

On the Global Minimizers of Real Robust Phase Retrieval with Sparse Noise

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Abstract—We study a class of real robust phase retrieval problems under a Gaussian assumption on the coding matrix when the received signal is sparsely corrupted by noise. The goal is to establish conditions on the sparsity under which the input vector can be exactly recovered. The recovery problem is formulated as residual minimization in the ℓ_1 -norm. The main contribution is a robust phase retrieval counterpart to the seminal paper by Candes and Tao on compressed sensing (ℓ_1 regression) [Decoding by linear programming. IEEE Transactions on Information Theory, 51(12):4203–4215, 2005]. The analysis depends on a key new property of the coding matrix called the *Absolute Range Property* (ARP) which is the analogue to the Null Space Property (NSP) in compressed sensing. When the residuals are computed using squared magnitudes, we show that ARP follows from a standard Restricted Isometry Property (RIP). However, when the residuals are computed using absolute magnitudes, a different kind of RIP or growth property is required. We conclude by showing that the robust phase retrieval objectives are sharp with respect to their minimizers with high probability.

I. INTRODUCTION

Phase retrieval has been widely studied in machine learning, signal processing and optimization. The goal of finite dimensional phase retrieval is to recover a signal $x \in \mathbb{C}^n$ or $\mathbb{R}^n$ given a finite number of observations of the amplitude of its linear measurements:

$$|\langle a_i, x \rangle| = b_i, \quad 1 \leq i \leq m \quad (\text{I.1})$$

where $a_i \in \mathbb{C}^n$ or $\mathbb{R}^n$, $b_i \in \mathbb{R}$ are observations, and x is an unknown signal we wish to recover (e.g. see [33]). When the b_i are observed with error, then we say that the phase retrieval problem is noisy and refer this as the noisy case. Observe that recovery is only unique with respect to equivalence in the metric

$$\phi_1(x, y) := \min \{ \|x - \mu y\| \mid |\mu| = 1 \} \quad (\text{I.2})$$

(called the *natural metric* in [4]). Consequently, in the noiseless case, recovery of the signal is equivalent to the injectivity of the signal modulus mapping $\mathcal{A}_1(x) = \sum_{j=1}^m |\langle a_j, x \rangle| e_j$ from $\mathbb{C}^n$ or $\mathbb{R}^n$ to the space of equivalence classes determined by this metric (x is equivalent to y if and only if $\phi_p(x, y) = 0$). If it is assumed that the measurement vectors $\{a_i\}_{i=1}^m$ form a *frame* (i.e. a spanning set), then much is known about the number of measurements required in order to assure

the injectivity of $\mathcal{A}_1$. For example, if $m \geq 2n - 1$, it is generically injective on $\mathbb{R}^n$ [3], and if $m \geq 4n - 4$, it is generically injective on $\mathbb{C}^n$ [22]. In addition, if the frame can be precisely specified, then recovery is possible in the noiseless case through finite recursions using as few as $3n - 2$ measurements ([6], [29], [30]). In [15], Candès et al. show that, under general distributional assumptions on A , the semi-definite programming relaxation called Phaselift can exactly recover the signal for $m \geq c_0 n \log n$ with high probability, where c_0 is a universal constant. This result is refined in [13] for exact recovery for $m \geq c_0 n$ measurements with high probability. In the noisy complex case, Bodmann and Hammen [6] give explicitly chosen frames $\{a_i\}_{i=1}^m$ of size $m \geq 6n - 3$ for which concrete error bounds or *stability bounds* on the signal recovery are established. In both the real and complex noisy cases, Candès and Li [13] show that if the measurement vectors $\{a_i\}_{i=1}^m$ are independently generated from either uniform or Gaussian distributions on the unit sphere, then an estimate of the signal obtained through Phaselift yields a stability bound on the error with high probability for $m \geq c_0 n$. By explicitly choosing a frame, Balan and Zou [4] obtain refined stability bounds in the noisy case and show the intimate connection between such error bounds and the following *bi-Lipschitzian* property of the modulus mapping $\mathcal{A}_1$: there exists positive real scalars $0, \mu_1 < \mu_2$ such that

$$\mu_1 \phi_1(x, y) \leq \|\mathcal{A}_1(x) - \mathcal{A}_1(y)\|_1 \leq \mu_2 \phi_1(x, y), \quad (\text{I.3})$$

for some choice of $p, q \in [1, \infty]$. This property is reminiscent of the *restricted isometry property* in compressed sensing and plays an essential role in our study as well. In its most general form, the phase retrieval problem is NP-hard [34].

In this paper we focus on the *sparse* real phase retrieval problem which has been well studied in the recent literature ([13], [14], [18], [24], [28]). In sparse phase retrieval one assumes that the true signal satisfies all but a few but unknown number of the equalities in (I.1). This is a special instance of the noisy case of phase retrieval. In particular, we address the question of *exact signal recovery* through ℓ_1 residual minimization problems of the form

$$\min_x f_p(x) := \|\mathcal{A}_p(x) - b\|_1 \quad \text{for } p = 1, 2, \quad (\text{I.4})$$

where $\mathcal{A}_1$ is the signal modulus mapping and $\mathcal{A}_2$ is its componentwise square. By exact signal recovery, we mean the recovery of solutions to the ℓ_0 problem

$$\{x_*, -x_*\} = \underset{x}{\operatorname{argmin}} \| |Ax|^p - b \|_0. \quad (\text{I.5})$$

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For convenience, let $A \in \mathbb{R}^{m \times n}$ be the matrix whose rows are the vectors $\{a_i\}_{i=1}^m$ in (I.1) so that $\mathcal{A}_1(x) = |Ax|$ and $\mathcal{A}_2(x) = |Ax|^2$, where, for any vector $z \in \mathbb{R}^m$, $|z|$ and z^2 are vectors in $\mathbb{R}^m$ whose components are the absolute value and squares of those in z , respectively. We assume that the matrix A satisfies the following Gaussian assumption:

G: The entries of A are i.i.d. standard Gaussians $N(0, 1)$.

Our goal is to establish a sparse phase retrieval counterpart to the seminal results by Candes and Tao on compressed sensing (ℓ_1 regression) [16]. Compressed sensing problems [26] take the form

$$\min_y \|y\|_1 \text{ such that } \Phi y = c, \quad (\text{I.6})$$

where $\Phi \in \mathbb{R}^{n \times N}$, $y \in \mathbb{R}^N$, $c \in \mathbb{R}^n$. This problem is known to be equivalent to the ℓ_1 linear regression problem

$$\min_x \|Ax - b\|_1, \quad (\text{I.7})$$

where $\Phi b = -c$ and $A \in \mathbb{R}^{N \times (N-n)}$ (e.g., the columns of A form basis of $\text{Null}(\Phi)$). In [16] it is shown that there is a universal constant $c_0 > 2$ such that, under suitable conditions on A (e.g., assumption G), if $m > c_0 n$, then the set of sparse residual solutions to (I.1) coincides with the set of solutions to the two robust phase retrieval problems (I.4), with high probability (Theorem III.2 coupled with Lemma IV.2 for $p = 2$ and Lemma IV.8 for $p = 1$).

A key underlying structural requirement used by [16] is the Restricted Isometry Property (RIP). The RIP also plays a role in our study when $p = 2$. However, in the $p = 1$ case a different property, which we call the p-Absolute Growth Property (p-AGP) (see Definition II.3), is required. When $p = 2$, RIP implies 2-AGP. The p-AGP holds under assumption G, with high probability (see Lemmas IV.1 and IV.7). A second key property, which mimics the so-called Null Space Property (NSP) in compressed sensing ([21], [23], [27], [31]) is also introduced. We call this the p-Absolute Range Property (p-ARP) (see Definition II.1), and show that p-AGP implies p-ARP under assumption G with high probability. In [12], it is shown that, for problem (I.6), if Φ satisfies RIP with parameter $\delta_{2s} < \sqrt{2} - 1$, then Φ satisfies NSP of order s . Correspondingly, we show that the p-AGP implies the p-ARP with high probability under assumption G. (see Lemmas IV.2 and IV.8).

The two most popular approaches to the real phase retrieval recovery are through semidefinite programming relaxations ([2], [13], [15], [20], [25], [32], [37]) and convex-composite optimization ([8], [24], [28]). These approaches begin by formulating the real phase retrieval problem as a regression problem of the form

$$\min_x \rho(|Ax|^2 - b), \quad (\text{I.8})$$

where ρ is chosen to be either the ℓ_1 -norm or the squared ℓ_2 -norm. This structure plays a key role in both optimality conditions and algorithm development [8].

In the noiseless case, when there exists a vector $x_* \in \mathbb{R}^n$ such that $|Ax_*|^2 = b$, a gradient based method called

Wirtinger Flow (WF) was introduced by [14] to solve the smooth problem

$$\min_x \||Ax|^2 - b\|_2^2.$$

WF admits a linear convergence rate when properly initialized. Further work along this line includes the Truncated Wirtinger Flow (TWF), e.g., see [20]. Truncated Wirtinger Flow requires $m \geq Cn$ measurements as opposed to the $m \geq Cn \log n$ measurements for WF to obtain a linear rate. A similar approach using sub-gradients is used to minimize $\min_x \||Ax| - b\|_2^2$ in [38] for the noiseless case.

Different solution methods are employed to solve (I.4) for $p = 2$ and $p = 1$. When $p = 1$, one can apply a smoothing method to the absolute value function ([1], [33]), or use other relaxation techniques that preserve the nonsmooth objective but introduce auxiliary variables [40]. When $p = 2$, the solution methods typically exploit the convex-composite structure of the objective f_2 in (I.4) and rely on two key conditions: weak convexity (i.e., $f_2 + \frac{\rho}{2} \|\cdot\|^2$ is convex for some $\rho > 0$) and sharpness (i.e., $f_2(x) - \min f_2 \geq c \cdot \text{dist}(x, \mathcal{X})$ for some $c > 0$ where $\mathcal{X}$ is the set of minimizers of f). Under these two properties, Duchi and Ruan [28], Drusvyatskiy et al. [24] and Charisopoulos, et al. [18] establish convergence and iteration complexity results for prox-linear and subgradient algorithms. Recently, Zhang et al. [39] and Chen et al. [19] considered gradient-based methods for the problem $\min_x f_2(x)$ when the noise is sm sparse for some $s < 1$. To establish locally linear convergence of their algorithms the authors of [39] require that the measurements satisfy $m \geq cn \log n$ for $c > 0$, while the authors of [19] require that $s < c/\log m$ for some $c > 0$. The results in [28] and [18] require $m \geq cn$ for some $c > 0$ and for some $s \in [0, \frac{1}{2})$ sufficiently small.

Conditions for the weak convexity of f_2 follow from results in [24] and [28] under assumptions weaker than assumption G. In both the noiseless and noisy cases, the sharpness of f_2 also follows from results in [24] and [28]. We establish sharpness for both f_1 and f_2 under assumption G uniformly for all possible supports of the sparse noise. Our result for the $p = 2$ case has a similar flavor to those in [28] and [18], but more closely parallels the result of Candes and Tao in the compressed sensing case. When $p = 1$, our result has no precedence in the literature and requires a new approach. The function f_1 is not weakly convex since it is not even subdifferentially regular [33].

The paper is organized as follows. In section 2, we introduce the new properties p-ARP and p-AGP and provide a detailed description of how our program of proof parallels the program used in compressed sensing. In Section 3, we show that if A satisfies p-ARP and the residual $||Ax_*|^p - b|$ is sufficiently sparse, then $\{\pm x_*\} \subset \text{argmin } f_p$ with equality under assumption G. In section 4, we show that assumption G implies that p-AGP implies p-ARP with high probability. In the final section we show that f_p is sharp with respect to $\text{argmin } f_p$, with high probability.

A. Notation

Lower case letters (i.e. x, y) denote vectors, while x_i denotes the i th component of the x . $c_0, c_1, c_2, \tilde{c}_0, \tilde{c}_1, C$ denote

universal constants. $\|x\|$, $\|x\|_1$ denote the Euclidean and ℓ_1 norms of vector x , while $\|x\|_0$ denotes the ℓ_0 ‘norm’ $|\{i|x_i \neq 0\}|$. For a matrix X , $\|X\|_F$ denotes the Frobenius, $\|X\|$ denotes the ℓ_2 operator norm, and X_i denotes the i th row of X . When $x = (x_i)_{1 \leq i \leq n}$ is a vector, $|x| := (|x_i|)_{1 \leq i \leq n}$ and $x^p := (x_i^p)_{1 \leq i \leq m}$. For a vector $v \in \mathbb{R}^m$, and $T \subset [m] := \{1, 2, \dots, m\}$, v_T is defined to be a vector in $\mathbb{R}^m$ where the i th entry is v_i if $i \in T$ and 0 else where. $\text{supp}(x) := \{i | x_i \neq 0\}$. We say a vector x is L sparse if $\|x\|_0 := |\text{supp}(x)| \leq L$.

II. THE ROADMAP

Recall from the compressed sensing literature ([21], [23]) that a matrix $\Phi \in \mathbb{R}^{m \times n}$ satisfies Null Space Property (NSP) of order L at $\psi \in (0, 1)$ if

$$\|y_T\|_1 \leq \psi \|y_{T^c}\|_1 \quad \forall y \in \text{Null}(\Phi) \text{ and } |T| \leq L. \quad (\text{II.1})$$

It is shown in [27] and [31] that every L -sparse signal $y_* \in \mathbb{R}^m$ is the unique minimizer of the compressed sensing problem (1.6) with $b = \Phi y_*$ if and only if $\Phi \in \mathbb{R}^{p \times m}$ satisfies NSP of order L for some $\psi \in (0, 1)$. NSP of order L is implied by the Restricted Isometry Property (RIP) for a sufficiently small RIP parameter δ_{2L} [12], where a matrix $\Phi \in \mathbb{R}^{p \times m}$ is said to satisfy RIP with constant δ_L if [16]

$$(1 - \delta_L) \|y\|_2^2 \leq \|\Phi y\|_2^2 \leq (1 + \delta_L) \|y\|_2^2 \quad \forall L\text{-sparse } y \in \mathbb{R}^m. \quad (\text{II.2})$$

It is known that RIP is satisfied under many distributional hypothesis on the matrix Φ , for example, random matrices Φ with entries i.i.d. Gaussian or Bernoulli random variables are known to satisfy RIP with high probability for $L \leq Cm/\log m$ for a universal constant C ([5], [16], [17], [35]). The general pattern of the proof for establishing that sufficiently sparse y_* is the unique minimizer of problem (1.6) using distributional assumptions on Φ is given in the following program:

$$\begin{array}{c} \text{Dist.} \xrightarrow{[16]} \text{RIP} \xrightarrow{[12]} \text{NSP} \xleftrightarrow{[27][31]} y_* \text{ minimizes (1.6).} \\ \text{Assu.} \end{array} \quad (\text{II.3})$$

We extend this program to the class of robust phase retrieval problems (1.4) and show that, under assumption G, when the residuals $|Ax_*|^p - b$ are sufficiently sparse, the vectors $\pm x_*$ are the global minimizers of the real robust phase retrieval problems (1.4) with high probability. In our program, we substitute NSP and RIP with new properties called the p -Absolute Range Property (p-ARP) and the p -Absolute Growth Property (p-AGP), respectively.

Definition II.1 (p-Absolute Range Property (p-ARP)). *For $p \in \{1, 2\}$, we say $A \in \mathbb{R}^{m \times n}$ satisfies the p -Absolute Range Property of order L_p for $\psi_p \in (0, 1)$ if, for any $x, y \in \mathbb{R}^n$ and for any $T \subseteq [m]$ with $|T| \leq L_p$,*

$$\begin{aligned} \|(|Ax|^p - |Ay|^p)_T\|_1 &\leq \psi_p \|(|Ax|^p - |Ay|^p)_{T^c}\|_1 \\ \forall x, y \in \mathbb{R}^n \text{ and } T \subseteq [m] \text{ with } |T| \leq L_p. \end{aligned} \quad (\text{II.4})$$

Definition II.1 only makes sense when m must be significantly larger than n as is illustrated by the following example.

Example II.2. *For $p \in \{1, 2\}$, an example in which ARP does not hold for any order L is $A = I_n$ for any $\psi \in (0, 1)$.*

An example in which ARP of order $L = 3$ holds is $A = (I_n, I_n, I_n)^T$ for any $\psi \in [\frac{1}{2}, 1)$.

The close connection between p -ARP and NSP is seen by observing that Φ satisfies NSP of order L for $\psi \in (0, 1)$ (II.1) if

$$\begin{aligned} \|(Ax - Ay)_T\|_1 &\leq \psi \|(Ax - Ay)_{T^c}\|_1 \\ \forall x, y \in \mathbb{R}^n \text{ and } T \subseteq [m] \text{ with } |T| \leq L, \end{aligned}$$

where the columns of A form a basis of $\text{Null}(\Phi)$.

Definition II.3 (p-Absolute Growth Property (p-AGP)). *For $p \in \{1, 2\}$, we say that the matrix $A \in \mathbb{R}^{m \times n}$ satisfies the p -Absolute Growth Property if there exists constants $0 < \mu_1 < \mu_2 < 2\mu_1$ for which the modulus mapping $\mathcal{A}_p$ satisfies the bi-Lipschitz property*

$$\mu_1 \phi_p(x, y) \leq \frac{1}{m} \| |Ax|^p - |Ay|^p \|_1 \leq \mu_2 \phi_p(x, y) \quad \forall x, y \in \mathbb{R}^n, \quad (\text{II.5})$$

where

$$\begin{aligned} \forall x, y \in \mathbb{R}^n, \phi_2(x, y) &:= \|xx^T - yy^T\|_F \text{ and} \\ \phi_1(x, y) &:= \min\{\|x + y\|, \|x - y\|\} \end{aligned} \quad (\text{II.6})$$

This definition for ϕ_1 coincides with (1.2) in the real case. The mapping ϕ_2 is called the *matrix-norm* metric in [4]. The matrix-norm metric is the most common and straightforward to use when $p = 2$ since $\phi_2(x, y) = \| |x|^2 - |y|^2 \|_1$.

The relationship between RIP and p-AGP is seen by comparing (II.2) with (II.5). The essential difference is that RIP for compressed sensing applies to any selection of L columns from Φ where L is considered to be small since it determines the sparsity of the solution. On the other hand, our p-AGP applies to the rows of A corresponding to the zero entries in the sparse residual vector $|Ax_*|^p - b$.

Our program of proof tracks that for compressed sensing (II.3) as follows.

For $p = 2$,

$$G \xrightarrow{\text{Lem IV.1}} \text{RIP} \Rightarrow 2\text{-AGP} \xrightarrow{\text{Lem IV.2}} 2\text{-ARP} \xrightarrow{\text{Thm III.2}} x_* \text{ minimizes } f_2(x).$$

For $p = 1$,

$$G \xrightarrow{\text{Lem IV.7}} 1\text{-AGP} \xrightarrow{\text{Lem IV.8}} 1\text{-ARP} \xrightarrow{\text{Thm III.3}} x_* \text{ minimizes } f_1(x).$$

III. GLOBAL MINIMIZATION UNDER P-ARP

In this section we parallel the discussion given in [12] and [23] with NSP replaced by p-ARP. The first step is to introduce a measure of residual sparsity. Given $y \in \mathbb{R}^n$, let $T \subseteq [m]$ be the set of indices corresponding to the L largest entries in the residual vector $\| |Ax|^p - b \|_1$ and define

$$\sigma_L^p(x) := \|(|Ax|^p - b)_{T^c}\|_1.$$

Note that $\sigma_L^p(x) = 0$ if and only if $\| |Ax|^p - b \|_0 \leq L$.

Lemma III.1. *Let $A \in \mathbb{R}^{m \times n}$, $p \in \{1, 2\}$ and $L \in (0, m)$. If the matrix A satisfies p -ARP of order L for $\psi \in (0, 1)$, then*

$$\begin{aligned} \| |Ax|^p - |Ay|^p \|_1 &\leq \frac{1 + \psi}{1 - \psi} (\| |Ax|^p - b \|_1 - \| |Ay|^p - b \|_1 \\ &\quad + 2\sigma_L^p(y)), \end{aligned} \quad (\text{III.1})$$

for all $x, y \in \mathbb{R}^n$.

Proof. For either $p = 1$ or $p = 2$, let T be the indices of the L largest entries in $\| |Ax|^p - b \|_1$. Then

$$\begin{aligned} \|(|Ax|^p - |Ay|^p)_{T^c}\|_1 &\leq \|(|Ax|^p - b)_{T^c}\|_1 + \|(|Ay|^p - b)_{T^c}\|_1 \\ &= \| |Ax|^p - b \|_1 - \|(|Ax|^p - b)_T\|_1 + \sigma_L^p(y) \\ &= \|(|Ay|^p - b)_T\|_1 - \|(|Ax|^p - b)_T\|_1 \\ &\quad + \| |Ax|^p - b \|_1 - \| |Ay|^p - b \|_1 + 2\sigma_L^p(y) \\ &\leq \|(|Ax|^p - |Ay|^p)_T\|_1 + \| |Ax|^p - b \|_1 \\ &\quad - \| |Ay|^p - b \|_1 + 2\sigma_L^p(y). \end{aligned} \quad (\text{III.2})$$

By p-ARP,

$$\|(|Ax|^p - |Ay|^p)_T\|_1 \leq \psi \|(|Ax|^p - |Ay|^p)_{T^c}\|_1. \quad (\text{III.3})$$

Consequently, by (III.2) and (III.3),

$$\begin{aligned} \|(|Ax|^p - |Ay|^p)_{T^c}\|_1 &\leq \frac{1}{1-\psi} (\| |Ax|^p - b \|_1 - \| |Ay|^p - b \|_1 \\ &\quad + 2\sigma_L^p(y)). \end{aligned} \quad (\text{III.4})$$

By (III.3), we know

$$\begin{aligned} \| |Ax|^p - |Ay|^p \|_1 &= \|(|Ax|^p - |Ay|^p)_T\|_1 + \|(|Ax|^p - |Ay|^p)_{T^c}\|_1 \\ &\leq (1+\psi) \|(|Ax|^p - |Ay|^p)_{T^c}\|_1. \end{aligned}$$

By combining this with (III.4), we obtain (III.1) which holds true for all $x, y \in \mathbb{R}^n$. $\square$

The main result of this section follows.

Theorem III.2. Let $L \in (0, m)$, $p \in \{1, 2\}$, and suppose $x_* \in \mathbb{R}^n$ is such that $(|Ax_*|^p - b)$ is L sparse. Let the assumptions of Lemma III.1 holds. Then x_* is a global minimizer of the robust phase retrieval problem (I.4). Moreover, for any x ,

$$\| |Ax|^p - |Ax_*|^p \|_1 \leq \frac{2(1+\psi)}{1-\psi} \sigma_L^p(x).$$

If $\tilde{x}$ is another global minimizer, then $|A\tilde{x}| = |Ax_*|$. If it is further assumed that the entries of A are i.i.d. standard Gaussians and $m \geq 2n - 1$, then, with probability 1, x_* is the unique solution of (I.4) up to multiplication by -1 .

Proof. By Lemma III.1, since $\sigma_L^p(x_*) = 0$,

$$\begin{aligned} \| |Ax|^p - |Ax_*|^p \|_1 &\leq \frac{1+\psi}{1-\psi} (\| |Ax|^p - b \|_1 - \| |Ax_*|^p - b \|_1) \\ &\quad \forall x \in \mathbb{R}^n, \end{aligned} \quad (\text{III.5})$$

and so $\| |Ax|^p - b \|_1 \geq \| |Ax_*|^p - b \|_1$ for all x , i.e., x_* is a global minimizer. Again by Lemma III.1,

$$\begin{aligned} \| |Ax|^p - |Ax_*|^p \|_1 &\leq \frac{1+\psi}{1-\psi} (\| |Ax_*|^p - b \|_1 - \| |Ax|^p - b \|_1 \\ &\quad + 2\sigma_L^p(x)) \\ &\leq \frac{2(1+\psi)}{1-\psi} \sigma_L^p(x) \end{aligned} \quad (\text{III.6})$$

Inequality (III.5) also implies that if there is another minimizer $\tilde{x}$, then $|A\tilde{x}| = |Ax_*|$. The final statement on the uniqueness of x_* is established in [2, Corollary 2.6]. $\square$

IV. ASSUMPTION G $\Rightarrow$ P-AGP $\Rightarrow$ P-ARP

We now use assumption G to show that p-AGP holds for A with high probability, and that p-AGP implies p-ARP of order $L := sm$ with high probability for a constant $s \in (0, 1)$. The cases $p = 2$ and $p = 1$ are treated separately since different techniques are required.

The $p = 2$ Case: We begin by re-stating [20, Lemma 1] in our notation, where the conclusion of [20, Lemma 1] is called RIP in [18].

Lemma IV.1 (Assumption G $\Rightarrow$ 2-AGP(RIP)). [20, Lemma 1] Under assumption G, there exists universal constants c_0, c_1, C such that for $\epsilon \in (0, 1)$, if $m > c_0 n \epsilon^{-2} \log \frac{1}{\epsilon}$, then with probability at least $1 - C \exp(-c_1 \epsilon^2 m)$,

$$0.9(1-\epsilon) \|M\|_F \leq \frac{1}{m} \sum_{i=1}^m |A_i M A_i^T| \leq \sqrt{2}(1+\epsilon) \|M\|_F \quad (\text{IV.1})$$

for all symmetric rank-2 matrices M . In particular, if $M = xx^T - yy^T$ and ϕ_2 is as give in (II.6), then inequality (IV.1) implies 2-AGP with $\mu_1 = 0.9(1-\epsilon)$ and $\mu_2 = \sqrt{2}(1+\epsilon)$.

Lemma IV.2 (Assumption G $\Rightarrow$ 2-AGP $\Rightarrow$ 2-ARP). Under assumption G, there exist universal constants $c_0, c_1, C > 0$, $s \in (0, 1)$, $\psi \in (0, 1)$ such that if $m > c_0 n$ and $A \in \mathbb{R}^{m \times n}$ satisfies G, then

$$\begin{aligned} \|(|Ax|^2 - |Ay|^2)_T\|_1 &\leq \psi \|(|Ax|^2 - |Ay|^2)_{T^c}\|_1 \\ &\quad \forall x, y \in \mathbb{R}^n \text{ and } T \subseteq [m] \text{ with } |T| \leq sm \end{aligned}$$

with probability at least $1 - C \exp(-c_1 m)$. Consequently, 2-ARP holds for m with high probability for m sufficiently large.

Proof. We first derive conditions on $\epsilon, s \in (0, 1)$ so that $\psi \in (0, 1)$ exists. To this end let $\epsilon, s \in (0, 1)$ be given. Let $T \subseteq [m]$ be any subset of sm indices and denote by A_{T^c} the $(1-s)m \times n$ sub-matrix of A whose rows correspond to the indices in T^c . With this notation, we have $|A_{T^c} x| = |Ax|_{T^c}$. Also note that the entries of the matrix A_{T^c} satisfy G. By Lemma IV.1, there exist universal constants c_0, c_1, C such that if $m > c_0 n \epsilon^{-2} \log \frac{1}{\epsilon}$, then, for $M = xx^T - yy^T$ and each subset $T \subseteq [m]$ with $|T| = sm$,

$$\begin{aligned} 0.9(1-\epsilon) \|xx^T - yy^T\|_F &\leq \frac{1}{(1-s)m} \|(|Ax|^2 - |Ay|^2)_{T^c}\|_1 \\ &\leq \sqrt{2}(1+\epsilon) \|xx^T - yy^T\|_F \end{aligned} \quad (\text{IV.2})$$

fails to hold with probability no greater than $C \exp(-c_1 \epsilon^2 (1-s)m)$, that is, 2-AGP holds for A_{T^c} . Since there are

$$\binom{m}{(1-s)m} = \binom{m}{sm} \leq \left(e \frac{m}{sm}\right)^{sm} = \left(\frac{e}{s}\right)^{sm}$$

such T 's, the event

$$B := \{(\text{IV.2}) \text{ holds for every } T \subseteq [m] \text{ with } |T| = sm\} \cap \{(\text{IV.1}) \text{ holds}\},$$

satisfies

$$\begin{aligned} \mathbb{P}(B) &\geq 1 - C(e/s)^{sm} \exp(-c_1 \epsilon^2 (1-s)m) - C \exp(-c_1 \epsilon^2 m) \\ &= 1 - C \exp((1+c_1 \epsilon^2)sm + sm \log(\frac{1}{s}) - c_1 \epsilon^2 m) \\ &\quad - C \exp(-c_1 \epsilon^2 m). \end{aligned}$$

Choose $\hat{s} > 0$ so that $(1 + c_1\epsilon^2)\hat{s} + \hat{s}\log(\frac{1}{\hat{s}}) < \frac{c_1}{2}\epsilon^2$. Then, for all $s \in (0, \hat{s})$, $\mathbb{P}(B) \geq 1 - 2C \exp(-(c_1/2)\epsilon^2 m)$. Thus, if event B occurs, we have

$$\begin{aligned} \|(|Ax|^2 - |Ay|^2)_T\|_1 &= \| |Ax|^2 - |Ay|^2 \|_1 - \|(|Ax|^2 - |Ay|^2)_{T^c}\|_1 \\ &\leq \sqrt{2}(1 + \epsilon)m \|xx^T - yy^T\|_F \\ &\quad - 0.9(1 - \epsilon)(1 - s)m \|xx^T - yy^T\|_F \\ &\leq \frac{\sqrt{2}(1 + \epsilon) - 0.9(1 - \epsilon)(1 - s)}{0.9(1 - \epsilon)(1 - s)} \|(|Ax|^2 - |Ay|^2)_{T^c}\|_1, \end{aligned} \quad (\text{IV.3})$$

where the first inequality follows from (IV.1) applied to the first term and (IV.2) applied to the second, and the second inequality follows by (IV.2). Consequently, as long as $s \in (0, \hat{s})$ is chosen so that $\psi := \frac{\sqrt{2}(1 + \epsilon) - 0.9(1 - \epsilon)(1 - s)}{0.9(1 - \epsilon)(1 - s)} < 1$, the conclusion follows. This can be accomplished by choosing ϵ so that $\frac{\sqrt{2}(1 + \epsilon)}{1.8(1 - \epsilon)} < 1$ (or equivalently, $0 < \epsilon < \frac{1.8 - \sqrt{2}}{1.8 + \sqrt{2}}$) and then choosing $s \in (0, \min\{\hat{s}, 1 - \frac{\sqrt{2}(1 + \epsilon)}{1.8(1 - \epsilon)}\})$. $\square$

The $p = 1$ Case: We begin with four technical lemmas whose proofs are given in the appendix (Section VII).

Lemma IV.3. *Under assumption G, there exist universal constants $\tilde{c}_0, \tilde{c}_1, \tilde{C}$ such that for $\epsilon > 0$ sufficiently small, if $m > \tilde{c}_0 n \epsilon^{-4} \log \frac{1}{\epsilon^2}$, then with probability at least $1 - C \exp(-\tilde{c}_1 \epsilon^4 m)$,*

$$(1 - \tilde{\epsilon}) \sqrt{\frac{2}{\pi}} \|h\| \leq \frac{1}{m} \sum_{i=1}^m |A_i h| \leq (1 + \tilde{\epsilon}) \sqrt{\frac{2}{\pi}} \|h\| \quad \forall h \in \mathbb{R}^n. \quad (\text{IV.4})$$

Lemma IV.4. *Under assumption G, there exists universal constants $\tilde{c}_0, \tilde{c}_1, \tilde{C}$ such that for $\tilde{\epsilon}$ sufficiently small, if $m > \tilde{c}_0 n \tilde{\epsilon}^{-4} \log \frac{1}{\tilde{\epsilon}^2}$, then with probability at least $1 - \tilde{C} \exp(-\tilde{c}_1 \tilde{\epsilon}^4 m)$,*

$$\begin{aligned} \frac{1}{m} \sum_{i=1}^m ||A_i x|^2 - |A_i y|^2|^{\frac{1}{2}} &\geq 0.77(1 - \tilde{\epsilon}) \|xx^T - yy^T\|_F^{\frac{1}{2}} \\ &\quad \forall x, y \in \mathbb{R}^n. \end{aligned} \quad (\text{IV.5})$$

Lemma IV.5. *For $x, y \in \mathbb{R}^n$, if $x^T y \geq 0$ (i.e. $\|x - y\| \leq \|x + y\|$), then*

$$\|x + y\| + (\sqrt{2} - 1) \|x - y\| \geq \|x\| + \|y\| \quad (\text{IV.6})$$

Lemma IV.6. *For $x, y \in \mathbb{R}^n$,*

$$\sqrt{2} \|xx^T - yy^T\|_F \geq \|x + y\| \|x - y\| \quad (\text{IV.7})$$

We now show that if the matrix A satisfies assumption G, then it satisfies 1-AGP with high probability.

Lemma IV.7 (Assumption G $\implies$ 1-AGP). *Under assumption G, there exist universal constants $\tilde{c}_0, \tilde{c}_1, \tilde{C} > 0$ such that for $\tilde{\epsilon} > 0$ sufficiently small, if $m > \tilde{c}_0 n \tilde{\epsilon}^{-4} \log \frac{1}{\tilde{\epsilon}^2}$, then with probability at least $1 - \tilde{C} \exp(-\tilde{c}_1 \tilde{\epsilon}^4 m)$,*

$$\mu_1 \phi_1(x, y) \leq \frac{1}{m} ||Ax| - |Ay||_1 \leq \mu_2 \phi_1(x, y) \quad \forall x, y \in \mathbb{R}^n, \quad (\text{IV.8})$$

where $\phi_1(x, y)$ is defined in (II.6), $\mu_1 = \sqrt{\frac{2}{\pi}}(2 - \sqrt{2} - \tilde{\epsilon})$ and $\mu_2 = \sqrt{\frac{2}{\pi}}(1 + \tilde{\epsilon})$. Consequently, 1-AGP holds with high probability for m sufficiently large.

Proof. By Lemma IV.3 and Lemma IV.4, there exist universal constant c_0, c_1, C such that for ϵ sufficiently small, if $m > c_0 n \epsilon^{-4} \log \frac{1}{\epsilon^2}$, then with probability at least $1 - C \exp(-c_1 \epsilon^4 m)$, (IV.4) and (IV.5) hold. Since we can substitute y by $-y$ if necessary, without loss of generality, we assume $\|x - y\| \leq \|x + y\|$.

The right hand inequality in (IV.8) easily follows by (IV.4) and triangle inequality

$$||Ax| - |Ay||_1 \leq \|A(x - y)\|_1.$$

For the left hand inequality in (IV.8), we consider two cases: (1) $\|x - y\| \leq \|x + y\| \leq 10 \|x - y\|$, and (2) $\|x + y\| \geq 10 \|x - y\|$.

(1) Assume $\|x - y\| \leq \|x + y\| \leq 10 \|x - y\|$. Then

$$\begin{aligned} \frac{1}{m} ||Ax| - |Ay||_1 &= \frac{1}{m} \sum_{i=1}^m |A_i x| - |A_i y| \\ &= \frac{1}{m} \sum_{i=1}^m |A_i(x + y)| + \frac{1}{m} \sum_{i=1}^m |A_i(x - y)| \\ &\quad - \frac{1}{m} \sum_{i=1}^m |A_i x| - \frac{1}{m} \sum_{i=1}^m |A_i y| \\ &\geq \sqrt{\frac{2}{\pi}} ((1 - \epsilon) \|x + y\| + (1 - \epsilon) \|x - y\| \\ &\quad - (1 + \epsilon) \|x\| - (1 + \epsilon) \|y\|) \\ &\geq \sqrt{\frac{2}{\pi}} ((2 - \sqrt{2} - \sqrt{2}\epsilon) \|x - y\| - 2\epsilon \|x + y\|) \\ &\geq \sqrt{\frac{2}{\pi}} (2 - \sqrt{2} - (\sqrt{2} + 20)\epsilon) \|x - y\|, \end{aligned} \quad (\text{IV.9})$$

where the second equality is from $|a| - |b| = |a + b| + |a - b| - |a| - |b|$ for $a, b \in \mathbb{R}$, the first inequality is from (IV.4) (with h successively set to $x + y$, $x - y$, x , and y), the second inequality uses (IV.6), and the last inequality follows from our assumption that $\|x + y\| \leq 10 \|x - y\|$.

(2) Assume $\|x + y\| \geq 10 \|x - y\|$. Then

$$\begin{aligned} \frac{1}{m} ||Ax| - |Ay||_1 &= \frac{1}{m} \sum_{i=1}^m |A_i x| - |A_i y| \\ &\geq \left(\frac{1}{m} \sum_{i=1}^m |A_i x|^2 - |A_i y|^2 \right)^{\frac{1}{2}} \left/ \left(\frac{1}{m} \sum_{i=1}^m (|A_i x| + |A_i y|) \right) \right. \\ &\geq \sqrt{\frac{\pi}{2}} \frac{0.77^2 (1 - \epsilon)^2 \|xx^T - yy^T\|_F}{(1 + \epsilon)(\|x\| + \|y\|)} \\ &\geq \frac{0.77^2 \sqrt{\pi} (1 - \epsilon)^2 \|x + y\| \|x - y\|}{2(1 + \epsilon)(\|x\| + \|y\|)} \\ &\geq \frac{0.77^2 \sqrt{\pi} (1 - \epsilon)^2 \|x + y\| \|x - y\|}{2(1 + \epsilon)(\|x + y\| + (\sqrt{2} - 1) \|x - y\|)} \\ &\geq \frac{5 \cdot 0.77^2 \sqrt{\pi} (1 - \epsilon)^2}{(\sqrt{2} + 9)(1 + \epsilon)} \|x - y\|, \end{aligned} \quad (\text{IV.10})$$

where the first inequality is by Cauchy-Schwarz inequality applied to the vectors with $u_i = ||A_i x| - |A_i y||^{\frac{1}{2}}$ and $v_i = ||A_i x| + |A_i y||^{\frac{1}{2}}$, the second inequality is by Lemma IV.4 and Lemma IV.3, the third inequality is by Lemma IV.6, the fourth inequality is by Lemma IV.5 and the last inequality follows from our assumption that $\|x + y\| \geq 10\|x - y\|$. When $0 < \epsilon < 0.01$, one can show by direct computation that

$$\frac{5 \cdot 0.77^2 \sqrt{\pi}(1 - \epsilon)^2}{(\sqrt{2} + 9)(1 + \epsilon)} > 0.02 + \sqrt{\frac{2}{\pi}}(2 - \sqrt{2}),$$

and so

$$\frac{1}{m} |||Ax| - |Ay|||_1 \geq \sqrt{\frac{2}{\pi}}(2 - \sqrt{2})\|x - y\| \quad (\text{IV.11})$$

Consequently, when $0 < \epsilon < 0.01$,

$$\frac{1}{m} |||Ax| - |Ay|||_1 \geq \sqrt{\frac{2}{\pi}} \left(2 - \sqrt{2} - (20 + \sqrt{2})\epsilon\right) \|x - y\|.$$

By substituting $\tilde{\epsilon}/(\sqrt{2} + 20)$ for ϵ and adjusting c_0 and c_1 accordingly, we arrived at the desired result. $\square$

Lemma IV.8 (Assumption G $\implies$ 1-AGP $\implies$ 1-ARP). *Under assumption G, there exist universal constants $c_0, c_1, C > 0, s \in (0, 1), \psi \in (0, 1)$ such that if $m > c_0 n$, then*

$$\begin{aligned} ||(|Ax| - |Ay|)_T||_1 &\leq \psi ||(|Ax| - |Ay|)_{T^c}||_1 \\ \forall x, y \in \mathbb{R}^n \text{ and } T \subseteq [m] \text{ with } |T| &\leq sm \end{aligned}$$

holds with probability at least $1 - C \exp(-c_1 m)$. Consequently, 1-ARP holds with high probability for m sufficiently large.

Proof. The proof strategy is similar to Lemma IV.2. Let $\phi_1(x, y)$ be as defined in (II.6). We begin by deriving conditions on $\epsilon, s \in (0, 1)$ so that $\psi \in (0, 1)$ exists. Let c_0, c_1, C be the universal constants given by Lemma IV.7 so that the implications of the lemma hold for $\epsilon \in (0, 1)$. Then choose $s \in (0, 1)$ and m so that $(1 - s)m > c_0 n \epsilon^{-4} \log \frac{1}{\epsilon^2}$. Then, for any $x, y \in \mathbb{R}^n$ and each subset $T \subseteq [m]$ with $|T| = sm$, the double sided inequality

$$\begin{aligned} \sqrt{\frac{2}{\pi}}(2 - \sqrt{2} - \epsilon)\phi_1(x, y) &\leq \frac{1}{(1 - s)m} ||(|Ax| - |Ay|)_{T^c}||_1 \\ &\leq \sqrt{\frac{2}{\pi}}(1 + \epsilon)\phi_1(x, y) \end{aligned} \quad (\text{IV.12})$$

fails to hold with probability no larger than $C \exp(-c_1 \epsilon^4(1 - s)m)$, that is, 1-AGP holds for A_{T^c} . Consider the event

$$\mathcal{B} := \{(\text{IV.12}) \text{ holds for every } T \subseteq [m] \text{ with } |T| = sm\} \cap \{(\text{IV.8}) \text{ holds}\}.$$

By taking s sufficiently small, there exist positive constants $\tilde{c}$ and $\tilde{C}$ such that $\mathbb{P}(\mathcal{B}) \geq 1 - \tilde{C} \exp(-\tilde{c}\epsilon^4 m)$. On the event $\mathcal{B}$, we obtain

$$\begin{aligned} ||(|Ax| - |Ay|)_T||_1 &= |||Ax| - |Ay|||_1 - ||(|Ax| - |Ay|)_{T^c}||_1 \\ &\leq \sqrt{\frac{2}{\pi}}(1 + \epsilon)m\phi_1(x, y) - \sqrt{\frac{2}{\pi}}(2 - \sqrt{2} - \epsilon)(1 - s)m\phi_1(x, y) \\ &\leq \frac{\sqrt{2} - 1 + 2\epsilon}{(2 - \sqrt{2} - \epsilon)(1 - s)} ||(|Ax| - |Ay|)_{T^c}||_1 \end{aligned}$$

Therefore, if $\epsilon, s \in (0, 1)$ are such that $\psi := \frac{\sqrt{2} - 1 + 2\epsilon}{(2 - \sqrt{2} - \epsilon)(1 - s)} < 1$, the conclusion follows. In particular, this inequality holds if $0 < \epsilon < \frac{3 - 2\sqrt{2}}{3}$ and $0 < s < \frac{3(1 - \epsilon) - 2\sqrt{2}}{2 - \sqrt{2} - \epsilon}$, where $\frac{3 - 2\sqrt{2}}{3} \geq 0.05$. $\square$

By combining the results of this section with those of Section III, we find that, under assumption G, the solutions to the ℓ_0 optimization problem (I.5) and ℓ_1 optimization problem (I.4) coincide with high probability when the residuals are sufficiently sparse.

V. SHARPNESS

Sharpness is an extremely useful tool for analyzing the convergence and the rate of convergence of optimization algorithms ([7], [9], [10], [11], [18], [24], [28]).

Definition V.1. [11] Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and set $\mathcal{X} := \text{argmin } f$. Then f is said to be sharp with respect to $\mathcal{X}$ if

$$f(x) \geq \min_x f + \mu \text{dist}(x | \mathcal{X}) \quad \forall x \in \mathbb{R}^n,$$

where $\text{dist}(x | \mathcal{X}) := \inf_{y \in \mathcal{X}} \|x - y\|$.

In our context, good convergence rates for methods solving (I.4) often require that the objective function f_p satisfies a sharpness condition. In this section we show that, under assumption G, if $|Ax_*|^p - b$ is sufficiently sparse, then the function

$$f_p(x) := \frac{1}{m} |||Ax|^p - b||_1$$

is sharp with respect to the solution set $\{x_*, -x_*\}$ with high probability, for $p = 1, 2$.

Theorem V.2. Let assumption G hold and let $p \in \{1, 2\}$. Then there exist constants $C_p, c_{p0}, c_{p1} > 0$ and $s_p \in (0, 1)$, such that if $|||Ax_*|^p - b||_0 \leq s_p m$, then, for $m \geq c_{p0} n$, f_p is sharp with probability at least $1 - C_p \exp(-c_{p1} m)$.

Proof. Let c_0, c_1 , and C be the maximum and $s, \epsilon \in (0, 1)$ the smallest of the corresponding constants from Lemmas IV.1, IV.2 and IV.8 so that the implications of all three of these results stand for these values. By either Lemma IV.2 ($p = 2$) or Lemma IV.8 ($p = 1$), A satisfies p -ARP of order sm for $\psi_p \in (0, 1)$ for $p = 1, 2$, where ψ_p are constants depending on p . Since $\sigma_{sm}(x_*) = 0$, Lemma III.1 tells us that

$$f_p(x) - f_p(x_*) \geq \frac{1 - \psi_p}{1 + \psi_p} |||Ax|^2 - |Ax_*|^2||_1. \quad (\text{V.1})$$

For $p = 2$, Lemma IV.1 tells us that if $m \geq c_0 \epsilon^{-2} \log(\frac{1}{\epsilon})n$, then, with probability at least $1 - C \exp(-c_1 \epsilon^{-2} \log \frac{1}{\epsilon} m)$,

$$\begin{aligned} \frac{1}{m} |||Ax|^2 - |Ax_*|^2||_1 &\geq 0.9(1 - \epsilon) \|xx^T - yy^T\|_F \\ &\geq 0.45\sqrt{2}(1 - \epsilon) \|x + x_*\| \|x - x_*\| \\ &= 0.45\sqrt{2}(1 - \epsilon) \max\{\|x - x_*\|, \|x + x_*\|\} \phi_1(x, x_*) \\ &\geq 0.45\sqrt{2}(1 - \epsilon) \|x_*\| \text{dist}(x | \{x_*, -x_*\}), \end{aligned} \quad (\text{V.2})$$

where $\phi_1(x, x_*)$ is defined in (II.6), the second inequality follows from Lemma IV.6, and the final inequality follows from the triangle inequality which tells us that

$\max\{\|x - x_*\|, \|x + x_*\|\} \geq \max\{\|x\|, \|x_*\|\}$. For $p = 1$, Lemma IV.7 tells us that, if $m \geq c_0 \epsilon^{-4} \log(\frac{1}{\epsilon^2})n$, then, with probability at least $1 - C \exp(-c_1 \epsilon^{-4} \log \frac{1}{\epsilon^2} m)$,

$$\frac{1}{m} \| |Ax| - |Ax_*| \|_1 \geq \sqrt{\frac{2}{\pi}} (2 - \sqrt{2} - \epsilon) \text{dist}(x | \{x_*, -x_*\}).$$

Thus, in either case, by taking an $0 < \epsilon < 1$ small enough and using (V.1), there is constant $\mu > 0$ such that

$$f_p(x) - f_p(x_*) \geq \mu \text{dist}(x | \mathcal{X}),$$

where $\mathcal{X}$ is $\arg\min f_p$. $\square$

It is shown in [18] and [28] that if f_2 is sharp and weakly convex at $\arg\min f_2$, then prox-linear method and subgradient descent method with geometrically decreasing stepsize converges locally quadratically and locally linearly, respectively. Since weak convexity of f_2 under assumption G ([28], [18], [24]), sharpness in this regime guarantees these two algorithms converge with the specified rate. In both algorithms proper initialization is needed (e.g., Section 5 of [39]).

VI. CONCLUDING REMARKS

There are several recent results discussing the nature of the solution set to the robust phase retrieval problem $\min_x f_2(x)$ with sparse noise under weaker distributional hypotheses than employed here ([18], [28], [39], [19]). The focus of these works are algorithmic. Their goal is to show their methods are robust to outliers, and, in addition, some establish the sharpness of f_2 in order to prove rates of convergence ([18], [28]). Although these works use weaker distributional hypotheses, the probability of successful recovery is an average over all possible subsets $T \subseteq [m]$ with $|T| = sm$ for some $s \in (0, \frac{1}{2})$. Consequently, the value of s in their results is larger than ours. The reason for this difference is that, in our result, successful recovery is valid for all possible subsets $T \subseteq [m]$ with $|T| = sm$ for some $s \in (0, 1)$, with uniformly high probability. A more precise description of difference between these results follows.

In [18] and [28], the random matrix A and the random index set $T \subseteq [m]$, with $|T| = sm$ for $s \in (0, \frac{1}{2})$, are drawn independently of each other. Let $w \in \{0, 1\}^m$ denote the random indicator vector of T , that is, $w_i = 1$ if $i \in T$ and $w_i = 0$ otherwise. Let $z \in \mathbb{R}^m$ be an arbitrary vector. The noisy model in [18] and [28] has the form

$$\min_x \tilde{f}_2(x) := \left\| |Ax|^2 - (\vec{1} - w) \odot b - w \odot z \right\|_1,$$

where $b = |Ax_*|^2$, $\vec{1}$ represents the vector with 1 in each entry and $\odot$ represents the elementwise product of vectors. The authors in [18] and [28] prove sharpness of $\tilde{f}_2$ with respect to x_* with high probability. Due to the independence of A and T , these results show that the probability

$$\mathbb{P}(\tilde{f}_2 \text{ is sharp}) = \frac{1}{\binom{m}{sm}} \sum_{T_0: |T_0|=sm} \mathbb{P}(\tilde{f}_2^{T_0} \text{ is sharp})$$

is high, where $\tilde{f}_2^{T_0}(x) := \left\| |Ax|^2 - (\vec{1} - w_0) \odot b - w_0 \odot z \right\|_1$ and w_0 is the indicator vector for a fixed index set T_0 . On the

other hand, we show that with high probability, $\tilde{f}_2^{T_0}$ is sharp for all possible T_0 with $|T_0| = sm$. However, this stronger implication comes at the expense of a smaller value for s . By design, our stronger result closely parallels the result in [16] for compressed sensing.

VII. APPENDIX

The proofs for Lemmas IV.3, IV.4, IV.5, and IV.6 are given below. These proofs make use of a Hoeffding-type inequality [36] explained below. A random variable X is said to be sub-gaussian [36, Definition 5.7] if

$$\|X\|_{\psi_2} := \sup_{p \geq 1} p^{-1/2} (\mathbb{E}|X|^p)^{1/p} \quad (\text{VII.1})$$

is finite, and is said to be centered if it has zero expectation. By [36, Proposition 5.10], there is a universal constant $c > 0$ such that if $X_1, \dots, X_N$ are independent centered sub-Gaussian random variables, then, for every $a = \{a_1, \dots, a_N\} \in \mathbb{R}^N$ and $t \geq 0$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^N a_i X_i\right| \geq t\right) \leq e \cdot \exp\left(-\frac{ct^2}{K^2 \|a\|^2}\right), \quad (\text{VII.2})$$

where $K := \max_i \|X_i\|_{\psi_2}$.

Proof of Lemma IV.3: First observe that the inequality (IV.4) is trivially true for $h = 0$. Next, let $h \in \mathbb{R}^n \setminus \{0\}$ and $0 < \epsilon < \sqrt{2} - 1$. Observe that $\frac{|A_i h|}{\|h\|}$ are independent sub-gaussian random variables with mean $\sqrt{\frac{2}{\pi}}$. Therefore, $\frac{|A_i h|}{\|h\|} - \sqrt{\frac{2}{\pi}}$ is a centered sub-gaussian random variable. Hence, (VII.2) tells us that there are universal constants $\hat{C} > 0$ and $\hat{c} > 0$ such that

$$\mathbb{P}\left(\left|\sum_{i=1}^m \left(\frac{|A_i h|}{\|h\|} - \sqrt{\frac{2}{\pi}}\right)\right| > m \sqrt{\frac{2}{\pi}} \epsilon\right) \leq \hat{C} \exp(-\hat{c} m \epsilon^2). \quad (\text{VII.3})$$

Therefore (IV.4) holds for each fixed $h \in \mathbb{R}^n \setminus \{0\}$ with probability $1 - \hat{C} \exp(-\hat{c} m \epsilon^2)$. We now show that there exist a universal event with large probability, in which (IV.4) holds for every h . On the unit sphere $S := \{x | \|x\| = 1\} \subset \mathbb{R}^n$ construct an ϵ -net $\mathcal{N}_\epsilon$ with $|\mathcal{N}_\epsilon| \leq (1 + \frac{2}{\epsilon})^n$ [36, Lemma 5.2], i.e., for any $h \in S$, there exists $h_0 \in \mathcal{N}_\epsilon \subseteq S$ such that $\|h - h_0\| \leq \epsilon$. Taking the probability of the union of the events in (VII.3) for all the points $h_0 \in \mathcal{N}_\epsilon$, we obtain the bound $\hat{C}(1 + \frac{2}{\epsilon})^n \exp(-\hat{c} m \epsilon^2)$. Hence, (IV.4) holds for each $h_0 \in \mathcal{N}_\epsilon$ with probability at least $1 - \hat{C}(1 + \frac{2}{\epsilon})^n \exp(-\hat{c} m \epsilon^2)$. Let $h_0 \in \mathcal{N}_\epsilon$ and $h \in S$ be such that $\|h - h_0\| \leq \epsilon$, and let c_0, c_1 , and C be the universal constants from Lemma IV.1. If $m \geq c_0 n \epsilon^{-2} \log(\frac{1}{\epsilon})$, then, with probability at least

$$1 - C \exp(-c_1 m \epsilon^2),$$

$$\begin{aligned} \frac{1}{m} \left| \sum_{i=1}^m |A_i h| - \sum_{i=1}^m |A_i h_0| \right| &\leq \frac{1}{m} \sum_{i=1}^m ||A_i h| - |A_i h_0|| \\ &\leq \frac{1}{m} \sum_{i=1}^m ||A_i h|^2 - |A_i h_0|^2|^{\frac{1}{2}} \\ &\leq \left(\frac{1}{m} \sum_{i=1}^m ||A_i h|^2 - |A_i h_0|^2| \right)^{\frac{1}{2}} \\ &\leq 2^{1/4} (1 + \epsilon)^{1/2} \|h h^T - h_0 h_0^T\|_F^{\frac{1}{2}} \\ &\leq 2^{1/4} (1 + \epsilon)^{1/2} (\|h - h_0\| \|h\| + \|h - h_0\| \|h_0\|)^{\frac{1}{2}} \\ &\leq 2\epsilon^{1/2}, \end{aligned} \quad (\text{VII.4})$$

where the second inequality follows since $||a| - |b|| \leq (|a| + |b|)|a| - |b||$, the third from the concavity of $(\cdot)^{1/2}$, the fourth is by Lemma IV.1, the fifth is by triangle inequality and the last inequality is from $\|h\| = \|h_0\| = 1$ and $\|h - h_0\| \leq \epsilon$. Therefore, on the intersection of these events we deduce that if $m \geq c_0 n \epsilon^{-2} \log(\frac{1}{\epsilon})$, then

$$(1 - (\epsilon + \sqrt{2\pi\epsilon})) \sqrt{2/\pi} \leq \frac{1}{m} \sum_{i=1}^m |A_i h| \leq (1 + (\epsilon + \sqrt{2\pi\epsilon})) \sqrt{2/\pi}$$

holds for all $h \in S$ with probability at least $1 - \widehat{C}(1 + \frac{2}{\epsilon})^n \exp(-\widehat{c} m \epsilon^2) - C \exp(-c_1 m \epsilon^2)$ for $m \geq c_1 n \epsilon^{-2} \log(\frac{1}{\epsilon})$. Since $0 < \epsilon < \sqrt{2} - 1$, we have $\epsilon^2 + 2\epsilon - 1 \leq 0$, or equivalently, $\log(1 + \frac{2}{\epsilon}) \leq 2 \log(1/\epsilon)$. Let $\bar{c}_0 > \max\{c_0, 2/\widehat{c}\}$ and set $\hat{c}_1 := \widehat{c} - 2/\bar{c}_0$ (which is positive because $\bar{c}_0 > 2/\widehat{c}$, or equivalently, $\widehat{c} > 2/\bar{c}_0$). Then, for $m \geq \bar{c}_0 n \epsilon^{-2} \log(\frac{1}{\epsilon})$, or equivalently, $m \epsilon^2 / \bar{c}_0 \geq n \log(1/\epsilon)$,

$$\begin{aligned} &1 - \widehat{C}(1 + 2/\epsilon)^n \exp(-\widehat{c} m \epsilon^2) - C \exp(-c_1 m \epsilon^2) \\ &= 1 - \widehat{C} \exp(-\widehat{c} m \epsilon^2 + n \log(1 + 2/\epsilon)) - C \exp(-c_1 m \epsilon^2) \\ &\geq 1 - \widehat{C} \exp(-\widehat{c} m \epsilon^2 + 2n \log(\frac{1}{\epsilon})) - C \exp(-c_1 m \epsilon^2) \\ &\geq 1 - \widehat{C} \exp(-\widehat{c} m \epsilon^2 + 2m \epsilon^2 / \bar{c}_0) - C \exp(-c_1 m \epsilon^2) \\ &= 1 - C \exp(-c_1 m \epsilon^2) - \widehat{C} \exp(-(\widehat{c} - \frac{2}{\bar{c}_0}) m \epsilon^2) \\ &= 1 - C \exp(-c_1 m \epsilon^2) - \widehat{C} \exp(-\hat{c}_1 m \epsilon^2) \\ &\geq 1 - \widetilde{C} \exp(-\bar{c}_1 m \epsilon^2), \end{aligned} \quad (\text{VII.5})$$

where $\widetilde{C} = 2 \max\{\widehat{C}, C\}$ and $\bar{c}_1 := \min\{c_1, \hat{c}_1\}$. Define $\tilde{\epsilon} := (1 + \sqrt{2\pi}) \epsilon^{1/2}$ so that $\epsilon = (1 + \sqrt{2\pi})^{-2} \tilde{\epsilon}^2$ and

$$(1 + \sqrt{2\pi})^{-1} \tilde{\epsilon}^2 = (1 + \sqrt{2\pi}) \epsilon \leq (\epsilon + \sqrt{2\pi\epsilon}) \leq (1 + \sqrt{2\pi}) \epsilon^{1/2} = \tilde{\epsilon}.$$

Also, for $\epsilon < e^{-1}$, $\log(\epsilon^{-1}) = \log(\frac{(1 + \sqrt{2\pi})^2}{\tilde{\epsilon}^2}) \leq (1 + \log(1 + \sqrt{2\pi})) \log(\tilde{\epsilon}^{-2})$ (here we use the fact that if $\alpha > 1$, $\beta < e^{-1}$, then $\ln(\alpha/\beta) \leq (1 + \ln(\alpha)) \ln(1/\beta)$). If we now set $\tilde{c}_1 := (1 + \sqrt{2\pi})^{-4} \bar{c}_1$ and $\tilde{c}_0 := \bar{c}_0 (1 + \sqrt{2\pi})^4 (1 + \log(1 + \sqrt{2\pi}))^2$, we obtain the result. $\square$

Proof of Lemma IV.4: We begin by showing that for all rank-2 matrix M

$$\frac{1}{m} \sum_{i=1}^m |A_i M A_i^T|^{\frac{1}{2}} \geq 0.77(1 - \epsilon) \|M\|_F^{\frac{1}{2}} \quad (\text{VII.6})$$

with high probability for all ϵ sufficiently small. Let $\epsilon \in (0, 1/9)$. Clearly inequality (VII.6) holds when $M = 0$ regardless of the value of $\epsilon \in (0, 1/9)$. Assume $M \neq 0$. Since we can divide (VII.6) by the square-root of the spectral norm $\|M\|^{\frac{1}{2}}$, we can assume $\|M\| = 1$. Using the eigenvalue decomposition of M , we can assume that $M = z_1 z_1^T - s z_2 z_2^T$ where $z_1^T z_2 = 0$, $\|z_1\| = \|z_2\| = 1$ and $s \in [-1, 1]$. Since for each i , $A_i z_1$ and $A_i z_2$ are independent standard Gaussians and

$$\begin{aligned} |A_i M A_i^T|^{\frac{1}{2}} &= |(A_i z_1)^2 - s(A_i z_2)^2|^{\frac{1}{2}} \\ &\leq ((A_i z_1)^2 + (A_i z_2)^2)^{\frac{1}{2}} \leq |A_i z_1| + |A_i z_2|, \end{aligned} \quad (\text{VII.7})$$

the random variables $|A_i M A_i^T|^{\frac{1}{2}}$ are sub-Gaussian. Set $e(s) := \mathbb{E} |A_i M A_i^T|^{\frac{1}{2}} = \mathbb{E} |Z_1^2 - s Z_2^2|^{\frac{1}{2}}$ where Z_1 and Z_2 are independent standard Gaussian scalar random variables. Notice $\|M\|_F = \|z_1 z_1^T - s z_2 z_2^T\|_F = \sqrt{1 + s^2}$ and

$$\begin{aligned} e(s) &= \mathbb{E} |Z_1^2 - s Z_2^2|^{\frac{1}{2}} \\ &= \frac{1}{2\pi} \int_0^\infty r^2 e^{-\frac{r^2}{2}} dr \int_0^{2\pi} |\cos^2 \theta - s \sin^2 \theta|^{-\frac{1}{2}} d\theta \\ &= \frac{1}{2\sqrt{2\pi}} \int_0^{2\pi} |\cos^2 \theta - s \sin^2 \theta|^{\frac{1}{2}} d\theta. \end{aligned} \quad (\text{VII.8})$$

In Figure 1, we plot

$$\frac{e(s)}{\|M\|_F^{1/2}} = \int_0^{2\pi} |\cos^2 \theta - s \sin^2 \theta|^{\frac{1}{2}} d\theta / (2\sqrt{2\pi}(1 + s^2)^{1/4})$$

for $s \in [-1, 1]$.

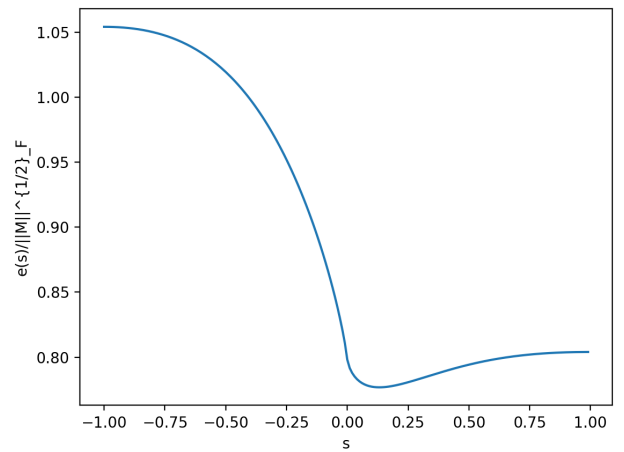


Fig. 1: Values of $\frac{e(s)}{\|M\|_F^{1/2}}$ when $s \in [-1, 1]$.

Numerically we find that $\frac{e(s)}{\|M\|_F^{1/2}} \geq 0.77$ for all $s \in [-1, 1]$.

Note that for each $i = 1, \dots, m$, $Y_i := \frac{|A_i M A_i^T|^{\frac{1}{2}}}{e(s)} - 1$ is a

centered sub-Gaussian random variable, since, by (VII.1) and (VII.7),

$$\begin{aligned} \|Y_i\|_{\psi_2} &\leq \sup_{p \geq 1} p^{-\frac{1}{2}} \left(\frac{2(\mathbb{E}|Z|^p)^{\frac{1}{p}}}{e(s)} + 1 \right) \\ &\leq \frac{2}{0.77 \|M\|_F^{1/2}} \|Z\|_{\psi_2} + 1 < +\infty, \end{aligned} \quad (\text{VII.9})$$

where Z is a standard Gaussian variable. Hence, (VII.2) tells us that there exist universal constants $\hat{C} > 0$ and $\hat{c}_1 > 0$ such that

$$\mathbb{P} \left(\left| \sum_{i=1}^m \left(\frac{|A_i M A_i^T|^{\frac{1}{2}}}{e(s)} - 1 \right) \right| > mt \right) \leq \hat{C} \exp(-\hat{c}_1 m t^2) \quad \forall t \geq 0. \quad (\text{VII.10})$$

Consequently, for fixed M and $t = \epsilon$,

$$\frac{1}{m} \sum_{i=1}^m |A_i M A_i^T|^{\frac{1}{2}} \geq (1 - \epsilon) e(s) \geq 0.77(1 - \epsilon) \|M\|_F^{1/2} \quad (\text{VII.11})$$

holds with probability at least $1 - \hat{C} \exp(-\hat{c}_1 m \epsilon^2)$.

Next we generalize (VII.11) to all rank-2 matrices M . Again, by scale invariance, we assume $\|M\|_F = 1$. Consequently, we only need to prove (VII.11) holds with high probability for all $M \in \mathcal{M} := \{\beta u u^T + \gamma v v^T \mid \|u\| = \|v\| = 1, u^T v = 0 \text{ and } \beta^2 + \gamma^2 = 1\}$. Set $\mathcal{S}_{\epsilon^2} := \mathcal{T}_{\epsilon^2} \times \mathcal{N}_{\epsilon^2} \times \mathcal{N}_{\epsilon^2}$ where $\mathcal{T}_{\epsilon^2}$ is an ϵ^2 -net of $[-1, 1]$ and $\mathcal{N}_{\epsilon^2}$ is an ϵ^2 -net of the unit sphere $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$. Since $|\mathcal{T}_{\epsilon^2}| \leq \frac{2}{\epsilon^2}$ and $|\mathcal{N}_{\epsilon^2}| \leq \left(\frac{3}{\epsilon^2}\right)^n$, we know $|\mathcal{S}_{\epsilon^2}| \leq \left(\frac{3}{\epsilon^2}\right)^{2n+1} \leq \left(\frac{3}{\epsilon^2}\right)^{4n+2}$. Let $\mathcal{E}$ denote the event that (VII.11) holds for every $(\beta_0, u_0, v_0) \in \mathcal{S}_{\epsilon^2}$. Consequently,

$$\begin{aligned} \mathbb{P}(\mathcal{E}) &\geq 1 - \hat{C}(3/\epsilon)^{4n+2} \exp(-\hat{c}_1 m \epsilon^2) \\ &= 1 - \hat{C} \exp(-\hat{c}_1 m \epsilon^2 + (4n+2) \log(3/\epsilon)). \end{aligned}$$

Since $\epsilon \leq 1/9$, $3n \log(1/\epsilon) \geq (4n+2) \log(3/\epsilon)$, and so

$$\mathbb{P}(\mathcal{E}) \geq 1 - \hat{C} \exp(-\hat{c}_1 m \epsilon^2 + 3n \log(1/\epsilon)). \quad (\text{VII.12})$$

For $M \in \mathcal{M}$, we want to approximate $M = \beta u u^T + \gamma v v^T$ by an element $M_0 = \beta_0 u_0 u_0^T + \gamma_0 v_0 v_0^T \in \mathcal{M}$ with $(\beta_0, u_0, v_0) \in \mathcal{S}_{\epsilon^2}$. More precisely, let $(\beta_0, u_0, v_0) \in \mathcal{S}_{\epsilon^2}$ and $M_0 = \beta_0 u_0 u_0^T + \text{sgn}(\gamma) \sqrt{1 - \beta_0^2} v_0 v_0^T$ be such that $|\beta - \beta_0| \leq \epsilon^2$, $\|u - u_0\| \leq \epsilon^2$ and $\|v - v_0\| \leq \epsilon^2$. Consequently, we have

$$\begin{aligned} |\gamma - \text{sgn}(\gamma) \sqrt{1 - \beta_0^2}| &= |\sqrt{1 - \beta^2} - \sqrt{1 - \beta_0^2}| \\ &\leq |\beta^2 - \beta_0^2|^{\frac{1}{2}} \leq \sqrt{2} |\beta - \beta_0|^{\frac{1}{2}} \leq \sqrt{2} \epsilon. \end{aligned}$$

Also note that

$$\begin{aligned} \|\beta u u^T - \beta_0 u_0 u_0^T\|_F &\leq |\beta - \beta_0| \|u u^T\|_F + \|\beta_0 u(u - u_0)^T\|_F \\ &\quad + \|\beta_0(u - u_0)u_0^T\|_F \\ &= |\beta - \beta_0| \|u\|^2 + |\beta_0| \|u - u_0\| (\|u\| + \|u_0\|) \\ &\leq 3\epsilon^2 < 4\epsilon \end{aligned} \quad (\text{VII.13})$$

Similarly $\|\gamma v v^T - \text{sgn}(\gamma) \sqrt{1 - \beta_0^2} v_0 v_0^T\| \leq 2\epsilon^2 + 2\epsilon < 4\epsilon$. Let C , c_0 , and c_1 be the universal constants from Lemma IV.1.

Then, on the intersection of events where (IV.1) holds and $\mathcal{E}$ we have

$$\begin{aligned} &\left| \frac{1}{m} \sum_{i=1}^m |A_i M A_i^T|^{\frac{1}{2}} - \frac{1}{m} \sum_{i=1}^m |A_i M_0 A_i^T|^{\frac{1}{2}} \right| \\ &\leq \frac{1}{m} \sum_{i=1}^m \left| |A_i M A_i^T|^{\frac{1}{2}} - |A_i M_0 A_i^T|^{\frac{1}{2}} \right| \\ &\leq \frac{1}{m} \sum_{i=1}^m \left| |A_i M A_i^T| - |A_i M_0 A_i^T| \right|^{\frac{1}{2}} \\ &\leq \left(\frac{1}{m} \sum_{i=1}^m \left| |A_i M A_i^T| - |A_i M_0 A_i^T| \right| \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{m} \sum_{i=1}^m |A_i (M - M_0) A_i^T| \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{m} \sum_{i=1}^m |A_i (\beta u u^T - \beta_0 u_0 u_0^T) A_i^T| + |A_i (\gamma v v^T - \gamma_0 v_0 v_0^T) A_i^T| \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{m} \sum_{i=1}^m |A_i (\beta u u^T - \beta_0 u_0 u_0^T) A_i^T| \right)^{\frac{1}{2}} \\ &\quad + \left(\frac{1}{m} \sum_{i=1}^m |A_i (\gamma v v^T - \gamma_0 v_0 v_0^T) A_i^T| \right)^{\frac{1}{2}} \\ &\leq 2^{\frac{1}{4}} (1 + \epsilon)^{\frac{1}{2}} \|\beta u u^T - \beta_0 u_0 u_0^T\|_F^{\frac{1}{2}} \\ &\quad + 2^{\frac{1}{4}} (1 + \epsilon)^{\frac{1}{2}} \|\gamma v v^T - \gamma_0 v_0 v_0^T\|_F^{\frac{1}{2}} \\ &\leq 2^{\frac{9}{4}} (1 + \epsilon)^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \\ &\leq 2^3 \epsilon^{\frac{1}{2}}, \end{aligned}$$

where the second inequality is by $\|a\| - \|b\| \leq \|a - b\|$ for any $a, b \in \mathbb{R}$, the third inequality is by concavity of $(\cdot)^{1/2}$, the fourth and the fifth inequalities are by triangle inequality, the sixth inequality is by $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$ for all $a, b \in \mathbb{R}_+$ and the seventh inequality follows from the right hand side of equation (IV.1) when $m > c_0 n \epsilon^{-2} \log(1/\epsilon)$. We now follow the argument for (VII.5). Let $\bar{c}_0 > \max\{c_0, \frac{3}{\epsilon}\}$ and set $\hat{c}_1 := \hat{c} - \frac{3}{\bar{c}_0} > 0$. Consequently, by (VII.12), if $m > \bar{c}_0 n \epsilon^{-2} \log \frac{1}{\epsilon}$, or equivalently, $\frac{3m\epsilon^2}{\bar{c}_0} > 3n \log(1/\epsilon)$, we have

$$\frac{1}{m} \sum_{i=1}^m |A_i M A_i^T|^{\frac{1}{2}} \geq 0.77(1 - \epsilon - 2^3 \epsilon^{1/2}) \quad (\text{VII.14})$$

holds with probability at least

$$\begin{aligned} &1 - \hat{C} \exp(-\hat{c}_1 m \epsilon^2 + 3n \log(1/\epsilon)) - C \exp(-c_1 \epsilon^2 m) \\ &\geq 1 - \hat{C} \exp(-\hat{c}_1 m \epsilon^2 + \frac{3m\epsilon^2}{\bar{c}_0}) - C \exp(-c_1 \epsilon^2 m) \\ &= 1 - C \exp(-c_1 \epsilon^2 m) - \hat{C} \exp(-\hat{c}_1 m \epsilon^2) \\ &\geq 1 - \tilde{C} \exp(-\bar{c}_1 \epsilon^2 m), \end{aligned}$$

where $\tilde{C} = 2 \max\{C, \hat{C}\}$ and $\bar{c}_1 := \min\{c_1, \hat{c}_1\}$. By repeating the adjustments made to ϵ , $\bar{c}_0$, and $\bar{c}_1$ as described at the end of the proof of Lemma IV.3, we obtain the result. $\square$

Proof of Lemma IV.5: If $x = 0$ or $y = 0$ or $x = y$, the inequality holds. Thus, in particular, by the symmetry of (IV.6)

in x and y , we can assume that $\|x\| \geq \|y\| > 0$. Dividing (IV.6) by $\|x\|$, tells us that we can assume $\|x\| = 1$ and $\|y\| = t$ for $t \in [0, 1]$. Set $\rho := \frac{x^T y}{\|y\|} \in [0, 1]$, and define $h(t, \rho) := \sqrt{t^2 - 2\rho t + 1} + \sqrt{t^2 + 2\rho t + 1} - 1 - t = \|x + y\| + \|x - y\| - \|x\| - \|y\|$. If $x = y$, we are done; otherwise, set $q(t, \rho) := \frac{h(t, \rho)}{\sqrt{t^2 - 2\rho t + 1}} = \frac{\|x + y\| + \|x - y\| - \|x\| - \|y\|}{\|x - y\|}$, for each $(t, \rho) \in [0, 1] \times [0, 1]$. We now show that the minimum value of q over $[0, 1] \times [0, 1]$ is $2 - \sqrt{2}$. For fixed $t \in [0, 1]$,

$$\begin{aligned} \frac{\partial q(t, \rho)}{\partial \rho} &= \frac{t(t^2 + 1)}{(t^2 - 2\rho t + 1)^{\frac{3}{2}}} \left[\frac{2}{(t^2 + 2\rho t + 1)^{\frac{1}{2}}} - \frac{t + 1}{t^2 + 1} \right] \\ &\geq \frac{t(t^2 + 1)}{(t^2 - 2\rho t + 1)^{\frac{3}{2}}} \left[\frac{2}{t + 1} - \frac{t + 1}{t^2 + 1} \right] \\ &\geq 0, \end{aligned}$$

where the first inequality follows since $t^2 + 2\rho t + 1 \leq (1 + t)^2$ as $\rho \in [0, 1]$, and the last inequality follows since $2(t^2 + 1) \geq (t + 1)^2$. That is, $q(t, \rho)$ is increasing with respect to ρ when $\rho \in [0, 1]$ for each fixed $t \in [0, 1]$. Also

$$\frac{dq(t, 0)}{dt} = -\frac{1 - t}{(1 + t^2)^{\frac{3}{2}}} \leq 0.$$

Hence $q(t, 0)$ is decreasing for $t \in [0, 1]$. We know for each $t \in [0, 1]$, $\rho \in [0, 1]$,

$$q(t, \rho) \geq q(t, 0) \geq q(1, 0) = 2 - \sqrt{2}$$

Thus $h(t, \rho) \geq (2 - \sqrt{2}) \|x - y\|$, which leads to the desired result. $\square$

Proof of Lemma IV.6: If $x = y = 0$, we are done. Next assume at least one of x and y is non-zero. Since the result is symmetric in x and y , we can assume that $\|x\| \geq \|y\|$ and $x \neq 0$. In addition, we can assume $\|x\| = 1$ and $\|y\| = t \in [0, 1]$ since we can divide (IV.7) by $\|x\|$ on both sides. Setting $\rho := \frac{x^T y}{\|y\|}$, We have

$$\begin{aligned} 2 \|xx^T - yy^T\|_F^2 &= 2 \left(\sum_{i,j} (x_i x_j - y_i y_j)^2 \right) \\ &= 2 \left(\left(\sum_i x_i^2 \right) \left(\sum_j x_j^2 \right) + \left(\sum_i y_i^2 \right) \left(\sum_j y_j^2 \right) \right. \\ &\quad \left. - 4 \left(\sum_i x_i y_i \right) \left(\sum_j x_j y_j \right) \right) \\ &= 2(1 + t^4) - 4\rho^2 t^2 \\ &\geq (1 + t^2)^2 - 4\rho^2 t^2 \\ &= (1 + t^2 + 2\rho t)(1 + t^2 - 2\rho t) \\ &= \|x + y\|^2 \|x - y\|^2, \end{aligned}$$

where the inequality follows since $(1 + t^2)^2 = (1 + t^4) + 2t^2 \leq 2(1 + t^4)$ for $t \in [0, 1]$. $\square$

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