

# Near-Quadratic Lower Bounds for Two-Pass Graph Streaming Algorithms\*

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**Abstract**—We prove that any two-pass graph streaming algorithm for the  $s$ - $t$  reachability problem in  $n$ -vertex directed graphs requires near-quadratic space of  $n^{2-o(1)}$  bits. As a corollary, we also obtain near-quadratic space lower bounds for several other fundamental problems including maximum bipartite matching and (approximate) shortest path in undirected graphs.

Our results collectively imply that a wide range of graph problems admit essentially no non-trivial streaming algorithm even when two passes over the input is allowed. Prior to our work, such impossibility results were only known for single-pass streaming algorithms, and the best two-pass lower bounds only ruled out  $o(n^{7/6})$  space algorithms, leaving open a large gap between (trivial) upper bounds and lower bounds.

**Keywords**—Graph streaming; communication complexity;  $s$ - $t$  reachability, multi pass streaming lower bounds

## I. INTRODUCTION

Graph streaming algorithms process the input graph with  $n$  known vertices by making one or a few passes over the sequence of its unknown edges (given in an arbitrary order) and using a limited memory (much smaller than the input size which is  $O(n^2)$  for a graph problem). In recent years, graph streaming algorithms and lower bounds for numerous problems have been studied extensively. In particular, we now have a relatively clear picture of the powers and limitations of *single-pass* algorithms. With a rather gross oversimplification, this can be stated as follows:

- The *exact* variant of most graph problems of interest are intractable: There are  $\Omega(n^2)$  space lower bounds for maximum matching and minimum vertex cover [1], [2], (directed) reachability and topological sorting [1], [3], [4], shortest path and diameter [1], [5], minimum or maximum cut [6], maximal independent set [7], [8], dominating set [9], [10], and many others.
- On the other hand, *approximate* variants of many graph problems are tractable: There are  $O(n \cdot \text{polylog}(n))$  space algorithms (often referred to as *semi-streaming* algorithms) for approximate (weighted) matching and vertex cover [1], [11]–[13], spanner computation and approximation for distance problems [5], [14]–[16],

cut or spectral sparsifiers and approximation for cut problems [17]–[20], large independent sets [8], [21], graph coloring [7], [22], and approximate dominating set [9], [10], among others<sup>1</sup>.

Recent years have also witnessed a surge of interest in designing *multi-pass* graph streaming algorithms (see, e.g. [1], [4], [23]–[36]); see, e.g., [1], [37] for discussions on practical applications of multi-pass streaming algorithms in particular in obtaining I/O-efficiency. These results suggest that allowing even just one more pass over the input greatly enhances the capability of the algorithms. For instance, while computing the *exact* global or  $s$ - $t$  minimum cut in undirected graphs requires  $\Omega(n^2)$  space in a single pass [6], perhaps surprisingly, one can solve both problems in only *two* passes with  $\tilde{O}(n)$  and  $\tilde{O}(n^{5/3})$  space, respectively [38] (see also [39] for an  $O(\log n)$ -pass algorithm for weighted minimum cut). Qualitatively similar separations are known for numerous other problems such as triangle counting [33], [40] (with two passes), approximate matching [2], [23], [26], [35] (with  $O(1)$  passes), maximal independent set [7], [8], [41] (with  $O(\log \log n)$  passes), approximate dominating set [10], [42], [43] (with  $O(\log n)$  passes), and exact shortest path [5], [36] (with  $O(\sqrt{n})$  passes).

Despite this tremendous progress, the general picture for the abilities and limitations of multi-pass algorithms is not so clear even when we focus on *two-pass* algorithms. What other problems beside minimum cut admit non-trivial two-pass streaming algorithms? For instance, can we obtain similar results for directed versions of these problems? What about closely related problems such as maximum bipartite matching or not-so-related problems such as shortest path? Currently, none of these problems admit any non-trivial two-pass streaming algorithm, while known lower bounds only rule out algorithms with  $o(n^{7/6})$  space [4], [5], [44] leaving a considerable gap between upper and lower bounds (see [45] for a discussion on the current landscape of multi-pass graph streaming lower bounds).

\*A full version of the paper including all technical proofs is available on arXiv: <https://arxiv.org/abs/2009.01161>.

<sup>1</sup>It should be noted that, in contrast, determining the *best* approximation ratio possible for many of these problems have remained elusive and is an active area of research.

### A. Our Contributions

We present near-quadratic space lower bounds for *two-pass* streaming algorithms for several fundamental graph problems including reachability, bipartite matching, and shortest path.

*Reachability and related problems in directed graphs:* We prove the following lower bound for the reachability problem in directed graphs.

**Result 1** (Formalized in [Theorem 4](#)). *Any two-pass streaming algorithm (deterministic or randomized) that given an  $n$ -vertex directed graph  $G = (V, E)$  with two designated vertices  $s, t \in V$  can determine whether or not  $s$  can reach  $t$  in  $G$  requires  $\Omega(\frac{n^2}{2^{\Theta(\sqrt{\log n})}})$  space.*

The reachability problem is one of the earliest problems studied in the graph streaming model [3]. Previously, Henzinger *et al.* [3] and Feigenbaum *et al.* [5] proved an  $\Omega(n^2)$  space lower bound for this problem for single-pass algorithms, and Guruswami and Onak [44] gave an  $\Omega_p(n^{1+1/(2p+2)})$  lower bound for  $p$ -pass algorithms which translates to  $\Omega(n^{7/6})$  space for two-pass algorithms; this lower bound was recently extended to random-order streams by Chakrabarti *et al.* [4]. Note that the *undirected* version of this problem has a simple  $O(n)$  space algorithm in one pass by maintaining a spanning forest of the input graph (see, e.g. [1]).

Using standard reductions, our results in this part can be extended to several other related problems on directed graphs such as estimating number of vertices reachable from a given source or approximating minimum feedback arc set, studied in [3] and [4], respectively.

*Matching and cut problems:* We have the following lower bound for bipartite matching.

**Result 2** (Formalized in [Theorem 5](#)). *Any two-pass streaming algorithm (deterministic or randomized) that given an  $n$ -vertex undirected bipartite graph  $G = (L \sqcup R, E)$  can determine whether or not  $G$  has a perfect matching requires  $\Omega(\frac{n^2}{2^{\Theta(\sqrt{\log n})}})$  space.*

Maximum matching problem is arguably the most studied problem in the graph streaming model. However, the main focus on this problem so far has been on approximation algorithms and not much is known for exact computation of this problem, beside that it can be done in  $\tilde{O}(k^2)$  space in a single pass where  $k$  is size of the maximum matching [46] (for the perfect matching problem, this gives an  $O(n^2)$  space algorithm which is the same as storing the entire input). Previously, Feigenbaum *et al.* [1] and Chitnis *et al.* [47] proved an  $\Omega(n^2)$  space lower bound for single-pass algorithms for this problem and Guruswami and Onak [44] extended the lower bound to  $\tilde{\Omega}_p(n^{1+1/(2p+2)})$  for  $p$ -pass algorithms.

Both the perfect matching problem and the  $s$ - $t$  reachability problem are simpler versions of the  $s$ - $t$  minimum cut problem in directed graphs. As such, our lower bounds imply that even though the  $s$ - $t$  minimum cut problem can be solved in undirected graphs in  $\tilde{O}(n^{5/3})$  space and two passes [38], its directed version requires  $n^{2-o(1)}$  space in two passes (for any multiplicative approximation). Previously, Assadi *et al.* [45] proved a lower bound of  $\Omega(n^2/p^5)$  for  $p$ -pass algorithms for the *weighted*  $s$ - $t$  minimum cut problem (with exponential-in- $p$  weights); for the unweighted problem, the previous best lower bound was still  $\tilde{\Omega}_p(n^{1+1/(2p+2)})$ .

*Shortest path problem:* Finally, we also prove a lower bound for the shortest path problem.

**Result 3** (Formalized in [Theorem 6](#)). *Any two-pass streaming algorithm (deterministic or randomized) that given an undirected graph  $G = (V, E)$  and two designated vertices  $s, t \in V$ , can output the length of the shortest  $s$ - $t$ -path in  $G$  requires  $\Omega(\frac{n^2}{2^{\Theta(\sqrt{\log n})}})$  space. The lower bound continues to hold even for approximation algorithms with approximation ratio better than  $9/7$ .*

Shortest path problem have also been extensively studied in graph streaming literature. For single-pass streams, the focus has been on maintaining *spanners* (subgraphs that preserve pairwise distances approximately) which allow for obtaining algorithms with different space-approximation tradeoffs [5], [14]–[16] (starting from 2-approximation in  $O(n^{3/2})$  space to  $O(\log n)$  approximation in  $O(n)$  space), which are known to be almost tight [5]. For multi-pass algorithms,  $\tilde{O}(n)$  space algorithms are known for  $(1 + \varepsilon)$ -approximation with  $\text{poly}(\log n, \frac{1}{\varepsilon})$  passes [28], [48], and exact algorithms with  $O(\sqrt{n})$  passes [36]. On the lower bound front, an  $\Omega(n^2)$  space lower bound is known for single-pass algorithms [5] and  $\tilde{\Omega}_p(n^{1+1/(2p+2)})$  for  $p$ -pass algorithms [44] (for exact answer or even some small approximation  $\approx (2p+4)/(2p+2)$ ); a stronger lower bound of  $\Omega(n^{1+1/2p})$  was proven earlier in [5] for algorithms that need to *find* the shortest path itself.

Our results show that a wide range of graph problems including directed reachability, cut and matching, and shortest path problems, admit essentially no non-trivial two-pass streaming algorithms: modulo the  $n^{o(1)}$ -term in our bounds, the best one could do to is to simply store the entire stream in  $O(n^2)$  space and solve the problem at the end using any offline algorithm.

### B. Our Techniques

We prove our main lower bound for the  $s$ - $t$  reachability problem; the other lower bounds then follow easily from this using standard ideas.

It helps to start the discussion with the lower bounds in [4], [5], [44]. These lower bounds work with *random graphs* wherein  $s$  can reach  $\Theta(\sqrt{n})$  random vertices  $S$  and

$t$  is independently reachable from  $\Theta(\sqrt{n})$  random vertices  $T$ ; thus, by Birthday Paradox,  $s$ - $t$  reachability can have either answer with constant probability. One then shows that to determine this, the algorithm needs to “find”  $S$  and  $T$  explicitly. The final part is then to use ideas from *pointer chasing* problems [49]–[53] to prove a lower bound for this task. The particular space-pass tradeoff is then determined as follows: (i) as a streaming algorithm can find the  $p$ -hop neighborhood of  $s$  and  $t$  in  $p$  passes (by BFS),  $S$  and  $T$  need to be  $(p+1)$ -hop away from  $s$  and  $t$ ; (ii) as we are working with random graphs, to achieve the bound of  $O(\sqrt{n})$  on size of  $S$  and  $T$ , we need the degree of the graph to be  $O(n^{1/2(p+1)})$ , leading to an  $O(n^{1+1/(2p+2)})$  space lower bound for  $p$ -pass algorithms. We note that the *limit* of these approaches based on random graphs seem to be  $\tilde{O}(n^{3/2})$ ; see [4, Section 5.2].

Our lower bound takes a different route and works with “more structured” graphs. We start with proving a *single-pass* streaming lower bound for an “algorithmically easier” variant of the reachability problem. In this problem, we are promised that  $s$  can reach a *unique* vertex  $s^*$  chosen uniformly at random from a set  $U$  of  $n^{1-o(1)}$  vertices and the goal is to “find” this vertex. Previous lower bounds [4], [5], [44] already imply that if our goal was to determine the identity of  $s^*$  *exactly*, we need  $\Omega(n^2)$  space. In this paper, we prove a stronger lower bound that an  $n^{2-o(1)}$ -space single-pass algorithm essentially cannot even change the distribution of  $s^*$  from uniform over  $U$ . The proof of this part is based on information theoretic arguments that rely on “embedding” multiple instances of the set intersection problem (see Section III) inside a *Ruzsa-Szemerédi* (RS) graph (see Section II-B), and proving a new lower bound for the set intersection problem.

We remark that our new lower bound for set intersection is related to the recent lower bound of [45] with a subtle technical difference that is explained in Section III and in more details in the full version. We also note that RS graphs have been used extensively for proving graph streaming lower bounds [2], [8], [26], [54]–[56] starting from [2], but this is their first application to the  $s$ - $t$  reachability problem.

In the next part of the argument, we construct a family of graphs in which the  $s$ - $t$  reachability is determined by existence of a single edge  $(s^*, t^*)$  in the graph, where  $s^*$  is the unique vertex reachable from  $s$  in a large set  $U$  and  $t^*$  is the unique vertex that can reach  $t$  in a large set  $W$  (see Figure 2 for an illustration). By exploiting our lower bound in the first part, we show that a  $n^{2-o(1)}$ -space algorithm cannot properly “find” the pairs  $s^*$  and  $t^*$  in the first pass. We then argue that this forces the algorithm to effectively “store” all the edges between  $U$  and  $W$  in the second pass to determine if  $(s^*, t^*)$  is an edge of the graph, leading to an  $n^{2-o(1)}$  space lower bound.

**Remark (More than two passes?).** The intermediate “sim-

pler” problem we considered in our proofs (part one above) is only hard in one pass (see Section IV) and thus our lower bound proof does not directly go beyond two passes. However, it appears that our techniques can be extended to multi-pass algorithms to prove lower bounds of the type  $n^{1+\Omega(1/p)}$  space for  $p$ -pass algorithms which are slightly better in terms of dependence on  $p$  in the exponent compared to [4], [5], [44]. Nevertheless, as unlike the case for two-pass algorithms, it is no longer clear whether such bounds are the “right” answer to the problems at hand, we opted to not pursue this direction in this paper.

## II. PRELIMINARIES

*Notation:* For any integer  $t \geq 1$ , we use  $[t] := \{1, \dots, t\}$ . For any  $k$ -tuple  $X = (X_1, \dots, X_k)$  and integer  $i \in [k]$ , we define  $X^{<i} := (X_1, \dots, X_{i-1})$ .

Throughout the paper, we use ‘sans serif’ letters to denote random variables (e.g.,  $A$ ), and the corresponding normal letters to denote their values (e.g.,  $A$ ). For brevity and to avoid the clutter in notation, in conditioning terms which involve assignments to random variables, we directly use the value of the random variable (with the same letter), e.g., write  $B \mid A$  instead of  $B \mid A = A$ .

For random variables  $A, B$ , we use  $\mathbb{H}(A)$  and  $\mathbb{I}(A; B) := \mathbb{H}(A) - \mathbb{H}(A \mid B)$  to denote the Shannon entropy and mutual information, respectively. Moreover, for two distributions  $\mu, \nu$  on the same support,  $\|\mu - \nu\|_{\text{tvd}}$  denotes the total variation distance, and  $\mathbb{D}(\mu \parallel \nu)$  is the KL-divergence. A summary of basic information theory facts that we use in our proofs appear in the full version.

### A. Communication Complexity and Information Complexity

We work with the two-party communication model of Yao [57]. See the excellent textbook by Kushilevitz and Nisan [58] for an overview of communication complexity.

Let  $P : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$  be a relation. Alice receives an input  $X \in \mathcal{X}$  and Bob receives  $Y \in \mathcal{Y}$ , where  $(X, Y)$  are chosen from a distribution  $\mathcal{D}$  over  $\mathcal{X} \times \mathcal{Y}$ . We allow players to have access to both public and private randomness. They communicate with each other by exchanging messages according to some *protocol*  $\pi$ . Each message in  $\pi$  depends only on the private input and random bits of the player sending the message, the already communicated messages, and the public randomness. At the end, one of the players outputs an answer  $Z$  such that  $Z \in P(X, Y)$ . For any protocol  $\pi$ , we use  $\Pi := \Pi(X, Y)$  to denote the messages and the public randomness used by  $\pi$  on the input  $(X, Y)$ .

We now define two measures of “cost” of a protocol.

**Definition II.1** (Communication cost). *The communication cost of a protocol  $\pi$ , denoted by  $\text{CC}(\pi)$ , is the worst-case length of the messages communicated between Alice and Bob in the protocol.*

**Definition II.2** (Information cost). *The information cost of a protocol  $\pi$ , when the inputs  $(X, Y)$  are drawn from a distribution  $\mathcal{D}$ , is  $\text{IC}_{\mathcal{D}}(\pi) := \mathbb{I}(\Pi; X | Y) + \mathbb{I}(\Pi; Y | X)$ .*

The internal information cost (introduced by [59]; see also [59]–[63]) measures the average amount of information each player learns about the input of the other player by observing the transcript of the protocol. As each bit of communication cannot reveal more than one bit of information, the internal information cost of a protocol lower bounds its communication cost [62].

*Communication complexity and streaming:* There is a standard connection between the communication cost of any protocol  $\pi$  for a communication problem  $P(X, Y)$  and the space of any streaming algorithm that can solve  $P(X, Y)$  on a stream  $X \circ Y$  (see Proposition VI.1); we use this connection to establish our streaming lower bounds.

### B. Ruzsa-Szemerédi Graphs

A graph  $G^{\text{RS}} = (V, E)$  is called an  $(r, t)$ -Ruzsa-Szemerédi (RS) graph iff its edge-set  $E$  can be partitioned into  $t$  induced matchings  $M_1^{\text{RS}}, \dots, M_t^{\text{RS}}$ , each of size  $r$ . We further define an  $(r, t)$ -RS digraph as a directed bipartite graph  $G^{\text{RS}} = (L, R, E)$  obtained by directing every edge of a bipartite  $(r, t)$ -RS graph from  $L$  to  $R$ .

We use the original construction of RS graphs due to Ruzsa and Szemerédi [64] based on the existence of large sets of integers with no 3-term arithmetic progression, proven by Behrend [65]. We note that there are multiple other constructions with different parameters (see, e.g. [2], [66]–[68] and references therein) but the following construction works best for our purpose.

**Proposition II.3** ([64]). *For infinitely many integers  $N$ , there are  $(r, t)$ -RS digraphs with  $N$  vertices on each side of the bipartition and parameters  $r = \frac{N}{e^{\Theta(\sqrt{\log N})}}$  and  $t = N/3$ .*

## III. A NEW LOWER BOUND FOR THE SET INTERSECTION PROBLEM

One key ingredient of our paper is a new lower bound for the set intersection problem, defined formally as follows.

**Problem 1** (set-intersection). *The set-intersection problem is a two-player communication problem in which Alice and Bob are given sets  $A$  and  $B$  from  $[m]$ , respectively, with the promise that there exists a **unique** element  $e^*$  such that  $\{e^*\} = A \cap B$ . The goal is to find the **target element**  $e^*$  using back and forth communication (i.e., in the two-way communication model).*

The set-intersection problem is closely related to the well-known *set disjointness* problem. It is in fact straightforward to prove an  $\Omega(m)$  lower bound on the communication complexity of set-intersection using a simple reduction from the set disjointness problem. However, in this paper,

we are interested in an algorithmically simpler variant of this problem which we define below.

**Definition III.1.** *Let  $\mathcal{D}$  be a distribution of inputs  $(A, B)$  for set-intersection (known to both players). A protocol  $\pi$  **internal  $\varepsilon$ -solves** set-intersection over  $\mathcal{D}$  iff at least one of the following holds:*

$$\begin{aligned} \mathbb{E}_{\Pi, A} \|\text{dist}(e^* | \Pi, A) - \text{dist}(e^* | A)\|_{\text{tvd}} &\geq \varepsilon \quad \text{or} \\ \mathbb{E}_{\Pi, B} \|\text{dist}(e^* | \Pi, B) - \text{dist}(e^* | B)\|_{\text{tvd}} &\geq \varepsilon, \end{aligned}$$

where all variables are defined with respect to the distribution  $\mathcal{D}$  and the internal randomness of  $\pi$  (recall that  $\Pi$  includes the transcript and the public randomness).

**Definition III.1** basically states that a protocol can internal  $\varepsilon$ -solve the set-intersection problem iff the transcript of the protocol can change the distribution of the target element  $e^*$  from the perspective of Alice or Bob by at least  $\varepsilon$  in the total variation distance on average.

Our definition is inspired but different from  $\varepsilon$ -solving in [45] (which we call *external  $\varepsilon$ -solving* to avoid ambiguity) which required the transcript to change the distribution of the target element by  $\varepsilon$  from the perspective of an *external observer* (who only sees the transcript but not the inputs of players). More formally, external  $\varepsilon$ -solving of set-intersection over a distribution  $\mu$ , as defined in [45], requires the protocol  $\pi$  to have the following property (compare this with Definition III.1),

$$\mathbb{E}_{\Pi} \|\text{dist}(e^* | \Pi) - \text{dist}(e^*)\|_{\text{tvd}} \geq \varepsilon.$$

The previous work in [45] has shown that there is a distribution  $\mu$  such that any protocol that external  $\varepsilon$ -solves set-intersection over  $\mu$  has information cost  $\Omega(\varepsilon^2 \cdot m)$ . This however does *not* imply a lower bound for the internal  $\varepsilon$ -solving problem. This is because, in principle, these two tasks can be different. For instance, (i) a protocol that reveals the entire set of Alice, changes the distribution of target for Bob dramatically but not so much for an external observer; or (ii) a protocol that reveals all the elements that are neither in  $A$  nor in  $B$  changes the distribution of the target for an external observer by a lot but does not change the distribution for either of the players at all.

We prove the following lower bound on the information cost of internal  $\varepsilon$ -solving of set-intersection.

**Theorem 1.** *There is a distribution  $\mathcal{D}_{\text{SI}}$  for set-intersection over the universe  $[m]$  such that:*

- 1) *For any  $A$  or  $B$  sampled from  $\mathcal{D}_{\text{SI}}$ , both  $\text{dist}(e^* | A)$  and  $\text{dist}(e^* | B)$  are uniform distributions on  $A$  and  $B$ , each of size  $m/4$ , respectively.*
- 2) *For any  $\varepsilon \in (0, 1)$ , any protocol  $\pi$  that internal  $\varepsilon$ -solves the set-intersection problem over the distribution  $\mathcal{D}_{\text{SI}}$  (Definition III.1) has internal information cost  $\text{IC}_{\mathcal{D}_{\text{SI}}}(\pi) = \Omega(\varepsilon^2 \cdot m)$ .*



#### IV. THE UNIQUE-REACH COMMUNICATION PROBLEM

We now start with our main lower bounds. Define the following two-player communication problem.

**Problem 2** (unique-reach). *The unique-reach problem is defined as follows. Consider a digraph  $G = (V, E)$  on  $n$  vertices where  $V := \{s\} \sqcup V_1 \sqcup V_2 \sqcup V_3$  and any edge  $(u, v) \in E$  is directed from  $s$  to  $V_1$  or some  $V_i$  to  $V_{i+1}$  for  $i \in [2]$  (we refer to each  $V_i$  as a layer). We are promised that there is a **unique** vertex  $s^*$  in the layer  $V_3$  reachable from  $s$ .*

Alice is given edges in  $E$  from  $V_1$  to  $V_2$ , denoted by  $E_A$ , and Bob is given the remainder of the edges in  $E$ , denoted by  $E_B$  (the partitioning of vertices of  $V$  is known to both players). The goal for the players is to **find**  $s^*$  by Alice sending a single message to Bob (i.e., in the one-way communication model).

It is easy to prove a lower bound of  $\Omega(n^2)$  on the one-way communication complexity of unique-reach using a reduction from the Index problem. It is also easy to see that this problem can be solved with  $O(n \log n)$  bits of communication, if we allow Bob to send a single message to Alice: By the uniqueness promise on  $s^*$ , no vertex with out-degree more than one in  $V_2$  should be reachable from  $s$  and thus Bob can communicate all the remaining edges in  $E_B$  to Alice.

Nevertheless, in this paper, we are interested in an algorithmically simpler variant of this problem similar-in-spirit to  $\varepsilon$ -solving for set-intersection (Definition III.1).

**Definition IV.1.** *Let  $\mathcal{D}$  be any distribution of valid inputs  $G = (V, E_A \sqcup E_B)$  for unique-reach (known to both players). We say that a protocol  $\pi$  **internal**  $\varepsilon$ -solves unique-reach over  $\mathcal{D}$  iff:*

$$\mathbb{E}_{\pi, E_B} \|\text{dist}(s^* \mid \Pi, E_B) - \text{dist}(s^* \mid E_B)\|_{\text{td}} \geq \varepsilon, \quad (1)$$

where all variables are defined with respect to the distribution  $\mathcal{D}$  and the internal randomness of  $\pi$  (recall that  $\Pi$  includes the transcript and the public randomness).

**Definition IV.1** basically states that a protocol can internal  $\varepsilon$ -solve the problem iff the message sent from Alice can change the distribution of the unique vertex  $s^*$  from the perspective of Bob by at least  $\varepsilon$  in the total variation distance (in expectation over Alice's message and Bob's input).

Our main theorem in this section is the following.

**Theorem 2.** *There is a distribution  $\mathcal{D}_{\text{UR}}$  for unique-reach and an integer  $b := \frac{n}{2^{\Theta(\sqrt{\log n})}}$  with the following properties:*

- 1) *For any  $E_B$  sampled from  $\mathcal{D}_{\text{UR}}$ ,  $\text{dist}(s^* \mid E_B)$  is a uniform distribution over a subset  $V_3^*$  of  $b$  vertices in the layer  $V_3$  of the input graph;*
- 2) *for any  $\varepsilon \in (0, 1)$ , any one-way protocol  $\pi$  that internal  $\varepsilon$ -solves unique-reach over the distribution  $\mathcal{D}_{\text{UR}}$*

(Definition IV.1) has communication cost  $\text{CC}(\pi) = \Omega(\varepsilon^2 \cdot n \cdot b)$ .

Proof of Theorem 2 is by a reduction from our Theorem 1 using a combinatorial construction based on Ruzsa-Szemerédi graphs (see Section II-B). In the following section, we first present our distribution  $\mathcal{D}_{\text{UR}}$  and then in the subsequent section prove the desired lower bound.

##### A. Distribution $\mathcal{D}_{\text{UR}}$ in Theorem 2

To continue, we need to set up some notation. Let  $G^{\text{RS}} = (L, R, E)$  be an  $(r, t)$ -RS digraph with induced matchings  $M_1^{\text{RS}}, \dots, M_t^{\text{RS}}$  as defined in Section II-B. For each induced matching  $M_i^{\text{RS}}$ , we assume an arbitrary ordering of edges  $e_{i,1}, \dots, e_{i,r}$  in the matching and for each  $j \in [r]$  denote  $e_{ij} := (u_{ij}, v_{ij})$  for  $u_{ij} \in L$  and  $v_{ij} \in R$ ; moreover, we let  $L(M_i^{\text{RS}}) := \{u_{i1}, \dots, u_{ir}\}$  and  $R(M_i^{\text{RS}}) := \{v_{i1}, \dots, v_{ir}\}$ . Based on these, we have the following definition:

- For any matching  $M_i^{\text{RS}}$  and any set  $S \subseteq [r]$ , we define  $M_i^{\text{RS}}|S$  as the matching in  $G^{\text{RS}}$  consisting of the edges  $e_{ij} \in M_i^{\text{RS}}$  for all  $j \in S$ .

We are now ready to define our distribution. See Figure 1 for an illustration.

**Distribution  $\mathcal{D}_{\text{UR}}$ .** An input distribution on graphs  $G = (\{s\} \sqcup V_1 \sqcup V_2 \sqcup V_3, E_A \sqcup E_B)$ .

- 1) Let  $G^{\text{RS}} = (L, R, E^{\text{RS}})$  be a fixed  $(r, t)$ -RS digraph on  $2N$  vertices from Proposition II.3 with parameters  $r = \frac{N}{2^{\Theta(\sqrt{\log N})}}$ , and  $t = \frac{N}{3}$ . We note that this graph is known to both players.
- 2) Let  $V_1 = L = \{u_1, \dots, u_N\}$ ,  $V_2 = R = \{v_1, \dots, v_N\}$ , and  $V_3$  be  $r$  new vertices  $\{w_1, \dots, w_r\}$ .
- 3) Sample  $t$  independent instances  $(S_1, T_1), \dots, (S_t, T_t)$  of set-intersection on the universe  $[r]$  from the distribution  $\mathcal{D}_{\text{SI}}$  in Theorem 1.
- 4) The input  $E_A$  to Alice is  $E_A := (M_1^{\text{RS}}|S_1) \cup \dots \cup (M_t^{\text{RS}}|S_t)$ .
- 5) Sample  $i^* \in [t]$  uniformly at random.
- 6) The input  $E_B$  to Bob is the set of edges  $(s, u_{i^*j})$  for  $j \in T_{i^*}$  and  $(v_{i^*j}, w_j)$  for  $j \in T_{i^*}$ .

**Observation IV.2.** *Several observations are in order:*

- 1) *For any  $G \sim \mathcal{D}_{\text{UR}}$ , there is a unique vertex  $s^*$  reachable from  $s$  in  $V_3$ . Moreover,  $s^* = w_{e^*}$  where  $e^* \in [r]$  is the unique element in the intersection of  $S_{i^*}$  and  $T_{i^*}$ .*
- 2) *For any  $E_B \sim \mathcal{D}_{\text{UR}}$ ,  $\text{dist}(s^* \mid E_B)$  is uniform over vertices  $w_j \in V_3$  for  $j \in T_{i^*}$ .*

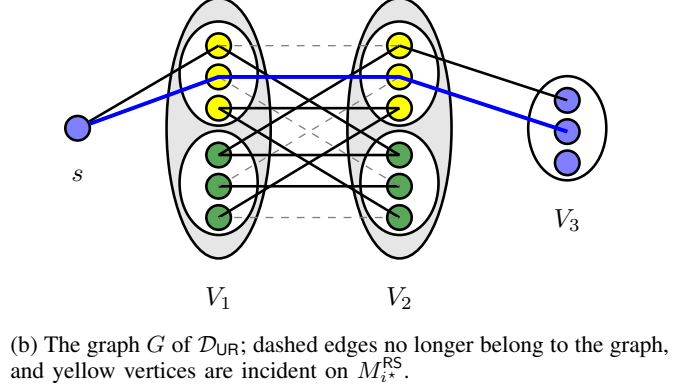
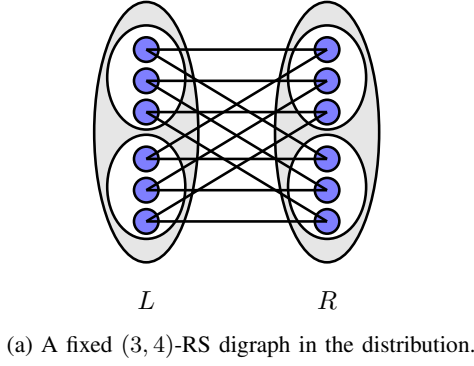


Figure 1: An illustration of the input distribution  $\mathcal{D}_{UR}$ . Here, directions of all edges are from left to right and hence omitted. The marked vertex (blue) in  $V_3$  denotes the unique vertex  $s^*$  in this example along with the path connecting  $s$  to  $s^*$ .

- 3) In  $\mathcal{D}_{UR}$ , the index  $i^* \in [t]$  is independent of the sets  $(S_1, T_1), \dots, (S_t, T_t)$ . Moreover, the pairs  $(S_1, T_1), \dots, (S_t, T_t)$  are mutually independent.
- 4) The input  $E_A$  to Alice in  $\mathcal{D}_{UR}$  is uniquely determined by  $S_1, \dots, S_t$ , and the input  $E_B$  to Bob is determined by  $i^*$  and  $T_{i^*}$ .

#### B. Proof Sketch of [Theorem 2](#)

Let  $\pi_{UR}$  be any one-way protocol that internal  $\varepsilon$ -solves unique-reach on the distribution  $\mathcal{D}_{UR}$ . We will prove that  $\text{CC}(\pi_{UR}) = \Omega(\varepsilon^2 \cdot r \cdot t)$  which proves [Theorem 2](#). The argument relies on the following two claims: (i) internal  $\varepsilon$ -solving of unique-reach on  $\mathcal{D}_{UR}$  is equivalent to internal  $\varepsilon$ -solving of set-intersection on  $\mathcal{D}_{SI}$  for the pair  $(S_{i^*}, T_{i^*})$ ; and (ii) the information revealed by  $\pi_{UR}$  about the instance  $(S_{i^*}, T_{i^*})$  is at least  $t$  times smaller than  $\text{CC}(\pi_{UR})$ . Having both these steps, we can then invoke [Theorem 1](#) to conclude the proof.

We shall emphasize that this is *not* an immediate reduction from [Theorem 1](#) as we are aiming to gain an *additional* factor of  $t$  in the information cost lower bound for  $\pi_{UR}$  compared to the lower bound for set-intersection. This part crucially relies on the fact that  $\pi_{UR}$  is a one-way protocol and that index  $i^* \in [t]$  in the distribution is independent of Alice's input (and thus her message).

We now present the formal proof. Consider the following protocol  $\pi_{SI}$  for set-intersection on the distribution  $\mathcal{D}_{SI}$  using  $\pi_{UR}$  as a subroutine.

*Protocol  $\pi_{SI}$ :* Given an instance  $(A, B) \sim \mathcal{D}_{SI}$  on universe  $[r]$ , Alice and Bob do as follows:

- 1) Alice and Bob sample  $i^* \in [t]$  using public randomness.
- 2) Alice sets  $S_{i^*} = A$  and samples the remaining sets  $S_i$  for  $i \neq i^* \in [t]$  independently from  $\mathcal{D}_{SI}$  using private randomness (this is doable by part (iii) of [Observation IV.2](#)). This allows Alice to generate

the set  $E_A$  of edges for  $\pi_{UR}$  as in  $\mathcal{D}_{UR}$  (by part (iv) of [Observation IV.2](#)).

- 3) Bob sets  $T_{i^*} = B$  and creates the set of edges  $E_B$  for  $\pi_{UR}$  as in  $\mathcal{D}_{UR}$  (again doable by part (iv) of [Observation IV.2](#) as Bob also knows  $i^*$ ).
- 4) The players then run the protocol  $\pi_{UR}$  on the input  $(E_A, E_B)$  with Alice sending the message in  $\pi_{UR}$  to Bob.

The first step of the proof is the following claim.

**Claim IV.3.**  $\pi_{SI}$  internal  $\varepsilon$ -solves set-intersection on  $\mathcal{D}_{SI}$ .

We can also bound the internal information cost of  $\pi_{SI}$  which allows us to apply [Theorem 1](#) and conclude the proof. The proof of this lemma is by a direct-sum style argument. We note that these arguments (based on information theory tools) are by now mostly standard in the literature.

**Lemma IV.4.**  $\text{IC}_{\mathcal{D}_{SI}}(\pi_{SI}) \leq \frac{1}{t} \cdot \text{CC}(\pi_{UR})$ .

The proofs of [Claim IV.3](#) and [Lemma IV.4](#) are deferred to the full version.

We now conclude the proof of [Theorem 2](#). By [Claim IV.3](#),  $\pi_{SI}$  internal  $\varepsilon$ -solves set-intersection and thus by [Theorem 1](#), we have  $\text{IC}_{\mathcal{D}_{SI}}(\pi_{SI}) = \Omega(\varepsilon^2 \cdot r)$ . Plugging in this bound in [Lemma IV.4](#), we obtain that

$$\text{CC}(\pi_{UR}) = \Omega(\varepsilon^2 \cdot r \cdot t) = \Omega\left(\varepsilon^2 \cdot \frac{n^2}{2^{\Theta(\sqrt{\log n})}}\right),$$

as the number of vertices  $n$  in the graph is  $O(N)$ . Setting  $b = r/4 = \frac{n}{2^{\Theta(\sqrt{\log n})}}$  concludes the proof of [Theorem 2](#).

#### C. The Inverse Unique-Reach Problem

In addition to the unique-reach problem, we also need another (almost identical) variant of this problem which we call the *inverse* of the unique-reach problem, denoted by unique-reach. This problem is basically what one would naturally expect if we *reverse* the direction of all edges in an instance of unique-reach and ask for finding the unique

vertex that can now reach the end-vertex  $t$  (corresponding to  $s$ ). Formally, we define this problem as follows.

In **unique-reach**, we have a digraph  $\overleftarrow{G} = (U, \overleftarrow{E})$  on  $n$  vertices where  $U := U_3 \sqcup U_2 \sqcup U_1 \sqcup \{t\}$ , all edges of the graph are directed from  $U_1$  to  $t$  or some  $U_{i+1}$  to  $U_i$  for  $i \in [2]$ , and we are promised that there is a **unique** vertex  $s^*$  in  $U_3$  that can reach  $t$ . The goal is to find this vertex  $s^*$ , or rather, internal  $\varepsilon$ -solve it exactly as in **Definition IV.1**. As before, the edges between  $U_2$  and  $U_1$ , denoted by  $\overleftarrow{E}_A$ , are given to Alice, and the remaining edges, denoted by  $\overleftarrow{E}_B$ , are given to Bob. The communication is also one-way from Alice to Bob.

We also define a hard input distribution for **unique-reach**, named  $\overleftarrow{\mathcal{D}}_{\text{UR}}$ , in exact analogy with  $\mathcal{D}_{\text{UR}}$  for **unique-reach**:  $\overleftarrow{\mathcal{D}}_{\text{UR}}$  is a distribution over graphs  $\overleftarrow{G} = (U_3 \sqcup U_2 \sqcup U_1 \sqcup \{t\}, \overleftarrow{E}_A \sqcup \overleftarrow{E}_B)$ , obtained by sampling a graph  $G = (\{s\} \sqcup \{t\} \sqcup V_3, E_A \sqcup E_B)$  from  $\mathcal{D}_{\text{UR}}$ , setting  $U_3 = V_3$ ,  $U_2 = V_2$ ,  $U_1 = V_1$ , and  $t = s$ , and **reversing** the direction of all edges in  $E_A$  and  $E_B$  to obtain  $\overleftarrow{E}_A$  and  $\overleftarrow{E}_B$ .

#### THE st-REACHABILITY COMMUNICATION PROBLEM V. T

We now define the main two-player communication problem (the setting of this problem is rather non-standard in terms of the communication model).

**Problem 3 (st-reachability).** Consider a digraph  $G = (V, E)_{E_3}$  with two designated vertices  $s, t$  and  $E := E_1 \sqcup E_2 \sqcup t$  in  $G$ .

Initially, Alice receives  $E_1$  and Bob receives  $E_2$  (the vertices  $s, t$  are known to both players). Next, Alice and Bob will have one round of communication by Alice sending a message  $\Pi_{A1}$  to Bob and Bob responding back with a message  $\Pi_{B1}$ . At this point, the edges  $E_3$  are revealed to both players. Finally, Alice is allowed to send yet another message  $\Pi_{A2}$  to Bob (which this time depends on  $E_3$  as well) and Bob outputs the answer (also a function of  $E_3$ ).

The following theorem is the main result of our paper.

**Theorem 3.** For any  $\varepsilon \in (n^{-1/2}, 1/2)$ , any communication protocol for **st-reachability** that succeeds with probability at least  $\frac{1}{2} + \varepsilon$  requires  $\Omega(\varepsilon^2 \cdot \frac{n^2}{2^{\Theta(\sqrt{\log n})}})$  bits of communication.

We note that the  $n^{-1/2}$  lower bound on  $\varepsilon$  in **Theorem 3** is not sacrosanct and any term which is  $\omega\left(\frac{\log n}{b}\right)$  still works where  $b = \frac{n}{2^{\Theta(\sqrt{\log n})}}$  is the parameter in **Theorem 2**.

#### A. A Hard Distribution for st-reachability

Recall the distributions  $\mathcal{D}_{\text{UR}}$ ,  $\overleftarrow{\mathcal{D}}_{\text{UR}}$  from **Section IV**. We will use them to define our distribution for **st-reachability**.

**Figure 2** for an illustration. See

**Distribution  $\mathcal{D}_{\text{ST}}$ .** A hard input distribution for the **st-reachability** problem.

- 1) Let  $V := \{s\} \sqcup V_1 \sqcup V_2 \sqcup V_3 \sqcup U_3 \sqcup U_2 \sqcup U_1 \sqcup \{t\}$  – each  $V_i$  or  $U_i$  is called a *layer* of  $G$  (this partitioning is known to both players).
- 2) Sample the graph  $G_1 := (V_3 \sqcup U_3, E_1)$  by picking each edge  $(v, u) \in V_3 \times U_3$  independently and with probability half.
- 3) Sample the following two graphs *independently*:
  - a)  $H := (\{s\} \sqcup V_1 \sqcup V_2 \sqcup V_3, E_A \sqcup E_B)$  sampled from the distribution  $\mathcal{D}_{\text{UR}}$ ;
  - b)  $\overleftarrow{H} := (U_3 \sqcup U_2 \sqcup U_1 \sqcup \{t\}, \overleftarrow{E}_A \sqcup \overleftarrow{E}_B)$  sampled from the distribution  $\overleftarrow{\mathcal{D}}_{\text{UR}}$ .
- 4) The initial input to Alice and Bob are, respectively,  $E_1$  and  $E_2 := E_A \sqcup \overleftarrow{E}_A$ , and the input revealed to both players in the second round is  $E_3 := E_B \sqcup \overleftarrow{E}_B$ .

To avoid potential confusion, we should note right away that Bob in the distribution  $\mathcal{D}_{\text{ST}}$  is receiving the input of Alice in  $\mathcal{D}_{\text{UR}}$  and  $\overleftarrow{\mathcal{D}}_{\text{UR}}$ .

**Observation V.1.** The following two remarks are in order:

- 1) The distributions of  $E_1$ ,  $H$ , and  $\overleftarrow{H}$  are mutually independent in  $\mathcal{D}_{\text{ST}}$ .
- 2)  $s$  can reach  $t$  in  $G$  iff the edge  $(s^*, t^*) \in E_1$ .  
(proof: the only vertex in  $V_3$  reachable from  $s$  is  $s^*$  and the only vertex in  $U_3$  that reaches  $t$  is  $t^*$ , thus the only potential  $s$ - $t$  path is  $s \rightsquigarrow s^* \rightarrow t^* \rightsquigarrow t$ .)

#### B. Setup and Notation

Let  $\pi_{\text{ST}}$  be any *deterministic* protocol for **st-reachability** over the distribution  $\mathcal{D}_{\text{ST}}$  with

$$\text{CC}(\pi_{\text{ST}}) = o(\varepsilon^2 \cdot b^2), \quad (2)$$

where  $b := \frac{n}{2^{\Theta(\sqrt{\log n})}}$  is the parameter in **Theorem 2** for instances of  $\mathcal{D}_{\text{UR}}$  and  $\overleftarrow{\mathcal{D}}_{\text{UR}}$ . We will prove that the probability that  $\pi_{\text{ST}}$  outputs the correct answer to **st-reachability** is  $\frac{1}{2} + o(\varepsilon)$ , hence proving **Theorem 3** for deterministic protocols. The results for randomized protocols follows immediately from this and an averaging argument (i.e., the easy direction of Yao's minimax principle [69]).

To facilitate our proofs, the following notation would be useful. For brevity, we use

$$\begin{aligned} \Pi &:= (\Pi_{A1}, \Pi_{B1}, \Pi_{A2}), \\ Z_1 &:= (\Pi_{A1}, \Pi_{B1}, E_3), \\ Z_2 &:= (\Pi, E_3, s^*, t^*). \end{aligned}$$

We also use  $O \in \{0, 1\}$  to denote the output Bob at the end of the protocol.

For any pair of vertices  $v, u \in V_3 \times U_3$ , we use the notation  $E_1(v, u) \in \{0, 1\}$  to denote whether or not the edge  $(v, u) \in$

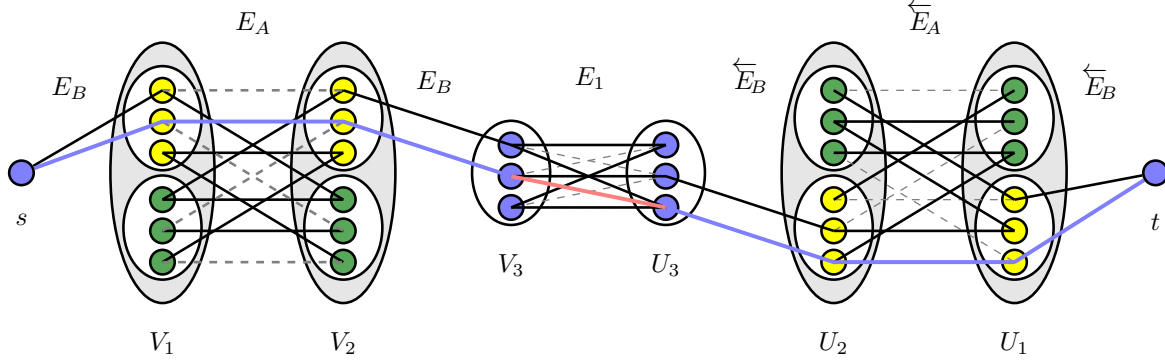


Figure 2: An illustration of the input distribution  $\mathcal{D}_{\text{ST}}$ . Here, the directions of all edges are from left to right and hence omitted. The vertices  $s^* \in V_3$  and  $t^* \in U_3$  are marked blue and the potential edge  $(s^*, t^*)$  is marked red—existence or non-existence of this edge uniquely determines whether or not  $s$  can reach  $t$  in  $G$ .

$E_1$ . For a fixed choice of  $E_3 = E_B \cup \overleftarrow{E}_B$  in  $\mathcal{D}_{\text{ST}}$ , we use  $V_3^*$  and  $U_3^*$  to denote the sets from which  $s^*$  and  $t^*$  are chosen uniformly at random from conditioned on  $E_B$  and  $\overleftarrow{E}_B$ , respectively (see part (i) of [Theorem 2](#)). We also define:

$$E_1(V_3^*, U_3^*) := \{E_1(v_i, u_i) \mid (v_i, u_i) \in V_3^* \times U_3^*\}.$$

We further assume a fixed arbitrary ordering of pairs  $v, u \in V_3 \times U_3$  and define:

$$E_1^{<(v,u)} := E_1(v_1, u_1), E_1(v_2, u_2), \dots$$

for all pairs  $(v_i, u_i) \in E_1(V_3^*, U_3^*)$  that appear before  $(v, u)$  in this ordering (note that we ignore the other edges of  $E_1$  that are not in  $E_1(V_3^*, U_3^*)$  here).

#### Crucial Independence Properties

The following independence properties are crucial for our proofs. They are all based on the rectangle property of communication protocols and part (i) of [Observation V.1](#).

$$\Pi_{A2} \perp s^*, t^* \mid Z_1 \quad (3)$$

$$E_1 \perp s^*, t^* \mid Z_1, \Pi_{A2} \quad (4)$$

$$E_2 \perp E_1(s^*, t^*) \mid Z_1, Z_2. \quad (5)$$

The proofs appear in full version.

#### C. Part One: The First Round of Communication

In the following lemma, we prove that after the first round of the protocol, the (joint) distribution of  $(s^*, t^*)$  conditioned on  $Z_1 = (\Pi_{A1}, \Pi_{B1}, E_3)$  is almost the same as if we only conditioned on  $E_3$ . This is basically through a reduction from [Theorem 2](#) considering  $s^*, t^*$  are distributed (originally) according to  $\mathcal{D}_{\text{UR}}$  and  $\overleftarrow{\mathcal{D}}_{\text{UR}}$  and the public information  $E_3$  provides the input of Bob in the instances of unique-reach and unique-reach in this reduction.

#### Lemma V.2.

$$\mathbb{E}_{Z_1} \|\text{dist}(s^*, t^* \mid Z_1) - \text{dist}(s^*, t^* \mid E_3)\|_{\text{tvd}} = o(\varepsilon).$$

#### D. Part Two: The Second Round of Communication

[Lemma V.2](#) implies that the extra information  $Z_1$  available to Alice at the beginning of the second round does not change the distribution of  $(s^*, t^*)$  by much. We use this to show that the message of Alice in the second round does not change the distribution of  $E_1(s^*, t^*) \in \{0, 1\}$  by much.

#### Lemma V.3.

$$\mathbb{E}_{Z_1, Z_2} \|\text{dist}(E_1(s^*, t^*) \mid Z_1, Z_2) - \text{dist}(E_1(s^*, t^*))\|_{\text{tvd}} = o(\varepsilon).$$

#### E. Concluding the Proof of [Theorem 3](#)

We are now ready to conclude the proof of [Theorem 3](#). [Lemma V.3](#) implies that conditioning on  $Z_1, Z_2$  does not change the distribution of  $E_1(s^*, t^*)$  by much. By the independence property of [Eq \(5\)](#), we know that this continues to hold even if we further condition on the input of Bob, i.e.,  $E_2$ . We use this to prove that the probability that  $\pi_{\text{ST}}$  outputs the correct answer is almost the same as random guessing.

**Claim V.4.**  $\Pr(\pi_{\text{ST}} \text{ outputs the correct answer}) = \frac{1}{2} + o(\varepsilon)$ .

To conclude, we have shown that for any deterministic protocol  $\pi_{\text{ST}}$  with  $\text{CC}(\pi_{\text{ST}}) = o(\varepsilon^2 \cdot b^2)$ , the probability that  $\pi_{\text{ST}}$  outputs the correct answer over the distribution  $\mathcal{D}_{\text{ST}}$  is only  $\frac{1}{2} + o(\varepsilon)$ . This can be extended directly to randomized protocols as by an averaging argument, we can always fix the randomness of any randomized protocol  $\pi_{\text{ST}}$  on the distribution  $\mathcal{D}_{\text{ST}}$  to obtain a deterministic protocol with the same error guarantee. Noting that  $b = \frac{n}{2^{\Theta(\sqrt{\log n})}}$  concludes the proof of [Theorem 3](#).

## VI. GRAPH STREAMING LOWER BOUNDS

We now obtain our graph streaming lower bounds by reductions from the **st-reachability** communication problem defined in [Section V](#). The first step of all these reductions is to show that one can simulate any two-pass graph streaming



algorithm on graphs  $G = (V, E)$  using a protocol in the setting of the st-reachability problem. The proof is via a standard simulation.

**Proposition VI.1.** *Any two-pass  $S$ -space streaming algorithm  $A$  on graphs  $G = (V, E_1 \sqcup E_2 \sqcup E_3)$  of st-reachability can be simulated exactly by a communication protocol  $\pi_A$  with  $\text{CC}(\pi_A) = O(S)$  and the communication-pattern restrictions of the st-reachability problem.*

#### A. Directed Reachability

We obtain the following theorem for the directed reachability problem.

**Theorem 4** (Formalization of [Result 1](#)). *Any streaming algorithm that makes two passes over the edges of any  $n$ -vertex directed graph  $G = (V, E)$  with two designated vertices  $s, t \in V$  and outputs whether or not  $s$  can reach  $t$  in  $G$  with probability at least  $2/3$  requires  $\Omega(\frac{n^2}{2^{\Theta(\sqrt{\log n})}})$  space.*

[Theorem 4](#) follows immediately from [Proposition VI.1](#) and our lower bound in [Theorem 3](#).

We also present some standard extension of this lower bound to other problems related to the directed reachability problem.

- **Estimating number of vertices reachable from a source:** Consider any instance of the problem in [Theorem 4](#) and connect  $t$  to  $2n$  new vertices. In the new graph, if  $s$  can reach  $t$ , then it can also reach at least  $2n$  other vertices, while if  $s$  does not reach  $t$ , it can reach at most  $n$  other vertices. Hence, the lower bound in [Theorem 4](#) extends to this problem as well which was studied (in a similar format) in [\[3\]](#).
- **Testing if  $G$  is acyclic or not:** Recall that the hard distribution of graphs in [Theorem 3](#) and hence [Theorem 4](#) is supported on acyclic graphs. If in these graphs, we connect  $t$  to  $s$  directly, then the graph remains acyclic iff  $s$  cannot reach  $t$ . Hence, the lower bound in [Theorem 4](#) extends to this problem as well.
- **Approximating minimum feedback arc set:** The lower bound for acyclicity implies the same bounds for any (multiplicative) approximation algorithm of minimum feedback arc set (the minimum number of edges to be deleted to make a graph acyclic) studied in [\[4\]](#).

#### B. Bipartite Perfect Matching

We obtain the following theorem for the bipartite perfect matching problem using a standard reduction.

**Theorem 5** (Formalization of [Result 2](#)). *Any streaming algorithm that makes two passes over the edges of any  $n$ -vertex undirected bipartite graph  $G = (L \sqcup R, E)$  and outputs whether or not  $G$  has a perfect matching with probability at least  $2/3$  requires  $\Omega(\frac{n^2}{2^{\Theta(\sqrt{\log n})}})$  space.*

#### C. Single-Source Shortest Path

Finally, we have the following theorem for the shortest path problem, again using a standard reduction.

**Theorem 6** (Formalization of [Result 3](#)). *Any streaming algorithm that makes two passes over the edges of any  $n$ -vertex undirected graph  $G = (V, E)$  with two designated vertices  $s, t \in V$  and outputs the length of the shortest  $s$ - $t$  path in  $G$  with probability at least  $2/3$  requires  $\Omega(\frac{n^2}{2^{\Theta(\sqrt{\log n})}})$  space.*

The lower bound continues to hold even if the algorithm is allowed to output an estimate which, with probability at least  $2/3$ , is as large as the length of the shortest  $s$ - $t$  path and strictly smaller than  $9/7$  times the length of the shortest  $s$ - $t$  path.

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