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INVERSE BOUNDARY PROBLEMS FOR BIHARMONIC OPERATORS IN TRANSVERSALLY ANISOTROPIC GEOMETRIES*

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Abstract. We study inverse boundary problems for first order perturbations of the biharmonic operator on a conformally transversally anisotropic Riemannian manifold of dimension $n \geq 3$. We show that a continuous first order perturbation can be determined uniquely from the knowledge of the set of the Cauchy data on the boundary of the manifold provided that the geodesic X-ray transform on the transversal manifold is injective.

Key words. inverse problems, biharmonic operators, conformally transversally geometries

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1. Introduction and statement of results. Let (M, g) be a smooth compact oriented Riemannian manifold of dimension $n \geq 3$ with smooth boundary ∂M. Let $-\Delta_g$ be the Laplace–Beltrami operator, and let $(-\Delta_g)^2$ be the biharmonic operator on M. Let $X \in C(M, TM)$ be a complex vector field and let $q \in C(M, \mathbb{C})$. In this paper we shall be concerned with an inverse boundary problem for the first order perturbation of the biharmonic operator,

$$L_{X,q} = (-\Delta_g)^2 + X + q.$$

Let us now introduce some notation and state the main result of the paper. Let $u \in H^3(M^{\text{int}})$ be a solution to

$$L_{X,q} u = 0 \quad \text{in} \quad M. \tag{1.1}$$

Here and in what follows $H^s(M^{\text{int}})$, $s \in \mathbb{R}$, is the standard Sobolev space on M^{int}, and $M^{\text{int}} = M \setminus \partial M$ stands for the interior of M. Let ν be the unit outer normal to ∂M. We shall define the trace of the normal derivative $\partial_\nu(\Delta_g u) \in H^{-1/2}(\partial M)$ as follows. Let $\varphi \in H^{1/2}(\partial M)$. Then letting $v \in H^1(M^{\text{int}})$ be a continuous extension of φ, we set

$$\langle \partial_\nu(-\Delta_g u), \varphi \rangle_{H^{-1/2}(\partial M) \times H^{1/2}(\partial M)} = \int_M \big(\langle \nabla_g(-\Delta_g u), \nabla_g v \rangle_g + X(u)v + quv \big) dV_g, \tag{1.2}$$

where dV_g is the Riemannian volume element on M. As u satisfies (1.1), the definition of the trace $\partial_\nu(\Delta_g u)$ on ∂M is independent of the choice of an extension v of φ. Associated to (1.1), we define the set of the Cauchy data,

$$\mathcal{C}_{X,q} = \{(u|_{\partial M}, (\Delta_g u)|_{\partial M}, \partial_\nu u|_{\partial M}, \partial_\nu(\Delta_g u)|_{\partial M}) : u \in H^3(M^{\text{int}}),\, L_{X,q} u = 0 \text{ in } M\}. \tag{1.3}$$

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Note that the first two elements in the set of the Cauchy data $\mathcal{C}_{X,q}$ correspond to the Navier boundary conditions for the first order perturbation of the biharmonic operator. Physically, such operators arise when considering the equilibrium configuration of an elastic plate which is hinged along the boundary; see [18]. One can also define the set of the Cauchy data for the first order perturbation of the biharmonic operator, based on the Dirichlet boundary conditions $(u|_{\partial M}, \partial_\nu u|_{\partial M})$, which corresponds to the clamped plate equation,

$$\widetilde{\mathcal{C}}_{X,q} = \{(u|_{\partial M}, \partial_\nu u|_{\partial M}, \partial_\nu^2 u|_{\partial M}, \partial_\nu^3 u|_{\partial M}) : u \in H^3(M^{\mathrm{int}}),\ L_{X,q}u = 0 \text{ in } M\}.$$

The explicit description for the Laplacian in the boundary normal coordinates shows that $\mathcal{C}_{X,q} = \widetilde{\mathcal{C}}_{X,q}$; see [35], [30].

The inverse boundary problem that we are interested in is to determine the vector field X and the potential q from the knowledge of the set of the Cauchy data $\mathcal{C}_{X,q}$.

This problem was studied extensively in the Euclidean setting, see [29], [30], [2], [4], [24] [25] [8], [7], [19], [20], [44]. Specifically, it was shown in [29] that the set of the Cauchy data $\mathcal{C}_{X,q}$ determines the vector field X and the potential q uniquely. Let us note that the unique determination of a first order perturbation of the Laplacian is not possible due to the gauge invariance of boundary measurements and in this case the first order perturbation can be recovered only modulo a gauge transformation; see [37], [42].

Going beyond the Euclidean setting, inverse boundary problems for lower order perturbations of the Laplacian were only studied in the case when (M, g) is CTA (conformally transversally anisotropic; see Definition 1.1 below) and under the assumption that the geodesic X-ray transform on the transversal manifold is injective; see the fundamental works [14] and [15] which initiated this study, and see also [12], [13], [32], [33], [10].

Definition 1.1. *A compact Riemannian manifold (M, g) of dimension $n \geq 3$ with boundary ∂M is called conformally transversally anisotropic (CTA) if $M \subset\subset \mathbb{R} \times M_0^{int}$ where $g = c(e \oplus g_0)$, $(\mathbb{R}, e)$ is the Euclidean real line, (M_0, g_0) is a smooth compact $(n-1)$-dimensional manifold with smooth boundary, called the transversal manifold, and $c \in C^\infty(\mathbb{R} \times M_0)$ is a positive function.*

The injectivity of the geodesic X-ray transform is known when the manifold (M_0, g_0) is simple, in the sense that any two points in M_0 are connected by a unique geodesic depending smoothly on the endpoints and that ∂M_0 is strictly convex (see [1], [36]), when M_0 has strictly convex boundary and is foliated by strictly convex hypersurfaces [41], [43], and also when M_0 has a hyperbolic trapped set and no conjugate points [21], [22]. An example of the latter occurs when M_0 is a negatively curved manifold.

Turning our attention to the inverse boundary problem of determining the first order perturbation of the biharmonic operator, this problem was solved in [5] in the case when (M, g) is CTA and the transversal manifold (M_0, g_0) is simple, extending the result of [14] to the case of biharmonic operators. To be on par with the best results available for the perturbations of the Laplacian in the context of Riemannian manifolds, the goal of this paper is to solve the inverse problem for the first order perturbation of the biharmonic operator in the case when (M, g) is CTA and the geodesic X-ray transform is injective on the transversal manifold (M_0, g_0), generalizing the result of [15] to the case of biharmonic operators.

Let us recall some definitions related to the geodesic X-ray transform following [21], [14]. The geodesics on M_0 can be parametrized by points on the unit sphere bundle $SM_0 = \{(x, \xi) \in TM_0 : |\xi| = 1\}$. Let

$$\partial_\pm SM_0 = \{(x,\xi) \in SM_0 : x \in \partial M_0, \pm\langle \xi, \nu(x)\rangle > 0\}$$

be the incoming $(-)$ and outgoing $(+)$ boundaries of SM_0. Here ν is the unit outer normal vector field to ∂M_0. Here and in what follows $\langle \cdot,\cdot\rangle$ is the duality between T^*M_0 and TM_0.

Let $(x,\xi) \in \partial_- SM_0$ and $\gamma = \gamma_{x,\xi}(t)$ be the geodesic on M_0 such that $\gamma(0) = x$ and $\dot\gamma(0) = \xi$. Let us denote by $\tau(x,\xi)$ the first time when the geodesic γ exits M_0 with the convention that $\tau(x,\xi) = +\infty$ if the geodesic does not exit M_0. We define the incoming tail by

$$\Gamma_- = \{(x,\xi) \in \partial_- SM_0 : \tau(x,\xi) = +\infty\}.$$

When $f \in C(M_0,\mathbb{C})$ and $\alpha \in C(M_0, T^*M_0)$ is a complex valued 1-form, we define the geodesic X-ray transform on (M_0, g_0) as follows:

$$I(f,\alpha)(x,\xi) = \int_0^{\tau(x,\xi)} \big[f(\gamma_{x,\xi}(t)) + \langle \alpha(\gamma_{x,\xi}(t)), \dot\gamma_{x,\xi}(t)\rangle\big]\,dt, \quad (x,\xi) \in \partial_- SM_0 \setminus \Gamma_-.$$

A unit speed geodesic segment $\gamma = \gamma_{x,\xi} : [0,\tau(x,\xi)] \to M_0$, $\tau(x,\xi) > 0$, is called nontangential if $\gamma(0), \gamma(\tau(x,\xi)) \in \partial M_0$, $\dot\gamma(0), \dot\gamma(\tau(x,\xi))$ are nontangential vectors on ∂M_0, and $\gamma(t) \in M_0^{\mathrm{int}}$ for all $0 < t < \tau(x,\xi)$.

Assumption 1. We assume that the geodesic X-ray transform on (M_0, g_0) is injective in the sense that if $I(f,\alpha)(x,\xi) = 0$ for all $(x,\xi) \in \partial_- SM_0 \setminus \Gamma_-$ such that $\gamma_{x,\xi}$ is a nontangential geodesic, then $f = 0$ and $\alpha = dp$ in M_0 for some $p \in C^1(M_0,\mathbb{C})$ with $p|_{\partial M_0} = 0$.

The main result of the paper is as follows.

THEOREM 1.2. *Let (M,g) be a CTA manifold of dimension $n \geq 3$ such that Assumption 1 holds for the transversal manifold. Let $X^{(1)}, X^{(2)} \in C(M,TM)$ be complex vector fields, and let $q^{(1)}, q^{(2)} \in C(M,\mathbb{C})$. If $\mathcal{C}_{X^{(1)},q^{(1)}} = \mathcal{C}_{X^{(2)},q^{(2)}}$, then $X^{(1)} = X^{(2)}$ in M. Assuming furthermore that*

$$q^{(1)}|_{\partial M} = q^{(2)}|_{\partial M}, \tag{1.4}$$

we have $q^{(1)} = q^{(2)}$ in M.

Remark 1.3. Examples of nonsimple manifolds M_0 satisfying Assumption 1 include in particular manifolds with a strictly convex boundary which are foliated by strictly convex hypersurfaces [41], [43], and manifolds with a hyperbolic trapped set and no conjugate points [21], [22].

Remark 1.4. To the best of our knowledge, Theorem 1.2 seems to be the first result where one recovers a vector field uniquely on general CTA manifolds.

Remark 1.5. The assumption (1.4) is made for simplicity only and can be removed by performing the boundary determination as done in Appendix A for the vector fields $X^{(1)}$ and $X^{(2)}$. This can be done by using the approach of [23] combined with its extensions in [34] and [17].

Let us proceed to describe the main ideas in the proof of Theorem 1.2. The key step in the proof is a construction of complex geometric optics solutions for the equations $L_{X,q}u = 0$ and $L_{-\overline{X}, -\operatorname{div}(\overline{X}) + \overline{q}}u = 0$ in M. Here the operator $L_{-\overline{X}, -\operatorname{div}(\overline{X}) + \overline{q}}$ represents the formal L^2 adjoint of the operator $L_{X,q}$. In contrast to the work [5], where one deals with the same inverse problem in the case of a simple transversal manifold, here without a simplicity assumption, complex geometric optics solutions cannot be easily constructed by means of a global WKB method, and following [15], we shall construct complex geometric optics solutions based on Gaussian beam quasimodes for the biharmonic operator $(-\Delta_g)^2$ conjugated by an exponential weight corresponding to the limiting Carleman weight $\phi(x) = \pm x_1$ for $-h^2\Delta_g$ on the CTA manifold (M, g); see [14]. To convert the Gaussian beam quasimodes to exact solutions, we shall rely on the corresponding Carleman estimate with a gain of two derivatives established in [32]; see also [14].

Remark 1.6. We would like to note that one can obtain Gaussian beam quasimodes for the biharmonic operator $(-\Delta_g)^2$ conjugated by an exponential weight as the Gaussian beam quasimodes for the Laplacian conjugated by an exponential weight. However, such quasimodes are not enough to prove Theorem 1.2 as in order to recover the vector field uniquely, one has to exploit a richer set of amplitudes which are not available for the Gaussian beam quasimodes for the Laplacian.

Remark 1.7. When constructing Gaussian beam quasimodes for the Laplacian conjugated by an exponential weight, one first reduces to the setting when the conformal factor $c = 1$ by using the following transformation:

$$c^{\frac{n+2}{4}} \circ (-\Delta_g) \circ c^{-\frac{(n-2)}{4}} = -\Delta_{\widetilde{g}} + \widetilde{q},$$

where

$$\widetilde{g} = e \oplus g_0, \quad \widetilde{q} = -c^{\frac{n+2}{4}}(-\Delta_g)(c^{-\frac{(n-2)}{4}});$$

see [15]. However, it seems that no such useful reduction is available for the biharmonic operator and therefore, when constructing Gaussian beam quasimodes for the biharmonic operator $(-\Delta_g)^2$ conjugated by an exponential weight, we shall proceed directly accommodating the conformal factor in the construction which makes it somewhat more complicated.

Once complex geometric optics solutions are constructed, the next step is to substitute them into a suitable integral identity which is obtained as a consequence of the equality $\mathcal{C}_{X^{(1)},q^{(1)}} = \mathcal{C}_{X^{(2)},q^{(2)}}$ for the Cauchy data sets. Exploiting the concentration properties of the corresponding Gaussian beam together with Assumption 1, we first show that there exists $\psi \in C^1(\mathbb{R} \times M_0)$ with compact support in x_1 such that $\psi(x_1, \cdot)|_{\partial M_0} = 0$ and $X^{(1)} - X^{(2)} = \nabla_g \psi$. To show that $\psi = 0$, i.e., $X^{(1)} = X^{(2)}$, we use the concentration properties of the Gaussian beam for the biharmonic operator with a richer set of amplitudes which are not available for the Laplacian, combining with Assumption 1. Finally, we show that $q^{(1)} = q^{(2)}$ by using the concentration properties of the Gaussian beam together with Assumption 1 once again.

The plan of the paper is as follows. In section 2 we construct Gaussian beam quasimodes for the biharmonic operator conjugated by an exponential weight corresponding to the limiting Carleman weight ϕ and establish some concentration properties of them. In section 3 we convert the Gaussian beam quasimodes to the exact complex geometric optics solutions. Section 4 is devoted to the proof of Theorem 1.2. Finally, in Appendix A the boundary determination of a continuous vector field on a compact manifold with boundary, from the set of the Cauchy data, is presented.

2. Gaussian beam quasimodes for biharmonic operators on conformally anisotropic manifolds. Let (M, g) be a CTA manifold so that $(M, g) \subset\subset (\mathbb{R} \times M_0^{\text{int}}, c(e \oplus g_0))$. Here $(\mathbb{R}, e)$ is the Euclidean real line, (M_0, g_0) is a smooth compact

$(n-1)$–dimensional manifold with smooth boundary, and $c \in C^\infty(\mathbb{R} \times M_0)$ is a positive function. Let us write $x = (x_1, x')$ for local coordinates in $\mathbb{R} \times M_0$. Note that $\phi(x) = \pm x_1$ is a limiting Carleman weight for $-h^2\Delta_g$; see Definition 3.1 in section 3, and see also [14].

In this section we shall construct Gaussian beam quasimodes for the biharmonic operator $(-\Delta_g)^2$ conjugated by an exponential weight corresponding to the limiting Carleman weight $\phi = \pm x_1$, i.e., suitable approximate solutions concentrated on a single curve; see [38], [39]. Due to the presence of the conformal factor c, our quasimodes will be constructed on the manifold M and will be localized to nontangential geodesics on the transversal manifold M_0.

The first main result of this section is as follows. In this result $H^1(M^{\mathrm{int}})$ stands for the standard Sobolev space, equipped with the semiclassical norm,

$$\|u\|^2_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})} = \|u\|^2_{L^2(M)} + \|h\nabla_g u\|^2_{L^2(M)}.$$

PROPOSITION 2.1. *Let $s = \mu + i\lambda$ with $1 \le \mu = 1/h$ and $\lambda \in \mathbb{R}$ being fixed, and let $\gamma : [0, L] \to M_0$ be a unit speed nontangential geodesic on M_0. Then there exist families of Gaussian beam quasimodes $v_s, w_s \in C^\infty(M)$ such that*

$$\|v_s\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(1), \quad \|e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s\|_{L^2(M)} = \mathcal{O}(h^{5/2}), \tag{2.1}$$

and

$$\|w_s\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(1), \quad \|e^{-sx_1}(-h^2\Delta_g)^2 e^{sx_1} w_s\|_{L^2(M)} = \mathcal{O}(h^{5/2}), \tag{2.2}$$

as $h \to 0$. Moreover, in a sufficiently small neighborhood U of a point $p \in \gamma([0, L])$, the quasimode v_s is a finite sum,

$$v_s|_U = v_s^{(1)} + \cdots + v_s^{(P)},$$

where $t_1 < \cdots < t_P$ are the times in $[0, L]$ where $\gamma(t_l) = p$. Each $v_s^{(l)}$ has the form

$$v_s^{(l)} = e^{is\varphi^{(l)}} a^{(l)}, \quad l = 1, \dots, P, \tag{2.3}$$

where $\varphi = \varphi^{(l)} \in C^\infty(\overline{U}; \mathbb{C})$ satisfies for t close to t_l,

$$\varphi(\gamma(t)) = t, \quad \nabla\varphi(\gamma(t)) = \dot\gamma(t), \quad \mathrm{Im}\,(\nabla^2\varphi(\gamma(t))) \ge 0, \quad \mathrm{Im}\,(\nabla^2\varphi)|_{\dot\gamma(t)^\perp} > 0,$$

and $a^{(l)} \in C^\infty(\mathbb{R} \times \overline{U})$ is of the form

$$a^{(l)}(x_1, t, y) = h^{-\frac{(n-2)}{4}} a_0^{(l)}(x_1, t)\chi\left(\frac{y}{\delta'}\right),$$

where for all $l = 1, \dots, P$, either $a_0^{(l)}$ is given by

$$a_0^{(l)} = e^{-\phi^{(l)}(x_1, t)}, \tag{2.4}$$

defining an amplitude of the first type, or $a_0^{(l)}$ satisfies the equation

$$\frac{1}{c(x_1, t, 0)}(\partial_{x_1} - i\partial_t)(e^{\phi^{(l)}(x_1, t)} a_0^{(l)}) = 1, \tag{2.5}$$

defining an amplitude of the second type. Here

$$\phi^{(l)}(x_1,t)=\log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}}+G^{(l)}(t),\quad \partial_t G^{(l)}(t)=\frac{1}{2}(\Delta_{g_0}\varphi^{(l)})(t,0), \tag{2.6}$$

(t,y) are the Fermi coordinates for γ for t close to t_l, $\chi\in C_0^\infty(\mathbb{R}^{n-2})$ is such that $0\le\chi\le 1$, $\chi=1$ for $|y|\le 1/4$ and $\chi=0$ for $|y|\ge 1/2$, and $\delta'>0$ is a fixed number that can be taken arbitrarily small.

In a sufficiently small neighborhood U of a point $p\in\gamma([0,L])$, the quasimode w_s is a finite sum,

$$w_s|_U=w_s^{(1)}+\cdots+w_s^{(P)},$$

where $t_1<\cdots<t_P$ are the times in $[0,L]$ where $\gamma(t_l)=p$. Each $w_s^{(l)}$ has the form

$$w_s^{(l)}=e^{is\varphi^{(l)}}b^{(l)},\quad l=1,\dots,P, \tag{2.7}$$

where $\varphi^{(l)}$ is the same as in (2.3)*, and $b^{(l)}\in C^\infty(\mathbb{R}\times\overline{U})$ is of the form*

$$b^{(l)}(x_1,t,y)=h^{-\frac{(n-2)}{4}}b_0^{(l)}(x_1,t)\chi\left(\frac{y}{\delta'}\right),$$

where

$$b_0^{(l)}=e^{-\widetilde{\phi}^{(l)}(x_1,t)}. \tag{2.8}$$

Here

$$\widetilde{\phi}^{(l)}(x_1,t)=\log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}}+F^{(l)}(t),\quad \partial_t F^{(l)}(t)=\frac{1}{2}(\Delta_{g_0}\varphi^{(l)})(t,0). \tag{2.9}$$

Remark 2.2. Note that the first type of the amplitudes, i.e., $a_0^{(l)}$ given by (2.4), will be used to recover the potential q as well as the vector field X up to a suitable gauge transformation, while to recover X uniquely, we shall have to work with the second type of amplitudes, i.e., $a_0^{(l)}$ solving (2.5).

Proof. To construct Gaussian beam quasimodes, we shall follow the standard approach; see [15], [32]. The novelty here is that when working with the biharmonic operator we have to accommodate the presence of the conformal factor c throughout the construction. We are also led to consider a richer class of amplitudes for the Gaussian beam quasimodes.

Step 1. *Preparation.* Let us isometrically embed the manifold (M_0,g_0) into a larger closed manifold $(\widehat{M_0},g_0)$ of the same dimension. This is possible as we can form the manifold $\widehat{M_0}=M_0\sqcup_{\partial M_0}M_0$, which is the disjoint union of two copies of M_0, glued along the boundary; see [15, Proof of Proposition 3.1]. We extend γ as a unit speed geodesic in $\widehat{M_0}$. Let $\varepsilon>0$ be such that $\gamma(t)\in\widehat{M_0}\setminus M_0$ and $\gamma(t)$ has no self-intersection for $t\in[-2\varepsilon,0)\cup(L,L+2\varepsilon]$. This choice of ε is possible since γ is nontangential.

Our aim is to construct Gaussian beam quasimodes near $\gamma([-\varepsilon,L+\varepsilon])$. We shall start by carrying out the quasimode construction locally near a given point $p_0=\gamma(t_0)$ on $\gamma([-\varepsilon,L+\varepsilon])$. Let $(t,y)\in U=\{(t,y)\in\mathbb{R}\times\mathbb{R}^{n-2}:|t-t_0|<\delta,|y|<\delta'\}$, $\delta,\delta'>0$, be Fermi coordinates near p_0; see [27]. We may assume that the coordinates (t,y) extend smoothly to a neighborhood of $\overline{U}$. The geodesic γ near p_0 is then given by $\Gamma=\{(t,y):y=0\}$, and

$$g_0^{jk}(t,0)=\delta^{jk},\quad \partial_{y_l}g_0^{jk}(t,0)=0.$$

Hence, near the geodesic

$$g_0^{jk}(t,y) = \delta^{jk} + \mathcal{O}(|y|^2). \tag{2.10}$$

Let us first construct the quasimode v_s in (2.1) for the operator $e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1}$. In doing so, we consider the following Gaussian beam ansatz:

$$v_s(x_1,t,y) = e^{is\varphi(t,y)} a(x_1,t,y;s). \tag{2.11}$$

Here $\varphi \in C^\infty(U,\mathbb{C})$ is such that

$$\operatorname{Im}\varphi \geq 0, \quad \operatorname{Im}\varphi|_\Gamma = 0, \quad \operatorname{Im}\varphi(t,y) \sim |y|^2 = \operatorname{dist}((y,t),\Gamma)^2, \tag{2.12}$$

and $a \in C^\infty(\mathbb{R}\times U,\mathbb{C})$ is an amplitude such that $\operatorname{supp}(a(x_1,\cdot))$ is close to Γ; see [39], [26]. Notice that here we choose φ to depend on the transversal variables (t,y) only while a is a function of all the variables.

Let us first compute $e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s$. To that end, letting

$$\widetilde{\varphi}(x_1,t,y) = x_1 - i\varphi(t,y), \quad \widehat{\varphi} = sh\widetilde{\varphi}, \tag{2.13}$$

we first get

$$e^{\frac{\widehat{\varphi}}{h}}(-h^2\Delta_g)e^{-\frac{\widehat{\varphi}}{h}} = -h^2\Delta_g + h(2\langle\nabla_g\widehat{\varphi},\nabla_g\cdot\rangle_g + \Delta_g\widehat{\varphi}) - \langle\nabla_g\widehat{\varphi},\nabla_g\widehat{\varphi}\rangle_g. \tag{2.14}$$

Here and in what follows we write $\langle\cdot,\cdot\rangle_g$ to denote the Riemannian scalar product on tangent and cotangent spaces. In view of (2.14), we see that

$$e^{s\widetilde{\varphi}}(-h^2\Delta_g)^2e^{-s\widetilde{\varphi}} = h^4\big(-\Delta_g + s(2\langle\nabla_g\widetilde{\varphi},\nabla_g\cdot\rangle_g + \Delta_g\widetilde{\varphi}) - s^2\langle\nabla_g\widetilde{\varphi},\nabla_g\widetilde{\varphi}\rangle_g\big)^2,$$

and therefore,

$$e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s = e^{is\varphi}h^4\big(-\Delta_g + s(2\langle\nabla_g\widetilde{\varphi},\nabla_g\cdot\rangle_g + \Delta_g\widetilde{\varphi}) - s^2\langle\nabla_g\widetilde{\varphi},\nabla_g\widetilde{\varphi}\rangle_g\big)^2 a. \tag{2.15}$$

Step 2. *Solving an eikonal equation to determine the phase function* $\varphi(t,y)$. Following the WKB method, we start by considering the eikonal equation

$$\langle\nabla_g\widetilde{\varphi},\nabla_g\widetilde{\varphi}\rangle_g = 0,$$

and we would like to find $\varphi = \varphi(t,y) \in C^\infty(U,\mathbb{C})$ such that

$$\langle\nabla_g\widetilde{\varphi},\nabla_g\widetilde{\varphi}\rangle_g = \mathcal{O}(|y|^3), \quad y \to 0, \tag{2.16}$$

and

$$\operatorname{Im}\varphi \geq d|y|^2, \tag{2.17}$$

with some $d > 0$. Using that $g = c(e\otimes g_0)$ and (2.13), we see that

$$\langle\nabla_g\widetilde{\varphi},\nabla_g\widetilde{\varphi}\rangle_g = c^{-1}(1 - \langle\nabla_{g_0}\varphi,\nabla_{g_0}\varphi\rangle_{g_0}),$$

and therefore, in view of (2.16), we have to find φ satisfying the standard eikonal equation,

$$1 - \langle\nabla_{g_0}\varphi,\nabla_{g_0}\varphi\rangle_{g_0} = \mathcal{O}(|y|^3), \quad y \to 0.$$

As in [15], [38], and [39], we can choose,

$$\varphi(t,y) = t + \frac{1}{2}H(t)y\cdot y, \tag{2.18}$$

where $H(t)$ is a unique smooth complex symmetric solution of the initial value problem for the matrix Riccati equation,

$$\dot{H}(t) + H(t)^2 = F(t), \quad H(t_0) = H_0, \tag{2.19}$$

with H_0 being a complex symmetric matrix with $\mathrm{Im}\,(H_0)$ positive definite and $F(t)$ being a suitable symmetric matrix, determined by the metric tensor; see [15, Proof of Proposition 3.1]. Hence, as explained in [15], [38], and [39], $\mathrm{Im}\,(H(t))$ is positive definite for all t.

Step 3. *Solving a transport equation to find an amplitude a.* We look for a smooth amplitude $a = a(x_1, x')$ satisfying the transport equation,

$$L^2 a = \mathcal{O}(|y|), \tag{2.20}$$

as $y \to 0$. Here

$$L := 2\langle \nabla_g \widetilde{\varphi}, \nabla_g \cdot\rangle_g + \Delta_g \widetilde{\varphi}. \tag{2.21}$$

To proceed let us first simplify the operator L. To that end, in view of (2.13), a direct computation shows that

$$\langle \nabla_g \widetilde{\varphi}, \nabla_g \cdot\rangle_g = \frac{1}{c}(\partial_{x_1} - i g_0^{-1}(x')\varphi'_{x'}\cdot\partial_{x'}), \tag{2.22}$$

$$\Delta_g \widetilde{\varphi} = \Delta_g x_1 - i\Delta_g \varphi(x'), \tag{2.23}$$

where

$$\Delta_g x_1 = \left(\frac{n}{2} - 1\right)\frac{1}{c^2}\partial_{x_1} c, \tag{2.24}$$

and

$$\Delta_g \varphi = \frac{1}{c}\Delta_{g_0}\varphi + \left(\frac{n}{2} - 1\right)\frac{1}{c^2}\langle \nabla_{g_0} c, \nabla_{g_0}\varphi\rangle_{g_0}. \tag{2.25}$$

In view of (2.22), (2.23), (2.24), (2.25), the operator L given by (2.21) becomes

$$L = \frac{2}{c}(\partial_{x_1} - i g_0^{-1}(x')\varphi'_{x'}\cdot\partial_{x'}) + \left(\frac{n}{2}-1\right)\frac{1}{c^2}\partial_{x_1} c - \frac{i}{c}\Delta_{g_0}\varphi - \left(\frac{n}{2}-1\right)\frac{i}{c^2}\langle \nabla_{g_0} c, \nabla_{g_0}\varphi\rangle_{g_0}. \tag{2.26}$$

Let us proceed to simplify the operator L further. Using (2.10) and (2.18), we see that

$$g_0^{-1}(x')\varphi'_{x'}\cdot\partial_{x'} = \partial_t + \mathcal{O}(|y|^2)\partial_t + H(t)y\cdot\partial_y + \mathcal{O}(|y|^2)\cdot\partial_y. \tag{2.27}$$

Using (2.10) and (2.18), we also have

$$\begin{aligned}(\Delta_{g_0}\varphi)(t,0) &= |g_0|^{-1/2}\partial_{x'_j}(|g_0|^{1/2} g_0^{jk}\partial_{x'_k}\varphi)|_{y=0} = \delta^{jk}\partial_{x'_j}\partial_{x'_k}\varphi|_{y=0} \\ &= \delta^{jk}H_{jk} = \mathrm{tr}\, H(t),\end{aligned}$$

and therefore

$$(\Delta_{g_0}\varphi)(t,y) = (\Delta_{g_0}\varphi)(t,0) + \mathcal{O}(|y|) = \operatorname{tr} H(t) + \mathcal{O}(|y|). \tag{2.28}$$

Finally, using (2.10) and (2.18), we get

$$\langle \nabla_{g_0} c, \nabla_{g_0}\varphi\rangle_{g_0} = \partial_t c + \mathcal{O}(|y|). \tag{2.29}$$

Using (2.27), (2.28), (2.29), the operator L in (2.26) becomes

$$\begin{aligned} L &= \frac{2}{c}\bigg[\partial_{x_1} - i\partial_t - iH(t)y\cdot\partial_y + \left(\frac{n}{4}-\frac{1}{2}\right)(\partial_{x_1} - i\partial_t)\log c - \frac{i}{2}\operatorname{tr} H(t) \\ &\quad + \mathcal{O}(|y|) + \mathcal{O}(|y|^2)\partial_t + \mathcal{O}(|y|^2)\partial_y\bigg] \\ &= \frac{2}{c(x_1,t,0)}\bigg[\partial_{x_1} - i\partial_t - iH(t)y\cdot\partial_y + (\partial_{x_1} - i\partial_t)\log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}} \\ &\quad - \frac{i}{2}\operatorname{tr} H(t) + \mathcal{O}(|y|) + \mathcal{O}(|y|)(\partial_{x_1},\partial_t) + \mathcal{O}(|y|^2)\partial_y\bigg]. \end{aligned} \tag{2.30}$$

Let $\chi \in C_0^\infty(\mathbb{R}^{n-2})$ be such that $\chi = 1$ for $|y| \le 1/4$ and $\chi = 0$ for $|y| \ge 1/2$. We look for the amplitude a in the form

$$a(x_1,t,y) = h^{-\frac{(n-2)}{4}} a_0(x_1,t)\chi\left(\frac{y}{\delta'}\right), \tag{2.31}$$

where $a_0(\cdot,\cdot) \in C^\infty(\mathbb{R}\times\{t : |t-t_0| < \delta\})$ is independent of y. In view of (2.20), a_0 should satisfy the equation

$$L^2 a_0 = \mathcal{O}(|y|), \tag{2.32}$$

as $y \to 0$. In view of (2.30), we write

$$L = \frac{2}{c(x_1,t,0)}(L_0 + R), \tag{2.33}$$

where

$$L_0 = (\partial_{x_1} - i\partial_t) + (\partial_{x_1} - i\partial_t)\log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}} - \frac{i}{2}\operatorname{tr} H(t) \tag{2.34}$$

and

$$R = -iH(t)y\cdot\partial_y + \mathcal{O}(|y|) + \mathcal{O}(|y|)(\partial_{x_1},\partial_t) + \mathcal{O}(|y|^2)\partial_y. \tag{2.35}$$

To solve our inverse problem, we need two types of amplitudes. Let us proceed to construct the first type of amplitudes. In doing so, first note that as a_0 is independent of y, if a_0 solves the equation

$$L_0 a_0 = 0, \tag{2.36}$$

then a_0 satisfies (2.32). Let us proceed to find a solution to (2.36). To that end, letting

$$\phi(x_1,t) = \log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}} + G(t), \quad \partial_t G(t) = \frac{1}{2}\operatorname{tr} H(t), \tag{2.37}$$

we see that

$$L_0 = e^{-\phi(x_1,t)}(\partial_{x_1} - i\partial_t)e^{\phi(x_1,t)}. \tag{2.38}$$

We solve (2.36) by taking

$$a_0 = e^{-\phi} = c(x_1,t,0)^{\frac{1}{2}-\frac{n}{4}}e^{-G(t)}, \quad \partial_t G(t) = \frac{1}{2}\operatorname{tr} H(t). \tag{2.39}$$

Now we proceed to find the second type of amplitudes, which is given by more general solutions to (2.32). As a_0 is independent of y, using (2.33), (2.34), and (2.35), equation (2.32) becomes

$$\frac{2}{c(x_1,t,0)}[L_0 + R]\left(\frac{2}{c(x_1,t,0)}L_0 a_0(x_1,t) + \mathcal{O}(|y|)\right) = \mathcal{O}(|y|),$$

or simply

$$L_0\left(\frac{1}{c(x_1,t,0)}L_0\right)a_0(x_1,t) = 0. \tag{2.40}$$

Using (2.38), we see that (2.40) becomes

$$(\partial_{x_1} - i\partial_t)\left(\frac{1}{c(x_1,t,0)}(\partial_{x_1} - i\partial_t)(e^{\phi(x_1,t)}a_0)\right) = 0. \tag{2.41}$$

To solve (2.41), we choose $a_0(x_1,t)$ to be a solution to

$$\frac{1}{c(x_1,t,0)}(\partial_{x_1} - i\partial_t)(e^{\phi(x_1,t)}a_0) = 1. \tag{2.42}$$

Note that (2.42) can be solved as it is a standard inhomogeneous $\overline{\partial}$ equation in the complex plane $z = x_1 - it$,

$$\overline{\partial}(e^{\phi(x_1,t)}a_0) = c/2. \tag{2.43}$$

Step 4. *Establishing the estimates* (2.1) *locally near the point* p_0. First it follows from (2.11) and (2.31) that

$$v_s(x_1,t,y) = e^{is\varphi(t,y)}h^{-\frac{(n-2)}{4}}a_0(x_1,t)\chi\left(\frac{y}{\delta'}\right). \tag{2.44}$$

Using (2.17), we have

$$|v_s(x_1,t,y)| \le \mathcal{O}(1)h^{-\frac{(n-2)}{4}}e^{-\frac{1}{h}d|y|^2}\chi\left(\frac{y}{\delta'}\right), \quad (x_1,t,y) \in J \times U, \tag{2.45}$$

and therefore,

$$\|v_s\|_{L^2(J\times U)} \le \mathcal{O}(1)\|h^{-\frac{(n-2)}{4}}e^{-\frac{1}{h}d|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(1), \quad h \to 0, \tag{2.46}$$

where $J \subset \mathbb{R}$ is a large fixed bounded open interval. Similarly, it follows from (2.44) that

$$\|\nabla v_s\|_{L^2(J\times U)} = \mathcal{O}(h^{-1}). \tag{2.47}$$

Let us next estimate $\|e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s\|_{L^2(J\times U)}$. To that end, letting

$$f = \langle \nabla_g\widetilde{\varphi}, \nabla_g\widetilde{\varphi}\rangle_g = \mathcal{O}(|y|^3) \tag{2.48}$$

(cf. (2.16)), we obtain from (2.15) with the help of (2.21) that

$$\begin{aligned} e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s &= e^{is\varphi}h^4\big((-\Delta_g)^2a - s\Delta_g(La) + s^2\Delta_g(fa) \\ &\quad + sL(-\Delta_g a) + s^2L^2a - s^3L(fa) + s^2f(\Delta_g a) - s^3fLa + s^4f^2a\big). \end{aligned} \tag{2.49}$$

We shall proceed to bound each term in (2.49) in $L^2(J\times U)$. First using (2.31) and (2.17), we get

$$\begin{aligned} \|e^{is\varphi}h^4(-\Delta_g)^2a\|_{L^2(J\times U)} &= h^4\|e^{is\varphi}h^{-\frac{(n-2)}{4}}(-\Delta_g)^2(a_0\chi)\|_{L^2(J\times U)} \\ &= \mathcal{O}(h^4)\|h^{-\frac{(n-2)}{4}}e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^4), \end{aligned} \tag{2.50}$$

and similarly,

$$\|e^{is\varphi}h^4s\Delta_g(La)\|_{L^2(J\times U)} = \mathcal{O}(h^3) \tag{2.51}$$

and

$$\|e^{is\varphi}h^4sL(\Delta_g a)\|_{L^2(J\times U)} = \mathcal{O}(h^3). \tag{2.52}$$

Now to bound $e^{is\varphi}h^4s^2\Delta_g(fa)$ in $L^2(J\times U)$ we note that the worst case occurs when Δ_g falls on f, and in this case we have, using (2.48) and (2.31),

$$\|e^{is\varphi}h^4s^2\Delta_g(f)a\|_{L^2(J\times U)} \le \mathcal{O}(h^2)\|h^{-\frac{(n-2)}{4}}|y|e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{5/2}),$$

and therefore,

$$\|e^{is\varphi}h^4s^2\Delta_g(fa)\|_{L^2(J\times U)} = \mathcal{O}(h^{5/2}). \tag{2.53}$$

Here we have used the following bound:

$$\|h^{-\frac{(n-2)}{4}}|y|^ke^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{k/2}), \quad k=1,2,\dots. \tag{2.54}$$

Similarly, using (2.32) and (2.54), we get

$$\|e^{is\varphi}h^4s^2L^2a\|_{L^2(J\times U)} \le \mathcal{O}(h^2)\|h^{-\frac{(n-2)}{4}}|y|e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{5/2}). \tag{2.55}$$

Using (2.48), (2.54), and the fact that $L(\mathcal{O}(|y|^3)) = \mathcal{O}(|y|^3)$, we obtain that

(2.56)

$$\begin{aligned} &\|e^{is\varphi}h^4s^3L(fa)\|_{L^2(J\times U)} \le \mathcal{O}(h)\|h^{-\frac{(n-2)}{4}}|y|^3e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{5/2}), \\ &\|e^{is\varphi}h^4s^2f(\Delta_g a)\|_{L^2(J\times U)} \le \mathcal{O}(h^2)\|h^{-\frac{(n-2)}{4}}|y|^3e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{7/2}), \\ &\|e^{is\varphi}h^4s^3fLa\|_{L^2(J\times U)} \le \mathcal{O}(h)\|h^{-\frac{(n-2)}{4}}|y|^3e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{5/2}), \\ &\|e^{is\varphi}h^4s^4f^2a\|_{L^2(J\times U)} \le \mathcal{O}(1)\|h^{-\frac{(n-2)}{4}}|y|^6e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^3). \end{aligned}$$

Combining (2.49), (2.50), (2.51), (2.52), (2.53), (2.55), (2.56), we get

$$\|e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s\|_{L^2(J\times U)} = \mathcal{O}(h^{5/2}). \tag{2.57}$$

This completes verification of (2.1) locally.

For later purposes we need estimates for $\|v_s(x_1,\cdot)\|_{L^2(\partial M_0)}$. If U contains a boundary point $x_0=(t_0,0)\in\partial M_0$, then $\partial_t|_{x_0}$ is transversal to ∂M_0. Let ρ be a boundary defining function for M_0 so that ∂M_0 is given by the zero set $\rho(t,y)=0$ near x_0. Then $\nabla\rho(x_0)$ is normal to ∂M_0, and hence, $\partial_t\rho(x_0)\neq 0$. By the implicit function theorem, there is a smooth function $y\mapsto t(y)$ near 0 such that ∂M_0 near x_0 is given by $\{(t(y),y):|y|<r_0\}$ for some $r_0>0$ small; see also [27]. Then using (2.45), we get

$$\begin{aligned}\|v_s(x_1,\cdot)\|^2_{L^2(\partial M_0\cap U)} &= \int_{|y|<r_0}|v_s(x_1,t(y),y)|^2dS(y)\\ &\le \mathcal{O}(1)\int_{\mathbb{R}^{n-2}}h^{-\frac{(n-2)}{2}}e^{-2\frac{d}{h}|y|^2}dy=\mathcal{O}(1).\end{aligned}\tag{2.58}$$

Step 5. *Establishing estimates* (2.1) *globally.* Now let us construct the quasimode v_s in M by gluing together quasimodes defined along small pieces of the geodesic. As $\gamma:(-2\varepsilon,L+2\varepsilon)\to\widehat{M_0}$ is a unit speed nontangential geodesic, an application of [27, Lemma 7.2] shows that $\gamma|_{[-\varepsilon,L+\varepsilon]}$ self-intersects only at finitely many times t_j with

$$0\le t_1<\cdots<t_N\le L.$$

We let $t_0=-\varepsilon$ and $t_{N+1}=L+\varepsilon$. By [15, Lemma 3.5], there exists an open cover $\{(U_j,\kappa_j)\}_{j=0}^{N+1}$ of $\gamma([-\varepsilon,L+\varepsilon])$ consisting of coordinate neighborhoods having the following properties:

(i) $\kappa_j(U_j)=I_j\times B$, where I_j are open intervals and $B=B(0,\delta')$ is an open ball in $\mathbb{R}^{n-2}$. Here $\delta'>0$ can be taken arbitrarily small and the same for each U_j,

(ii) $\kappa_j(\gamma(t))=(t,0)$ for each $t\in I_j$,

(iii) t_j only belongs to I_j and $\overline{I_j}\cap\overline{I_k}=\emptyset$ unless $|j-k|\le 1$,

(iv) $\kappa_j=\kappa_k$ on $\kappa_j^{-1}((I_j\cap I_k)\times B)$.

To construct the quasimode v_s globally, we first find a function $v_s^{(0)}=e^{is\varphi^{(0)}}a^{(0)}$, $a^{(0)}=h^{-\frac{(n-2)}{4}}a_0^{(0)}\chi$, in U_0 as above. Choose some t_0' with $\gamma(t_0')\in U_0\cap U_1$. To construct the phase $\varphi^{(1)}$ in U_1, we solve the Riccati equation (2.19) with the initial condition $H^{(1)}(t_0')=H^{(0)}(t_0')$. Continuing in this way, we obtain the phases $\varphi^{(0)},\varphi^{(1)},\dots,\varphi^{(N+1)}$ such that $\varphi^{(j)}=\varphi^{(j+1)}$ on $U_j\cap U_{j+1}$. In a similar way, by solving ODE in (2.37) with prescribed initial conditions we get $\phi^{(0)},\dots,\phi^{(N+1)}$, and therefore, in view of (2.39) we obtain $a_0^{(0)},a_0^{(1)},\dots,a_0^{(N+1)}$, and hence, we construct the amplitude of the first type globally.

To construct the amplitude of the second type, we need to solve the inhomogeneous $\bar\partial$–type equations (2.43). To that end, we first find $a_0^{(0)}$ and $a_0^{(1)}$ which are solutions of (2.43) on $\widetilde{J}\times I_0$ and on $\widetilde{J}\times I_1$, respectively. Here $\widetilde{J}\subset\mathbb{R}$ is a bounded open interval. Then we see that $e^{\phi^{(1)}}a_0^{(1)}-e^{\phi^{(0)}}a_0^{(0)}$ is holomorphic on $\widetilde{J}\times(I_0\cap I_1)$. By [6, Example 3.25], there are holomorphic functions g_1,g_0 on $\widetilde{J}\times I_1$ and $\widetilde{J}\times I_0$, respectively, such that $e^{\phi^{(1)}}a_0^{(1)}-e^{\phi^{(0)}}a_0^{(0)}=g_0-g_1$ on $\widetilde{J}\times(I_0\cap I_1)$. Thus, modifying $a_0^{(0)}$ and $a_0^{(1)}$, we can always arrange so that $a_0^{(0)}=a_0^{(1)}$ on $\widetilde{J}\times(I_0\cap I_1)$. Proceeding in the same way, we can find $a_0^{(2)},\dots,a_0^{(N+1)}$ so that $a_0^{(j)}=a_0^{(j+1)}$ on $\widetilde{J}\times(I_j\cap I_{j+1})$, and hence, we construct the amplitude of the second type globally.

Thus, we obtain the quasimodes $v_s^{(0)},\dots,v_s^{(N+1)}$ such that

$$v_s^{(j)}(x_1,\cdot)=v_s^{(j+1)}(x_1,\cdot)\quad\text{in}\quad U_j\cap U_{j+1}\tag{2.59}$$

for all x_1. Let $\chi_j = \chi_j(t) \in C_0^\infty(I_j)$ be such that $\sum_{j=0}^{N+1} \chi_j = 1$ near $[-\varepsilon, L+\varepsilon]$, and define our quasimode v globally by

$$v_s = \sum_{j=0}^{N+1} \chi_j v_s^{(j)}.$$

Let us next give a local description of the quasimode v_s near self-intersecting points of the geodesic γ and near the other points of γ. To that end, let $p_1, \dots, p_R \in M_0$ be the distinct points where the geodesic self-intersects, and let $0 \le t_1 < \cdots < t_{R'}$ be the times of self-intersections. Let $V_1, \dots, V_R$ be small neighborhoods in $\widehat{M}_0$ around p_j, $j = 1, \dots, R$. Then choosing δ' small enough we obtain an open cover in $\widehat{M}_0$,

$$\text{supp } (v_s(x_1, \cdot)) \cap M_0 \subset (\cup_{j=1}^R V_j) \cup (\cup_{k=1}^S W_k), \tag{2.60}$$

where in each V_j, the quasimode is a finite sum,

$$v_s(x_1, \cdot)|_{V_j} = \sum_{l:\gamma(t_l)=p_j} v_s^{(l)}(x_1, \cdot), \tag{2.61}$$

and in each W_k (where there are no self-intersecting points), in view of (2.59), there is some $l(k)$ so that the quasimode is given by

$$v_s(x_1, \cdot)|_{W_k} = v_s^{l(k)}(x_1, \cdot). \tag{2.62}$$

We also have

$$\text{supp } (v_s) \cap M \subset (\cup_{j=1}^R \widetilde{J} \times V_j) \cup (\cup_{k=1}^S \widetilde{J} \times W_k),$$

where $\widetilde{J} \subset \mathbb{R}$ is a bounded open interval.

Finally, the bounds in (2.1) follow from the bounds (2.46), (2.47), (2.57), and the representations (2.61) and (2.62) of v.

Step 6. *Construction of the Gaussian beam quasimodes w_s.* Now look for a Gaussian beam quasimode for the operator $e^{-sx_1}(-h^2\Delta_g)^2 e^{sx_1}$ in the form

$$w_s(x_1, t, y) = e^{is\varphi(t,y)} b(x_1, t, y; s), \tag{2.63}$$

where $\varphi \in C^\infty(U)$ is the phase function given by (2.18), and $b \in C^\infty(\mathbb{R} \times U)$ is an amplitude, which we shall proceed to determine. To that end, first, similarly to (2.15), we get

(2.64)
$$e^{-sx_1}(-h^2\Delta_g)^2 e^{sx_1} w_s = e^{is\varphi} h^4 \big(-\Delta_g - s(2\langle \nabla_g \widetilde{\widetilde{\varphi}}, \nabla_g \cdot \rangle_g + \Delta_g \widetilde{\widetilde{\varphi}}) - s^2 \langle \nabla_g \widetilde{\widetilde{\varphi}}, \nabla_g \widetilde{\widetilde{\varphi}} \rangle_g \big)^2 b,$$

where

$$\widetilde{\widetilde{\varphi}}(x_1, t, y) = x_1 + i\varphi(t, y). \tag{2.65}$$

With φ given by (2.18), we have

$$\langle \nabla_g \widetilde{\widetilde{\varphi}}, \nabla_g \widetilde{\widetilde{\varphi}} \rangle_g = \mathcal{O}(|y|^3),$$

as $y \to 0$. We thus look for the smooth amplitude $b = b(x_1, x')$ satisfying the transport equation,

$$\widetilde{L}^2 b = \mathcal{O}(|y|), \tag{2.66}$$

where

$$\widetilde{L} = 2\langle \nabla_g \widetilde{\widetilde{\varphi}}, \nabla_g \cdot \rangle_g + \Delta_g \widetilde{\widetilde{\varphi}}. \tag{2.67}$$

Let us simplify the operator $\widetilde{L}$. First using (2.65), we get

$$\langle \nabla_g \widetilde{\widetilde{\varphi}}, \nabla_g \cdot \rangle_g = \frac{1}{c}(\partial_{x_1} + i g_0^{-1}(x')\varphi'_{x'} \cdot \partial_{x'}), \tag{2.68}$$

$$\Delta_g \widetilde{\varphi} = \Delta_g x_1 + i\Delta_g \varphi(x'). \tag{2.69}$$

Hence, using (2.68), (2.69), (2.24), and (2.25), the operator $\widetilde{L}$ given by (2.67) becomes

$$\widetilde{L} = \frac{2}{c}(\partial_{x_1} + i g_0^{-1}(x')\varphi'_{x'} \cdot \partial_{x'}) + \left(\frac{n}{2} - 1\right)\frac{1}{c^2}\partial_{x_1} c + \frac{i}{c}\Delta_{g_0}\varphi + \left(\frac{n}{2} - 1\right)\frac{i}{c^2}\langle \nabla_{g_0} c, \nabla_{g_0}\varphi\rangle_{g_0}. \tag{2.70}$$

Using (2.27), (2.28), (2.29), the operator $\widetilde{L}$ in (2.70) becomes

$$\begin{aligned} \widetilde{L} = &= \frac{2}{c(x_1,t,0)}\Big[\partial_{x_1} + i\partial_t + iH(t)y \cdot \partial_y + (\partial_{x_1} + i\partial_t)\log c(x_1,t,0)^{\frac{n}{4} - \frac{1}{2}} \\ &+ \frac{i}{2}\operatorname{tr} H(t) + \mathcal{O}(|y|) + \mathcal{O}(|y|)(\partial_{x_1}, \partial_t) + \mathcal{O}(|y|^2)\partial_y\Big]. \end{aligned} \tag{2.71}$$

We look for the amplitude b in the form

$$b(x_1,t,y) = h^{-\frac{(n-2)}{4}} b_0(x_1,t)\chi\left(\frac{y}{\delta'}\right), \tag{2.72}$$

where $b_0(\cdot,\cdot) \in C^\infty(\mathbb{R} \times \{t : |t - t_0| < \delta\})$ is independent of y, and in view of (2.66), b_0 should satisfy

$$\widetilde{L}^2 b_0 = \mathcal{O}(|y|), \quad y \to 0. \tag{2.73}$$

It follows from (2.70) that

$$\widetilde{L} = \frac{2}{c(x_1,t,0)}(\widetilde{L}_0 + \widetilde{R}), \tag{2.74}$$

where

$$\widetilde{L}_0 = (\partial_{x_1} + i\partial_t) + (\partial_{x_1} + i\partial_t)\log c(x_1,t,0)^{\frac{n}{4} - \frac{1}{2}} + \frac{i}{2}\operatorname{tr} H(t), \tag{2.75}$$

and

$$\widetilde{R} = iH(t)y \cdot \partial_y + \mathcal{O}(|y|) + \mathcal{O}(|y|)(\partial_{x_1}, \partial_t) + \mathcal{O}(|y|^2)\partial_y. \tag{2.76}$$

In contrast to the construction of the Gaussian beam quasimodes v_s, we shall only need amplitudes of the first type. To construct such amplitudes, we note that as b_0 is independent of y, if b_0 solves the equation

$$\widetilde{L}_0 b_0 = 0, \tag{2.77}$$

then b_0 satisfies (2.73). To find a solution to (2.77), we note that

$$\widetilde{L}_0 = e^{-\widetilde{\phi}(x_1,t)}(\partial_{x_1} + i\partial_t)e^{\widetilde{\phi}(x_1,t)}, \tag{2.78}$$

where $\widetilde{\phi}(x_1,t)$ is given by

$$\widetilde{\phi}(x_1,t) = \log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}} + F(t), \quad \partial_t F(t) = \frac{1}{2}\operatorname{tr} H(t). \tag{2.79}$$

We solve (2.77) by taking

$$b_0 = e^{-\widetilde{\phi}} = c(x_1,t,0)^{\frac{1}{2}-\frac{n}{4}}e^{-F(t)}. \tag{2.80}$$

Proceeding further as in the construction of the quasimode v_s above, we obtain the quasimode $w_s \in C^\infty(M)$ such that (2.2) holds. ∎

We shall need the following result.

PROPOSITION 2.3. *Let $X \in C(M, TM)$ be a complex vector field, let $\psi \in C(M_0)$, and let $x_1' \in \mathbb{R}$. Then there exist the Gaussian beam quasimodes v_s and w_s given by Proposition* 2.1 *such that v_s is obtained using amplitudes of the first type and we have*

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} v_s\overline{w_s}\psi dV_{g_0} = \int_0^L e^{-2\lambda t}c(x_1,\gamma(t))^{1-\frac{n}{2}}\psi(\gamma(t))dt \tag{2.81}$$

and

$$\lim_{h\to 0} h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} = i\int_0^L X_t(x_1',\gamma(t))e^{-2\lambda t}c(x_1,\gamma(t))^{1-\frac{n}{2}}\psi(\gamma(t))dt. \tag{2.82}$$

Here $X_t(x_1',\gamma(t)) = \langle X(x_1',\gamma(t)), (0,\dot\gamma(t))\rangle_g$.

Proof. Step 1. *Proof of* (2.81). Let $\psi \in C(M_0)$, $x_1' \in \mathbb{R}$. Using a partition of unity, in view of (2.60), it suffices to establish (2.81) for ψ having compact support in one of the sets V_j or W_k. First, assume that $\psi \in C_0(M_0)$, supp $(\psi) \subset W_k$. Thus, in view of (2.62), (2.44), (2.63), (2.72), on supp (ψ), we have

$$v_s = e^{is\varphi}h^{-\frac{(n-2)}{4}}a_0(x_1',t)\chi\left(\frac{y}{\delta'}\right), \quad w_s = e^{is\varphi}h^{-\frac{(n-2)}{4}}b_0(x_1',t)\chi\left(\frac{y}{\delta'}\right). \tag{2.83}$$

To proceed, we shall need the consequence of (2.10),

$$|g_0|^{1/2} = 1 + \mathcal{O}(|y|^2), \tag{2.84}$$

as well as

$$is\varphi - i\overline{s\varphi} = -2\frac{1}{h}\mathrm{Im}\varphi - 2\lambda\mathrm{Re}\varphi. \tag{2.85}$$

Using (2.83), (2.84), (2.85), (2.18), we get

$$\begin{aligned}
&\int_{\{x_1'\}\times M_0} v_s\overline{w_s}\psi dV_{g_0}\\
&= \int_0^L\int_{\mathbb{R}^{n-2}} e^{-2\frac{1}{h}\mathrm{Im}\varphi}e^{-2\lambda\mathrm{Re}\varphi}h^{-\frac{(n-2)}{2}}a_0(x_1',t)\overline{b_0(x_1',t)}\chi^2\left(\frac{y}{\delta'}\right)\psi(t,y)|g_0|^{\frac{1}{2}}dydt\\
&= \int_0^L\int_{\mathbb{R}^{n-2}} e^{-\frac{1}{h}\mathrm{Im}H(t)y\cdot y}e^{-2\lambda t}e^{\lambda\mathcal{O}(|y|^2)}h^{-\frac{(n-2)}{2}}a_0(x_1',t)\overline{b_0(x_1',t)}\chi^2\left(\frac{y}{\delta'}\right)\\
&\quad \psi(t,y)(1+\mathcal{O}(|y|^2))dydt.
\end{aligned} \tag{2.86}$$

Making the change of variable $y = h^{1/2}\widetilde{y}$ in (2.86), we obtain that

$$\int_{\{x_1'\}\times M_0} v_s\overline{w_s}\psi dV_{g_0} = \int_0^L\int_{\mathbb{R}^{n-2}} e^{-\mathrm{Im}H(t)\widetilde{y}\cdot\widetilde{y}}e^{-2\lambda t}e^{\lambda h\mathcal{O}(|\widetilde{y}|^2)}a_0(x_1',t)\overline{b_0(x_1',t)} \chi^2\left(\frac{h^{1/2}\widetilde{y}}{\delta'}\right)\psi(t,h^{1/2}\widetilde{y})(1+h\mathcal{O}(|\widetilde{y}|^2))dtd\widetilde{y}. \tag{2.87}$$

Using that

$$\int_{\mathbb{R}^{n-2}} e^{-\mathrm{Im}H(t)y\cdot y}dy = \frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}, \tag{2.88}$$

and the dominated covergence theorem, we get from (2.87) that

$$\begin{aligned}\lim_{h\to 0}&\int_{\{x_1'\}\times M_0} v_s\overline{w_s}\psi dV_{g_0}\\ &= \int_0^L e^{-2\lambda t}a_0(x_1',t)\overline{b_0(x_1',t)}\psi(t,0)\int_{\mathbb{R}^{n-2}} e^{-\mathrm{Im}H(t)y\cdot y}dydt\\ &= \int_0^L e^{-2\lambda t}a_0(x_1',t)\overline{b_0(x_1',t)}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}\psi(t,0)dt.\end{aligned} \tag{2.89}$$

Let us proceed to simplify the expression in (2.89) in the case when a_0 is the amplitude of the first type, i.e., a_0 be given by (2.39), and let b_0 be given by (2.80). Then

$$a_0(x_1',t)\overline{b_0(x_1',t)}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}} = c(x_1,t,0)^{1-\frac{n}{2}}e^{-(G(t)+\overline{F(t)})}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}. \tag{2.90}$$

Now it follows from (2.39) and (2.79) that

$$G(t)+\overline{F(t)} = G(t_0)+\overline{F(t_0)}+\int_{t_0}^t \mathrm{tr}\,\mathrm{Re}(H(s))ds. \tag{2.91}$$

Using (2.91) and the property of solutions of the matrix Riccati equation [26, Lemma 2.58],

$$\det(\mathrm{Im}H(t)) = \det(\mathrm{Im}H(t_0))e^{-2\int_{t_0}^t \mathrm{tr}\,\mathrm{Re}(H(s))ds},$$

we see that

$$e^{-(G(t)+\overline{F(t)})}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}} = e^{-(G(t_0)+\overline{F(t_0)})}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t_0))}} \tag{2.92}$$

is a constant in t. To fix this constant, when constructing the amplitude a_0 and b_0, specifically, when solving (2.39) and (2.79) in U_0, we choose initial conditions for G and F so that the constant in (2.92) is equal to 1. With this choice, it follows from (2.89), (2.90), (2.92) that

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} v_s\overline{w_s}\psi dV_{g_0} = \int_0^L e^{-2\lambda t}c(x_1,t,0)^{1-\frac{n}{2}}\psi(t,0)dt. \tag{2.93}$$

This completes the proof of (2.81) in the case when supp $(\psi)\subset W_k$.

Let us now establish (2.81) when supp $(\psi) \subset V_j$. Here on supp (ψ) we have

$$v_s = \sum_{l:\gamma(t_l)=p_j} v_s^{(l)}, \quad w_s = \sum_{l:\gamma(t_l)=p_j} w_s^{(l)}, \tag{2.94}$$

and hence,

$$v_s\overline{w_s} = \sum_{l:\gamma(t_l)=p_j} v_s^{(l)}\overline{w_s^{(l)}} + \sum_{l\neq l',\gamma(t_l)=\gamma(t_{l'})=p_j} v_s^{(l)}\overline{w_s^{(l')}}. \tag{2.95}$$

We shall use a nonstationary phase argument as in [15, end of proof Proposition 3.1] to show that the contribution of the mixed terms vanishes in the limit $h \to 0$, i.e., if $l \neq l'$,

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} v_s^{(l)}\overline{w_s^{(l')}}\psi dV_{g_0} = 0. \tag{2.96}$$

In doing so, write

$$v_s^{(l)} = e^{i\frac{1}{h}\mathrm{Re}\,\varphi^{(l)}}p^{(l)}, \quad p^{(l)} = e^{-\lambda \mathrm{Re}\,\varphi^{(l)}}e^{-s\mathrm{Im}\,\varphi^{(l)}}a^{(l)}$$

and

$$w_s^{(l')} = e^{i\frac{1}{h}\mathrm{Re}\,\varphi^{(l')}}q^{(l')}, \quad q^{(l')} = e^{-\lambda \mathrm{Re}\,\varphi^{(l')}}e^{-s\mathrm{Im}\,\varphi^{(l')}}b^{(l')},$$

and therefore,

$$v_s^{(l)}\overline{w_s^{(l')}} = e^{i\frac{1}{h}\phi}p^{(l)}\overline{q^{(l')}}, \tag{2.97}$$

where

$$\phi = \mathrm{Re}\,\varphi^{(l)} - \mathrm{Re}\,\varphi^{(l')}.$$

Thus, in view of (2.96) and (2.97) we shall show that for $l \neq l'$,

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} e^{i\frac{1}{h}\phi}p^{(l)}\overline{q^{(l')}}\psi dV_{g_0} = 0. \tag{2.98}$$

Since $\partial_t\varphi^{(l)}(t,0) = \partial_t\varphi^{(l')}(t,0) = 1$ and the geodesic intersects itself transversally, as explained in [27, Lemma 7.2], we see that $d\phi(p_j) \neq 0$. By decreasing the set V_j if necessary, we may assume that $d\phi \neq 0$ in V_j.

To prove (2.98), we shall integrate by parts and in doing so, we let $\varepsilon > 0$ be fixed, and decompose $\psi = \psi_1 + \psi_2$, where $\psi_1 \in C^\infty(M_0)$, supp $(\psi_1) \subset V_j$ and and $\|\psi_2\|_{L^\infty(V_j\cap M_0)} \leq \varepsilon$. Notice that ψ may be nonzero on ∂M_0. We have

$$\left|\int_{\{x_1'\}\times M_0} e^{i\frac{1}{h}\phi}p^{(l)}\overline{q^{(l')}}\psi_2 dV_{g_0}\right| \leq \|v_s^{(l)}\|_{L^2}\|w_s^{(l)}\|_{L^2}\|\psi_2\|_{L^\infty} \leq \mathcal{O}(\varepsilon). \tag{2.99}$$

For the smooth part ψ_1, we integrate by parts using that

$$e^{i\frac{1}{h}\phi} = \frac{h}{i}L(e^{i\frac{1}{h}\phi}), \quad L = \frac{1}{|d\phi|^2}\langle d\phi, d\cdot\rangle_{g_0}.$$

We have

$$\begin{aligned}\int_{\{x_1'\}\times M_0} e^{i\frac{1}{h}\phi}p^{(l)}\overline{q^{(l')}}\psi_1 dV_{g_0} &= \int_{\{x_1'\}\times (V_j\cap\partial M_0)} h\frac{\partial_\nu\phi}{i|d\phi|^2}e^{i\frac{1}{h}\phi}p^{(l)}\overline{q^{(l')}}\psi_1 dS \\ &\quad + h\frac{1}{i}\int_{\{x_1'\}\times M_0} e^{i\frac{1}{h}\phi}L^t(p^{(l)}\overline{q^{(l')}}\psi_1)dV_{g_0},\end{aligned} \tag{2.100}$$

where $L^t = -L - \mathrm{div}\, L$ is the transpose of L.

In view of (2.58), the boundary term is of $\mathcal{O}(h)$ as $h \to 0$. To estimate the second term in the right-hand side of (2.100), we recall that

$$p^{(l)}\overline{q^{(l')}} = e^{-\lambda(\operatorname{Re}\varphi^{(l)}+\operatorname{Re}\varphi^{(l')})}e^{-i\lambda(\operatorname{Im}\varphi^{(l)}-\operatorname{Im}\varphi^{(l')})}e^{-\frac{1}{h}(\operatorname{Im}\varphi^{(l)}+\operatorname{Im}\varphi^{(l')})}h^{-\frac{(n-2)}{2}} a_0^{(l)}(x_1',t)\overline{b_0^{(l')}(x_1',t)}\chi^2\left(\frac{y}{\delta'}\right).$$

This shows that to bound the second term in the right-hand side of (2.100), it is enough to analyze the contributions occurring when differentiating

$$e^{-\frac{1}{h}(\operatorname{Im}\varphi^{(l)}+\operatorname{Im}\varphi^{(l')})},$$

as all the other contributions are of $\mathcal{O}(h)$, as $h \to 0$.

As in [15], using (2.17), we have

$$|L(e^{-\frac{1}{h}(\operatorname{Im}\varphi^{(l)}+\operatorname{Im}\varphi^{(l')})})| \le \mathcal{O}(h^{-1})|d(\operatorname{Im}\varphi^{(l)}+\operatorname{Im}\varphi^{(l')})|e^{-\frac{1}{h}d|y|^2} \le \mathcal{O}(h^{-1}|y|)e^{-\frac{1}{h}d|y|^2},$$

which shows that the corresponding contribution to the second term in the right-hand side of (2.100) is of $\mathcal{O}(h^{1/2})$. This shows that the integral in the left-hand side of (2.100) goes to 0 as $h \to 0$, and this together with (2.99) establishes (2.96).

Using (2.93) for each of the factors $v_s^{(l)}\overline{w_s^{(l)}}$ in (2.95), we get

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} v_s^{(l)}\overline{w_s^{(l)}}\psi dV_{g_0} = \int_{I_l} e^{-2\lambda t}c(x_1,t,0)^{1-\frac{n}{2}}\psi(t,0)dt.$$

Summing over I_l, appearing in the Fermi coordinates, such that $t_l \in I_l$ and $\gamma(t_l) = p_j$, we get (2.81) when supp $(\psi) \subset V_j$ and hence, in general.

Step 2. *Establishing* (2.82). Let $X \in C(M, TM)$ be a complex vector field, $\psi \in C(M_0)$, and $x_1' \in \mathbb{R}$. Using a partition of unity, it is enough to verify (2.82) in the following two cases: supp $(\psi) \subset W_k$ and supp $(\psi) \subset V_j$. Assume first that supp $(\psi) \subset W_k$. Using (2.83), we get

$$h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} = I_{1,1} + I_{1,2} + I_2, \tag{2.101}$$

where

$$I_{1,1} = \int_{\{x_1'\}\times M_0} iX(\varphi)v_s\overline{w_s}\psi dV_{g_0}, \tag{2.102}$$

$$I_{1,2} = -h\int_{\{x_1'\}\times M_0} \lambda X(\varphi)v_s\overline{w_s}\psi dV_{g_0}, \tag{2.103}$$

$$I_2 = h\int_{\{x_1'\}\times M_0} h^{-\frac{(n-2)}{4}}e^{is\varphi}X(a_0\chi)\overline{w_s}\psi dV_{g_0}. \tag{2.104}$$

Using (2.1) and (2.2), we have

$$\begin{aligned} |I_{1,2}| &\le \mathcal{O}(h)\|v_s(x',\cdot)\|_{L^2(M_0)}\|w_s(x_1',\cdot)\|_{L^2(M_0)} = \mathcal{O}(h),\\ |I_2| &\le \mathcal{O}(h)\|e^{is\varphi}h^{-\frac{(n-2)}{4}}\|_{L^2(\{|y|\le\delta'/2\})}\|w_s(x_1',\cdot)\|_{L^2(M_0)} = \mathcal{O}(h). \end{aligned} \tag{2.105}$$

Let us now compute $\lim_{h\to 0} I_{1,1}$. To that end, we write

$$X = X_1\partial_{x_1} + X_t\partial_t + X_y\cdot\partial_y, \quad x=(x_1,t,y). \tag{2.106}$$

Using (2.18), we get

$$\partial_t\varphi = 1+\mathcal{O}(|y|^2), \quad \partial_y\varphi = \mathcal{O}(|y|). \tag{2.107}$$

As X is continuous, it follows from (2.106) and (2.107) that

$$X(\varphi) = (X_t(x_1,t,0)+o(1))(1+\mathcal{O}(|y|^2)) + \mathcal{O}(|y|) = X_t(x_1,t,0)+o(1), \tag{2.108}$$

as $y\to 0$, uniformly in x_1 and t. Using (2.108), as in (2.86), we obtain from (2.102) that

$$\begin{aligned} I_{1,1} = \int_0^L \int_{\mathbb{R}^{n-2}} & i(X_t(x_1',t,0)+o(1))h^{-\frac{(n-2)}{2}} e^{-\frac{1}{h}\mathrm{Im}H(t)y\cdot y} e^{-2\lambda t} e^{\lambda\mathcal{O}(|y|^2)} \\ & a_0(x_1',t)\overline{b_0(x_1',t)}\chi^2\left(\frac{y}{\delta'}\right)\psi(t,y)(1+\mathcal{O}(|y|^2))dydt. \end{aligned} \tag{2.109}$$

We first observe that

$$\lim_{h\to 0} I_{1,1,2} = 0, \tag{2.110}$$

uniformly in x_1' and t, where

$$\begin{aligned} I_{1,1,2} = \int_{\mathbb{R}^{n-2}} g(x_1',t,y)dy, \quad & g(x_1',t,y) = o(1)h^{-\frac{(n-2)}{2}} e^{-\frac{1}{h}\mathrm{Im}H(t)y\cdot y} e^{-2\lambda t} \\ & e^{\lambda\mathcal{O}(|y|^2)} a_0(x_1',t)\overline{b_0(x_1',t)}\chi^2\left(\frac{y}{\delta'}\right)\psi(t,y)(1+\mathcal{O}(|y|^2)). \end{aligned}$$

Indeed, let $\varepsilon>0$ and let $\delta>0$ be such that $|o(1)|\le\varepsilon$ when $|y|\le\delta$. Then

$$\begin{aligned} |I_{1,1,2}| &\le \left|\int_{|y|\le\delta} g(x_1',t,y)dy\right| + \left|\int_{|y|\ge\delta} g(x_1',t,y)dy\right| \\ &\le \varepsilon\mathcal{O}(1)\left|\int_{\mathbb{R}^{n-2}} h^{-\frac{(n-2)}{2}} e^{-\frac{1}{h}\mathrm{Im}H(t)y\cdot y}dy\right| + \mathcal{O}(e^{-d\delta^2/h}) \le \varepsilon\mathcal{O}(1) + \mathcal{O}(e^{-d\delta^2/h}), \end{aligned}$$

showing (2.110).

Using (2.110), making the change of variables $y = h^{1/2}\widetilde{y}$ in (2.109), using the dominated convergence theorem, and (2.88), we get

$$\lim_{h\to 0} I_{1,1} = i\int_0^L X_t(x_1',t,0)e^{-2\lambda t}a_0(x_1',t)\overline{b_0(x_1',t)}\psi(t,0)\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}dt. \tag{2.111}$$

It follows from (2.101) with the help of (2.105) and (2.111) that

$$\begin{aligned} &\lim_{h\to 0} h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} \\ &\quad = i\int_0^L X_t(x_1',t,0)e^{-2\lambda t}a_0(x_1',t)\overline{b_0(x_1',t)}\psi(t,0)\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}dt. \end{aligned} \tag{2.112}$$

When a_0 is the amplitude of the first type, i.e. a_0 be given by (2.39), and b_0 be given by (2.80), using (2.90), (2.92), we get from (2.112) that

(2.113)
$$\lim_{h\to 0} h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} = i\int_0^L X_t(x_1',t,0)e^{-2\lambda t}c(x_1,t,0)^{1-\frac{n}{2}}\psi(t,0)dt.$$

This establishes (2.82) when supp $(\psi)\subset W_k$.

Assume now that supp $(\psi)\subset V_j$, and therefore, on supp (ψ), v_s and w_s are given by (2.94). Then

(2.114)
$$\begin{aligned} h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} &= h\sum_{l:\gamma(t_l)=p_j}\int_{\{x_1'\}\times M_0} X(v_s^{(l)})\overline{w_s^{(l)}}\psi dV_{g_0} \\ &\quad + h\sum_{l\neq l':\gamma(t_l)=\gamma(t_{l'})=p_j}\int_{\{x_1'\}\times M_0} X(v_s^{(l)})\overline{w_s^{(l')}}\psi dV_{g_0}. \end{aligned}$$

As before, we shall show that the mixed terms, i.e., $l\neq l'$, vanish in the limit as $h\to 0$,

$$\lim_{h\to 0} h\int_{\{x_1'\}\times M_0} X(v_s^{(l)})\overline{w_s^{(l')}}\psi dV_{g_0} = 0. \tag{2.115}$$

It follows from (2.101), (2.102), (2.103), (2.104), (2.105) that we only have to prove that

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} iX(\varphi^{(l)})v_s^{(l)}\overline{w_s^{(l')}}\psi dV_{g_0} = 0. \tag{2.116}$$

Now (2.116) follows by repeating a nonstationary phase argument as in the proof of (2.96) replacing ψ by $X(\varphi^{(l)})\psi\in C(M_0)$. Thus, using (2.114) and (2.116), we see that

$$\begin{aligned} &\lim_{h\to 0} h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} \\ &\qquad = \sum_{l:\gamma(t_l)=p_j} i\int_{I_l} X_t(x_1',t,0)e^{-2\lambda t}c(x_1,t,0)^{1-\frac{n}{2}}\psi(t,0)dt, \end{aligned}$$

completing the proof of (2.82) when supp $(\psi)\subset V_j$. ∎

We shall also need the following result.

PROPOSITION 2.4. *Let $\psi\in C^1(\mathbb{R}\times M_0)$ be such that $\psi(x_1,\cdot)|_{\partial M_0}=0$ and with compact support in x_1. Then there exist Gaussian beam quasimodes v_s and w_s given by Proposition* 2.1 *such that v_s is obtained using amplitudes of the second type and*

$$\begin{aligned} \lim_{h\to 0}\Big[& h\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0}(\nabla_g\psi)(v_s)\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1 \\ & - \int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0}(\nabla_g\psi)_1 v_s\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1\Big] \\ & = \int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1-it)}\psi(x_1,\gamma(t))c(x_1,\gamma(t))dtdx_1. \end{aligned} \tag{2.117}$$

Proof. In view of (2.60), using a partition of unity, it suffices to check (2.117) for ψ such that supp $(\psi(x_1,\cdot))$ is in one of the sets V_j or W_k. Let us first consider the case when supp $(\psi(x_1,\cdot)) \subset W_k$. Thus, on supp $(\psi(x_1,\cdot))$, v_s and w_s are given by (2.83) with a_0 being an amplitude of type two. To proceed, we note that

$$\nabla_g\psi = \frac{1}{c}(\partial_{x_1}\psi\partial_{x_1} + g_0^{-1}\partial_{x'}\psi\cdot\partial_{x'}), \tag{2.118}$$

and therefore, using (2.10), we see that

$$(\nabla\psi)_t(x_1,t,0) = \frac{\partial_t\psi(x_1,t,0)}{c(x_1,t,0)}. \tag{2.119}$$

Using (2.83), (2.118), and (2.119), a computation similar to that in the proof of Proposition 2.3 (cf. (2.89) and (2.112)) gives

$$\begin{aligned} I = \lim_{h\to 0}\Big[h\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0}(\nabla_g\psi)(v_s)\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1 \\ -\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0}(\nabla_g\psi)_1 v_s\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1\Big] \\ = -\int_{\mathbb{R}}\int_0^L e^{-2i\lambda x_1}e^{-2\lambda t}((\partial_{x_1}-i\partial_t)\psi(x_1,t,0))a_0(x_1,t)\overline{b_0(x_1,t)} \\ \frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}c(x_1,t,0)^{\frac{n}{2}-1}dtdx_1. \end{aligned} \tag{2.120}$$

When solving (2.37) and (2.79) for G and F, respectively, we choose the initial conditions $G(t_0)$ and $F(t_0)$ so that the constant in (2.92) is equal to 1. Then using (2.80), (2.37), (2.92), we see that

$$\begin{aligned} a_0(x_1,t)\overline{b_0(x_1,t)}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}c(x_1,t,0)^{\frac{n}{2}-1} \\ = a_0(x_1,t)c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}}e^{-\overline{F(t)}}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}} \\ = a_0(x_1,t)c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}}e^{G(t)} = a_0(x_1,t)e^{\phi(x_1,t)}. \end{aligned} \tag{2.121}$$

Combining (2.120) and (2.121), integrating by parts, using the fact that ψ compact support in x_1 and $\psi(x_1,\cdot)|_{\partial M_0} = 0$, and using (2.42), we get

$$\begin{aligned} I = -\int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1-it)}((\partial_{x_1}-i\partial_t)\psi(x_1,t,0))a_0(x_1,t)e^{\phi(x_1,t)}dtdx_1 \\ = \int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1-it)}\psi(x_1,t,0)(\partial_{x_1}-i\partial_t)(a_0(x_1,t)e^{\phi(x_1,t)})dtdx_1 \\ = \int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1-it)}\psi(x_1,t,0)c(x_1,t,0)dtdx_1. \end{aligned} \tag{2.122}$$

This completes the proof of (2.117) in the case when supp $(\psi(x_1,\cdot)) \subset W_k$.

Let us now show (2.117) when supp $(\psi(x_1,\cdot)) \subset V_j$. Then on supp (ψ), v_s and w_s are given by (2.94), and we have

$$
\begin{aligned}
&\int_{\mathbb{R}} e^{-2i\lambda x_1} \int_{M_0} (h(\nabla_g\psi)(v_s) - (\nabla_g\psi)_1 v_s)\overline{w_s} c(x_1,x')^{\frac{n}{2}} dV_{g_0} dx_1 \\
&= \sum_{l:\gamma(t_l)=p_j} \int_{\mathbb{R}} e^{-2i\lambda x_1} \int_{M_0} (h(\nabla_g\psi)(v_s^{(l)}) - (\nabla_g\psi)_1 v_s^{(l)})\overline{w_s^{(l)}} c(x_1,x')^{\frac{n}{2}} dV_{g_0} dx_1 + \\
&\sum_{l\neq l':\gamma(t_l)=\gamma(t_{l'})=p_j} \int_{\mathbb{R}} e^{-2i\lambda x_1} \int_{M_0} (h(\nabla_g\psi)(v_s^{(l)}) - (\nabla_g\psi)_1 v_s^{(l)})\overline{w_s^{(l')}} c(x_1,x')^{\frac{n}{2}} dV_{g_0} dx_1.
\end{aligned} \tag{2.123}
$$

Now when $l \neq l'$, as in (2.96) and (2.115), by a nonstationary phase argument we see that

$$
\lim_{h\to 0} \int_{M_0} (h(\nabla_g\psi)(v_s^{(l)}) - (\nabla_g\psi)_1 v_s^{(l)})\overline{w_s^{(l')}} c(x_1,x')^{\frac{n}{2}} dV_{g_0} = 0,
$$

uniformly in x_1, and therefore, the limit $h \to 0$ of the second sum in (2.123) is equal to 0. Hence,

$$
\begin{aligned}
&\lim_{h\to 0} \int_{\mathbb{R}} e^{-2i\lambda x_1} \int_{M_0} (h(\nabla_g\psi)(v_s) - (\nabla_g\psi)_1 v_s)\overline{w_s} c(x_1,x')^{\frac{n}{2}} dV_{g_0} dx_1 \\
&= \sum_{l:\gamma(t_l)=p_j} \int_{\mathbb{R}} \int_{I_l} e^{-2i\lambda(x_1-it)} \psi(x_1,t,0) c(x_1,t,0) dt dx_1,
\end{aligned}
$$

showing (2.117) when supp $(\psi(x_1,\cdot)) \subset V_j$. ∎

3. Construction of complex geometric optics solutions based on Gaussian beam quasimodes. Let (M,g) be a CTA manifold so that $(M,g) \subset\subset (\mathbb{R}\times M_0^{\mathrm{int}}, c(e\oplus g_0))$. Let $X, Y \in L^\infty(M,TM)$ be complex vector fields, and let $q \in L^\infty(M,\mathbb{C})$. Consider the following operator:

$$
P_{X,Y,q} = (-\Delta_g)^2 + X + \operatorname{div}(Y) + q. \tag{3.1}
$$

Note that the operator $P_{X,Y,q}$ comprises both the operator $L_{X,q}$ as well as its formal adjoint $L^*_{X,q} = (-\Delta_g)^2 - \overline{X} - \operatorname{div}(\overline{X}) + \overline{q}$. Here $\operatorname{div}(Y) \in H^{-1}(M^{\mathrm{int}})$ is given by

$$
\langle \operatorname{div}(Y), \varphi\rangle_{M^{\mathrm{int}}} := -\int Y(\varphi) dV, \quad \varphi \in C_0^\infty(M^{\mathrm{int}}), \tag{3.2}
$$

where $\langle\cdot,\cdot\rangle_{M^{\mathrm{int}}}$ is a distributional duality on M^{int}. We shall also view $\operatorname{div}(Y)$ as multiplication operator,

$$
\operatorname{div}(Y) : C_0^\infty(M^{\mathrm{int}}) \to H^{-1}(M^{\mathrm{int}}). \tag{3.3}
$$

Therefore, it follows from (3.1) that

$$
P_{X,Y,q} : C_0^\infty(M^{\mathrm{int}}) \to H^{-1}(M^{\mathrm{int}}).
$$

In this section, we will construct complex geometric optics solutions to the equation $P_{X,Y,q}u = 0$ in M based on the Gaussian beam quasimodes for the conjugated biharmonic operator, constructed in section 2.

Assume, as we may, that (M,g) is embedded in a compact smooth manifold (N,g) without boundary of the same dimension, and let U be open in N such that $M \subset U$. Let $\varphi \in C^\infty(U,\mathbb{R})$ and let us consider the conjugated operator

$$P_\varphi = e^{\frac{\varphi}{h}}(-h^2\Delta_g)e^{-\frac{\varphi}{h}} = -h^2\Delta_g - |\nabla\varphi|_g^2 + 2\langle\nabla\varphi, h\nabla\rangle_g + h\Delta_g\varphi$$

with the semiclassical principal symbol

$$p_\varphi = |\xi|_g^2 - |d\varphi|_g^2 + 2i\langle\xi, d\varphi\rangle_g \in C^\infty(T^*U).$$

Following [28], [14], we have the following definition.

DEFINITION 3.1. *We say that $\varphi \in C^\infty(U,\mathbb{R})$ is a limiting Carleman weight for $-h^2\Delta_g$ on (U,g) if $d\varphi \neq 0$ on U, and the Poisson bracket of $Re\, p_\varphi$ and $Im\, p_\varphi$ satisfies*

$$\{Re\, p_\varphi, Im\, p_\varphi\} = 0 \quad \textit{when} \quad p_\varphi = 0.$$

We refer to [14] for a characterization of Riemannian manifolds admitting limiting Carleman weights as well as for examples of limiting Carleman weights. In particular, note that $\phi(x) = \pm x_1$ is a limiting Carleman weight for $-h^2\Delta_g$ on a CTA manifold; see [14].

Our starting point is the following Carleman estimates for $-h^2\Delta_g$ with a gain of two derivatives, established in [32]; see also [14] and [40].

PROPOSITION 3.2. *Let ϕ be a limiting Carleman weight for $-h^2\Delta_g$ on U. Then for all $0 < h \ll 1$ and $t \in \mathbb{R}$, we have*

$$h\|u\|_{H^{t+2}_{\mathrm{scl}}(N)} \leq C\|e^{\frac{\phi}{h}}(-h^2\Delta_g)e^{-\frac{\phi}{h}}u\|_{H^t_{\mathrm{scl}}(N)}, \quad C > 0, \tag{3.4}$$

for all $u \in C_0^\infty(M^{int})$.

Here $H^t(N)$, $t \in \mathbb{R}$, is the standard Sobolev space, equipped with the natural semiclassical norm,

$$\|u\|_{H^t_{\mathrm{scl}}(N)} = \|(1 - h^2\Delta_g)^{\frac{t}{2}}u\|_{L^2(N)}.$$

Iterating (3.4), we get the following Carleman estimates for $(-h^2\Delta_g)^2$, for $0 < h \ll 1$ and $t \in \mathbb{R}$:

$$h^2\|u\|_{H^{t+4}_{scl}(N)} \leq C\|e^{\frac{\phi}{h}}(-h^2\Delta_g)^2e^{-\frac{\phi}{h}}u\|_{H^t_{scl}(N)}, \quad C > 0, \tag{3.5}$$

for all $u \in C_0^\infty(M^{\mathrm{int}})$.

To construct complex geometric optics solutions for $P_{X,Y,q}u = 0$, we shall need the following Carleman estimates for the operator $P_{X,Y,q}$. In what follows we extend X, Y, and q to N by zero and we denote these extensions by the same letters so that $X, Y \in L^\infty(N, TN)$ and $q \in L^\infty(N, \mathbb{C})$.

PROPOSITION 3.3. *Let ϕ be a limiting Carleman weight for $-h^2\Delta_g$ on U. Then for all $0 < h \ll 1$, we have*

$$h^2\|u\|_{H^1_{scl}(N)} \leq C\|e^{\frac{\phi}{h}}(h^4P_{X,Y,q})e^{-\frac{\phi}{h}}u\|_{H^{-3}_{scl}(N)}, \quad C > 0, \tag{3.6}$$

for all $u \in C_0^\infty(M^{int})$.

Proof. First letting $t=-3$ in (3.5), we get for all $0<h\ll 1$,

$$h^2\|u\|_{H^1_{scl}(N)} \le C\|e^{\frac{\phi}{h}}(-h^2\Delta_g)^2 e^{-\frac{\phi}{h}}u\|_{H^{-3}_{scl}(N)} \tag{3.7}$$

for all $u\in C_0^\infty(M^{\mathrm{int}})$. We also have

$$\|e^{\frac{\phi}{h}}h^4X(e^{-\frac{\phi}{h}}u)\|_{H^{-3}_{scl}(N)} \le \|h^4X(u)-h^3X(\phi)u\|_{L^2(N)} = \mathcal{O}(h^3)\|u\|_{H^1_{scl}(N)}. \tag{3.8}$$

In order to estimate $\|h^4\operatorname{div}(Y)u\|_{H^{-3}_{\mathrm{scl}}(N)}$, we shall use the following characterization of the semiclassical norm in the Sobolev space $H^{-3}(N)$:

$$\|v\|_{H^{-3}_{\mathrm{scl}}(N)} = \sup_{0\ne\psi\in C^\infty(N)} \frac{|\langle v,\psi\rangle_N|}{\|\psi\|_{H^3_{\mathrm{scl}}(N)}}.$$

Using (3.2), for $0\ne\psi\in C^\infty(N)$, we get

$$|\langle h^4e^{\frac{\phi}{h}}\operatorname{div}(Y)e^{-\frac{\phi}{h}}u,\psi\rangle_N| \le \int_N h^4|Y(u\psi)|dV \le \mathcal{O}(h^3)\|u\|_{H^1_{\mathrm{scl}}(N)}\|\psi\|_{H^3_{\mathrm{scl}}(N)},$$

and therefore,

$$\|h^4\operatorname{div}(Y)u\|_{H^{-3}_{\mathrm{scl}}(N)} \le \mathcal{O}(h^3)\|u\|_{H^1_{\mathrm{scl}}(N)}. \tag{3.9}$$

Finally, we have

$$\|h^4qu\|_{H^{-3}_{\mathrm{scl}}(N)} \le \mathcal{O}(h^4)\|u\|_{H^1_{\mathrm{scl}}(N)}. \tag{3.10}$$

Combining (3.7), (3.8), (3.9), and (3.10), we obtain (3.6) for all $0<h\ll 1$ and $u\in C_0^\infty(M^{\mathrm{int}})$. ∎

Note that the formal L^2 adjoint of $P_{X,Y,q}$ is given by $P_{-\overline{X},-\overline{X}+\overline{Y},\overline{q}}$. Using the fact that if ϕ is a limiting Carleman weight then so is $-\phi$, we obtain the following solvability result; see [14] and [31] for the details.

PROPOSITION 3.4. *Let $X,Y\in L^\infty(M,TM)$ be complex vector fields, and let $q\in L^\infty(M,\mathbb{C})$. Let ϕ be a limiting Carleman weight for $-h^2\Delta_g$ on (U,g). If $h>0$ is small enough, then for any $v\in H^{-1}(M^{int})$, there is a solution $u\in H^3(M^{int})$ of the equation*

$$e^{\frac{\phi}{h}}(h^4P_{X,Y,q})e^{-\frac{\phi}{h}}u = v \quad in \quad M^{int},$$

which satisfies

$$\|u\|_{H^3_{\mathrm{scl}}(M^{int})} \le \frac{C}{h^2}\|v\|_{H^{-1}_{\mathrm{scl}}(M^{int})}.$$

Let

$$s=\mu+i\lambda, \quad 1\le\mu=\frac{1}{h}, \quad \lambda\in\mathbb{R}, \quad \lambda \quad \text{fixed}.$$

We shall construct complex geometric optics solutions to the equation

$$P_{X,Y,q}u=0 \quad \text{in} \quad M^{\mathrm{int}} \tag{3.11}$$

of the form

$$u=e^{-sx_1}(v_s+r_s), \tag{3.12}$$

where v_s is a Gaussian beam quasimode for $(-h^2\Delta_g)^2$, constructed in Proposition 2.1. Thus, u is a solution to (3.11) provided that

$$\begin{aligned} e^{sx_1}h^4P_{X,Y,q}e^{-sx_1}r_s = &- e^{sx_1}h^4P_{X,Y,q}e^{-sx_1}v_s = -e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s \\ &- e^{sx_1}h^4X(e^{-sx_1}v_s) - e^{sx_1}h^4\operatorname{div}(Y)(e^{-sx_1}v_s) - h^4qv_s =: F. \end{aligned} \tag{3.13}$$

Let us estimates the terms in the right-hand side of (3.13) in $H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})$. First, it follows from (2.1) that

$$\|e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} \le \|e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s\|_{L^2(M)} = \mathcal{O}(h^{5/2}) \tag{3.14}$$

and

$$\|e^{sx_1}h^4X(e^{-sx_1}v_s)\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} \le \|h^4X(v_s) - h^4sX(x_1)v_s\|_{L^2(M)} = \mathcal{O}(h^3). \tag{3.15}$$

Letting $0 \ne \rho \in C_0^\infty(M^{\mathrm{int}})$ and using (3.2), we obtain that

$$\begin{aligned} |\langle e^{sx_1}h^4\operatorname{div}(Y)(e^{-sx_1}v_s), \rho\rangle_{M^{\mathrm{int}}}| &\le h^4\int |Y(v_s\rho)|dV \\ &= \mathcal{O}(h^3)\|v_s\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})}\|\rho\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(h^3)\|\rho\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})}, \end{aligned}$$

and therefore,

$$\|e^{sx_1}h^4\operatorname{div}(Y)(e^{-sx_1}v_s)\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(h^3). \tag{3.16}$$

We also have

$$\|h^4qv_s\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(h^4). \tag{3.17}$$

Using (3.14), (3.15), (3.16), (3.17), we get from (3.13) that $\|F\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(h^{5/2})$. An application of Proposition 3.4 to (3.13) gives that for all $h > 0$ small enough, there exists $r_s \in H^3(M^{\mathrm{int}})$ such that $\|r_s\|_{H^3_{scl}(M^{\mathrm{int}})} = \mathcal{O}(h^{1/2})$. To summarize, we have proven the following result.

PROPOSITION 3.5. *Let $X, Y \in L^\infty(M, TM)$ be complex vector fields, and let $q \in L^\infty(M, \mathbb{C})$. Let $s = \frac{1}{h} + i\lambda$ with $\lambda \in \mathbb{R}$ being fixed. For all $h > 0$ small enough, there is a solution $u_1 \in H^3(M^{int})$ of $P_{X,Y,q}u_1 = 0$ in M^{int} having the form*

$$u_1 = e^{-sx_1}(v_s + r_1),$$

where $v_s \in C^\infty(M)$ is the Gaussian beam quasimode given in Proposition 2.1 and $r_1 \in H^3(M^{int})$ such that $\|r_1\|_{H^3_{\mathrm{scl}}(M^{int})} = \mathcal{O}(h^{1/2})$ as $h \to 0$.

Similarly, for all $h > 0$ small enough, there is a solution $u_2 \in H^3(M^{int})$ of $P_{X,Y,q}u_2 = 0$ in M^{int} having the form

$$u_2 = e^{sx_1}(w_s + r_2),$$

where $w_s \in C^\infty(M)$ is the Gaussian beam quasimode given in Proposition 2.1 and $r_2 \in H^3(M^{int})$ such that $\|r_2\|_{H^3_{\mathrm{scl}}(M^{int})} = \mathcal{O}(h^{1/2})$ as $h \to 0$.

4. Proof of Theorem 1.2. Our starting point is the following integral identity.

PROPOSITION 4.1. *Let $X^{(1)}, X^{(2)} \in C(M, TM)$ with complex valued coefficients, and $q^{(1)}, q^{(2)} \in C(M, \mathbb{C})$. If $\mathcal{C}_{X^{(1)},q^{(1)}} = \mathcal{C}_{X^{(2)},q^{(2)}}$, then*

$$\int_M \big((X^{(1)} - X^{(2)})(u_1)\overline{u_2} + (q^{(1)} - q^{(2)})u_1\overline{u_2}\big)dV_g = 0 \tag{4.1}$$

for $u_1, u_2 \in H^3(M^{int})$ satisfying

$$L_{X^{(1)},q^{(1)}}u_1 = 0 \quad \text{and} \quad L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}}u_2 = 0. \tag{4.2}$$

Proof. First, using that $\overline{u_2}$ solves the equation

$$L_{-X^{(2)},-\operatorname{div}(X^{(2)})+q^{(2)}}\overline{u_2} = 0, \tag{4.3}$$

similar to (1.2), we define the boundary trace $\partial_\nu(\Delta_g\overline{u_2}) \in H^{-1/2}(\partial M)$ as follows. Letting $\varphi \in H^{1/2}(\partial M)$ and letting $v \in H^1(M^{\text{int}})$ be a continuous extension of φ, we set

$$\begin{aligned}\langle \partial_\nu(-\Delta_g\overline{u_2}), \varphi\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} = &-\int_{\partial M}(X^{(2)}\cdot\nu)\overline{u_2}v\,dS_g\\ &+\int_M \big(\langle\nabla_g(-\Delta_g\overline{u_2}), \nabla_g v\rangle_g + \overline{u_2}X^{(2)}(v) + q^{(2)}\overline{u_2}v\big)dV_g.\end{aligned} \tag{4.4}$$

It follows from (4.3) that the definition of the trace $\partial_\nu(\Delta_g\overline{u_2})$ is independent of the choice of extension v of φ.

As $\mathcal{C}_{X^{(1)},q^{(1)}} = \mathcal{C}_{X^{(2)},q^{(2)}}$, there exists $v_2 \in H^3(M^{\text{int}})$ such that

$$L_{X^{(2)},q^{(2)}}v_2 = 0 \quad \text{in} \quad M \tag{4.5}$$

and

$$\begin{aligned}u_1|_{\partial M} = v_2|_{\partial M}, \quad (\Delta_g u_1)|_{\partial M} = (\Delta_g v_2)|_{\partial M}, \quad \partial_\nu u_1|_{\partial M} = \partial_\nu v_2|_{\partial M},\\ \partial_\nu(\Delta_g u_1)|_{\partial M} = \partial_\nu(\Delta_g v_2)|_{\partial M}.\end{aligned} \tag{4.6}$$

It follows from (4.6) in particular that

$$\langle\partial_\nu(\Delta_g u_1), \overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} = \langle\partial_\nu(\Delta_g v_2), \overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)}. \tag{4.7}$$

Using that v_2 solves (4.5) and (1.2), we get

$$\begin{aligned}&\langle\partial_\nu(-\Delta_g v_2), \overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)}\\ &\qquad = \int_M \big(\langle\nabla_g(-\Delta_g v_2), \nabla_g\overline{u_2}\rangle_g + X^{(2)}(v_2)\overline{u_2} + q^{(2)}v_2\overline{u_2}\big)dV_g.\end{aligned} \tag{4.8}$$

Using (4.4) and integration by parts, we obtain that

$$\begin{aligned}\langle\partial_\nu(-\Delta_g\overline{u_2}), v_2\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} = &-\int_{\partial M}(X^{(2)}\cdot\nu)\overline{u_2}v_2\,dS_g\\ &+\int_M \big(\langle\nabla_g\overline{u_2}, \nabla_g(-\Delta_g)v_2\rangle_g + \overline{u_2}X^{(2)}(v_2) + q^{(2)}\overline{u_2}v_2\big)dV_g\\ &+\int_{\partial M}(\partial_\nu\overline{u_2})\Delta_g v_2\,dS_g - \int_{\partial M}(\Delta_g\overline{u_2})\partial_\nu v_2\,dS_g.\end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), using (4.6), we obtain that

$$
\begin{aligned}
&\langle \partial_\nu(-\Delta_g v_2),\overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} = \langle \partial_\nu(-\Delta_g \overline{u_2}), v_2\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)}\\
&\qquad + \int_{\partial M}(X^{(2)}\cdot\nu)\overline{u_2}v_2 dS_g - \int_{\partial M}(\partial_\nu\overline{u_2})\Delta_g v_2 dS_g + \int_{\partial M}(\Delta_g\overline{u_2})\partial_\nu v_2 dS_g\\
&\qquad = \langle \partial_\nu(-\Delta_g\overline{u_2}), u_1\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} + \int_{\partial M}(X^{(2)}\cdot\nu)\overline{u_2}u_1 dS_g\\
&\qquad - \int_{\partial M}(\partial_\nu\overline{u_2})\Delta_g u_1 dS_g + \int_{\partial M}(\Delta_g\overline{u_2})\partial_\nu u_1 dS_g\\
&\qquad = \int_M \big(\langle\nabla_g\overline{u_2}, \nabla_g(-\Delta_g)u_1\rangle_g + \overline{u_2}X^{(2)}(u_1) + q^{(2)}\overline{u_2}u_1\big)dV_g.
\end{aligned} \tag{4.10}
$$

On the other hand, using (4.2) for u_1 and (1.2), we get

$$
\begin{aligned}
&\langle \partial_\nu(-\Delta_g u_1),\overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)}\\
&\qquad = \int_M \big(\langle\nabla_g(-\Delta_g)u_1, \nabla_g\overline{u_2}\rangle_g + X^{(1)}(u_1)\overline{u_2} + q^{(1)}u_1\overline{u_2}\big)dV_g.
\end{aligned} \tag{4.11}
$$

The claim follows from (4.7), (4.10), and (4.11). ∎

Now by Proposition 3.5, for $h>0$ small enough, there are $u_1,u_2\in H^3(M^{\mathrm{int}})$ solutions to $L_{X^{(1)},q^{(1)}}u_1=0$ and $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}}u_2=0$ in M^{int}, of the form

$$
u_1 = e^{-sx_1}(v_s+r_1),\quad u_2 = e^{sx_1}(w_s+r_2), \tag{4.12}
$$

where $v_s,w_s\in C^\infty(M)$ are the Gaussian beam quasimode given in Proposition 2.1 and

$$
\|r_1\|_{H^1_{scl}(M^{\mathrm{int}})} = \mathcal{O}(h^{1/2}),\quad \|r_2\|_{H^1_{scl}(M^{\mathrm{int}})} = \mathcal{O}(h^{1/2}), \tag{4.13}
$$

as $h\to 0$.

Let us denote $X=X^{(1)}-X^{(2)}$ and $q=q^{(1)}-q^{(2)}$. By the boundary determination of Proposition A.1, we have that $X^{(1)}|_{\partial M}=X^{(2)}|_{\partial M}$, and therefore, we may extend X by zero to the complement of M in $\mathbb{R}\times M_0$ so that the extension $X\in C(\mathbb{R}\times M_0, T(\mathbb{R}\times M_0))$.

Step 1. *Proving that there exists $\psi\in C^1(\mathbb{R}\times M_0)$ with compact support in x_1 such that $\psi(x_1,\cdot)|_{\partial M_0}=0$ and $\nabla_g\psi = X$.* In this step, we shall work with solutions u_1 and u_2 given by (4.12) with v_s and w_s being the Gaussian beam quasimode for which Proposition 2.3 holds. In particular, here v_s has an amplitude of the first type. Next, we would like to substitute u_1 and u_2 into the integral identity (4.1), multiply it by h, and let $h\to 0$. To that end, first using (4.13), (2.1), and (2.2), we get

$$
\left|h\int_M qu_1\overline{u_2}dV_g\right| = \left|h\int_M qe^{-2i\lambda x_1}(v_s+r_1)(\overline{w_s}+\overline{r_2})dV_g\right| = \mathcal{O}(h). \tag{4.14}
$$

Writing $x=(x_1,x')$, $x'\in M_0$, and $X = X_1\partial_{x_1}+\widetilde{X}\cdot\partial_{x'}$, we obtain that

$$
h\int_M X(u_1)\overline{u_2}dV_g = I_1+I_2+I_3+I_4, \tag{4.15}
$$

where

$$I_1 = h\int_M e^{-2i\lambda x_1} X(v_s)\overline{w_s}dV_g - \int_M X_1(x_1,x')e^{-2i\lambda x_1} v_s\overline{w_s}dV_g, \tag{4.16}$$

$$I_2 = -hi\lambda\int_M X_1(x_1,x')e^{-2i\lambda x_1}(v_s+r_1)(\overline{w_s}+\overline{r_2})dV_g, \tag{4.17}$$

$$I_3 = -\int_M X_1(x_1,x')e^{-2i\lambda x_1}(v_s\overline{r_2}+\overline{w_s}r_1+r_1\overline{r_2})dV_g, \tag{4.18}$$

$$I_4 = h\int_M e^{-2i\lambda x_1}(X(v_s)\overline{r_2}+X(r_1)\overline{w_s}+X(r_1)\overline{r_2})dV_g. \tag{4.19}$$

Using (4.13), (2.1), and (2.2), we get

$$|I_2| = \mathcal{O}(h), \quad |I_3| = \mathcal{O}(h^{1/2}), \quad |I_4| = \mathcal{O}(h^{1/2}). \tag{4.20}$$

It follows from (4.1) with the help of (4.14), (4.15), and (4.20) that

$$\lim_{h\to 0} I_1 = 0. \tag{4.21}$$

Using that $X = 0$ outside of M, $dV_g = c^{\frac{n}{2}}dx_1 dV_{g_0}$, Fubini's theorem, and Proposition 2.3, we obtain from (4.21) that

$$\begin{aligned} 0 &= \lim_{h\to 0} h\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0} X(v_s)\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1 \\ &\quad - \lim_{h\to 0}\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0} X_1(x_1,x')v_s\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1 \\ &= -\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_0^L \big(X_1(x_1,\gamma(t)) - iX_t(x_1,\gamma(t))\big)c(x_1,\gamma(t))e^{-2\lambda t}dt dx_1. \end{aligned} \tag{4.22}$$

Now the Riemmanian metric g on M induces a natural isomorphism between the tangent and cotangent bundles given by

$$TM \to T^*M, \quad (x,X) \mapsto (x, X^b), \tag{4.23}$$

where $X^b(Y) = \langle X, Y\rangle$. In local coordinates, $X^b = \sum_{j,k=1}^n g_{jk}X_j dx_k$, and using that $g = c(e\oplus g_0)$, and (2.10), we get

$$X_1^b(x_1,\gamma(t)) = c(x_1,\gamma(t))X_1(x_1,\gamma(t)), \quad X_t^b(x_1,\gamma(t)) = c(x_1,\gamma(t))X_t(x_1,\gamma(t)).$$

Hence, it follows from (4.22), replacing 2λ by λ, that

$$\int_{\mathbb{R}}\int_0^L e^{-i\lambda x_1-\lambda t}(X_1^b(x_1,\gamma(t)) - iX_t^b(x_1,\gamma(t)))dt dx_1 = 0. \tag{4.24}$$

Letting

$$\begin{aligned} f(\lambda,x') &= \int_{\mathbb{R}} e^{-i\lambda x_1}X_1^b(x_1,x')dx_1, \quad x'\in M_0, \\ \alpha(\lambda,x') &= \sum_{j=2}^n\Big(\int_{\mathbb{R}} e^{-i\lambda x_1}X_j^b(x_1,x')\Big)dx_j, \end{aligned} \tag{4.25}$$

we have $f(\lambda,\cdot)\in C(M_0)$, $\alpha(\lambda,\cdot)\in C(M_0,T^*M)$, and (4.24) implies that

$$\int_0^L [f(\lambda,\gamma(t))-i\alpha(\lambda,\dot\gamma(t))]e^{-\lambda t}dt=0, \tag{4.26}$$

along any unit speed nontangential geodesic $\gamma:[0,L]\to M_0$ on M_0 and any $\lambda\in\mathbb{R}$. Arguing as in [32, section 7], [10], using the injectivity of the geodesic X-ray transform on functions and 1-forms, we conclude from (4.26) that there exist $p_l\in C^1(M_0)$, $p_l|_{\partial M_0}=0$, such that

$$\partial_\lambda^l f(0,x')+lp_{l-1}(x')=0,\quad \partial_\lambda^l\alpha(0,x')=idp_l(x'),\quad l=0,1,2,\dots. \tag{4.27}$$

To proceed we shall follow [16, section 5] and let

$$\psi(x_1,x')=\int_{-a}^{x_1}X_1^b(y_1,x')dy_1, \tag{4.28}$$

where supp $(X^b(\cdot,x'))\subset(-a,a)$. It follows from (4.27), (4.25) that

$$0=f(0,x')=\int_{\mathbb{R}}X_1^b(y_1,x')dy_1,$$

and therefore, ψ has compact support in x_1. Thus, the Fourier transform of ψ with respect to x_1, which we denote by $\widehat{\psi}(\lambda,x')$, is real analytic with respect to λ, and therefore, we have

$$\widehat{\psi}(\lambda,x')=\sum_{k=0}^\infty\frac{\psi_k(x')}{k!}\lambda^k, \tag{4.29}$$

where $\psi_k(x')=(\partial_\lambda^k\widehat{\psi})(0,x')$. It follows from (4.28) that

$$\partial_{x_1}\psi(x_1,x')=X_1^b(x_1,x'), \tag{4.30}$$

and therefore, taking the Fourier transform with respect to x_1, and using (4.25)

$$i\lambda\psi(\lambda,x')=f(\lambda,x'). \tag{4.31}$$

Differentiating (4.31) $(l+1)$-times in λ, letting $\lambda=0$, and using (4.27), we get

$$\partial_\lambda^l\widehat{\psi}(0,x')=ip_l(x'),\quad l=0,1,2,\dots. \tag{4.32}$$

Substituting (4.32) into (4.29), we obtain that

$$\widehat{\psi}(\lambda,x')=\sum_{k=0}^\infty\frac{ip_l(x')}{k!}\lambda^k,$$

and taking the differential in x' in the sense of distributions, and using (4.27), (4.25), we see that

$$d_{x'}\widehat{\psi}(\lambda,x')=\sum_{k=0}^\infty\frac{idp_l(x')}{k!}\lambda^k=\sum_{k=0}^\infty\frac{\partial_\lambda^k\alpha(0,x')}{k!}\lambda^k=\alpha(\lambda,x')=\sum_{j=2}^n\widehat{X}_j^b(\lambda,x')dx_j. \tag{4.33}$$

Taking the inverse Fourier transform $\lambda \mapsto x_1$ in (4.33), we get

$$d_{x'}\psi(x_1, x') = \sum_{j=2}^{n} X_j^b(x_1, x')dx_j. \tag{4.34}$$

We also have from (4.30) that

$$d_{x_1}\psi(x_1, x') = X_1^b(x_1, x')dx_1. \tag{4.35}$$

It follows from (4.34) and (4.35) that

$$d\psi = X^b. \tag{4.36}$$

Using the inverse of (4.23), we see from (4.36) that

$$\nabla_g \psi = X. \tag{4.37}$$

Recall that $\psi \in C(\mathbb{R} \times M_0)$ with compact support in x_1 and $\psi(x_1, \cdot)|_{\partial M_0} = 0$. It follows from (4.37) that $\psi \in C^1(\mathbb{R} \times M_0)$.

Step 2. *Showing that* $X = 0$. Returning to (4.1) and using (4.37), we get

$$\int_M \big((\nabla_g \psi)(u_1)\overline{u_2} + qu_1\overline{u_2}\big)dV_g = 0 \tag{4.38}$$

for $u_1, u_2 \in H^3_{scl}(M^{\text{int}})$ satisfying $L_{X^{(1)},q^{(1)}}u_1 = 0$ and $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}}u_2 = 0$. Let now u_1 and u_2 be given by (4.12) with v_s and w_s being the Gaussian beam quasimode for which Proposition 2.4 holds. In particular, here v_s has an amplitude of the second type. We would like to substitute u_1 and u_2 into the integral identity (4.38), multiply it by h, and let $h \to 0$. Similar to (4.21), using (4.14) and (4.20), we get

$$\lim_{h\to 0} h \int_M e^{-2i\lambda x_1}(\nabla_g \psi)(v_s)\overline{w_s}dV_g - \int_M (\nabla_g \psi)_1 e^{-2i\lambda x_1} v_s\overline{w_s}dV_g = 0. \tag{4.39}$$

It follows from (4.39) with the help of Proposition 2.4,

$$\int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1 - it)}\psi(x_1, \gamma(t))c(x_1, \gamma(t))dtdx_1 = 0. \tag{4.40}$$

Now (4.40) can be written as

$$\int_\gamma \widehat{\psi c}(2\lambda, \gamma(t))e^{-2\lambda t}dt = 0 \tag{4.41}$$

for any $\lambda \in \mathbb{R}$ and any nontangential geodesic γ in M_0, where

$$\widehat{\psi c}(2\lambda, x') = \int_{-\infty}^{\infty} e^{-2i\lambda x_1}(\psi c)(x_1, x')dx_1.$$

Equation (4.41) says that the attenuated geodesic ray transform of $\widehat{\psi c}$ with constant attenuation -2λ vanishes along all nontangential geodesics in M_0. Arguing as in [15, Proof of Theorem 1.2] and using the injectivity of the geodesic X-ray transform on functions, we conclude that $\psi c = 0$, and therefore $\psi = 0$, and hence $X = 0$.

Step 3. *Proving that* $q = 0$. Returning to (4.1) and substituting $X^{(1)} = X^{(2)}$, we get

$$\int_M q u_1 \overline{u_2} dV_g = 0 \tag{4.42}$$

for $u_1, u_2 \in H^3_{scl}(M^{\text{int}})$ satisfying $L_{X^{(1)},q^{(1)}} u_1 = 0$ and $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}} u_2 = 0$. Let now u_1 and u_2 be given by (4.12) with v_s and w_s being the Gaussian beam quasimode for which Proposition 2.3 holds. In particular, here v_s has an amplitude of the first type. Substituting u_1 and u_2 into (4.42), we obtain that

$$0 = \int_M q u_1 \overline{u_2} dV_g = I_1 + I_2, \tag{4.43}$$

where

$$I_1 = \int_M e^{-2i\lambda x_1} q v_s \overline{w_s} dV_g = \int_{\mathbb{R}} e^{-2i\lambda x_1} \int_{M_0} q v_s \overline{w_s} c^{\frac{n}{2}} dV_{g_0} dx_1,$$
$$I_2 = \int_M e^{-2i\lambda x_1} q (v_s \overline{r_2} + r_1 \overline{w_s} + r_1 \overline{r_2}) dV_g.$$

Here in view of the assumption (1.4), we extended q by zero to the complement of M in $\mathbb{R} \times M_0$ so that the extension $q \in C(\mathbb{R} \times M_0, \mathbb{C})$.

Using (4.13), (2.1), and (2.2), we see that

$$|I_2| = \mathcal{O}(h^{1/2}). \tag{4.44}$$

Letting $h \to 0$, we obtain from (4.43), (4.44) with the help of Proposition 2.3 that

$$\int_{\mathbb{R}} e^{-2i\lambda x_1} \int_0^L e^{-2\lambda t} (qc)(x_1, \gamma(t)) dt dx_1 = 0.$$

Arguing as in [15, Proof of Theorem 1.2] and using the injectivity of the geodesic X-ray transform on functions, we conclude that $qc = 0$, and therefore $q = 0$. This complete the proof of Theorem 1.2.

Appendix A. Boundary determination of a first order perturbation of the biharmonic operator. When proving Theorem 1.2, an important step consists in determining the boundary values of the first order perturbation of the biharmonic operator. The purpose of this section is to carry out this step by adapting the method of [9], [32].

PROPOSITION A.1. *Let* (M, g) *be a CTA manifold of dimension* $n \geq 3$. *Let* $X^{(1)}, X^{(2)} \in C(M, TM)$ *with complex vector fields and* $q^{(1)}, q^{(2)} \in L^\infty(M, \mathbb{C})$. *If* $\mathcal{C}_{g,X^{(1)},q^{(1)}} = \mathcal{C}_{g,X^{(2)},q^{(2)}}$, *then* $X^{(1)}|_{\partial M} = X^{(2)}|_{\partial M}$.

Proof. We shall follow [9], [32] closely. We shall construct some special solutions to the equations $L_{X^{(1)},q^{(1)}} u_1 = 0$ and $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}} u_2 = 0$, whose boundary values have an oscillatory behavior while becoming increasingly concentrated near a given point on the boundary of M. Substituting these solutions into the integral identity (4.1) will allow us to prove that $X^{(1)}|_{\partial M} = X^{(2)}|_{\partial M}$.

In doing so, let $x_0 \in \partial M$ and let $(x_1, \dots, x_n)$ be the boundary normal coordinates centered at x_0 so that in these coordinates, $x_0 = 0$, the boundary ∂M is given by $\{x_n = 0\}$, and M^{int} is given by $\{x_n > 0\}$. We shall assume, as we may, that

$$g^{\alpha\beta}(0) = \delta^{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n - 1, \tag{A.1}$$

and therefore $T_0\partial M = \mathbb{R}^{n-1}$, equipped with the Euclidean metric. The unit tangent vector τ is then given by $\tau = (\tau', 0)$ where $\tau' \in \mathbb{R}^{n-1}$, $|\tau'| = 1$. Associated to the tangent vector τ' is the covector $\xi'_\alpha = \sum_{\beta=1}^{n-1} g_{\alpha\beta}(0)\tau'_\beta = \tau'_\alpha \in T^*_{x_0}\partial M$.

Let $\eta \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ be a function such that supp (η) is in a small neighborhood of 0, and

$$\int_{\mathbb{R}^{n-1}} \eta(x', 0)^2 dx' = 1. \tag{A.2}$$

Following [9], in the boundary normal coordinates, we set

$$v_0(x) = \eta\left(\frac{x}{\lambda^{1/2}}\right) e^{\frac{i}{\lambda}(\tau'\cdot x' + ix_n)}, \quad 0 < \lambda \ll 1, \tag{A.3}$$

so that $v_0 \in C^\infty(M)$ with supp (v_0) in $\mathcal{O}(\lambda^{1/2})$ neighborhood of $x_0 = 0$. Here τ' is viewed as a covector.

Let $v_1 \in H_0^1(M^{\text{int}})$ be the solution to the following Dirichlet problem for the Laplacian:

$$\begin{aligned} -\Delta_g v_1 &= \Delta_g v_0 \quad \text{in} \quad M, \\ v_1|_{\partial M} &= 0. \end{aligned} \tag{A.4}$$

Let $\delta(x)$ be the distance from $x \in M$ to the boundary of M. As proved in the [32, Appendix], the following estimates hold:

$$\|v_0\|_{L^2(M)} \le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}), \tag{A.5}$$

$$\|v_1\|_{L^2(M)} \le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}), \tag{A.6}$$

$$\|dv_1\|_{L^2(M)} \le \mathcal{O}(\lambda^{\frac{n-1}{4}}), \tag{A.7}$$

$$\|dv_0\|_{L^2(M)} \le \mathcal{O}(\lambda^{\frac{n-1}{4}-\frac{1}{2}}), \tag{A.8}$$

$$\|\delta d(v_0 + v_1)\|_{L^2(M)} \le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}), \tag{A.9}$$

$$\|v_0\|_{L^2(\partial M)} \le \mathcal{O}(\lambda^{\frac{n-1}{4}}). \tag{A.10}$$

We shall also need Hardy's inequality,

$$\int_M |f(x)/\delta(x)|^2 dV_g \le C \int_M |df(x)|^2 dV_g, \tag{A.11}$$

where $f \in H_0^1(M^{\text{int}})$; see [11].

Next we would like to show the existence of a solution $u_1 \in H^3(M^{\text{int}})$ to the equation

$$L_{X^{(1)},q^{(1)}} u_1 = 0 \quad \text{in} \quad M, \tag{A.12}$$

of the form

$$u_1 = v_0 + v_1 + r_1, \tag{A.13}$$

with

$$\|r_1\|_{H^3(M^{\text{int}})} \le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}). \tag{A.14}$$

To that end, plugging (A.13) into (A.12), we obtain the following equation of r_1:

(A.15)
$$L_{X^{(1)},q^{(1)}} r_1 = -((-\Delta_g)^2 + X^{(1)} + q^{(1)})(v_0 + v_1) = -(X^{(1)} + q^{(1)})(v_0 + v_1) \quad \text{in} \quad M.$$

Applying Proposition 3.4 with $h > 0$ small but fixed, we conclude the existence of $r_1 \in H^3(M^{\text{int}})$ such that

$$\|r_1\|_{H^3(M^{\text{int}})} \le \mathcal{O}(1)\|(X^{(1)} + q^{(1)})(v_0 + v_1)\|_{H^{-1}(M^{\text{int}})}. \tag{A.16}$$

Let us now bound the norm in the right-hand side of (A.16). To that end, letting $\psi \in C_0^\infty(M^{\text{int}})$ and using (A.11), (A.9), we get

$$\begin{aligned} |\langle X^{(1)}(v_0 + v_1), \psi\rangle_{M^{\text{int}}}| &\le \mathcal{O}(1)\|X^{(1)}\|_{L^\infty(M)}\|\delta d(v_0 + v_1)\|_{L^2(M)}\|\psi\|_{H^1(M^{\text{int}})} \\ &\le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}})\|\psi\|_{H^1(M^{\text{int}})}. \end{aligned} \tag{A.17}$$

By (A.5) and (A.6), we have

$$\begin{aligned} |\langle q^{(1)}(v_0 + v_1), \psi\rangle_{M^{\text{int}}}| &\le \|q^{(1)}\|_{L^\infty(M^0)}\|v_0 + v_1\|_{L^2(M)}\|\psi\|_{L^2(M)} \\ &\le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}})\|\psi\|_{H^1(M^{\text{int}})}. \end{aligned} \tag{A.18}$$

The estimate (A.14) follows from (A.16), (A.17), and (A.18).

Let us show that there exists a solution $u_2 \in H^3(M^{\text{int}})$ of $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}} u_2 = 0$ in M of the form

$$u_2 = v_0 + v_1 + r_2, \tag{A.19}$$

where $r_2 \in H^3(M^{\text{int}})$ with

$$\|r_2\|_{H^3(M^{\text{int}})} \le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}). \tag{A.20}$$

Applying Proposition 3.4 with $h > 0$ small but fixed to the equation,

$$L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}} r_2 = (\overline{X^{(2)}} + \operatorname{div}(\overline{X^{(2)}}) - \overline{q^{(2)}})(v_0 + v_1) \quad \text{in} \quad M, \tag{A.21}$$

we conclude the existence of $r_2 \in H^1(M^{\text{int}})$ such that

$$\|r_2\|_{H^3(M^{\text{int}})} \le \mathcal{O}(1)\|(\overline{X^{(2)}} + \operatorname{div}(\overline{X^{(2)}}) - \overline{q^{(2)}})(v_0 + v_1)\|_{H^{-1}(M^{\text{int}})}. \tag{A.22}$$

To bound the norm in the right-hand side of (A.22), we let $\psi \in C_0^\infty(M^{\text{int}})$, and using (A.11), (3.2), (A.5), (A.6), (A.9), we get

$$\begin{aligned} |\langle \operatorname{div}(\overline{X^{(2)}})(v_0 + v_1), \psi\rangle_{M^{\text{int}}}| &= \left|\int \overline{X^{(2)}}((v_0 + v_1)\psi) dV_g\right| \\ &\le \left|\int \psi \overline{X^{(2)}}(v_0 + v_1) dV_g\right| + \left|\int (v_0 + v_1)\overline{X^{(2)}}(\psi) dV_g\right| \\ &\le \mathcal{O}(1)\|\delta d(v_0 + v_1)\|_{L^2(M)}\|\psi\|_{H^1(M^{\text{int}})} + \mathcal{O}(1)\|v_0 + v_1\|_{L^2(M)}\|\psi\|_{H^1(M^{\text{int}})} \\ &\le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}})\|\psi\|_{H^1(M^{\text{int}})}. \end{aligned} \tag{A.23}$$

The bound (A.20) follows from (A.22), (A.23), (A.17), (A.18).

The next step is to substitute the solution u_1 and u_2, given in (A.13) and (A.19), into the integral identity (4.1), multiply by $\lambda^{-\frac{(n-1)}{2}}$, and compute the limit as $\lambda \to 0$. In doing so, we write

$$I := \lambda^{-\frac{(n-1)}{2}} \int_M X(u_1)\overline{u_2} + q u_1 \overline{u_2} dV_g = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{A.24}$$

where

$$\begin{aligned} I_1 &= \lambda^{-\frac{(n-1)}{2}} \int_M X(v_0)\overline{v_0} dV_g, \quad I_2 = \lambda^{-\frac{(n-1)}{2}} \int_M X(v_0)\overline{v_1} dV_g, \\ I_3 &= \lambda^{-\frac{(n-1)}{2}} \int_M X(v_0)\overline{r_2} dV_g, \quad I_4 = \lambda^{-\frac{(n-1)}{2}} \int_M X(v_1)\overline{u_2} dV_g, \\ I_5 &= \lambda^{-\frac{(n-1)}{2}} \int_M X(r_1)\overline{u_2} dV_g, \quad I_6 = \lambda^{-\frac{(n-1)}{2}} \int_M q u_1 \overline{u_2} dV_g. \end{aligned}$$

Let us compute $\lim_{\lambda\to 0} I_1$. To that end, writing $X = X_j \partial_{x_j}$, we have

$$X v_0 = e^{\frac{i}{\lambda}(\tau'\cdot x' + i x_n)} \left[\lambda^{-\frac{1}{2}}(X\eta)\left(\frac{x}{\lambda^{\frac{1}{2}}}\right) + i\lambda^{-1} X(x)\cdot(\tau', i)\eta\left(\frac{x}{\lambda^{\frac{1}{2}}}\right)\right] \tag{A.25}$$

and

$$X v_0 \overline{v_0} = e^{-\frac{2x_n}{\lambda}} \left[\lambda^{-\frac{1}{2}}(X\eta)\left(\frac{x}{\lambda^{\frac{1}{2}}}\right)\eta\left(\frac{x}{\lambda^{\frac{1}{2}}}\right) + i\lambda^{-1} X(x)\cdot(\tau', i)\eta^2\left(\frac{x}{\lambda^{\frac{1}{2}}}\right)\right]. \tag{A.26}$$

Making the change of variable $y' = \frac{x'}{\lambda^{1/2}}$, $y_n = \frac{x_n}{\lambda}$, using that $X \in C(M, TM)$, η has compact support, (A.1) and (A.2), we get

$$\begin{aligned} \lim_{\lambda\to 0} I_1 &= \lim_{\lambda\to 0} \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{-2y_n} \lambda^{\frac{1}{2}} (X\eta)(y', \lambda^{\frac{1}{2}} y_n)\eta(y', \lambda^{\frac{1}{2}} y_n) |g(\lambda^{\frac{1}{2}} y', \lambda y_n)|^{\frac{1}{2}} dy_n dy' \\ &\quad + \lim_{\lambda\to 0} \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{-2y_n} i X(\lambda^{\frac{1}{2}} y', \lambda y_n)\cdot(\tau', i)\eta^2(y', \lambda^{\frac{1}{2}} y_n) |g(\lambda^{\frac{1}{2}} y', \lambda y_n)|^{\frac{1}{2}} dy_n dy' \\ &= \frac{i}{2} X(0)\cdot(\tau', i). \end{aligned} \tag{A.27}$$

The fact that $v_1 \in H_0^1(M^{\mathrm{int}})$ together with the estimates (A.11), (A.9), (A.7) gives that

$$|I_2| \le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}}) \|X\|_{L^\infty(M)} \|\delta d v_0\|_{L^2(M)} \|\frac{v_1}{\delta}\|_{L^2(M)} = \mathcal{O}(\lambda^{\frac{1}{2}}). \tag{A.28}$$

To estimate I_3, first assume that (M, g) is embedded in a compact smooth manifold (N, g) without boundary of the same dimension. Let us extend $X \in C(M, TM)$ to a continuous vector field on N, and still write $X \in C(N, TN)$. Using a partition of unity argument together with a regularization in each coordinate patch, we see that there exists a family $X_\tau \in C^\infty(N, TN)$ such that

$$\|X - X_\tau\|_{L^\infty} = o(1), \quad \|X_\tau\|_{L^\infty} = \mathcal{O}(1), \quad \|\nabla X_\tau\|_{L^\infty} = \mathcal{O}(\tau^{-1}), \quad \tau \to 0. \tag{A.29}$$

We write

$$I_3 = I_{3,1} + I_{3,2}, \tag{A.30}$$

where

$$I_{3,1}=\lambda^{-\frac{(n-1)}{2}}\int_M (X-X_\tau)(v_0)\overline{r_2}dV_g,\quad I_{3,2}=\lambda^{-\frac{(n-1)}{2}}\int_M X_\tau(v_0)\overline{r_2}dV_g. \tag{A.31}$$

Using (A.29), (A.8), (A.20), we get

$$|I_{3,1}|\le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|X-X_\tau\|_{L^\infty(M)}\|dv_0\|_{L^2(M)}\|r_2\|_{L^2(M)}=o(1), \tag{A.32}$$

as $\tau\to 0$. To estimate $I_{3,2}$, integrating by parts, we obtain that

$$I_{3,2}=J_1+J_2+J_3, \tag{A.33}$$

where

$$\begin{aligned} J_1=-\lambda^{-\frac{(n-1)}{2}}\int_M v_0X_\tau(\overline{r_2})dV_g,\quad J_2=-\lambda^{-\frac{(n-1)}{2}}\int_M \operatorname{div}(X_\tau)v_0\overline{r_2}dV_g,\\ J_3=\lambda^{-\frac{(n-1)}{2}}\int_{\partial M}(\nu\cdot X_\tau)v_0\overline{r_2}dS_g. \end{aligned} \tag{A.34}$$

Using (A.29), (A.20), (A.5), we get

$$\begin{aligned} |J_1|\le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|X_\tau\|_{L^\infty(M)}\|v_0\|_{L^2(M)}\|dr_2\|_{L^2(M)}=\mathcal{O}(\lambda),\\ |J_2|\le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|\operatorname{div}X_\tau\|_{L^\infty(M)}\|v_0\|_{L^2(M)}\|r_2\|_{L^2(M)}=\mathcal{O}(\tau^{-1}\lambda). \end{aligned} \tag{A.35}$$

Using (A.10), (A.29), (A.20), and the trace theorem, we obtain that

$$|J_3|\le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|\nu\cdot X_\tau\|_{L^\infty(M)}\|v_0\|_{L^2(\partial M)}\|r_2\|_{H^1(M)}=\mathcal{O}(\lambda^{1/2}). \tag{A.36}$$

Choosing $\tau=\lambda^{1/2}$, we conclude from (A.30), (A.31), (A.32), (A.33), (A.34), (A.35), (A.36) that

$$|I_3|=o(1),\quad \lambda\to 0. \tag{A.37}$$

Now (A.5), (A.6), (A.20) imply that

$$\|u_2\|_{L^2}=\mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}). \tag{A.38}$$

Using (A.38) together with (A.7), we have

$$|I_4|\le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|dv_1\|_{L^2(M)}\|u_2\|_{L^2(M)}=\mathcal{O}(\lambda^{\frac{1}{2}}). \tag{A.39}$$

Using (A.38) together with (A.14), we get

$$|I_5|\le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|dr_1\|_{L^2(M)}\|u_2\|_{L^2(M)}=\mathcal{O}(\lambda). \tag{A.40}$$

Last let us estimate $|I_6|$. Using (A.38) and a similar bound for u_1, we see that

$$|I_6|\le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|q\|_{L^\infty(M)}\|u_1\|_{L^2(M)}\|u_2\|_{L^2(M)}=\mathcal{O}(\lambda). \tag{A.41}$$

Now it follows from (A.24), (A.27), (A.28), (A.37), (A.39), (A.40), and (A.41) that

$$\lim_{\lambda\to 0} I=\frac{i}{2}X(0)\cdot(\tau',i)=0,$$

and therefore,

$$X^{(1)}(0)\cdot(\tau',i)=X^{(2)}(0)\cdot(\tau',i),$$

for all $\tau'\in\mathbb{R}^{n-1}$. This completes the proof of Proposition A.1. □

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REFERENCES

[1] Yu. Anikonov, *Some Methods for the Study of Multidimensional Inverse Problems for Differential Equations*, Nauka Sibirsk, Otdel, Novosibirsk, 1978.

[2] Y. Assylbekov, *Inverse problems for the perturbed polyharmonic operator with coefficients in Sobolev spaces with non-positive order*, Inverse Problems, 32 (2016), 105009.

[3] Y. Assylbekov, Corrigendum: *Inverse problems for the perturbed polyharmonic operator with coefficients in Sobolev spaces with non-positive order*, Inverse Problems, 33 (2017), 099501.

[4] Y. Assylbekov and K. Iyer, *Determining rough first order perturbations of the polyharmonic operator*, Inverse Problems Imaging, 13 (2019), pp. 1045–1066.

[5] Y. Assylbekov and Y. Yang, *Determining the first order perturbation of a polyharmonic operator on admissible manifolds*, J. Differential Equations, 262 (2017), pp. 590–614.

[6] C. Berenstein and R. Gay, *Complex Variables. An Introduction*, Grad. Texts in Math. 125, Springer-Verlag, New York, 1991.

[7] S. Bhattacharyya and T. Ghosh, *An Inverse Problem on Determining Second Order Symmetric Tensor for Perturbed Biharmonic Operator*, preprint, https://arxiv.org/abs/2010.00192, 2020.

[8] S. Bhattacharyya and T. Ghosh, *Inverse boundary value problem of determining up to a second order tensor appear in the lower order perturbation of a polyharmonic operator*, J. Fourier Anal. Appl., 25 (2019), pp. 661–683.

[9] R. Brown and M. Salo, *Identifiability at the boundary for first-order terms*, Appl. Anal., 85 (2006), pp. 735–749.

[10] M. Cekić, *The Calderón problem for connections*, Comm. Partial Differential Equations, 42 (2017), pp. 1781–1836.

[11] E. B. Davies, *Sharp boundary estimates for elliptic operators*, Math. Proc. Cambridge Philos. Soc., 129 (2000), pp. 165–178.

[12] D. Dos Santos Ferreira, C. Kenig, J. Sjöstrand, and G. Uhlmann, *Determining a magnetic Schrödinger operator from partial Cauchy data*, Comm. Math. Phys., 271 (2007), pp. 467–488.

[13] D. Dos Santos Ferreira, C. Kenig, and M. Salo, *Determining an unbounded potential from Cauchy data in admissible geometries*, Comm. Partial Differential Equations, 38 (2013), pp. 50–68.

[14] D. Dos Santos Ferreira, C. Kenig, M. Salo, and G. Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, Invent. Math., 178 (2009), pp. 119–171.

[15] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas, and M. Salo, *The Calderón problem in transversally anisotropic geometries*, J. Eur. Math. Soc., 18 (2016), pp. 2579–2626.

[16] A. Feizmohammadi, J. Ilmavirta, Y. Kian, and L. Oksanen, *Recovery of time dependent coefficients from boundary data for hyperbolic equations*, J. Spectral Theory, 11 (2021), pp. 1107–1143.

[17] A. Feizmohammadi, K. Krupchyk, L. Oksanen, and G. Uhlmann, *Reconstruction in the Calderón Problem on Conformally Transversally Anisotropic Manifolds*, preprint, https://arxiv.org/abs/2009.10280, 2020.

[18] F. Gazzola, H.-C. Grunau, and G. Sweers, *Polyharmonic Boundary Value Problems*, Springer-Verlag, Berlin, 2010.

[19] T. Ghosh, *An inverse problem on determining upto first order perturbations of a fourth order operator with partial boundary data*, Inverse Problems, 31 (2015).

[20] T. Ghosh and V. Krishnan, *Determination of lower order perturbations of the polyharmonic operator from partial boundary data*, Appl. Anal., 94 (2015).

[21] C. Guillarmou, *Lens rigidity for manifolds with hyperbolic trapped sets*, J. Amer. Math. Soc., 30 (2017), pp. 561–599.

[22] C. Guillarmou, M. Mazzucchelli, and L. Tzou, *Boundary and lens rigidity for non-convex manifolds*, Amer. J. Math., 143 (2021), pp. 533–575.

[23] C. Guillarmou and L. Tzou, *Calderón inverse problem with partial data on Riemann surfaces*, Duke Math. J., 158 (2011), pp. 83–120.

[24] M. Ikehata, *A special Green's function for the biharmonic operator and its application to an inverse boundary value problem*, Comput. Math. Appl., 22 (1991), pp. 53–66.

[25] V. Isakov, *Completeness of products of solutions and some inverse problems for PDE*, J. Differential Equations, 92 (1991), pp. 305–316.

[26] A. Katchalov, Y. Kurylev, and M. Lassas, *Inverse Boundary Spectral Problems*, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math. 123, CRC Press, Boca Raton, FL, 2001.

[27] C. Kenig and M. Salo, *The Calderón problem with partial data on manifolds and applications*, Anal. PDE, 6 (2013), pp. 2003–2048.

[28] C. Kenig, J. Sjöstrand, and G. Uhlmann, *The Calderón problem with partial data*, Ann. of Math. (2), 165 (2007), pp. 567–591.

[29] K. Krupchyk, M. Lassas, and G. Uhlmann, *Inverse boundary value problems for the perturbed polyharmonic operator*, Trans. Amer. Math. Soc., 366 (2014), pp. 95–112.

[30] K. Krupchyk, M. Lassas, and G. Uhlmann, *Determining a first order perturbation of the biharmonic operator by partial boundary measurements*, J. Funct. Anal., 262 (2012), pp. 1781–1801.

[31] K. Krupchyk and G. Uhlmann, *Uniqueness in an inverse boundary problem for a magnetic Schrödinger operator with a bounded magnetic potential*, Comm. Math. Phys., 327 (2014), pp. 993–1009.

[32] K. Krupchyk and G. Uhlmann, *Inverse problem for magnetic Schrödinger operators in transversally anisotropic geometries*, Comm. Math. Phys., 361 (2018), pp. 525–582.

[33] K. Krupchyk and G. Uhlmann, *Inverse problems for advection diffusion equations in admissible geometries*, Comm. Partial Differential Equations, 43 (2018), pp. 585–615.

[34] K. Krupchyk and G. Uhlmann, *Inverse Problems for Nonlinear Magnetic Schrödinger Equations on Conformally Transversally Anisotropic Manifolds*, preprint, https://arxiv.org/abs/2009.05089, 2020.

[35] J. Lee and G. Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math., 42 (1989), pp. 1097–1112.

[36] R. G. Muhometov, *The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry*, Dokl. Akad. Nauk SSSR, 232 (1977), pp. 32–35 (in Russian).

[37] G. Nakamura, Z. Sun, and G. Uhlmann, *Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field*, Math. Ann., 303 (1995), pp. 377–388.

[38] J. Ralston, *Approximate eigenfunctions of the Laplacian*, J. Differential Geometry, 12 (1977), pp. 87–100.

[39] J. Ralston, *Gaussian beam and the propagation of singularities*, in Studies in Partial Differential Equations, MAA Stud. Math. 23, Mathematical Association of America, Washington, DC, 1982, pp. 206–248.

[40] M. Salo and L. Tzou, *Carleman estimates and inverse problems for Dirac operators*, Math. Ann., 344 (2009), pp. 161–184.

[41] P. Stefanov, G. Uhlmann, and A. Vasy, *Inverting the local geodesic X-ray transform on tensors*, J. Anal. Math., 136 (2018), pp. 151–208.

[42] Z. Sun, *An inverse boundary value problem for Schrödinger operators with vector potentials*, Trans. Amer. Math. Soc., 338 (1993), pp. 953–969.

[43] G. Uhlmann and A. Vasy, *The inverse problem for the local geodesic ray transform*, Invent. Math., 205 (2016), pp. 83–120.

[44] Y. Yang, *Determining the first order perturbation of a bi-harmonic operator on bounded and unbounded domains from partial data*, J. Differential Equations, 257 (2014), pp. 3607–3639.