

# Minimal Taylor Algebras as a Common Framework for the Three Algebraic Approaches to the CSP

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**Abstract**—This paper focuses on the algebraic theory underlying the study of the complexity and the algorithms for the Constraint Satisfaction Problem (CSP). We unify, simplify, and extend parts of the three approaches that have been developed to study the CSP over finite templates – absorption theory that was used to characterize CSPs solvable by local consistency methods (JACM’14), and Bulatov’s and Zhuk’s theories that were used for two independent proofs of the CSP Dichotomy Theorem (FOCS’17, JACM’20).

As the first contribution we present an elementary theorem about primitive positive definability and use it to obtain the starting points of Bulatov’s and Zhuk’s proofs as corollaries. As the second contribution we propose and initiate a systematic study of minimal Taylor algebras. This class of algebras is broad enough so that it suffices to verify the CSP Dichotomy Theorem on this class only, but still is unusually well behaved. In particular, many concepts from the three approaches coincide in the class, which is in striking contrast with the general setting.

We believe that the theory initiated in this paper will eventually result in a simple and more natural proof of the Dichotomy Theorem that employs a simpler and more efficient algorithm, and will help in attacking complexity questions in other CSP-related problems.

## I. INTRODUCTION

The Constraint Satisfaction Problem (CSP) has attracted much attention from researchers in various disciplines. One direction of the CSP research has been greatly motivated by the so-called Dichotomy Conjecture of Feder and Vardi [1], [2] that concerns the computational complexity of CSPs over finite relational structures. The *Constraint Satisfaction Problem over a finite relational structure*  $\mathbb{A}$  of finite signature (also called a *template*), in its logical formulation, is the problem to decide the validity of a given *primitive positive sentence* (*pp-sentence*), i.e., a sentence that is an existentially quantified conjunction of atomic formulas over  $\mathbb{A}$  –

the *constraints*. Examples of problems in this class include satisfiability problems, graph coloring problems, and solving systems of equations over finite algebraic structures (see [3], [4], [5], [6]), the CSP is also ubiquitous in artificial intelligence [7].

A classic result in the field is a theorem by Schaefer [8] that completely classifies the complexity of CSPs over relational structures with a two-element domain, so-called *Boolean structures*, by providing a dichotomy theorem: each such a CSP is either solvable in polynomial time or is NP-complete. The Dichotomy Conjecture of Feder and Vardi then states that Schaefer’s result extends to arbitrary finite domains. This conjecture inspired a very active research program in the last 20 years, culminating in a positive resolution independently obtained by Bulatov [9] and Zhuk [10], [11]. The exact borderline between tractability and hardness can be formulated as follows [12], [13], [5].

**Theorem I.1.** *Let  $\mathbb{A}$  be a finite relational structure over a finite signature.*

- *If every finite structure is homomorphically equivalent to a finite structure pp-interpretable in  $\mathbb{A}$ , then the CSP over  $\mathbb{A}$  is NP-hard,*
- *otherwise it is solvable in polynomial time.*

It was already recognized in Schaefer’s work (in fact, it was the basis of his approach) that the complexity of a CSP depends only on the set of relations that are *pp-definable* (i.e., definable by a primitive positive formula) from the template. Such sets of relations are now usually referred to as *relational clones*. The impetus of rapid development in the area after Feder and Vardi’s seminal work [2] was a series of papers [4], [14] that brought to attention and applied a Galois connection between operations and relations studied in the sixties [15], [16], which gives a bijective correspondence between relational clones and *clones* – sets of term operations of algebras.

One way to phrase this core fact is as follows: for any finite algebra  $\mathbf{A}$ , its set of invariant relations (*subuniverses of powers* or *subpowers* in algebraic terminology) is always a relational clone; every relational clone is of this form; and two algebras have the same relational clone of subpowers if and only if they

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have the same set of term operations. For instance, a Boolean CSP, say over the domain  $\{0, 1\}$ , is solvable in polynomial time if and only if the relations of the template are subpowers of one of four types of algebras – an algebra with a single constant operation, a semilattice, the majority algebra, or the affine Mal’cev algebra of  $\mathbb{Z}/2$  (see Subsection III-A).

This connection between relations and operations allowed researchers to apply techniques from Universal Algebra. Application of these techniques became known as the *algebraic approach to the CSP*, although one may argue that the term misses the point a little – the success of the approach lies mostly in combining and moving back and forth between the relational and algebraic side, and this is the case for this paper as well. The general theory of the CSP was further refined in subsequent papers [12], [13] and turned out to be an efficient tool in other types of constraint problems including the Quantified CSP [17], [18], [19], the Counting CSP [20], [21], some optimization problems, e.g. the Valued CSP [22] and robust approximability [23], infinite-domain CSPs [24], and related promise problems such as “approximate coloring” and the Promise CSP [25], [26], and many others.

One useful technical finding of [12] is that every CSP is equivalent to a CSP over an *idempotent template*, i.e. a template that contains all the singleton unary relations. This allows us to use parameters in pp-definitions and omit homomorphic equivalence in the first item of Theorem I.1. On the algebraic side, this allows us to concentrate on so-called idempotent algebras (see Subsection II-A). Another important contribution of that paper was a conjecture postulating, for idempotent structures, the exact borderline between polynomial solvability and NP-hardness, which coincides with the borderline stated in Theorem I.1. The hardness part was already dealt with in the same paper and what was left was the tractability part. Within the realm of idempotent structures, the algebras corresponding to the second item of Theorem I.1 are so-called Taylor algebras (see Subsection III-A). The following theorem is therefore the core of the two proofs of the Dichotomy Conjecture.

**Theorem I.2** ([9], [10], [11]). *Let  $\mathbb{A}$  be an idempotent structure. If there exists an idempotent Taylor algebra  $\mathbf{A}$  such that all relations in  $\mathbb{A}$  are subpowers of  $\mathbf{A}$ , then the CSP over  $\mathbb{A}$  is solvable in polynomial time.*

Partial results toward Theorem I.2 include dichotomies for various classes of relational structures and algebras (e.g. the class of 3-element algebras [27] and the class of structures containing all unary relations [28]), understanding of the limits of algorithmic techniques (e.g. local consistency methods [29] and describing generating sets of solutions [30]), and finding potentially useful characterizations of Taylor algebras (e.g. by means of weak near-unanimity operations [31] and by means of cyclic operations [32]). The papers [29] and [32] initiated a technique, now referred to as the *absorption theory* [33]. It is one of the fruits of CSP-motivated research that impacted also other CSP-related problems as well as universal algebra (e.g. [34]) and it is one of the three theories this paper is concerned with.

Bulatov and Zhuk in their resolution of the Dichotomy Conjecture (and their prior and subsequent work) developed novel techniques, which we refer to as *Bulatov’s theory* and *Zhuk’s theory* in this paper. These theories are (understandably) mostly focused on the task at hand, to prove Theorem I.2, and as such have several shortcomings. First, some of the new concepts are still evolving as the need arises and they do not yet feel quite elegant and settled. Moreover, the theories are technically complex which makes it difficult to master them and to apply them in different contexts. This is best witnessed by the absence of results from different authors that would employ the theories (needless to say they have already clearly witnessed their potential). Second, they both employ the following trick. Instead of studying a general, possibly wild Taylor algebra, one can first tame it by taking a certain *Taylor reduct* – an algebra whose operations are only some of the term operations but which is still Taylor. Taking reducts does not result in any loss of generality in Theorem I.2, since reducts keep all the original invariant relations, so proving tractability for a reduct is sufficient for tractability for the original problem. However, taking reducts does result in loss of generality of the theory and it is not yet clear to which natural classes of algebras the theories apply. Moreover, these reducts are different in the two approaches. Third, connections between Bulatov’s and Zhuk’s theories were not understood at all. While Zhuk’s theory and absorption theory at least had some concepts in common, Bulatov’s theory seemed quite orthogonal to the rest.

The contributions of this paper unify, simplify, and extend parts of these three theories, making them, we hope, more accessible and reducing the prerequisites for the dichotomy proofs. In particular, we initiate a systematic study of *minimal Taylor algebras*, i.e., those algebras that are Taylor but such that none of their proper reducts is Taylor. Thus, we employ the above trick to the extreme and study, in a sense, the tamest algebras or, in other words, “hardest” tractable CSPs. This restriction, on the one hand, limits the scope of the theory but, on the other hand, gives us a framework in which the three theories do not look separate at all anymore, as we shall see.

Even though our unifications, simplifications, and extensions do not cover some advanced parts of the three theories (more on this in due course and Section VII), we believe that they have the potential to evolve into one coherent theory of finite algebras that would make the CSP Dichotomy Theorem an exercise (albeit hard) and that would have applications well beyond constraint problems.

The contributions can be divided into two groups, results for (all finite) Taylor algebras stated in Section IV and results for minimal Taylor algebras in Sections V and VI. We now describe them in more detail together with more background.

#### A. Taylor algebras

The central concept in absorption theory is that of *absorbing subuniverses* introduced formally in Subsection III-C. These are invariant subsets of algebras with an additional property

resembling ideals in rings. A fundamental theorem, the absorption theorem, shows that nontrivial absorbing subuniverses in Taylor algebras exist under rather mild conditions and this fact makes the theory applicable in many situations. For instance, the strategy in [29] to provide a global solution to a locally consistent instance is to propagate local consistency into proper absorbing subuniverses. The abundance of absorption provided by absorption theorem makes this propagation often possible, and if it is not, gives us sufficient structural and algebraic information about the instance which makes the propagation possible nevertheless, until the instance becomes trivially solvable.

Zhuk’s starting point is a theorem stating that every Taylor algebra has a proper subuniverse of one of four special types (see Subsections III-C and IV-C). Zhuk derives the *four types theorem* from a complicated result in clone theory, Rosenberg’s classification of maximal clones [35] (the dependence of this approach on Rosenberg’s result is removed in [36]). Given the four types theorem, the overall strategy for the polynomial algorithm for Theorem I.2 is natural and similar in spirit to the absorption technique – to keep reducing to one of such subuniverses until the problem becomes trivial. Although Zhuk’s theory has a nontrivial intersection with the absorption theory, these connections were not properly explored and verbalized.

Bulatov’s algorithm in his proof of Theorem I.2 employs a similar general idea, he reduces the instance to certain subuniverses. However, these special subuniverses are defined, as opposed to absorption and Zhuk’s theories, in a very local way. They are sets that are, in a sense, closed under edges (e.g. strong components) of a labeled directed graph whose vertices are the elements of the algebra. Bulatov introduces three basic kinds of edges (see Subsection III-D), whose presence indicates that the local structure around the adjacent vertices, namely the subuniverse generated by the two vertices, somewhat resembles the three interesting tractable cases in Schaefer’s Boolean dichotomy. What makes this approach work is a fundamental theorem (Theorem 1 [37], see also [38]), which says that the edges sufficiently approximate the algebra in the sense that the directed graph is connected. The proof uses rather technically challenging constructions involving operations in the algebra.

In Section IV we first describe some of the connections between absorption theory and Zhuk’s theory, and explain simplifications and refinements that were scattered across literature, including a refinement of the absorption theorem that follows from [10], [11]. We also give two new results improving pieces of the two theories. The major novel contribution of Section IV is Theorem IV.7, a purely relational fact which roughly states that each “interesting” relation that uses all the domain elements in every coordinate pp-defines a binary relation with the same properties or a ternary relation of a very particular shape. Although the proof is elementary and not very long, it enables us to derive both Zhuk’s four types theorem and Bulatov’s connectivity theorems as corollaries. It may be also of interest for some readers to note that theorems

in this section often even do not require the algebra to be Taylor – they concern all finite idempotent algebras.

## B. Minimal Taylor algebras

The advantage of studying minimal reducts within a class of interest was clearly demonstrated in the work of Brady [39]. He concentrated on so-called bounded width algebras – these are algebras that play the same role in solvability of CSPs by local consistency methods [29] as Taylor algebras do for polynomial time solvability. The theory he developed enabled him to classify all the minimal bounded width algebras on small domains. Our first contributions in Section V show that the basic facts for minimal bounded width algebras have their counterparts for minimal Taylor algebras. For instance, Proposition V.2 shows that every Taylor algebra does have a minimal Taylor reduct, and so minimal Taylor algebras are indeed sufficiently general, e.g., in the CSP context.

The authors find the extent, to which the notions of the three theories simplify and unify in minimal Taylor algebras, truly striking. Our major results in this direction are Theorems V.7, V.9, V.12, V.18, V.20, V.22, V.23 in Section V and their consequences stated in Section VI, where various classes of algebras are characterized in terms of types of edges, types of operations, and types of absorption present in the algebras. We now discuss a sample of the obtained results.

Edges, as we already mentioned, are pairs of elements for which the local structure around the pair resembles one of the three interesting polynomially solvable cases in Schaefer’s Boolean dichotomy [8]. More precisely, and specializing to one kind of edges, we say that  $(a, b)$  is a *majority edge* if the subalgebra  $\mathbf{E}$  generated by  $a$  and  $b$  has a proper congruence (i.e., invariant equivalence relation)  $\theta$  and a term operation  $t$  that acts as the majority operation on the blocks  $a/\theta$  and  $b/\theta$ . The resemblance of the two-element majority algebra is in general quite loose – the equivalence  $\theta$  can have many more blocks and there may be many more operations in  $\mathbf{E}$  other than  $t$ . However, in minimal Taylor algebras,  $\mathbf{E}$  modulo  $\theta$  is always term equivalent to the two element majority algebra.

The second sample concerns the simplest absorbing subuniverses, the 2-absorbing ones, which constitute one of the four types of Zhuk’s fundamental theorem. The 2-absorption of a subuniverse  $B$  is a relatively strong property that requires the existence of *some* binary term operation  $t$  whose result is always in  $B$  provided at least one of the arguments is in  $B$ . An extreme further strengthening is as follows: the result of applying *any* operation  $f$  to an argument that contains an element in  $B$  in *any* essential coordinate is in  $B$ . It turns out that these notions actually coincide for minimal Taylor algebras. What is perhaps even more surprising is the connection to Bulatov’s theory: 2-absorbing sets are exactly subsets stable (in a certain sense) under all the three kinds of edges.

Finally, we mention that the clone of any minimal Taylor algebra is generated by a single ternary operation. This, together with other structural results in this paper, may help in enumerating Taylor algebras – at the very least we know that

there are at most  $n^{n^3}$  of them over a domain of size  $n$ . Such a catalogue could be a valuable source of examples for CSP-related problems as well as universal algebra. Additionally, having a complete catalogue of minimal Taylor algebras for a given domain allows you to write down an explicit, concrete generalization of Schaefer's Dichotomy Theorem [8] for a domain of that size, with as few cases as possible.

Brady has already initiated this project and has found all the three-element minimal Taylor algebras in unpublished work based on the results of this paper. Up to term-equivalence and permutations of the domain, there are exactly 24 minimal Taylor algebras on a domain of size 3. This gives us a concrete list of the hardest tractable CSPs on the 3-element domain, refining the main result of [27].

The full version of this paper, which contains all the proofs, is [40].

## II. PRELIMINARIES

### A. Algebras

Algebras, i.e. structures with purely functional signature, will be denoted by boldface capital letters (e.g.,  $\mathbf{A}$ ) and their universes (also called domains) typically by the same letter in the plain font (e.g.,  $A$ ). The basic general algebraic concepts, such as subuniverses, subalgebras, products, and quotients modulo congruences are used in the standard way (see, e.g. [41]). An algebra is *nontrivial* if it has more than two elements, otherwise it is *trivial*. We use  $B \leq \mathbf{A}$  to mean that  $B$  is a subuniverse of  $\mathbf{A}$ . By a *subpower* we mean a subuniverse (or a subalgebra) of a finite power. Recall that subpowers are the same as *invariant relations* and we may also call them *compatible relations*. The set of all subpowers is denoted  $\text{Inv}(\mathbf{A})$ . The subuniverse (or the subalgebra) of  $\mathbf{A}$  generated by a set  $X \subseteq A$  is denoted  $\text{Sg}_{\mathbf{A}}(X)$  or  $\text{Sg}_{\mathbf{A}}(x_1, \dots, x_n)$  when  $X = \{x_1, \dots, x_n\}$ .

**All theorems in this paper concern algebras that are finite and idempotent**, that is,  $f(x, x, \dots, x) = x$  for every operation  $f$  in the algebra and every element  $x$  of the universe. Recall that this is not a severe restriction, at least in the area of finite-template CSPs. **We do not explicitly mention this assumption in the statements of theorems or definitions.**

A (function) *clone* is a set of operations  $\mathcal{C}$  on a set  $A$  which contains all the projections  $\text{proj}_i^n$  (the  $n$ -ary projection to the  $i$ -th coordinate) and is closed under composition, i.e.,  $f(g_1, \dots, g_n) \in \mathcal{C}$  whenever  $f \in \mathcal{C}$  is  $n$ -ary and  $g_1, \dots, g_n \in \mathcal{C}$  are all  $m$ -ary, where  $f(g_1, \dots, g_n)$  denotes the operation defined by  $f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$ . By  $\text{Clo}(\mathbf{A})$  ( $\text{Clo}_n(\mathbf{A})$ , respectively), we denote the clone of all term operations (the set of all  $n$ -ary term operations, respectively) of  $\mathbf{A}$ . An algebra  $\mathbf{B}$  is a *reduct* of  $\mathbf{A}$  if they have the same universe  $A = B$  and  $\text{Clo}(\mathbf{B}) \subseteq \text{Clo}(\mathbf{A})$ . Algebras  $\mathbf{A}$  and  $\mathbf{B}$  are *term-equivalent* if each of them is a reduct of the other, i.e.,  $\text{Clo}(\mathbf{A}) = \text{Clo}(\mathbf{B})$ .

A coordinate  $i$  of an operation  $f : A^n \rightarrow A$  is *essential* if  $f$  depends on the  $i$ th coordinate, i.e.,  $f(\mathbf{a}) \neq f(\mathbf{b})$  of some tuples  $\mathbf{a}, \mathbf{b} \in A^n$  that differ only at the  $i$ th coordinate.

### B. Relations

A relation on  $A$  is a subset of  $A^n$ , but we often work with more general “multisorted” relations  $R \subseteq A_1 \times A_2 \times \dots \times A_n$ . We call such an  $R$  *proper* if  $R \neq A_1 \times \dots \times A_n$  and *nontrivial* if it is nonempty and proper. Tuples are written in boldface and components of  $\mathbf{x} \in A_1 \times \dots \times A_n$  are denoted  $x_1, x_2, \dots$ . Both  $\mathbf{x} \in R$  and  $R(\mathbf{x})$  are used to denote the fact that  $\mathbf{x}$  is in  $R$ . The projection of  $R$  onto the coordinates  $i_1, \dots, i_k$  is denoted  $\text{proj}_{i_1, \dots, i_k}(R)$ . The relation  $R$  is *subdirect*, denoted  $R \subseteq_{sd} A_1 \times \dots \times A_n$ , if  $\text{proj}_{i_i}(R) = A_i$  for each  $i$ . We call  $R$  *redundant*, if there exist coordinates  $i \neq j$  such that  $\text{proj}_{ij}(R)$  is a graph of bijection from  $A_i$  to  $A_j$ ; otherwise  $R$  is *irredundant*.

We say that a set of relations  $\mathcal{R}$  *pp-defines*  $S$  if  $S$  can be defined from  $\mathcal{R}$  by a primitive positive formula with parameters, that is, using the existential quantifier, relations from  $\mathcal{R}$ , the equality relation, and the **singleton unary relations**. Recall that the set of subpowers of an algebra is closed under pp-definitions.

For binary relations we write  $-R$  instead of  $R^{-1}$  and  $R + S$  for the relational composition of  $R$  and  $S$ , that is  $R + S = \{(a, c) : (\exists b) R(a, b) \wedge R(b, c)\}$ . For a unary relation  $B$  we write  $B + S$  to denote the set  $\{c : (\exists b) B(b) \wedge S(b, c)\}$  and if  $B$  is a singleton we often write  $b + S$  instead of  $\{b\} + S$ . Also, we set  $R - S = R + (-S) = R \circ S^{-1}$ . A relation  $R \subseteq A \times B$  is *linked* if  $(R - R) + (R - R) + \dots + (R - R)$  is equal to  $(\text{proj}_1(R))^2$  for some number of summands. In other words,  $R$  is connected when viewed as a bipartite graph between  $A$  and  $B$  (with possible isolated vertices). The *left center* of  $R \subseteq A \times B$  is the set  $\{a \in A : a + R = B\}$ . If  $R$  has a nonempty left center, it is called *left central*. *Right center* and *right central* relations are defined analogically. A relation is *central* if it is left central and right central. Note that  $R + S$ ,  $-R$ , and the left (right) center of  $R$  are pp-definable from  $\{R, S\}$ .

## III. BASIC CONCEPTS

### A. Taylor algebras

First we define the central concept of the algebraic theory of the CSP, Taylor algebra. From the many equivalent definitions (e.g., the one using so-called Taylor operations – that's where the name comes from) we present a direct algebraic counterpart of the first item in Theorem I.1 specialized to idempotent structures.

**Definition III.1.** *An (idempotent, finite) algebra  $\mathbf{A}$  is a Taylor algebra if no quotient of a subpower of  $\mathbf{A}$  is a two-element algebra whose every operation is a projection.*

We will often take advantage of the characterization by means of *cyclic operation*. Its original proof is via the absorption theory, an alternative proof is now available using Zhuk's approach [36].

**Theorem III.2** ([32]). *The following are equivalent for any algebra.*

- $\mathbf{A}$  is Taylor.

- There exists  $n > 1$  such that  $\mathbf{A}$  has a term operation  $t$  of arity  $n$  which is cyclic, that is, for any  $\mathbf{x} \in A^n$ ,  $t(x_1, x_2, \dots, x_n) = t(x_2, \dots, x_n, x_1)$ .
- For every prime  $p > |A|$ ,  $\mathbf{A}$  has a term operation  $t$  of arity  $p$  which is cyclic.

Several further types of operations are significant for this paper:

- *Semilattice operation* is a binary operation  $\vee$  which is commutative, idempotent, and associative.
- *Majority operation* is a ternary operation  $m$  satisfying  $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$  (for any  $x, y$  in the universe).
- *Mal'cev operation* is a ternary operation  $p$  satisfying  $p(y, x, x) = p(x, x, y) = y$ .

Any algebra with a semilattice, or majority, or Mal'cev operation is Taylor. The following algebras are particularly important for our purposes (recall also the discussion about Schaefer's result in the introduction):

- *Two-element semilattice*: a two-element set together with one of the two semilattice operations, e.g.,  $(\{0, 1\}; \vee)$  where  $\vee$  is the maximum operation,
- *Two-element majority algebra*: a two element set together with the unique majority operation, e.g.,  $(\{0, 1\}; \text{maj})$ . We also use  $\text{maj}_p$ , for odd  $p$ , to denote the  $p$ -ary majority operation on  $\{0, 1\}$ , that is,  $\text{maj}(\mathbf{a}) = 1$  iff the majority of the  $a_i$  is 1.
- *Affine Mal'cev algebra*: a set together with the Mal'cev operation  $x - y + z$ , where  $+$  and  $-$  is computed with respect to a fixed abelian group structure on the universe, e.g.,  $(\{0, 1, \dots, p-1\}; x - y + z \pmod{p})$ . More generally, an *affine module* is an algebra whose term operations are exactly the idempotent term operations of a module over a unital ring.

## B. Abelian algebras

The last example falls into a larger class of algebras, which is also significant in the algebraic theory of CSPs and Universal Algebra in general, so-called *abelian algebras*.

**Definition III.3.** An algebra  $\mathbf{A}$  is abelian if the diagonal  $\Delta_A = \{(a, a) : a \in A\}$  is a block of a congruence of  $\mathbf{A}^2$ .

As an example, for an affine Mal'cev algebra, a congruence satisfying the definition is the congruence  $\alpha$  defined by  $((x_1, x_2), (y_1, y_2)) \in \alpha$  iff  $x_1 - x_2 = y_1 - y_2$ . Note that an abelian algebra does not need to be Taylor, e.g., an algebra with no operations is such (except in the pathological, one-element case). However, for (finite, idempotent) Taylor algebras, abelian algebras admit a complete description up to term equivalence. The original proof of this result is using the tame congruence theory (a developed theory of finite algebras that we have not mentioned yet), an alternative proof using absorption is available [32].

**Theorem III.4** ([42]). Every (finite, idempotent) abelian Taylor algebra is an affine module.

In light of this theorem, readers interested solely in Taylor algebras may safely replace the phrase “abelian algebra” with “affine module”. In fact, it follows from our results that in minimal Taylor algebras “affine module” can be further simplified to “affine Mal'cev algebra”, up to term equivalence.

## C. Absorption

Now we introduce absorbing subuniverses, centers, and projective subuniverses, central concepts in the absorption theory and Zhuk's theory.

**Definition III.5.** Let  $\mathbf{A}$  be an algebra and  $B \subseteq A$ . We call  $B$  an  $n$ -absorbing set of  $\mathbf{A}$  if there is a term operation  $t \in \text{Clo}_n(\mathbf{A})$  such that  $t(\mathbf{a}) \in B$  whenever  $\mathbf{a} \in A^n$  and  $|\{i : a_i \in B\}| \geq n - 1$ .

If, additionally,  $B$  is a subuniverse of  $\mathbf{A}$ , we write  $B \leq_n \mathbf{A}$ , or  $B \leq \mathbf{A}$  when the arity is not important.

We also say “ $B$  absorbs  $\mathbf{A}$  (by  $t$ )” in the situation of Definition III.5. Of particular interest for us are  $n$ -absorbing subuniverse with  $n = 2$ , e.g.,  $\{1\}$  in the semilattice  $(\{0, 1\}; \vee)$ , or  $n = 3$ , e.g.,  $\{0\}$  and  $\{1\}$  in the two-element majority algebra.

The concept of a center is still evolving and it is not yet clear what the best version would be for general algebras. Our definition follows [11], although a more recent paper [36] made an adjustment motivated by this work. As we shall see in Theorem V.9, the situation is much cleaner for minimal Taylor algebras.

**Definition III.6.** A subset  $B \subseteq A$  is a center of  $\mathbf{A}$  if there exists an algebra  $\mathbf{C}$  (of the same signature) with no nontrivial 2-absorbing subuniverse and  $R \leq_{sd} \mathbf{A} \times \mathbf{C}$  such that  $B$  is the left center of  $R$ . The relation  $R$  is called a witnessing relation. If  $\mathbf{C}$  can be chosen Taylor, we call  $B$  a Taylor center of  $\mathbf{A}$ .

The final concept that we introduce in this section is a projective subuniverse. It appeared in [43] in connection with so-called cube operations, which characterize the limit of the *few subpowers algorithm* that finds generating set of all solutions to a CSP [30]. These subuniverses were called *cube term blockers* in [43] but it became clear that the concept is significant beyond this context [44], [36] and for this reason we prefer the terminology from the latter paper.

**Definition III.7.** Let  $\mathbf{A}$  be an algebra and  $B \subseteq A$ . We say that  $B$  is a projective subuniverse if for every  $f \in \text{Clo}_n(\mathbf{A})$  there exists a coordinate  $i$  of  $f$  such that  $f(\mathbf{a}) \in B$  whenever  $\mathbf{a} \in A^n$  is such that  $a_i \in B$ .

Note that a projective subuniverse of  $\mathbf{A}$  is, indeed, a subuniverse. Also note that centers are automatically subuniverses as well.

Many of the algebraic concepts that we introduce (such as absorbing subuniverses or strongly projective subuniverses from Section V) have a useful equivalent characterizations in terms of relations. Such a characterization for projective subuniverses is especially elegant and we state it here for reference.

**Proposition III.8** (Lemma 3.2 in [43]). *Let  $\mathbf{A}$  be an algebra and  $B \subseteq A$ . Then  $B$  is a projective subuniverse of  $\mathbf{A}$  if and only if, for every  $n$ , the relation  $B(x_1) \vee B(x_2) \vee \dots \vee B(x_n)$  is a subpower of  $\mathbf{A}$ .*

#### D. Edges

Finally we introduce the three types of edges used in Bulatov's approach to the CSP.

**Definition III.9.** *Let  $\mathbf{A}$  be an algebra. A pair  $(a, b) \in A^2$  is an edge if there exists a proper congruence  $\theta$  on  $\text{Sg}_{\mathbf{A}}(a, b)$  (a witness for the edge) such that one of the following happens:*

- (semilattice edge) *There is a term operation  $f \in \text{Clo}_2(\mathbf{A})$  acting as a join semilattice operation on  $\{a/\theta, b/\theta\}$  with top element  $b/\theta$ .*
- (majority edge) *There is a term operation  $m \in \text{Clo}_3(\mathbf{A})$  acting as a majority operation on  $\{a/\theta, b/\theta\}$ .*
- (abelian edge) *The algebra  $\text{Sg}_{\mathbf{A}}(a, b)/\theta$  is abelian.*

An edge  $(a, b)$  is called *minimal* if for some maximal congruence  $\theta$  witnessing the edge and every  $a', b' \in A$  such that  $(a, a'), (b, b') \in \theta$ , we have  $\text{Sg}_{\mathbf{A}}(a', b') = \text{Sg}_{\mathbf{A}}(a, b)$ .

A witnessing congruence  $\theta$  for an edge  $(a, b)$  necessarily separates  $a$  and  $b$ , i.e.,  $(a, b) \notin \theta$ , since each congruence block of an idempotent algebra is a subuniverse. Moreover, if  $\theta$  is a witness for an edge  $(a, b)$ , then any proper congruence of  $\text{Sg}_{\mathbf{A}}(a, b)$  containing  $\theta$  witnesses the same edge.

Note that if  $(a, b)$  is an edge of majority or abelian type, then so is  $(b, a)$ . If  $(a, b)$  is a semilattice edge it can happen that  $(b, a)$  is not an edge at all, in fact this is always the case for minimal edges in a minimal Taylor algebra.

In order to make the concepts in this paper elegant and theorems more general, we deviate from the definition given in e.g. [38], [45]. There, majority edges have an additional requirement that the same congruence does not witness the semilattice type, and abelian edges (called affine) required the quotient to be an affine module. Also note that the definition of abelian edges (as well as the original affine edges) is of a different type: it restricts the set of term operations from above, as opposed to semilattice and majority edges that restrict them from below. We shall see in Theorem V.12 that these differences disappear in minimal Taylor algebras.

Minimal edges do not appear in Bulatov's theory in this form. Somewhat related are thin edges, which at present have rather technical definitions with the exception of thin semilattice edges. We show in Proposition V.14 that minimal semilattice edges and thin semilattice edges coincide in minimal Taylor algebras.

## IV. TAYLOR ALGEBRAS

This section presents unifications, simplifications, and refinements of the three algebraic theories in the setting of Taylor algebras (still finite and idempotent) that are not necessarily minimal. In Subsection IV-A we discuss the already existing refinements to the proof of the absorption theorem (and provide two additional new refinements in Proposition IV.2 and Proposition IV.4). This gives tight links to Zhuk's theory,

in particular, centers and projective subuniverses. Subsection IV-B contains the main contribution of this section, Theorem IV.7. This theorem together with additional technical contributions, Theorems IV.10 and IV.11, directly imply the fundamental facts in the two proofs of the CSP Dichotomy Theorem – the four types theorem and the connectivity theorem, discussed in Subsection IV-C.

#### A. Absorption theorem

We phrase the absorption theorem in a slightly simplified form to keep the presentation compact.

**Theorem IV.1** (Absorption Theorem). [32] *If  $\mathbf{A}$  is Taylor and  $R \leq_{sd} \mathbf{A}^2$  is proper and linked, then  $\mathbf{A}$  has a nontrivial absorbing subuniverse.*

The original proof can be divided into 3 steps.

- (1) From  $\mathbf{A}$  being Taylor it is derived that  $\mathbf{A}$  either has a nontrivial binary absorbing subuniverse or a *transitive* term operation  $t$  of some arity  $n$ , i.e., for each  $b, c \in A$  and every coordinate  $i$  of  $t$ , there exists a tuple  $\mathbf{a} \in A^n$  with  $a_i = b$  such that  $t(\mathbf{a}) = c$ .
- (2) Using the transitive operation, it is proved that if  $\mathbf{A}$  has no nontrivial absorbing subuniverses, then  $R$  is left or right central.
- (3) It is shown that the transitive operation witnesses that the left (right) center absorbs  $\mathbf{A}$ .

We now comment on subsequent improvements and simplifications.

The first step was explored in more detail in [44]. Lemma 2.7. in [44] shows that each algebra has a nontrivial projective subuniverse or a transitive term operation. A simple argument then shows that every projective subuniverse in a Taylor algebra is 2-absorbing, a witness is, e.g., any operation of the form  $t(x, \dots, x, y, \dots, y)$  where  $t$  is cyclic.

As for the second step, it has turned out that left (or right) central relations can be very easily obtained from linked relations by means of pp-definitions, avoiding algebraic considerations altogether. We give a refined version that derives central relations with further properties.

**Proposition IV.2.** *Let  $R \subseteq_{sd} A^2$  be linked and proper. Then  $R$  pp-defines a subdirect proper central relation on  $A$  which is symmetric or transitive.*

The third step, that a transitive operation witnesses absorption of left centers, is straightforward. A significant refinement, Corollary 7.10.2 in [11] shows that left centers are, in fact, ternary absorbing. An adjustment of the proof will also help us in proving Theorem V.9.

**Proposition IV.3.** [11] *If  $B$  is a Taylor center of an algebra  $\mathbf{A}$ , then  $B \leq_3 \mathbf{A}$ .*

Note that, in the previous theorem  $\mathbf{A}$  need not be a Taylor algebra, but  $\mathbf{C}$  (where the witnessing relation is  $R \leq_{sd} \mathbf{A} \times \mathbf{C}$ ) must be. The following proposition states that we can switch the condition:

**Proposition IV.4.** *If  $B$  is a center of a Taylor algebra  $\mathbf{A}$ , then  $B \leq_3 \mathbf{A}$ .*

In the remainder of the paper, the assumptions of the latter proposition are easier to satisfy — the algebra  $\mathbf{A}$  is usually Taylor by default.

Altogether, either of the propositions above provides the following improvement of the absorption theorem, which does not seem to be explicitly stated in the literature.

**Corollary IV.5.** *If  $\mathbf{A}$  is Taylor and  $R \leq_{sd} \mathbf{A}^2$  proper and linked, then  $\mathbf{A}$  has a nontrivial 3-absorbing subuniverse.*

#### B. Subdirect irredundant subpowers

We now present the unification result. It says that any “interesting” (subdirect irredundant proper) relation either pp-defines an interesting binary relation or pp-defines (it is even inter-pp-definable with) ternary relations of very particular shape — they are graphs of quasigroup operations.

**Definition IV.6.** *A relation  $R \subseteq A^3$  is called strongly functional if*

- *binary projections of  $R_i$  are equal to  $A^2$ , and*
- *a tuple in  $R_i$  is determined by values on any two coordinates.*

**Theorem IV.7.** *Let  $R \subseteq_{sd} A^n$  be an irredundant proper relation. Then either*

- *$R$  pp-defines an irredundant and proper  $R' \subseteq_{sd} A^2$ , or*
- *there exist strongly functional ternary relations  $R_1, \dots, R_n \subseteq_{sd} A^3$  such that the set  $\{R_1, \dots, R_n\}$  is inter-pp-definable with  $R$  (i.e., the  $R_i$ ’s pp-define  $R$  and, conversely,  $R$  pp-defines all the  $R_i$ ’s).*

Theorem IV.7 implies that every algebra  $\mathbf{A}$  has at least one of the following properties of its invariant relations.

- (1)  $\mathbf{A}$  has no proper irredundant subdirect subpowers.
- (2)  $\mathbf{A}$  has a proper irredundant binary subdirect subpower.
- (3)  $\mathbf{A}$  has a ternary strongly functional subpower.

In the last case, it is easy to pp-define a congruence on  $\mathbf{A}^2$  such that the diagonal is one of its blocks, so  $\mathbf{A}$  is abelian in this case. If  $\mathbf{A}$  is Taylor, Theorem III.4 then gives a good understanding of  $\mathbf{A}$  — it is an affine module.

**Proposition IV.8.** *If  $R \leq \mathbf{A}^3$  is a strongly functional relation, then  $\mathbf{A}$  is abelian.*

In case (1), subdirect relations have a very simple structure; for instance, any constraint  $R(x_1, \dots, x_n)$  with subdirect  $R$  is effectively a conjunction of bijective dependencies  $x_i = f(x_j)$ . It is also immediate that  $\mathbf{A}$  is *polynomially complete*, that is, every operation on  $A$  is in the clone generated by  $\mathbf{A}$  together with the constant operations. Indeed, polynomial completeness is equivalent to having no proper reflexive (that is, containing all the tuples  $(a, a, \dots, a)$ ) irredundant subpowers. Less trivially, case (1) often leads to majority edges, as we show in Theorem IV.10 below. However, we require the following definition first.

**Definition IV.9.** *Let  $\mathbf{A}$  be an algebra. By the connected-by-subuniverses equivalence, denoted  $\mu_{\mathbf{A}}$ , we mean the smallest equivalence containing all the pairs  $(a, b)$  such that  $\text{Sg}_{\mathbf{A}}(a, b) \neq A$ .*

We remark that the equivalence  $\mu_{\mathbf{A}}$  is not, in general, a congruence of  $\mathbf{A}$ , so this concept may seem somewhat unnatural from the algebraic perspective.

**Theorem IV.10.** *Suppose that  $\mathbf{A}$  is simple and has no subdirect proper irredundant subpowers. Then there exists a term operation  $t \in \text{Clo}_3(\mathbf{A})$  such that for any  $(a, b) \notin \mu_{\mathbf{A}}$ ,  $t(a, a, b) = t(a, b, a) = t(b, a, a) = a$ .*

In case (2) and when  $\mathbf{A}$  is simple, a binary irredundant relation is necessarily linked. Then we get a central relation, e.g., by Proposition IV.2, and often also semilattice edges (please note the important, but easy-to-miss condition on the size of the algebra).

**Theorem IV.11.** *Suppose  $\mathbf{A}$  with  $|A| > 2$  is simple and there exists a proper irredundant subdirect binary subpower. Then there exists  $\mu_{\mathbf{A}}$ -class  $B$  such that, for every  $b \in B, a \notin B$ , the pair  $(a, b)$  is a semilattice edge witnessed by the identity congruence.*

#### C. Fundamental theorems of dichotomy proofs

Zhuk’s four types theorem is now a consequence of Theorem IV.7, Proposition IV.8, and Proposition IV.2. Indeed, one simply applies these facts to  $\mathbf{A}$  factored by a maximal congruence, which is a simple algebra, and then lifts 2-absorbing subuniverses and centers back to  $\mathbf{A}$ .

**Corollary IV.12.** *[The Four Types Theorem] Let  $\mathbf{A}$  be an algebra, then*

- (a)  *$\mathbf{A}$  has a nontrivial 2-absorbing subuniverse, or*
- (b)  *$\mathbf{A}$  has a nontrivial center, (which is a Taylor center in the case that  $\mathbf{A}$  is a Taylor algebra), or*
- (c)  *$\mathbf{A}/\alpha$  is abelian for some proper congruence  $\alpha$  of  $\mathbf{A}$ , or*
- (d)  *$\mathbf{A}/\alpha$  is polynomially complete for some proper congruence  $\alpha$  of  $\mathbf{A}$ .*

In the introduction we referred to four types of *subuniverses* whereas cases (c) and (d) talk about congruences — the subuniverses used in [11] are obtained from blocks of such congruences.

Examples of simple Taylor algebras, for which one of the cases takes place and no other, are (a) a two-element semilattice, (b) a two-element majority algebra, (c) an affine Mal’cev algebra, and (d) the three element rock-paper-scissors algebra  $(\{\text{paper, rock, scissors}\}; \text{winner}(x, y))$ . Note, however, that Corollary IV.12 does not require that  $\mathbf{A}$  is Taylor. If it is, then we get additional properties: centers are 3-absorbing by Proposition IV.3 and abelian algebras are term equivalent to affine modules by Theorem III.4. For non-Taylor idempotent algebras, [36] suggests a similar five type theorem, which also follows immediately from the presented results.

The connectivity theorem of Bulatov is also a straightforward consequence of the obtained results, Theorem IV.7,

Proposition IV.8, Theorem IV.10, and Theorem IV.11. In fact, a little additional effort gives a stronger statement – for minimal edges instead of edges.

**Corollary IV.13.** *[The Connectivity Theorem] The directed graph formed by the minimal edges of any algebra is connected.*

Notice that the last theorem also does not require the algebra to be Taylor. Outside Taylor algebras, it makes sense to separate abelian edges into two types: affine that are the same as abelian edges in the Taylor case, and sets whose only term operations are projections, as is done in [45].

## V. MINIMAL TAYLOR ALGEBRAS

We start this section by recalling the central definition and giving some examples.

**Definition V.1.** *An algebra  $\mathbf{A}$  is called a minimal Taylor algebra if it is Taylor but no proper reduct of  $\mathbf{A}$  is.*

Examples of minimal Taylor algebras include two-element semilattices, two-element majority algebras, and affine Mal'cev algebras. This follows from the description of their term operations: the term operations of the two-element semilattice  $(\{0,1\}; \vee)$  are exactly the operations of the form  $x_{i_1} \vee x_{i_2} \vee \dots \vee x_{i_k}$ ; the term operations of the two-element majority algebra  $(\{0,1\}; \text{maj})$  are exactly the idempotent, monotone (i.e., compatible with the inequality relation  $\leq$ ), and self-dual (i.e., compatible with the disequality relation  $\neq$ ) operations; the term operations of an affine Mal'cev algebra over an abelian group are exactly the operations of the form  $a_1x_1 + a_2x_2 + \dots + a_nx_n$ , where  $a_i$  are integers that sum up to one. (Each of the mentioned facts is either simple or follows from [46].) In fact, there are exactly four minimal Taylor algebras on a two-element set: the two semilattices, the majority algebra and the two-element affine Mal'cev algebra.

A nice example of a minimal Taylor algebra on a three-element domain is the rock-paper-scissors algebra mentioned after The Four Types Theorem. To see that this algebra is minimal Taylor observe that any term operation behaves on any two-element set like the term operation of a two-element semilattice with the same set of essential coordinates. Therefore, the original operation can be obtained by identifying variables in any term operation having at least two essential coordinates. The same argument shows that any semilattice, not necessarily two-element, is minimal Taylor.

In Subsection V-A we give the basic general theorems that were proved in [39] in the context of minimal bounded width algebras. Subsection V-B concentrates on absorption and related concepts in Zhuk's theory. It turns out that 2-absorbing sets are exactly projective subuniverses (Theorem V.7) and 3-absorbing sets are exactly centers (Theorem V.9). Subsection V-C shows that edges substantially simplify in minimal Taylor algebras (Theorem V.12) and gives additional information for minimal edges; in particular, minimal semilattice edges coincide with thin semilattice edges as defined in [37], [45] (Proposition V.14). Finally, in Subsection V-D,

we demonstrate a strong interaction between absorption and edges. We show that 2-absorbing subuniverses are exactly subsets that are, in some sense, stable under all the edges (Theorem V.18), we provide somewhat weaker interaction between absorbing subuniverses and subsets stable under semilattice and abelian edges (Theorem V.20), we give a common witnessing operation for all the edges as well as all the 2- and 3-absorbing subuniverses (Theorem V.22), and we show that each such a witnessing operation generates the whole clone of term operations (Theorem V.23).

### A. General facts

It is not immediate from the definitions that each Taylor algebra has a minimal Taylor reduct. Nevertheless, this fact easily follows from the characterization of Taylor algebras by means of cyclic operations.

**Proposition V.2.** *Every Taylor algebra has a minimal Taylor reduct.*

Another simple, but important consequence of cyclic operations is the following proposition. The result is slightly more technical than most of the others, but it is in the core of many strong properties of minimal Taylor algebras.

**Proposition V.3.** *Let  $\mathbf{A}$  be a minimal Taylor algebra and  $B \subseteq A$  be closed under an operation  $f \in \text{Clo}(\mathbf{A})$  such that  $B$  together with the restriction of  $f$  to  $B$  forms a Taylor algebra. Then  $B$  is a subuniverse of  $\mathbf{A}$ .*

A similar method based on cyclic operations prove that the class of minimal Taylor algebras is closed under the standard constructions.

**Proposition V.4.** *Any subalgebra, finite power, or quotient of a minimal Taylor algebra is a minimal Taylor algebra.*

### B. Absorption

The goal of this section is to show that absorbing subsets, which are abundant in general Taylor algebras by Corollary IV.12 and Proposition IV.4, have strong properties in minimal Taylor algebras. We start with a surprising fact, which clearly fails in general Taylor algebras.

**Theorem V.5.** *Let  $\mathbf{A}$  be a minimal Taylor algebra and  $B$  an absorbing set of  $\mathbf{A}$ . Then  $B$  is a subuniverse of  $\mathbf{A}$ .*

Now we move on to 2-absorption. We have already mentioned in Subsection IV-A that projectivity is a stronger form of absorption in Taylor algebras, but we can go even further.

**Definition V.6.** *Let  $\mathbf{A}$  be an algebra and  $B \subseteq A$ . The set  $B$  is a strongly projective subuniverse of  $\mathbf{A}$  if for every  $f \in \text{Clo}_n(\mathbf{A})$  and every essential coordinate  $i$  of  $f$ , we have  $f(\mathbf{a}) \in B$  whenever  $\mathbf{a} \in A^n$  is such that  $a_i \in B$ .*

The property of being a strong projective subuniverse is indeed very strong. For example, in any non-trivial clone, strong projective subuniverse is 2-absorbing and every binary operation of the clone, except for projections, witnesses the absorption. The next theorem states that strong projectivity in

minimal Taylor algebras is equivalent to 2-absorption, which in general is a much weaker concept.

**Theorem V.7.** *The following are equivalent for any minimal Taylor algebra  $\mathbf{A}$  and a set  $B \subseteq A$ .*

- (a)  $B$  2-absorbs  $\mathbf{A}$ .
- (b)  $R(x, y, z) = B(x) \vee B(y) \vee B(z)$  is a subuniverse of  $\mathbf{A}^3$ .
- (c)  $B$  is a projective subuniverse of  $\mathbf{A}$ .
- (d)  $B$  is a strongly projective subuniverse of  $\mathbf{A}$ .

The main value of this theorem is the implication showing that, in minimal Taylor algebras, every 2-absorption, i.e. (a), is as strong as possible (d). Moreover (c) provides a nice relational description of 2-absorption, which collapses the general condition from Proposition III.8 for projectivity to arity 3. We note that none of the conditions are equivalent in Taylor algebras without the minimality assumption.

The following proposition collects some strong and unusual properties of 2-absorbing subuniverses in minimal Taylor algebras. Already the first item might be surprising since the union of two subuniverses is rarely a subuniverse.

**Proposition V.8.** *Let  $\mathbf{A}$  be a minimal Taylor algebra and  $B \leq_2 \mathbf{A}$ .*

- 1) *If  $C \leq \mathbf{A}$  then  $B \cup C \leq \mathbf{A}$ .*
- 2) *If  $C$  is proper and  $C \leq \mathbf{A}$  by  $f$ , then*
  - a)  *$B \cup C \leq \mathbf{A}$  by  $f$ , and*
  - b)  *$B \cap C \neq \emptyset$  and  $B \cap C \leq \mathbf{A}$  by  $f$ .*
- 3) *Every minimal Taylor algebra  $\mathbf{A}$  has a unique minimal 2-absorbing subalgebra  $\mathbf{B}$ . Moreover, this algebra  $\mathbf{B}$  does not have any nontrivial 2-absorbing subuniverse.*

As for absorption of higher arity, we have already shown in Proposition IV.4 that centers are 3-absorbing. Next theorem says that, in minimal Taylor algebras, the converse is true as well.

**Theorem V.9.** *The following are equivalent for any minimal Taylor algebra  $\mathbf{A}$  and a set  $B \subseteq A$ .*

- (a)  $B$  3-absorbs  $\mathbf{A}$ .
  - (b)  $R(x, y) = B(x) \vee B(y)$  is a subuniverse of  $\mathbf{A}^2$ .
  - (c)  $B$  is a (Taylor) center of  $\mathbf{A}$ .
  - (d) *there exists  $\mathbf{C}$  with  $\text{Clo}(\mathbf{C}) \subseteq \text{Clo}(\{0, 1\}; \text{maj})$  such that  $R(x, y) = B(x) \vee (y = 0)$  is a centrality witness.*
- Moreover, if  $B = \{b\}$ , then these items are equivalent to*
- (e)  $B$  absorbs  $\mathbf{A}$ .

Just like in Theorem V.7 we have that a relatively weak notion of 3-absorption implies a very strong type of centrality which is (d). Let us investigate (d) in greater detail. To every operation of  $\mathbf{A}$ , say  $f$ , we associate an operation  $f' \in \text{Clo}(\{0, 1\}; \text{maj})$  such that  $f(\mathbf{a}) \in B$  whenever  $f'(\mathbf{x}) = 1$  and  $\mathbf{x}$  is the characteristic tuple of  $\mathbf{a}$  with respect to  $B$  (i.e.  $x_i = 1$  if and only if  $a_i \in B$ ). That is, from the viewpoint of “being outside  $B$ ” vs. “being inside  $B$ ” every operation outputs “inside  $B$ ” every time the corresponding operation of  $\text{Clo}(\{0, 1\}; \text{maj})$  outputs 1.

In fact, there exists a cyclic  $t$  in  $\mathbf{A}$  (say,  $p$ -ary) such that  $R(x, y) = B(x) \vee (y = 0)$  is a subuniverse of  $(A; t) \times (\{0, 1\}; \text{maj}_p)$  for every 3-absorbing  $B$ , where  $\text{maj}_p$  denotes the  $p$ -ary majority function. This translates to a simpler statement: for every 3-absorbing  $B$  we have  $t(\mathbf{a}) \in B$  whenever majority of the  $a_i$  belong to  $B$ , and we cannot expect more, as witnessed by the 2-element majority algebra. Since  $t$  is cyclic it generates the whole clone and, for example, 3) in Proposition V.11 below becomes obvious.

Item (b) provides a relational description of 3-absorption, while item (c) provides a connection with the notion of center (whether it is Taylor or not). We now give an example that (e) and (a) are not equivalent even in minimal Taylor algebras if  $B$  has more than one element.

**Example V.10.** *Consider the algebra  $\mathbf{A} = (\{0, 1, 2\}, m)$  where  $m$  is the majority operation such that  $m(a, b, c) = a$  whenever  $|\{a, b, c\}| = 3$ . This algebra is minimal Taylor and the set  $C = \{0, 1\}$  is an absorbing subuniverse of  $\mathbf{A}$ . However,  $C$  is not a center of  $\mathbf{A}$ .*

Finally, we list some strong and unusual properties of 3-absorbing subuniverses. They are not as strong as in the case of 2-absorbing subuniverses, which is to be expected since every 2-absorbing subuniverse is 3-absorbing but not vice versa.

**Proposition V.11.** *Let  $\mathbf{A}$  be a minimal Taylor algebra and  $B, C \leq_3 \mathbf{A}$ .*

- 1)  $B \cup C \leq \mathbf{A}$
- 2) *If  $B \cap C \neq \emptyset$  then  $B \cap C \leq_3 \mathbf{A}$ .*
- 3) *If  $B \cap C = \emptyset$  then  $B^2 \cup C^2$  is a congruence on the algebra with universe  $B \cup C$  and the quotient is term-equivalent to a two-element majority algebra.*

### C. Edges

The next theorem says that, in minimal Taylor algebras, every “thick” edge, in the terminology of [37], [47], is automatically a subuniverse. This property is a simple consequence of the result we have already stated, whereas it was relatively painful to achieve using the original approach. We additionally obtain that semilattice and majority edges have unique witnessing congruences.

**Theorem V.12.** *Let  $(a, b)$  be an edge (semilattice, majority, or abelian) of a minimal Taylor algebra  $\mathbf{A}$  and  $\theta$  a witnessing congruence of  $\mathbf{E} = \text{Sg}_{\mathbf{A}}(a, b)$ .*

- (a) *If  $(a, b)$  is a semilattice edge, then  $\mathbf{E}/\theta$  is term equivalent to a two-element semilattice with absorbing element  $b/\theta$ .*
- (b) *If  $(a, b)$  is a majority edge, then  $\mathbf{E}/\theta$  is term equivalent to a two-element majority algebra.*
- (c) *if  $(a, b)$  is an abelian edge, then  $\mathbf{E}/\theta$  is term equivalent to an affine Mal’cev algebra of an abelian group isomorphic to  $\mathbb{Z}/p_1^{k_1} \times \cdots \times \mathbb{Z}/p_i^{k_i}$  for distinct primes  $p_1, \dots, p_i$  and positive integers  $k_1, \dots, k_i$ , where  $\mathbb{Z}/m$  denotes the group of integers modulo  $m$ .*

Moreover a semilattice edge is witnessed by exactly one congruence of  $\mathbf{E}$ , and that congruence is maximal. The same holds for majority edges.

For minimal edges we can say a bit more. If  $(a, b)$  is a minimal edge witnessed by  $\theta$ , a congruence on  $\mathbf{E} = \text{Sg}_{\mathbf{A}}(a, b)$ , then  $\mathbf{E}/\theta$  is simple. In particular, for abelian edges,  $\mathbf{E}/\theta$  is an affine Mal'cev algebra of a group isomorphic to  $\mathbb{Z}/p$ . Moreover, such an  $\mathbf{E}$  has a unique maximal congruence as shown in the next proposition. This implies that the type of a minimal edge is unique and so is the direction of a semilattice minimal edge and the prime  $p$  associated to an abelian minimal edge.

**Proposition V.13.** *Let  $(a, b)$  be a minimal edge in a minimal Taylor algebra. Then  $\mathbf{E} = \text{Sg}_{\mathbf{A}}(a, b)$  has a unique maximal congruence equal to  $\mu_{\mathbf{E}}$ . In particular, minimal edges have unique types.*

The structure of minimal semilattice edges is especially simple.

**Proposition V.14.** *Let  $(a, b)$  be a minimal semilattice edge in a minimal Taylor algebra. Then  $\{a, b\}$  is a subuniverse of  $\mathbf{A}$ , so  $\text{Sg}_{\mathbf{A}}(a, b) = \{a, b\}$  and the witnessing congruence is the equality.*

Unfortunately, majority and abelian edges do not simplify in a similar way; see Example V.15 and Example V.16. Weaker versions of Proposition V.14 have been developed by Bulatov (comp. Lemma 12 and Corollary 13 in [37]) to deal with this problem.

**Example V.15.** *Let  $A = \{0, 1, 2, 3\}$  and  $\alpha$  the equivalence relation on  $A$  with blocks  $\{0, 2\}$  and  $\{1, 3\}$ . Define a symmetric ternary operation  $g$  on  $A$  as follows. When two of the inputs to  $g$  are equal,  $g$  is given by  $g(a, a, a+1) = a$ ,  $g(a, a, a+2) = g(a, a, a+3) = a+2$  (all modulo 4) and when all three inputs to  $g$  are distinct,  $g$  is given by  $g(a, b, c) = d-1 \pmod{4}$  where  $a, b, c, d$  are any permutation of  $0, 1, 2, 3$ . Then  $\mathbf{A} = (A; g)$  is a minimal Taylor algebra,  $\alpha$  is a congruence on  $\mathbf{A}$ , and each of pair of elements in different  $\alpha$ -blocks is a minimal majority edge with witnessing congruence  $\alpha$ .*

**Example V.16.** *Let  $\mathbf{A} = (\{a, b, c, d\}, p)$ , where  $p$  is a Mal'cev operation with the following properties. The operation  $p$  commutes with the permutations  $\sigma = (a\ c)$  and  $\tau = (b\ d)$ . The polynomials  $+_a = p(\cdot, a, \cdot)$ ,  $+_b = p(\cdot, b, \cdot)$  define abelian groups:*

| $+_a$ | $a$ | $b$ | $c$ | $d$ | $+_b$ | $a$ | $b$ | $c$ | $d$ |
|-------|-----|-----|-----|-----|-------|-----|-----|-----|-----|
| $a$   | $a$ | $b$ | $c$ | $d$ | $a$   | $b$ | $a$ | $d$ | $c$ |
| $b$   | $b$ | $c$ | $d$ | $a$ | $b$   | $a$ | $b$ | $c$ | $d$ |
| $c$   | $c$ | $d$ | $a$ | $b$ | $c$   | $d$ | $c$ | $b$ | $a$ |
| $d$   | $d$ | $a$ | $b$ | $c$ | $d$   | $c$ | $d$ | $a$ | $b$ |

Then  $\mathbf{A}$  is a minimal Taylor algebra, with a unique maximal congruence  $\theta$  whose congruence classes are  $\{a, c\}$  and  $\{b, d\}$ . Each pair of elements of  $\mathbf{A}$  in different congruence classes of  $\theta$  is a minimal abelian edge of  $\mathbf{A}$  with witnessing congruence  $\theta$ .

We can also provide nontrivial information about  $\text{Sg}(a, b)$  in case that  $(a, b)$  is not necessarily an edge, and this information helps in proving Theorem V.23 in the next subsection (and shows that case (d) in Corollary IV.12 is never necessary for two-generated algebras). However, the following fundamental question remains open: Is there a minimal Taylor algebra such that, for some  $a, b$ , neither  $(a, b)$  nor  $(b, a)$  is an edge?

#### D. Absorption and edges

We start this subsection with a definition that will connect absorption with edges.

**Definition V.17.** *Let  $\mathbf{A}$  be an algebra, let  $B \subseteq A$  and let  $(b, a)$  be an edge. We say that  $B$  is stable under  $(b, a)$  if, for every witnessing congruence  $\theta$  of  $\text{Sg}_{\mathbf{A}}(b, a)$  such that  $b/\theta$  intersects  $B$ , each  $\theta$ -block intersects  $B$ .*

As the next theorem states, stability under every edge can be added as a next item to Theorem V.7. This direct connection of absorption, which is a global property, to local concepts in Bulatov's theory is among the most surprising phenomena that the authors have encountered in this work.

**Theorem V.18.** *The following are equivalent for any minimal Taylor algebra  $\mathbf{A}$  and a set  $B \subseteq A$ .*

- (a)  $B$  2-absorbs  $\mathbf{A}$ .
- (b)  $B$  is stable under all the edges.

The implication from (b) to (a) does not require the full strength of stability for semilattice and majority edges. It is enough to require that for a minimal semilattice or a majority edge  $(b, a)$  it is never the case that  $b/\theta \subseteq B$  and  $a/\theta \cap B = \emptyset$ , where  $\theta$  is the edge-witnessing congruence of  $\text{Sg}(b, a)$  (which is the equality relation on  $\{a, b\}$  in case of semilattice edges). The following example shows that stability under abelian edges cannot be significantly weakened.

**Example V.19.** *We consider the four-element algebra  $\mathbf{A} = (\{0, 1, 2, *\}, \cdot)$  with binary operation  $\cdot$  given by*

| $\cdot$ | 0 | 1 | 2 | * |
|---------|---|---|---|---|
| 0       | 0 | 2 | 1 | * |
| 1       | 2 | 1 | 0 | 2 |
| 2       | 1 | 0 | 2 | 1 |
| *       | * | 2 | 1 | * |

Then  $\mathbf{A}$  is a minimal Taylor algebra, with a semilattice edge  $(0, *)$ , with  $\{0, 1, 2\}$  an affine subalgebra, and with a congruence  $\theta$  corresponding to the partition  $\{0, *\}, \{1\}, \{2\}$  such that  $\mathbf{A}/\theta$  is affine. The set  $\{*\}$  is stable under semilattice and majority edges and there is no minimal abelian edge  $(*, a)$  with  $a \neq *$ . But  $\{*\}$  is not an absorbing subalgebra of  $\mathbf{A}$ .

For absorption of higher arity the connection to edges is not as tight as for 2-absorption. Nevertheless, one direction still works and both directions work for singletons.

**Theorem V.20.** *Any absorbing set of a minimal Taylor algebra  $\mathbf{A}$  is stable under semilattice and abelian edges. Moreover, for any  $b \in A$  the following are equivalent.*

- (a)  $\{b\}$  absorbs  $\mathbf{A}$
- (b)  $\{b\}$  is stable under semilattice and abelian edges.

Stability under semilattice edges for the implication from (b) to (a) can be again replaced by the requirement that there is no minimal semilattice edge  $(b, a)$  with  $b \in B$  and  $a \notin B$ . Example V.19 shows that this is not the case for abelian edges.

The following example shows that the implication from (b) to (a) does not hold for non-singleton subuniverses.

**Example V.21.** Consider the algebra  $\mathbf{A} = (\{0, 1, 2\}, m)$  where  $m$  is the majority operation such that  $m(a, b, c) = 2$  whenever  $|\{a, b, c\}| = 3$ . This algebra is minimal Taylor, every pair of distinct elements forms a subuniverse, and every pair is a minimal majority edge. So there are no semilattice or abelian edges. However, the subuniverse  $\{0, 1\}$  is not absorbing.

An important fact for the edge approach is that semilattice, majority, and Mal'cev operations coming from edges can be unified, see Theorem 7 in [37]. In minimal Taylor algebras, a simple consequence of the already stated results is that we not only have a common ternary witness for all the edges but also for all the binary and ternary absorptions.

**Theorem V.22.** Every minimal Taylor algebra  $\mathbf{A}$  has a ternary term operation  $f$  such that if  $(a, b)$  is an edge witnessed by  $\theta$  on  $\mathbf{E} = \text{Sg}_{\mathbf{A}}(a, b)$ , then

- if  $(a, b)$  is a semilattice edge, then  $f(x, y, z) = x \vee y \vee z$  on  $\mathbf{E}/\theta$  (where  $b/\theta$  is the top);
- if  $(a, b)$  is a majority edge, then  $f$  is the majority operation on  $\mathbf{E}/\theta$  (which has two elements);
- if  $(a, b)$  is an abelian edge, then  $f(x, y, z) = x - y + z$  on  $\mathbf{E}/\theta$ ;
- $f$  witnesses all the ternary absorptions  $B \trianglelefteq_3 \mathbf{A}$ ;
- any binary operation obtained from  $f$  by identifying two arguments witnesses all the binary absorptions  $B \trianglelefteq_2 \mathbf{A}$ .

In fact, any ternary operation  $f$  defined from a cyclic term operation  $t$  of odd arity  $p \geq 3$  by

$$f(x, y, z) = t(\underbrace{x, x, \dots, x}_{k \times}, \underbrace{y, y, \dots, y}_{l \times}, \underbrace{z, z, \dots, z}_{m \times}),$$

where  $k + l + m, k + m > p/2$ , satisfies all the items in Theorem V.22 except possibly the third one (which can be obtained by picking  $k, l$ , and  $m$  a bit more carefully).

We finish this section with a theorem stating that any ternary witness of edges generates the whole clone of the algebra. In particular, the number of minimal Taylor clones on a domain of size  $n$  is at most  $n^{n^3}$ .

**Theorem V.23.** If  $\mathbf{A}$  is a minimal Taylor algebra, then  $\text{Clo}(\mathbf{A}; f) = \text{Clo}(\mathbf{A})$  for any operation  $f$  satisfying the first three items in Theorem V.22.

## VI. OMITTING TYPES

In this section we consider classes of algebras whose graph only contains edges of certain types. We say that an algebra is *a-free* if it has no abelian edges. More generally, an algebra is *x-free* or is *xy-free*, where  $x, y \in \{(a)belian, (m)ajority, (s)emilattice\}$  if it has no edges of type  $x$  (of types  $x, y$ ).

It turns out that within minimal Taylor algebras these “omitting types” conditions are often equivalent to important properties of algebras. In the theorems below we prove the equivalence of the following four types of conditions: (i) the absence of edges of a certain type (equivalently, minimal edges of the same type); (ii) properties of absorption and the four types in Zhuk’s approach; (iii) the existence of a certain special term operations; (iv) algorithmic properties of the CSP. Here recall that the properties of “having bounded width” and “having few subpowers” characterize the applicability of the two basic algorithmic ideas in the CSP – local propagation algorithms [48], [29] and finding a generating set of all solutions [49], [30]. Theorems in this section are consequences of the theory we have already built in the previous section and known results (see [5]).

The first theorem concerns the class of algebras omitting abelian edges. Numerous characterizations of this class are known for general algebras and we do not add a new one, but we state the characterization for comparison with the other classes. In order to state a characterization in terms of identities we recall that an operation  $f$  is a *weak near unanimity* operation (or *wnu* for short) if it satisfies  $f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, \dots, x, y)$  for every  $x, y$  in the algebra.

**Theorem VI.1.** The following are equivalent for any algebra  $\mathbf{A}$ .

- (i)  $\mathbf{A}$  is *a-free*.
- (ii) No subalgebra of  $\mathbf{A}$  falls into case (c) in Corollary IV.12, i.e., no subalgebra of  $\mathbf{A}$  has a nontrivial affine quotient.
- (iii)  $\mathbf{A}$  has a *wnu* term operation of every arity  $n \geq 3$ .
- (iv)  $\mathbf{A}$  has bounded width.

Minimal Taylor algebras omitting other types of edges *do* have significantly stronger properties than general Taylor algebras omitting those edges. Minimal *s-free* algebras are exactly those for which option (a) in Corollary IV.12 does not hold, and that have the few subpowers property [47]. The few subpowers property, i.e., that the number of subuniverses of  $\mathbf{A}^n$  is  $2^{O(n)}$ , can be characterized by the existence of an edge term operation [49] in general. In minimal Taylor algebras, the second strongest edge operation always exists – the 3-edge operation defined by the identities  $e(y, y, x, x) = e(y, x, y, x) = e(x, x, x, y) = x$ . This is significant, because the exponent in the running time of the few subpowers algorithm depends on the least  $k$  such that the algebra has a  $k$ -edge term operation. The number 3 here is best possible: a 2-edge operation is the same as a Mal'cev operation appearing in Theorem VI.6.

**Theorem VI.2.** The following are equivalent for any minimal Taylor algebra  $\mathbf{A}$ .

- (i)  $\mathbf{A}$  is *s-free*.
- (ii) Case (a) in Corollary IV.12 does not hold, that is, no subalgebra of  $\mathbf{A}$  has a nontrivial 2-absorbing subuniverse.
- (iii)  $\mathbf{A}$  has a 3-edge term operation.
- (iv)  $\mathbf{A}$  has few subpowers.

For the remaining omitting-single-type condition, m-freeness, we do not provide a natural condition in terms of identities, and we are not aware of algorithmic implications of this condition. Nevertheless, it can be characterized by means of absorption.

**Theorem VI.3.** *The following are equivalent for any minimal Taylor algebra  $\mathbf{A}$ .*

- (i)  $\mathbf{A}$  is m-free.
- (ii) Every center (3-absorbing subuniverse of)  $\mathbf{B} \leq \mathbf{A}$  2-absorbs  $\mathbf{B}$ , i.e., (b) implies (a) in Corollary IV.12 in all the subalgebras of  $\mathbf{A}$ .
- (ii') Every subalgebra of  $\mathbf{A}$  has a unique minimal 3-absorbing subuniverse

Surprisingly, if along with m-freeness we also limit the type of abelian edges allowed in an algebra, the resulting condition is equivalent to the existence of a binary commutative term operation. This is interesting, since the existence of a commutative term operation was not considered to be a natural requirement for the CSP (see the discussion in [5]) or in Universal Algebra. We call an abelian edge  $(a, b)$  a  $\mathbb{Z}/2$ -edge if the corresponding affine Mal'cev algebra  $\text{Sg}(a, b)/\theta$  is isomorphic to the affine Mal'cev algebra of  $\mathbb{Z}/2$ .

**Theorem VI.4.** *The following are equivalent for any minimal Taylor algebra  $\mathbf{A}$ .*

- (i)  $\mathbf{A}$  is m-free and has no  $\mathbb{Z}/2$ -edges.
- (iii)  $\mathbf{A}$  has a binary commutative term operation
- (iii')  $\text{Clo}(\mathbf{A})$  can be generated by a collection of binary operations.

Properties of minimal Taylor algebras having edges of only one type can be derived as conjunctions of the properties stated above. For two of these cases, sm-free and as-free, we provide additional information.

Minimal Taylor am-free algebras are exactly those which have wnu operations of every arity  $n \geq 2$ . These are exactly the minimal *spirals* in the terminology of [39] and a significant property is that for every  $(a, b)$  such that neither  $(a, b)$  nor  $(b, a)$  is a minimal semilattice edge, there is a surjective homomorphism from  $\text{Sg}\{a, b\}$  onto the (three-element) free semilattice on two generators.

The sm-free minimal Taylor algebras are those where cases (a) and (b) in Corollary IV.12 do not occur. Additionally, these are exactly the hereditarily absorption free algebras studied in [5] and, also, the algebras with a Mal'cev term operation – a type of operation that played a significant role in the CSP [50].

**Theorem VI.5.** *The following are equivalent for any minimal Taylor algebra  $\mathbf{A}$ .*

- (i)  $\mathbf{A}$  is sm-free.
- (ii) No subalgebra of  $\mathbf{A}$  has a nontrivial absorbing subuniverse.
- (iii)  $\mathbf{A}$  has a Mal'cev term operation.

Finally, the as-free algebras are those where cases (a) and (c) in Corollary IV.12 do not occur and those that have bounded

width and few subpowers. It is known [42], [49] that the latter property in general implies having a near-unanimity term operation of some arity. Surprisingly, in minimal Taylor algebras, the arity goes down directly to three. In the algorithmic language, these algebras have strict width two [2], [5].

**Theorem VI.6.** *The following are equivalent for any minimal Taylor algebra  $\mathbf{A}$ .*

- (i)  $\mathbf{A}$  is as-free.
- (iii)  $\mathbf{A}$  has a near unanimity term operation.
- (iii')  $\mathbf{A}$  has a majority term operation.

## VII. CONCLUSION

We have introduced the concept of minimal Taylor algebras and used it to significantly unify, simplify, and extend the three main algebraic approaches to the CSP – via absorption, via four types, and via edges. We believe that the theory started in this paper will help in attacking further open problems in computational complexity of CSP-related problems and Universal Algebra. There are, however, many directions which call for further exploration.

First, several technical questions naturally arise from the presented results: Do every two elements of a minimal Taylor algebra form an edge? How to characterize sets stable under affine and semilattice edges in a global way? Is it possible to characterize (3-)absorption in terms of edges? Does stability under other edge-types correspond to a global property? Is every minimal bounded width algebra a minimal Taylor algebra? Are the equivalent characterizations in Theorem VI.3 equivalent to “every subalgebra has a unique minimal absorbing (rather than 3-absorbing) subuniverse”?

Second, both CSP dichotomy proofs [9], [11] require and develop more advanced Commutator Theory [51], [52] concepts and results, while in this paper we have merely used some fundamental facts about the basic concept, the abelian algebra. Is it possible to develop our theory in this direction as well, potentially providing sufficient tools for the dichotomy result? Also, is there a natural concept that would replace thin edges in Bulatov's approach?

Third, Brady in [39] provided a complete classification of minimal bounded width algebras of small size. Can such a detailed analysis be made also for minimal Taylor algebras? Is it possible to develop a strong theory or even full classification for minimal algebras in other classes, such as the algebras conjectured to characterize CSPs in log-space or nondeterministic log-space?

Fourth, which of the facts presented in the paper have their counterpart for non-minimal Taylor algebras or even general finite idempotent algebras? Here we would like to mention Ross Willard's work (unpublished) that provides a generalization for some of the advanced facts in Zhuk's approach.

Finally, there is yet another, older, and highly developed theory of finite algebras, the Tame Congruence Theory started in [42]. What are the connections to the theory initiated in this paper?

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