathcal{C}_q} \le C z^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{5-31}$$

To that end, the pairs (i, j) will be divided into three cases as in Proposition 5.1.

In the first case, (2.i), in view of (5-3) and Corollary B.5, we know that the Schwartz kernel of the operator $Q_i(\eta)^* \phi(\Delta_g/z)(\Delta_g - z - i0)^{-1} Q_j(\eta)$ is $\mathcal{O}(z^{-N})$ in $L^{2p'}(M \times M)$ with $\frac{1}{p'} + \frac{1}{p} = 1$. Using this together with the fact that $q \geq 2$ and Hölder's inequality, we get

$$\left\| W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g-(z+i0))^{-1} Q_j(\eta) W_2 \right\|_{\mathcal{C}_q} \leq \left\| W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g-(z+i0))^{-1} Q_j(\eta) W_2 \right\|_{\mathcal{C}_2}$$
$$\leq \mathcal{O}(z^{-N}) \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}$$

for any $N \in \mathbb{N}$, showing (5-31).

In the second case, (2.ii), using Stone's formula, we write

$$W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2$$
$$= W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z-i0))^{-1} Q_j(\eta) W_2$$
$$+ \frac{\pi i}{\lambda} W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2, \quad \lambda = \sqrt{z}. \tag{5-32}$$

Then the estimate for the term involving the incoming resolvent in (5-32) follows exactly as in case (2.i). On the other hand, we have already proved the corresponding estimate (3-9) for the spectral measure, which leads to the estimate (5-31) in this case.

In the third case, (2.iii), we get

$$W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2 = W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2, \tag{5-33}$$

where the operator $Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta)$ is defined in (5-15). The required estimate for this term therefore is a consequence of Proposition 5.7.

In the low-energy case, $0 < z \leq 1$, the argument is similar. In cases (3.i) and (3.ii) we use Corollary B.5 together with Lemma 5.4 and the bound (3-9) for the spectral measure to deduce the Schatten norm estimate. In case (3.iii), the argument is the same as for case (2.iii). This concludes the proof of the proposition. □

6. Bounds on individual eigenvalues: proof of Theorem 4

In this section we shall follow some of the arguments of [Frank 2018; Frank and Simon 2017], making some necessary changes due to the fact that we are no longer in the Euclidean setting.

Let us recall that $n = \dim(M) \geq 3$. We have the following result which is a generalization of [Frank 2018, Lemma 4.2] to the case of the Laplace operator on asymptotically conic manifolds.

Proposition 6.1. *Let $V \in L^p(M)$ with $\frac{n}{2} \leq p < \infty$. The operator $\sqrt{|V|}(\Delta_g+1)^{-1/2}$ is compact on $L^2(M)$.*

Proof. We follow [Frank 2018, Lemma 4.2]. First we shall show that

$$\|W(\Delta_g+1)^{-\frac{1}{2}}\|_{\mathcal{L}(L^2(M),L^2(M))} \leq C\|W\|_{L^{2p}(M)}, \quad W \in L^{2p}(M). \tag{6-1}$$

Indeed, we have

$$(\Delta_g+1)^{-\frac{1}{2}} : L^2(M) \to H^1(M) \tag{6-2}$$

is bounded, and therefore, by Sobolev's embedding $H^1(M) \subset L^{2n/(n-2)}(M)$, which is valid on an asymptotically conic manifold of dimension $n \geq 3$, see [Guillarmou and Hassell 2014, Proposition 2.1], we get

$$(\Delta_g + 1)^{-\frac{1}{2}} : L^2(M) \to L^{\frac{2n}{n-2}}(M) \tag{6-3}$$

is also bounded. Using Hölder's inequality, the logarithmic convexity of L^p norms, and (6-2), (6-3), we obtain

$$\begin{aligned}\|W(\Delta_g + 1)^{-\frac{1}{2}} f\|_{L^2(M)} &\leq \|W\|_{L^{2p}(M)} \|(\Delta_g + 1)^{-\frac{1}{2}} f\|_{L^{2p/(p-1)}(M)} \\ &\leq \|W\|_{L^{2p}(M)} \|(\Delta_g + 1)^{-\frac{1}{2}} f\|_{L^2(M)}^{1-\frac{n}{2p}} \|(\Delta_g + 1)^{-\frac{1}{2}} f\|_{L^{2n/(n-2)}(M)}^{\frac{n}{2p}} \\ &\leq C \|W\|_{L^{2p}(M)} \|f\|_{L^2(M)},\end{aligned}$$

showing (6-1).

Let $W_j \in C_0^\infty(M)$ be such that $W_j \to \sqrt{|V|}$ in $L^{2p}(M)$. By Rellich's compactness theorem, the operator $W_j(\Delta_g + 1)^{-1/2}$ is compact on $L^2(M)$, and it follows from (6-1) that $W_j(\Delta_g + 1)^{-1/2} \to \sqrt{|V|}(\Delta_g + 1)^{-1/2}$ in $\mathcal{L}(L^2(M), L^2(M))$. □

Setting

$$\sqrt{V(x)} = \begin{cases} V(x)/\sqrt{|V(x)|}, & V(x) \neq 0, \\ 0, & V(x) = 0, \end{cases}$$

and combining Proposition 6.1 with [Frank 2018, Lemma B.1], we get that the quadratic form

$$\|(\Delta_g)^{\frac{1}{2}} u\|_{L^2(M)}^2 + (\sqrt{V} u, \sqrt{|V|} u)_{L^2(M)},$$

equipped with the domain $H^1(M)$, is closed and sectorial. Associated to the quadratic form is an m–sectorial operator with domain $\subset H^1(M)$, which we shall denote by $\Delta_g + V$. The spectrum of $\Delta_g + V$ in $\mathbb{C} \setminus [0, \infty)$ consists of isolated eigenvalues of finite algebraic multiplicity; see [Frank 2018, Proposition B.2].

Now interpolating between the estimate, valid for $z \in \mathbb{C} \setminus [0, \infty)$,

$$\|(\Delta_g - z)^{-1}\|_{L^2(M) \to L^2(M)} = \frac{1}{d(z)},$$

and the uniform estimate (1-8), with $p = \frac{2(n+1)}{n+3}$, we obtain the following result.

Corollary 6.2. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Then for all $p \in \left[\frac{2(n+1)}{n+3}, 2\right]$ there is a constant $C > 0$ such that for all $z \in \mathbb{C} \setminus [0, \infty)$,*

$$\|(\Delta_g - z)^{-1}\|_{L^p(M) \to L^{p'}(M)} \leq C d(z)^{(n+1)(\frac{1}{p} - \frac{1}{2}) - 1} |z|^{\frac{1}{2} - \frac{1}{p}}. \tag{6-4}$$

We shall now proceed to prove Theorem 4. In doing so we shall follow [Frank and Simon 2017, Theorem 3.2]. Let $\lambda \in \mathbb{C}$ be an eigenvalue and $\psi \in H^1(M)$ be the corresponding eigenfunction of $\Delta_g + V$,

$$(\Delta_g + V)\psi = \lambda \psi.$$

(i) Let $0 < \gamma \le \frac{1}{2}$. Assume first that $\lambda \in \mathbb{C} \setminus [0, \infty)$. Let us choose $p > 1$ such that

$$\gamma + \tfrac{n}{2} = \tfrac{p}{2-p}, \tag{6-5}$$

and notice that then $\frac{2n}{n+2} < p \le \frac{2(n+1)}{n+3}$ and $\frac{2(n+1)}{n-1} \le p' < \frac{2n}{n-2}$.

By Sobolev's embedding, we have $\psi \in L^{2n/(n-2)}(M)$, and thus, $\psi \in L^r(M)$ for $r \in \bigl[2, \frac{2n}{n-2}\bigr]$, by interpolation. In particular, $\psi \in L^{p'}(M)$, and by Hölder's inequality, we get

$$\|V\psi\|_{L^p(M)} \le \|V\|_{L^{p/(2-p)}(M)} \|\psi\|_{L^{p'}(M)} = \|V\|_{L^{\gamma+n/2}(M)} \|\psi\|_{L^{p'}(M)}.$$

We have

$$\psi = (\Delta_g - \lambda)^{-1}(\Delta_g - \lambda)\psi = -(\Delta_g - \lambda)^{-1}(V\psi).$$

Hence, using (1-8), we get

$$\begin{aligned} \|\psi\|_{L^{p'}(M)} &\le \|(\Delta_g - \lambda)^{-1}\|_{L^p(M) \to L^{p'}(M)} \|V\psi\|_{L^p(\mathbb{R}^n)} \\ &\le C|\lambda|^{\frac{n}{2}(\frac{2}{p}-1)-1} \|V\|_{L^{\gamma+n/2}(M)} \|\psi\|_{L^{p'}(M)}, \end{aligned} \tag{6-6}$$

which implies (1-11) in view of

$$\tfrac{n}{2}\bigl(\tfrac{2}{p} - 1\bigr) - 1 = -\frac{\gamma}{\gamma + \frac{n}{2}}.$$

Assume now that $\lambda \in (0, \infty)$. Then for $\varepsilon > 0$, we set

$$\psi_\varepsilon = (\Delta_g - \lambda - i\varepsilon)^{-1}(\Delta_g - \lambda)\psi = f_\varepsilon(\Delta_g)\psi,$$

where

$$f_\varepsilon(t) = \frac{t - \lambda}{t - \lambda - i\varepsilon}, \quad t \in \mathbb{R}.$$

By the spectral theorem, we have

$$\|\psi_\varepsilon - \psi\|^2_{L^2(M)} = \|f_\varepsilon(\Delta_g)\psi - \psi\|^2_{L^2(M)} = \int |f_\varepsilon(t) - 1|^2 \, d(E_{\Delta_g}(t)\psi, \psi)_{L^2(M)},$$

where $dE_{\Delta_g}(t)$ is the spectral measure of Δ_g. Using the dominated convergence theorem together with the fact that $f_\varepsilon(t) \to 1$ as $\varepsilon \to 0$ for all $t \ne \lambda$, and that $E_\lambda = 0$ as λ is not an eigenvalue of Δ_g, we conclude that $\psi_\varepsilon \to \psi$ in $L^2(M)$.

On the other hand, we have

$$\psi_\varepsilon = -(\Delta_g - \lambda - i\varepsilon)^{-1}(V\psi).$$

Choosing $p > 1$ satisfying (6-5) and using (1-8), we obtain

$$\|\psi_\varepsilon\|_{L^{p'}(M)} \le C|\lambda|^{\frac{n}{2}(\frac{2}{p}-1)-1} \|V\|_{L^{\gamma+n/2}(M)} \|\psi\|_{L^{p'}(M)}, \tag{6-7}$$

i.e., ψ_ε is uniformly bounded in $L^{p'}(M)$. Passing to a subsequence, we may assume that there exists $\tilde\psi \in L^{p'}(M)$ such that $\psi_\varepsilon \to \tilde\psi$ in the weak-$*$ topology of $L^{p'}(M)$. It follows that $\psi = \tilde\psi \in L^{p'}(M)$. By the lower semicontinuity of the norm and (6-7), we get

$$\|\psi\|_{L^{p'}(M)} \le \liminf_{\varepsilon \to 0} \|\psi_\varepsilon\|_{L^{p'}(M)} \le C|\lambda|^{\frac{n}{2}(\frac{2}{p}-1)-1} \|V\|_{L^{\gamma+n/2}(M)} \|\psi\|_{L^{p'}(M)}, \tag{6-8}$$

which shows (1-11) when $\lambda \in (0, \infty)$.

(ii) Let $V \in L^{n/2}(M)$. Setting $p = \frac{2n}{n+2}$, and arguing as in the case (i) above, for $\lambda \in \mathbb{C} \setminus \{0\}$, we obtain

$$\|\psi\|_{L^{p'}(M)} \le C\|V\|_{L^{n/2}(M)}\|\psi\|_{L^{p'}(M)}.$$

The case $\lambda = 0$ is handled similarly using that

$$\|(\Delta_g - i\varepsilon)^{-1}\|_{L^p(M)\to L^{p'}(M)} \le \mathcal{O}(1),$$

in view of (1-8). The claim (ii) follows.

(iii) Let $\gamma > \frac{1}{2}$, and let $\lambda \in \mathbb{C} \setminus [0,\infty)$ be an eigenvalue of $\Delta_g + V$, and $\psi \in H^1(M)$ be the corresponding eigenfunction. Choosing $p > 1$ satisfying (6-5), we have $\frac{2(n+1)}{n+3} < p < 2$ and $2 < p' < \frac{2(n+1)}{n-1}$. Using that $\psi \in L^{p'}(M)$ and (6-4), similarly to above, we obtain

$$\begin{aligned}\|\psi\|_{L^{p'}(M)} &\le \|(\Delta_g - \lambda)^{-1}\|_{L^p(M)\to L^{p'}(M)}\|V\psi\|_{L^p(M)}\\ &\le C\delta(\lambda)^{(n+1)(\frac{1}{p}-\frac{1}{2})-1}|\lambda|^{\frac{1}{2}-\frac{1}{p}}\|V\|_{L^{\gamma+n/2}(M)}\|\psi\|_{L^{p'}(M)},\end{aligned}$$

which implies (1-12) in view of the fact that

$$\frac{1}{p} = \frac{1+\gamma+\frac{n}{2}}{2\left(\gamma+\frac{n}{2}\right)}.$$

This completes the proof of Theorem 4.

7. Bounds on sums of eigenvalues for Schrödinger operators with complex potentials

7A. ***Short-range potentials: proof of Theorem 5.*** Let $V \in L^p(M)$, $\frac{n}{2} \le p \le \frac{n+1}{2}$, and let $q = \frac{p(n-1)}{n-p}$. Then Theorem 2 implies that for $z \in \mathbb{C} \setminus [0,\infty)$, we have $\sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|} \in \mathcal{C}_q(L^2(M))$ and

$$\|\sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}\|_{\mathcal{C}_q(L^2(M))} \le C|z|^{-1+\frac{n}{2p}}\|V\|_{L^p(M)}. \tag{7-1}$$

We claim that the map

$$\mathbb{C} \setminus [0,\infty) \ni z \mapsto \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|} \tag{7-2}$$

is holomorphic with values in $\mathcal{C}_q(L^2(M))$. First let us check that (7-2) is holomorphic with values in $\mathcal{L}(L^2(M), L^2(M))$. Indeed, letting $z_0 \in \mathbb{C} \setminus [0,\infty)$, we write

$$\sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|} = \sqrt{V}\sum_{j=0}^{\infty}(z-z_0)^j(\Delta_g - z_0)^{-j-1}\sqrt{|V|} \tag{7-3}$$

and notice that

$$\begin{aligned}\|\sqrt{V}(\Delta_g - z_0)^{-j-1}\sqrt{|V|}\|_{\mathcal{L}(L^2(M),L^2(M))} &\le \|\sqrt{V}(\Delta_g - z_0)^{-1}\|_{\mathcal{L}(L^2(M),L^2(M))},\\ \|(-\Delta - z_0)^{-1}\sqrt{|V|}\|_{\mathcal{L}(L^2(M),L^2(M))}&\|(\Delta_g - z_0)^{-1}\|^{j-1}_{\mathcal{L}(L^2(M),L^2(M))} \le C^{j+1}\end{aligned}$$

for some $C > 0$. Here we have used that the operators $\sqrt{V}(-\Delta - z_0)^{-1}$, $(\Delta_g - z_0)^{-1}\sqrt{|V|}$ are bounded on $L^2(M)$, as seen by arguing as in the proof of (6-1). This shows that the series (7-3) converges in $\mathcal{L}(L^2(M), L^2(M))$ for $|z-z_0|$ small, and therefore, the map (7-2) is holomorphic with values in $\mathcal{L}(L^2(M), L^2(M))$. In particular, if $T \in \mathcal{C}_1(L^2(M))$, i.e., of trace class, the map

$$\mathbb{C}\setminus[0,\infty) \ni z \mapsto \langle \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}, T\rangle \tag{7-4}$$

is holomorphic. Using the density of $\mathcal{C}_1(L^2(M))$ in $\mathcal{C}_{q'}(L^2(M))$, the bound (7-1), and Hölder's inequality in Schatten classes, we conclude that the map (7-4) is holomorphic for all $T \in \mathcal{C}_{q'}(L^2(M))$, establishing the claim.

Consider the holomorphic function

$$h(z) := \det_{\lceil q\rceil}(1 + \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}), \quad z \in \mathbb{C}\setminus[0,\infty),$$

where $\lceil q\rceil$ is the smallest integer $\geq q$, and $\det_{\lceil q\rceil}$ is the regularized determinant; see [Simon 1979, Chapter 9]. As explained in [Frank and Sabin 2017, proof of Theorem 16], using (7-1), we get

$$\log|h(z)| \leq C\|\sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}\|^q_{\mathcal{C}_q} \leq C|z|^{(-1+\frac{n}{2p})q}\|V\|^q_{L^p(M)}, \tag{7-5}$$

uniformly in $z \in \mathbb{C}\setminus[0,\infty)$.

Combining Proposition 6.1 and Lemma B.1 of [Frank 2018], we conclude that the following version of the Birman–Schwinger principle holds: $z \in \mathbb{C}\setminus[0,\infty)$ is an eigenvalue of $\Delta_g + V$ if and only if

$$\operatorname{Ker}(1 + \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}) \neq \{0\}. \tag{7-6}$$

An application of Lemma 3.2 of [Frank 2018] gives that (7-6) is equivalent to the fact that $h(z) = 0$ and that the order of vanishing of h at z agrees with the algebraic multiplicity of z as an eigenvalue of $\Delta_g + V$.

At this point we are exactly in the same situation as in [Frank and Sabin 2017, Theorem 16]. Here we may remark that the proof of that result is based on a result of Borichev, Golinskii and Kupin [Borichev et al. 2009] concerning the distribution of zeros of a holomorphic function in the unit disc growing rapidly at a boundary point. The proof of Theorem 5 is therefore complete.

7B. *Long-range potentials*: *proof of Theorem 6.* First we have the following result: Let $\gamma \geq \frac{1}{2}$. Then there exists a constant $C > 0$ such that for all $W \in L^{2(\gamma+n/2)}(M)$ and all $z \in \mathbb{C}\setminus[0,\infty)$,

$$\|W(\Delta_g - z)^{-1}W\|_{\mathcal{C}_{2(\gamma+n/2)}} \leq C d(z)^{-1+\frac{n+1}{2(\gamma+n/2)}}|z|^{-\frac{1}{2(\gamma+n/2)}}\|W\|^2_{L^{2(\gamma+n/2)}(M)}. \tag{7-7}$$

Indeed this follows as in [Frank 2018, Proposition 2.1] by interpolation between (1-9) with $p = \frac{n+1}{2}$ and the standard bound

$$\|W(\Delta_g - z)^{-1}W\|_{L^2(M)\to L^2(M)} \leq d(z)^{-1}\|W\|^2_{L^\infty(M)}.$$

Now an application of [Frank 2018, Theorem 3.1] to the holomorphic family $K(z) = \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}$ completes the proof of Theorem 6 exactly in the same way as in [Frank 2018, Theorem 1.2].

Appendix A: Proof of Lemma 5.5

We shall follow the proof of Lemma 3.3 in [Guillarmou et al. 2013b] closely. Let $a < b < c \leq 0$ and let $\alpha := a - c - 1 < -1$ and $\beta := b - c - 1 < -1$. We shall show the estimate (5-16) for $\|(\lambda - i\varepsilon)^{b+it} * f\|_{L^\infty_\lambda}$, as the bound (5-16) for $\|(\lambda + i\varepsilon)^{b+it} * f\|_{L^\infty_\lambda}$ can be proved similarly.

To that end, let χ_-^z be the family of distributions on $\mathbb{R}$ holomorphic in $z \in \mathbb{C}$ given by

$$\chi_-^z(\lambda) = \frac{\lambda_-^z}{\Gamma(z+1)}, \quad \operatorname{Re} z > -1,$$

where

$$\lambda_-^z = \begin{cases} 0 & \text{if } \lambda > 0, \\ |\lambda|^z & \text{if } \lambda < 0. \end{cases}$$

We have $\chi_-^z(-\lambda) = \chi_+^z(\lambda)$. Recall from [Hörmander 1990, Section 3.2] that when $\operatorname{Re} z > -1$, we have

$$(\lambda - i0)^z = \lambda_+^z + e^{-i\pi z}\lambda_-^z \tag{A-1}$$

and from [Hörmander 1990, Example 7.1.17] that for $\varepsilon > 0$ and $z \in \mathbb{C}$, we have

$$\mathcal{F}((\lambda - i\varepsilon)^{-z})(\xi) = 2\pi e^{\frac{iz\pi}{2}} e^{\varepsilon\xi}\chi_-^{z-1}(\xi), \tag{A-2}$$

and

$$\mathcal{F}(\chi_+^z)(\xi) = e^{-i(z+1)\frac{\pi}{2}}(\xi - i0)^{-z-1}. \tag{A-3}$$

Consider the family of operators A_t for $t \in \mathbb{R}$ given by

$$A_t : C_0^\infty(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}), \quad A_t f := \eta_t * f, \tag{A-4}$$

where

$$\hat{\eta}_t(\xi) = \frac{2\pi e^{i(-\beta-it)\frac{\pi}{2} - i\pi(c+1)} e^{\varepsilon\xi}\xi_-^{-\beta-1-it}}{\Gamma(-b-it)(\sigma + e^{-i(\alpha+1)\frac{\pi}{2}}(\xi - i0)^{-\alpha-1})} \tag{A-5}$$

when $c < 0$, and

$$\hat{\eta}_t(\xi) = \frac{2\pi e^{-i(b-1+it)\frac{\pi}{2}} e^{\varepsilon\xi}\xi_-^{-b-it}}{\Gamma(-b-it)(\sigma - e^{-\frac{i\pi a}{2}}(\xi - i0)^{-a})} \tag{A-6}$$

when $c = 0$, and $\sigma \in \mathbb{C}$, $|\sigma| = 1$ and $\sigma \notin \{ie^{-i\alpha\pi/2}, -ie^{i\alpha\pi/2}, e^{ia\pi/2}\}$. In view of (A-1), we see that $\hat{\eta}_t \in \mathcal{S}'(\mathbb{R})$.

We notice that for all $t \in \mathbb{R}$, $\hat{\eta}_t \in L^1_{\mathrm{loc}}(\mathbb{R})$. Furthermore, using that

$$\left|\frac{1}{\Gamma(-b-it)}\right| \leq Ce^{\pi|t|},$$

we have, for $|\xi| \geq 1$,

$$|\partial_\xi\hat{\eta}_t(\xi)| \leq Ce^{\frac{3\pi|t|}{2}}(1+|t|)|\xi|^{-\beta+\alpha-1}, \tag{A-7}$$

and for $|\xi| \leq 1$ we get

$$|\partial_\xi\hat{\eta}_t(\xi)| \leq Ce^{\frac{3\pi|t|}{2}}(1+|t|)|\xi|^{-\beta-2}, \tag{A-8}$$

and therefore,

$$\partial_\xi\hat{\eta}_t \in L^p(\mathbb{R}) \cap L^1(\mathbb{R}, \langle\xi\rangle^\delta d\xi) \quad \text{for some } p \in (1,2),\ \delta > 0.$$

By the Hausdorff–Young inequality, we see that $u(\lambda) := \lambda\eta_t(\lambda) \in L^{p'}(\mathbb{R})$ with $p' \in (2, \infty)$ being the dual exponent to p. We also have

$$|u(\lambda) - u(\lambda')| \le (2\pi)^{-1} \int |e^{i\xi\lambda} - e^{i\xi\lambda'}||\hat{u}(\xi)|\, d\xi \le C \int |\xi|^\delta |\lambda - \lambda'|^\delta |\hat{u}(\xi)|\, d\xi$$
$$\le C|\lambda - \lambda'|^\delta \|\hat{u}\|_{L^1(\mathbb{R}, \langle\xi\rangle^\delta d\xi)}, \tag{A-9}$$

showing that $u = \lambda\eta_t \in C^\delta(\mathbb{R})$. Thus, by the Hölder inequality, we get

$$\int_{\mathbb{R}} |\eta_t(\lambda)|\, d\lambda \le C \left(\int_{|\lambda|>1} |\lambda\eta_t|^{p'}\, d\lambda \right)^{\frac{1}{p'}} + \|\lambda\eta_t\|_{C^\delta} \int_{|\lambda|<1} |\lambda|^{-1+\delta}\, d\lambda < \infty. \tag{A-10}$$

It follows from (A-10) combined with the Hausdorff–Young inequality, (A-7), (A-8) and (A-9) that

$$\|\eta_t\|_{L^1(\mathbb{R})} \le C(1 + |t|)e^{\frac{3\pi|t|}{2}},$$

and therefore, A_t extends as a bounded operator on L^∞ with norm

$$\|A_t\|_{L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})} \le C(1 + |t|)e^{\frac{3\pi|t|}{2}},$$

where the constant $C > 0$ is independent of ε and t.

Next let B be the operator

$$B : C_0^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad Bf := (\sigma\chi_+^c + \chi_+^a) * f,$$

which is also equal to

$$B = \mathcal{F}^{-1}\mu\mathcal{F}, \tag{A-11}$$

with

$$\mu(\xi) := \sigma e^{-i(c+1)\frac{\pi}{2}}(\xi - i0)^{-c-1} + e^{-i(a+1)\frac{\pi}{2}}(\xi - i0)^{-a-1}, \tag{A-12}$$

in view of (A-3).

If $c < 0$ then $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$. Using also the fact that the distribution $(\xi - i0)^z$ is of polynomial growth when $\operatorname{Re} z > -1$, we have $\mu\hat{f} \in L^1(\mathbb{R})$ for any $f \in C_0^\infty(\mathbb{R})$. Thus, the operator $B : C_0^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded.

Now if $c = 0$ then $Bf := \sigma H * f + \chi_+^a * f$, where H is the Heaviside function. The fact that the convolution with the Heaviside function maps C_0^∞ functions into L^∞ functions implies that the operator $B : C_0^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded also in the case $c = 0$.

Thus, the composition $A_t B : C_0^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded in all cases $c \le 0$. We claim that

$$A_t Bf = (\lambda - i\varepsilon)^{b+it} * f, \quad f \in C_0^\infty(\mathbb{R}). \tag{A-13}$$

Indeed, (A-13) follows from (A-4), (A-11), and the equality

$$\hat{\eta}_t\mu = \mathcal{F}((\lambda - i\varepsilon)^{b+it})$$

obtained from (A-5), (A-6) (A-12), and (A-2). In the case $c = 0$, we also use that

$$\xi_-^{-b-it}(\xi - i0)^{-1} = \xi^{-b-1-it}, \quad b < 0.$$

We thus get for all $\varepsilon > 0$ and $t \in \mathbb{R}$

$$\|(\lambda - i\varepsilon)^{b+it} * f\|_{L^\infty} \le C(1+|t|)e^{\frac{3\pi|t|}{2}}(\|\chi_+^c * f\|_{L^\infty} + \|\chi^a * f\|_{L^\infty}). \tag{A-14}$$

Now a scaling argument as in the proof of Lemma 3.3 of [Guillarmou et al. 2013b] finishes the proof. Indeed, letting $f_\tau(\lambda) = f(\tau\lambda)$, we have

$$\chi_+^z * f_\tau(\lambda) = \tau^{-z-1}(\chi_+^z * f)(\tau\lambda), \quad (\lambda - i\varepsilon)^z * f_\tau(\lambda) = \tau^{-z-1}((\lambda - i\tau\varepsilon)^z * f)(\tau\lambda) \tag{A-15}$$

for all $\tau > 0$ and $z \in \mathbb{C}$. It follows from (A-14) and (A-14) that for each $\tau > 0$

$$\tau^{-b}\|(\lambda - i\tau\varepsilon)^{b+it} * f\|_{L^\infty} \le C(1+|t|)e^{\frac{3\pi|t|}{2}}(\tau^{-c}\|\chi_+^c * f\|_{L^\infty} + \tau^{-a}\|\chi_+^a * f\|_{L^\infty})$$

and choosing $\tau := \|\chi_+^a * f\|_{L^\infty}^{1/(a-c)}\|\chi_+^c * f\|_{L^\infty}^{-1/(a-c)}$, we obtain the desired estimate (5-16). The proof of Lemma 5.5 is complete.

Appendix B: Microlocal structure of the spectrally localized resolvent

We now analyze the microlocal structure of the spectrally localized resolvent $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$, where $z > 0$ and $\phi \in C_0^\infty\big(\big((1-\frac{\delta}{4})^2, (1+\frac{\delta}{4})^2\big)\big)$ is such that $\phi(t) = 1$ for $t \in \big((1-\frac{\delta}{8})^2, (1+\frac{\delta}{8})^2\big)$, for $\delta > 0$ small. In doing so, we use the notation and results established in [Guillarmou et al. 2013a; 2013b; Hassell and Wunsch 2008].

Proposition B.1. *Let ϕ be as above. For all $\mu > 0$, the operator $\phi(\Delta_g/\mu^2)$ is a pseudodifferential operator in the following senses*:

(i) ***High-energy case***. *For $h = \mu^{-1} \le 2$, the operator $\phi(h^2\Delta_g)$ is a semiclassical scattering pseudodifferential operator with microsupport in $\{(z,\zeta) \mid |\zeta|_g \in \big((1-\frac{\delta}{4})^2, (1+\frac{\delta}{4})^2\big)\}$, where ζ is the semiclassically rescaled cotangent variable*; *i.e., ζ_i is the symbol of $-ih\partial_{z_i}$.*

(ii) ***Low-energy case***. *For $\mu \in (0,2)$, the operator $\phi(\Delta_g/\mu^2)$ is a pseudodifferential operator in the class $\Psi_k^0(M, \Omega_{k,b}^{1/2}) + \mathcal{A}^{\mathcal{E}}(M_{k,b}^2, \Omega_{k,b}^{1/2})$ where $\mathcal{E}$ is an index family for the boundary hypersurfaces of $M_{k,b}^2$, satisfying $\mathcal{E}_{\mathrm{bf}_0} = 0$, $\mathcal{E}_{\mathrm{zf}} = n$, $\mathcal{E}_{\mathrm{lb}_0} = \mathcal{E}_{\mathrm{rb}_0} = \frac{n}{2}$, $\mathcal{E}_{\mathrm{lb}} = \mathcal{E}_{\mathrm{rb}} = \mathcal{E}_{\mathrm{bf}} = \infty$. That is, it is the sum of a pseudodifferential operator in the class defined in [Guillarmou et al. 2013a, Section 5] and a conormal function which is smooth across the diagonal, but has nontrivial behavior at the boundary hypersurfaces* lb_0 *and* rb_0.

Proof. (i) This follows by expressing the operator $\phi(h^2\Delta_g)$ using the Helffer–Sjöstrand formula for the self-adjoint functional calculus,

$$\phi(h^2\Delta_g) = \frac{1}{2\pi i}\int_{\mathbb{C}} \bar\partial\tilde\phi(z)(h^2\Delta_g - z)^{-1} d\bar z \wedge dz,$$

where $\tilde\phi$ is an almost holomorphic extension of ϕ; see [Dimassi and Sjöstrand 1999, Theorem 8.1]. In terms of the notation for the spaces of semiclassical scattering pseudodifferential operators used in [Vasy and Zworski 2000], we have $\phi(h^2\Delta_g) \in \Psi_{\mathrm{sc},h}^{-\infty,0,0}(M)$.

(ii) The same argument applies to show that the operator $\phi(\Delta_g/\mu^2)$ is pseudodifferential in a neighborhood of the diagonal on the space $M^2_{k,\mathrm{sc}}$. We also need to understand the behavior of the kernel of this operator away from the diagonal. Here, we recall from [Guillarmou et al. 2013a] that the spectral measure is conormal and vanishes to order $n-1$ at zf, order $\frac{n}{2}-1$ at lb_0 and rb_0 and order -1 at bf_0 as a b-half-density on $M^2_{k,b}$, while it is Legendrian (oscillatory) at lb, rb and bf. As a result, the integral

$$\phi\left(\frac{\Delta_g}{\mu^2}\right) = \int \phi\left(\frac{\lambda^2}{\mu^2}\right) dE_{\sqrt{\Delta_g}}(\lambda)\, d\lambda \tag{B-1}$$

is conormal on $M^2_{k,b}$ and vanishes to order n at zf, order $\frac{n}{2}$ at lb_0 and rb_0, order 0 at bf_0 and order ∞ at lb, rb and bf. □

Remark B.2. The pseudodifferential nature of $\phi(h^2\Delta_g)$ can also be proved via the spectral measure using the results of [Guillarmou et al. 2013a]. Recall from this article that the spectral measure $dE_{\sqrt{\Delta_g}}(\lambda)$ for $\lambda \geq 1$ is a Legendre distribution associated to a pair of Legendre submanifolds $(L, L^\#_2)$, where L is the flowout by (left) bicharacteristic flow starting from $N^*\mathrm{Diag}_b \cap \Sigma_l$, where $N^*\mathrm{Diag}_b$ is the conormal bundle to the diagonal in M^2_b. Here Σ_l denotes the "left" characteristic variety of the operator $h^2\Delta_g - 1$, that is, the set $\{(z,\zeta,z',\zeta') \mid |\zeta|_g = 1\}$ where the semiclassical symbol of $h^2\Delta_g - 1$, acting in the left variable z, vanishes. Being a Legendre distribution, the spectral measure may be expressed (up to a trivial kernel, that is, one that is smooth and rapidly vanishing both as $h \to 0$ and as one approaches the boundary of M^2_b) as a finite sum of oscillatory integrals associated to neighborhoods of the submanifold L. The phase function for this oscillatory integral takes the form $\lambda\Phi$, where Φ is independent of λ. If we then integrate in the λ-variable as in (B-1) (with $h = \mu^{-1}$ in the high-energy case), then it is straightforward to check that the phase function $\lambda\Phi$ parametrizes the conormal bundle to the diagonal, and the result is a semiclassical scattering pseudodifferential operator of order 0.

Remark B.3. It is not hard to see that the operator $\phi(\Delta_g/\mu^2)$ is microlocally equal to the identity for $|\zeta|_g \in \big((1-\frac{\delta}{8})^2, (1+\frac{\delta}{8})^2\big)$, where ζ is the rescaled cotangent variable. First, the operator $\phi(\Delta_g/\mu^2)$ is elliptic in this region. Next, choose a function ϕ_1 supported in the interior of the region where $\phi = 1$. Then by functional calculus, $\phi_1(\Delta_g/\mu^2) = \phi(\Delta_g/\mu^2)\phi_1(\Delta/\mu^2)$, from which it follows that $\phi(\Delta_g/\mu^2)$ is microlocally equal to the identity on the elliptic set of $\phi_1(\Delta_g/\mu^2)$, which is an arbitrary subset of $\big\{(z,\zeta) \,\big|\, |\zeta|_g \in \big((1-\frac{\delta}{8})^2, (1+\frac{\delta}{8})^2\big)\big\}$.

We next consider the microlocal structure of the spectrally localized resolvent.

Proposition B.4. *The microlocal structure of the operator* $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$, $z > 0$, *is as follows*:

(i) ***High-energy case.*** *Here we use semiclassical notation and we write* $z = h^{-2}$. *The operator* $\phi(h^2\Delta_g)(h^2\Delta_g - (1 \pm i0))^{-1}$, *acting on half-densities, lies in the same microlocal space as the semiclassical resolvent* (as detailed in [Hassell and Wunsch 2008, Theorem 1.1]), *indeed in a "better" space as the differential order is* $-\infty$ *rather than* -2. *That is, the spectrally localized resolvent is a sum of three terms* $S_1 + S_2 + S_3$, *where*

- S_1 *is a semiclassical pseudodifferential operator of differential order* $-\infty$ *and semiclassical order* 0,

- *S_2 is an intersecting Legendre distribution associated to the conormal bundle $N^*\mathrm{Diag}_b$ and to the propagating Legendrian L, and*
- *S_3 is a conic Legendre pair associated to L and to the outgoing Legendrian $L_2^\#$.*

Moreover, $S_2 + S_3$ are microlocally identical to the full resolvent in a neighborhood of the characteristic variety Σ_l of $h^2\Delta_g - 1$.

(ii) ***Low-energy case.*** *Let $z \in (0,2)$. The operator $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$, acting on half-densities, lies in the same microlocal space as the resolvent* (as detailed in [Guillarmou et al. 2013a, Theorem 3.9]), *indeed in a better space as the differential order is $-\infty$ rather than -2. In detail, the operator $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$ can be decomposed as $S_1 + S_2 + S_3 + S_4$ (with $\sqrt{z}$ playing the role of the spectral parameter on $M^2_{k,b}$), where*

- *$S_1 \in \Psi^{-\infty}(M, \Omega^{1/2}_{k,b})$ is a pseudodifferential operator of order $-\infty$ in the calculus of operators defined in [Guillarmou et al. 2013b],*
- *$S_2 \in I^{-1/2,\mathcal{B}}(M^2_{k,b}, ({}^{\mathrm{sc}}N^*_{\mathrm{Diag}_b}, L^{\mathrm{bf}}_+); \Omega^{1/2}_{k,b})$ is an intersecting Legendre distribution on $M^2_{k,b}$, microsupported close to ${}^{\mathrm{sc}}N^*_{\mathrm{Diag}_b}$,*
- *$S_3 \in I^{-1/2,(n-2)/2;(n-1)/2,(n-1)/2;\mathcal{B}}(M^2_{k,b}, (L^{\mathrm{bf}}_+, L^\#_+); \Omega^{1/2}_{k,b})$ is a Legendre distribution on $M^2_{k,b}$ associated to the intersecting pair of Legendre submanifolds with conic points $(L^{\mathrm{bf}}_+, L^\#_+)$, microsupported away from ${}^{\mathrm{sc}}N^*_{\mathrm{Diag}_b}$,*
- *S_4 is supported away from* bf *and is such that $e^{\pm i\lambda r}e^{\pm i\lambda r'}R_4$ is polyhomogeneous conormal on $M^2_{k,b}$.*

Here $\mathcal{B} = (\mathcal{B}_{\mathrm{bf}_0}, \mathcal{B}_{\mathrm{lb}_0}, \mathcal{B}_{\mathrm{rb}_0}, \mathcal{B}_{\mathrm{zf}})$ is an index family with minimal exponents (i.e., order of vanishing) $\min\mathcal{B}_{\mathrm{bf}_0} = -2,\ \min\mathcal{B}_{\mathrm{lb}_0} = \min\mathcal{B}_{\mathrm{rb}_0} = \frac{n}{2} - 2,\ \min\mathcal{B}_{\mathrm{zf}} = 0$. *In addition S_4 vanishes to order ∞ at* lb *and* bf *and to order $\frac{n-1}{2}$ at* rb.

Corollary B.5. *The estimates* (5-3), (5-4), (5-9) *and* (5-10) *hold if the resolvent $(\Delta_g - (z \pm i0))^{-1}$ is replaced by the spectrally localized resolvent $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$.*

Proof of Corollary B.5. The proofs of these estimates only used the location of the wavefront set of the resolvent kernel, together with the vanishing orders of the resolvent on the boundary hypersurfaces of $M^2_{k,b}$ at $z = 0$. In view of Proposition B.4, the same proof applies verbatim to the spectrally localized resolvent. □

Proof of Proposition B.4. (i) We study the composition of the operator $\phi(h^2\Delta_g)$ with the incoming or outgoing resolvent, $(h^2\Delta_g - (1 \pm i0))^{-1}$. We know from [Hassell and Wunsch 2008, Theorem 1.1] that the actual resolvent can be decomposed into a sum of three terms $R_1 + R_2 + R_3$ as in the proposition (except that R_1 will have differential order -2). We may assume that R_2 and R_3 are microsupported in the region where $|\zeta|_g \in \big((1 - \frac{\delta}{8})^2, (1 + \frac{\delta}{8})^2\big)$, and R_1 is microsupported in the region where $|\zeta|_g \notin \big((1 - \frac{\delta}{16})^2, (1 + \frac{\delta}{16})^2\big)$. The composition $S_1 := \phi(h^2\Delta_g)R_1$ is another semiclassical pseudodifferential operator, of semiclassical order 0 and differential order $-\infty$. On the other hand, the operator $\phi(h^2\Delta_g)$ is microlocally equal to the identity on the microsupport of R_2 and R_3, so using [Guillarmou et al. 2013b, Section 7], we find that the composition of $\phi(h^2\Delta_g)$ with R_2+R_3 is equal to R_2+R_3 up to an operator that is residual in all senses, that is, a smooth kernel that vanishes rapidly as $h\to 0$ or upon approach to the boundary of M_b^2. So we can take $S_2=R_2$ and $S_3=R_3$ up to a residual kernel.

(ii) Similarly, in the low-energy case the actual resolvent has a decomposition into $R_1+R_2+R_3+R_4$ having properties as in the proposition (with R_1 of differential order -2). We also need to decompose the operator $\phi(\Delta_g/z)=B_1+B_2$ into two parts, where B_1 is supported close to the diagonal on the space $M^2_{k,b}$, and B_2 has empty wavefront set. This second piece B_2 can be taken to vanish to infinite order at bf, lb and rb, and to be polyhomogeneous conormal to $\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0$ and zf vanishing to order 0 at bf_0, order $\frac{n}{2}$ at lb_0 and rb_0 and order n at zf. When we apply B_1 to the resolvent, the argument is just as in the high-energy case, using [Guillarmou et al. 2013b, Section 5] instead of Section 7 of that work.

To understand what happens when we apply B_2 to the resolvent, we view the composition of operators as the pushforward of the product of the Schwartz kernels on a "triple space" $M^3_{k,b}$ down to $M^2_{k,b}$, as was done in the appendix of [Guillarmou and Hassell 2008]. As a multiple of a nonvanishing b-half-density on $M^2_{k,b}$ we find that B_2 (multiplied by $\left|\frac{dk}{k}\right|^{1/2}$, $k=\sqrt{z}$, which is a purely formal factor) is polyhomogeneous conormal, with no log terms at leading order, and vanishes to order n at zf, 0 at bf_0 and $\frac{n}{2}$ at lb_0 and rb_0. On the other hand, we can decompose the resolvent kernel as the sum of R_1+R_2, supported near the diagonal, and R_3+R_4, which is microsupported in the set where $|\zeta|_g\in\left(\left(1-\frac{\delta}{8}\right)^2,\left(1+\frac{\delta}{8}\right)^2\right)$, where ζ is the cotangent variable rescaled by a factor $\sqrt{z}$.

The composition of B_2 with R_1+R_2 can be treated by lifting both kernels to the space $M^3_{k,b}$ and pushing forward. Since B_2 has no wavefront set, the composition has no wavefront set, so it is polyhomogeneous conormal, and the order of vanishing can be read off as n at zf, $\frac{n}{2}$ at lb_0, $\frac{n}{2}-2$ at rb_0, -2 at bf_0, and ∞ at lb, rb and bf. This lies in a better space than claimed in the proposition.

The composition of B_2 with R_3+R_4 can also be analyzed by lifting both kernels to $M^3_{k,b}$ and then pushing forward. Although R_3+R_4 is not polyhomogeneous conormal at the boundary hypersurfaces bf, lb and rb, when lifted to $M^3_{k,b}$ and multiplied by the lift of B_2, the rapid vanishing of B_2 at bf and rb means that the product of the two kernels is rapidly decreasing as the "middle variable" (the right variable of B_2 and the left variable of R_3+R_4) tends to the boundary. As for the right variable of R_3+R_4, after multiplying the kernel of R_3+R_4 by $e^{\mp i\lambda r'}$ (where $r'=\frac{1}{x'}$ is the right radial variable) it becomes polyhomogeneous conormal also at rb. So the product of the kernels B_2 (in the left and middle variables) and $(R_3+R_4)e^{\mp i\lambda r'}$ (in the middle and right variables) on $M^3_{k,b}$ *is* polyhomogeneous conormal. After pushing forward to $M^2_{k,b}$ a calculation similar to that done in [Guillarmou and Hassell 2008, Appendix] shows that the result is $e^{\mp i\lambda r'}$ times a polyhomogeneous kernel which vanishes to order $n-2$ at zf, -2 at bf_0, $\min\left(\frac{n}{2},n-2\right)$ at lb_0, $\frac{n}{2}-2$ at rb_0, $\frac{n-1}{2}$ at rb and ∞ at lb and bf, with no log terms to leading order except possibly at lb_0 in the case $n=4$. Again this is in a better space than is claimed in the proposition. □

Acknowledgements

We are grateful to the referees for their very helpful comments and suggestions, and we thank Julien Sabin and Adam Sikora for useful discussions. This project has received funding from the European

Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 725967). The research of Guillarmou was also partially supported by ANR grant 13-BS01-0007-01. Hassell acknowledges the support of the Australian Research Council through Discovery Grants DP150102419 and DP160100941. The research of Krupchyk is partially supported by the National Science Foundation (DMS 1500703, DMS 1815922). Guillarmou finally thanks the hospitality of the Mathematical Sciences Institute at ANU where part of this work was done, and ARC grant DP160100941 for supporting the visit.

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Received 19 Oct 2017. Revised 29 Apr 2019. Accepted 13 Aug 2019.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 13 No. 6 2020

2157-5045(2020)13:6;1-8