

Discrete and higher-form symmetries in SCFTs from wrapped M5-branes

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ABSTRACT: We analyze topological mass terms of BF type arising in supersymmetric M-theory compactifications to AdS_5 . These describe spontaneously broken higher-form gauge symmetries in the bulk. Different choices of boundary conditions for the BF terms yield dual field theories with distinct global discrete symmetries. We discuss in detail these symmetries and their 't Hooft anomalies for 4d $\mathcal{N} = 1$ SCFTs arising from M5-branes wrapped on a Riemann surface without punctures, including theories from M5-branes at a $\mathbb{Z}_2$ orbifold singularity. The anomaly polynomial is computed via inflow and contains background fields for discrete global 0-, 1-, and 2-form symmetries and continuous 0-form symmetries, as well as axionic background fields. The latter are properly interpreted in the context of anomalies in the space of coupling constants.

KEYWORDS: Anomalies in Field and String Theories, Conformal Field Theory, Discrete Symmetries, M-Theory

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1 Introduction and summary

't Hooft anomalies are robust and useful observables in quantum field theory. They are invariant under renormalization group flow and can be used to constrain the phases of theories at long distances. The most familiar type of 't Hooft anomalies are arguably perturbative anomalies for continuous, ordinary (0-form) symmetries, which only occur in even spacetime dimension. The full set of anomalies, however, is much richer. This work is concerned with anomalies for discrete symmetries, generalized or higher-form symmetries [1], and anomalies in the space of coupling constants of a quantum field theory [2, 3].

't Hooft anomalies are particularly helpful in the study of the dynamics of strongly-coupled field theories in the framework of geometric engineering. Moreover, anomalies provide an organizing principle in exploring the landscape of such theories. Discrete higher-form symmetries for field theories engineered by M-theory on a singular local geometry have been recently studied in [4, 5].

This work focuses on field theories engineered with M5-branes. Using M5-branes, one can realize 6d (2,0) theories of type A_{N-1} [6, 7], as well as 6d (1,0) theories obtained by putting the M5-brane stack on top of an orbifold singularity [8]. A vast class of 4d theories is realized by further compactification on a Riemann surface, possibly with punctures, as first studied for $\mathcal{N} = 2$ theories in [9, 10], and further extended to $\mathcal{N} = 1$ theories [11–15]. It is beneficial to develop tools to extract 't Hooft anomalies of a field theory engineered using branes directly from the topology and geometry of the brane configuration. Anomaly inflow provides the framework to address this problem. Building on the results of [16–19] about anomaly inflow onto a stack of M5-branes, systematic tools have been developed to compute perturbative 't Hooft anomalies for 0-form symmetries via inflow for setups engineered with M5-branes [20–22] and D3-branes [23].

A more complete understanding of the space of quantum field theories would require one to extend the scope of this program to include other types of 't Hooft anomalies. In this paper, we address a class of discrete and higher-form symmetries for 4d SCFTs engineered with wrapped M5-branes. In particular, we perform a detailed analysis for M5-branes probing a $\mathbb{Z}_2$ singularity, further wrapped on a Riemann surface. This case study furnishes a controlled example that exhibits interesting features. Our strategy and results are summarized below.

Summary of results. For a 4d SCFT engineered with wrapped M5-branes, non-trivial information about 't Hooft anomalies for discrete symmetries, higher-form symmetries, and anomalies in the space of coupling constants [2] can be extracted via anomaly inflow. This is done by studying the topological couplings in the 5d low-energy effective action originating from reduction of M-theory on M_6 , the compact space that encodes the geometry transverse to the four extended directions of the M5-branes worldvolume.

In our analysis, we include all 5d 0-, 1-, 2- and 3-form gauge fields associated to expansion of the M-theory 3-form C_3 onto cohomology classes of M_6 , as well as 1-form gauge fields associated to isometries of M_6 . A crucial role is played by 5d topological mass terms of BF type between a 1-form gauge field $\mathcal{A}_1$ and a 3-form gauge field c_3 , and between pairs $(B_{2i}, \tilde{B}_2^i)$ of 2-form gauge fields,

$$S = \int_{\mathcal{M}_5} \left[-\frac{1}{2\pi} k c_3 \wedge d\mathcal{A}_1 - \frac{1}{2\pi} N \tilde{B}_2^i \wedge dB_{2i} \right], \quad (1.1)$$

where $\mathcal{M}_5$ is 5d spacetime, and i labels the pairs of 2-form gauge fields. As we shall see, for setups with wrapped M5-branes the integers k and N are determined by the G_4 -flux quanta of the system, and $i = 1, \dots, g$ where g is the genus of the Riemann surface. The BF term $\frac{1}{2\pi} k \mathcal{A}_1 \wedge dc_3$ implies that the 5d U(1) 0-form gauge symmetry associated to $\mathcal{A}_1$ is spontaneously broken to a $\mathbb{Z}_k$ 0-form gauge symmetry, and the 5d U(1) 2-form gauge symmetry associated to c_3 is spontaneously broken to a $\mathbb{Z}_k$ 2-form gauge symmetry (see e.g. [24, 25] for reviews). In a similar way, for each i the term $\frac{1}{2\pi} N \tilde{B}_2^i \wedge dB_{2i}$ signals the spontaneous breaking of a bulk U(1)² 1-form gauge symmetry to a $(\mathbb{Z}_N)^2$ 1-form gauge symmetry. After a choice of topological boundary conditions for the BF terms is made, the discrete gauge symmetries in the bulk are mapped to discrete global symmetries of the 4d SCFT. Moreover, the extended operators of the 5d BF theory are mapped to defects in the 4d SCFT, which are charged under the discrete global symmetries. A similar analysis in the context of AdS₄/CFT₃ has been recently performed for ABJM-type theories [26].

In order to compute the full set of topological terms in five dimensions, including the contributions of gauge fields associated to isometries of M_6 and an arbitrary external space-time metric, we use the tools developed in [22]. The 5d topological terms are conveniently encoded in a gauge-invariant closed 6-form I_6^{inflow} , which is a polynomial in the 5d gauge field strengths. As concrete examples, we consider 4d SCFTs engineered by M5-branes wrapped on a Riemann surface [14, 15], as well as theories from M5-branes wrapped on a Riemann surface and probing a $\mathbb{Z}_2$ singularity — in this case the gravity dual was identified in [27] to be one of the solutions first discussed in [28]. The 6-form I_6^{inflow} for these setups are given in (5.7), (5.9), respectively.

The 6-form I_6^{inflow} encodes the 't Hooft anomalies of the 4d SCFT, together with the anomalies of modes that decouple in the IR. Since the 5d bulk theory contains massive gauge fields, care has to be taken in reading off 4d 't Hooft anomalies from I_6^{inflow} .

If one is interested in perturbative anomalies for continuous global symmetries of the 4d field theory, the topologically massive gauge fields in five dimensions must be integrated out. A similar mechanism is at play for 6d (1,0) SCFTs engineered with M5-branes probing

an ALE singularity, and clarifies how the Green-Schwarz terms in the 8-form anomaly polynomial [29] are reproduced by inflow.

The perturbative anomaly polynomial for wrapped M5-branes probing a $\mathbb{Z}_2$ singularity, recorded in (5.14), contains several terms with 0-form gauge fields (i.e. axions), with 1-form field strengths. Following [2], we interpret such terms as anomalies in the space of coupling constants. The couplings in question are associated to exactly marginal operators of the 4d SCFT. We argue that these operators can be thought of as dimensional reduction on the Riemann surface of the 6d conserved $U(1)$ currents associated to the Cartan $U(1)_N \times U(1)_S$ of the $SU(2)_N \times SU(2)_S$ flavor symmetry of the 6d (1,0) theory engineered by M5-branes on a $\mathbb{Z}_2$ singularity.

After the continuous part of a topologically massive gauge field is integrated out, a discrete gauge field is left over, whose precise features depend on the choice of boundary conditions for the BF terms. Hence, in order to extract 4d 't Hooft anomalies for discrete symmetries from the 6-form I_6^{inflow} , we need to specify the boundary conditions. For definiteness, we focus on the case in which we assign Dirichlet boundary conditions to the fields $\mathcal{A}_1$ and B_{2i} (and free boundary conditions to c_3 and $\tilde{B}_2^i$), so that the 4d field theory admits a $\mathbb{Z}_k$ global 0-form symmetry and a $(\mathbb{Z}_N)^g$ global 1-form symmetry. The field c_3 acts as a Lagrange multiplier that imposes a constraint on $\mathcal{A}_1$. If we write $\mathcal{A}_1 = \mathcal{A}_1^{\text{cont}} + \mathbf{A}_1$, the constraint fixes $\mathcal{A}_1^{\text{cont}}$ in terms of gauge fields for continuous symmetries, and forces $\mathbf{A}_1$ to be a flat 1-form gauge field with holonomies that are k -th roots of unity. Similarly, the Lagrange multiplier $\tilde{B}_2^i$ imposes a constraint on $B_{2i} = B_{2i}^{\text{cont}} + \mathbf{B}_{2i}$, which determines B_{2i}^{cont} in terms of continuous gauge fields, and forces $\mathbf{B}_{2i}$ to be a flat 2-form gauge field with holonomies that are N -th roots of unity.

By substituting $\mathcal{A}_1 = \mathcal{A}_1^{\text{cont}} + \mathbf{A}_1$, $B_{2i} = B_{2i}^{\text{cont}} + \mathbf{B}_{2i}$ into the 6-form I_6^{inflow} , we obtain a formal expression that encodes 't Hooft anomalies for both the continuous symmetries and the $\mathbb{Z}_k$ 0-form symmetry and $(\mathbb{Z}_N)^g$ 1-form symmetry. Discrete anomalies are read off from terms in I_6^{inflow} with $d\mathbf{A}_1$, $d\mathbf{B}_{2i}$. These objects are zero as differential forms. To circumvent this difficulty, we reinterpret the quantity I_6^{inflow} in the framework of differential cohomology (see e.g. [2, 30] and appendix D for some background material). Differential forms are regarded as a proxy for classes in differential cohomology, and their wedge product is a proxy for the product in differential cohomology. A crucial feature of the latter is that the product of a flat gauge field with other gauge fields is not necessarily zero. This approach dates back to Dijkgraaf and Witten [31] and has also been recently used in [32].

We apply the recipe outlined in the previous paragraphs to the setup with wrapped M5-branes probing a $\mathbb{Z}_2$ singularity. The terms in I_6^{inflow} involving $d\mathbf{A}_1$, $d\mathbf{B}_{2i}$ are collected schematically in (5.33), while the full result is recorded in appendix E. We encounter a rich variety of 't Hooft anomalies involving the discrete symmetries, including: a cubic term in $d\mathbf{A}_1$; terms mixing $d\mathbf{A}_1$ and $d\mathbf{B}_{2i}$ to the other continuous symmetries, including a gravitational term $d\mathbf{A}_1 p_1(T)$; a mixed anomaly between the two discrete symmetries and a coupling constant.

Finally, we observe that the BF couplings in the 5d topological bulk theory can also be used to identify some of the singleton modes of the 5d supergravity theory. (By “singleton modes” we mean modes that are pure gauge in the 5d bulk, but propagate on the conformal

boundary; they are holographically dual to modes in the 4d field theory that decouple in the IR.) For setups with wrapped M5-branes with 4d $\mathcal{N} = 2$ supersymmetry, the knowledge of singleton modes from BF terms, combined with supersymmetry, is sufficient to reconstruct from the gravity side the entire set of modes that decouple on the field theory side. This offers a proof of principle that one can compute the exact anomaly polynomial from the gravity dual, including $\mathcal{O}(1)$ terms in the number of M5-branes.

2 Topological mass terms in 5d supergravity

Let us consider a supersymmetric AdS_5 solution of M-theory with internal space M_6 . These solutions were classified in [28]. We study the 5d supergravity theory obtained from reduction of M-theory on a warped product of the form $\mathcal{M}_5 \times_w M_6$, where external spacetime $\mathcal{M}_5$ is negatively curved. The case $\mathcal{M}_5 = AdS_5$ is recovered as the vacuum solution of the 5d supergravity theory. We restrict our attention to solutions where the space M_6 is compact and smooth, and the warp factor is smooth and non-vanishing. In this section we focus on the topological couplings in the low-energy effective action of the 5d supergravity.

In particular, we are interested in identifying the topological mass terms for the p -form gauge fields that arise from Kaluza-Klein expansion of the M-theory 3-form C_3 onto a basis of non-trivial cohomology classes on M_6 . If the internal space M_6 has isometries, the 5d supergravity theory contains additional (possibly non-Abelian) gauge fields. For the remainder of this section, these gauge fields associated to isometries of M_6 are turned off, since it can be checked that they do not contribute to the topological terms of interest. They will be reinstated in section 5.

2.1 Ansatz for G_4 and dimensional reduction

The spectrum of the 5d supergravity obtained from reduction of M-theory on M_6 contains massless Abelian p -form gauge fields coming from the Kaluza-Klein expansion of the M-theory 3-form C_3 . These massless p -form gauge fields are in 1-to-1 correspondence with non-trivial cohomology classes of M_6 .

For each $q = 0, \dots, 6$ we choose a basis in the lattice $H^q(M_6, \mathbb{Z})_{\text{free}}$,¹ which has rank given by the Betti number $b^q(M_6)$. The Betti numbers of M_6 satisfy $b^0(M_6) = b^6(M_6) = 1$, $b^1(M_6) = b^5(M_6)$, $b^2(M_6) = b^4(M_6)$. Elements of $H^q(M_6, \mathbb{Z})_{\text{free}}$ can be identified with de Rham cohomology classes of closed q -forms with integral periods. As a result, we can represent a basis of $H^q(M_6, \mathbb{Z})_{\text{free}}$ using a set of closed (but not exact) q -forms on M_6 with integral periods. We use the following notation for these forms,

$$\begin{aligned} \text{1-forms:} & \quad \lambda_{1u}, & u = 1, \dots, b^1(M_6), \\ \text{2-forms:} & \quad \omega_{2\alpha}, & \alpha = 1, \dots, b^2(M_6), \\ \text{3-forms:} & \quad \Lambda_{3x}, & x = 1, \dots, b^3(M_6), \\ \text{4-forms:} & \quad \Omega_4^\alpha, & \alpha = 1, \dots, b^2(M_6). \end{aligned} \tag{2.1}$$

¹This is the finitely generated free Abelian group defined by the short exact sequence

$$0 \rightarrow \text{Tor } H^q(M_6, \mathbb{Z}) \rightarrow H^q(M_6, \mathbb{Z}) \rightarrow H^q(M_6, \mathbb{Z})_{\text{free}} \rightarrow 0,$$

where $\text{Tor } H^q(M_6, \mathbb{Z})$ is the torsion subgroup of $H^q(M_6, \mathbb{Z})$.

The 5d gauge fields originating from C_3 and their field strengths are denoted as follows,

$$\begin{aligned}
 \text{0-form potentials:} \quad & a_0^x, \quad f_1^x = da_0^x, \quad x = 1, \dots, b^3(M_6), \\
 \text{1-form potentials:} \quad & A_1^\alpha, \quad F_2^\alpha = dA_1^\alpha, \quad \alpha = 1, \dots, b^2(M_6), \\
 \text{2-form potentials:} \quad & B_2^u, \quad H_3^u = dB_2^u, \quad u = 1, \dots, b^1(M_6), \\
 \text{3-form potential:} \quad & c_3, \quad \gamma_4 = dc_3.
 \end{aligned} \tag{2.2}$$

Throughout this work, we adopt conventions in which the periods of the field strength of an Abelian p -form gauge fields are quantized in units of 2π . (A 0-form gauge field whose field strength is quantized in units of 2π is the same as a compact scalar field with period 2π .)

In string/M-theory compactifications, torsion cycles in the internal space can be a source of discrete gauge symmetries [33, 34]. In this work, we do not study the effects of torsion in the homology of M_6 . The geometries M_6 that are relevant for the setups with wrapped M5-branes studied in this paper do not have torsion in homology.

With the notation introduced in (2.1) and (2.2), the M-theory 4-form field strength $G_4 = dC_3$, including both its background value and fluctuations associated to cohomology classes on M_6 , is given by

$$\frac{G_4}{2\pi} = N_\alpha \Omega_4^\alpha + \frac{F_2^\alpha}{2\pi} \wedge \omega_{2\alpha} + \frac{f_1^x}{2\pi} \wedge \Lambda_{3x} + \frac{H_3^u}{2\pi} \wedge \lambda_{1u} + \frac{\gamma_4}{2\pi}. \tag{2.3}$$

The integers N_α specify the background flux that threads M_6 . The periods of G_4 in (2.3) are quantized in units of 2π .²

In our normalization conventions for G_4 , the topological terms of the low-energy effective action of M-theory are³

$$S_{\text{top}} = \int_{M_{11}} \left[-\frac{1}{6(2\pi)^2} C_3 \wedge G_4 \wedge G_4 - \frac{1}{2\pi} C_3 \wedge X_8 \right], \quad X_8 = \frac{p_1^2(TM_{11}) - 4p_2(TM_{11})}{192}. \tag{2.4}$$

The low-energy effective action for the 5d p -form gauge fields listed in (2.2) is computed via standard Kaluza-Klein reduction. Recall that external metric fluctuations and gauge fields associated to isometries of M_6 are turned off in this section. For the purpose of

²The flux quantization condition in M-theory on an orientable spacetime M_{11} can be written as [35]

$$\int_{C_4} \frac{G_4}{2\pi} = \frac{1}{2} \int_{C_4} w_4(TM_{11}) \mod 1, \quad \text{for any 4-cycle } C_4 \text{ in } M_{11}.$$

where w_4 denotes the fourth Stiefel-Whitney class. It is known that w_4 is zero for a spin manifold of dimension ≤ 7 (the argument can be found for instance on page 65 of [36]). In our setups the internal space M_6 and external spacetime are spin manifolds, hence the shift in the quantization condition of G_4 is not important. This holds true also for the purposes of writing the anomaly polynomial of a 4d theory using descent: in that case external spacetime is effectively six-dimensional.

³In these conventions, the Einstein-Hilbert term and the kinetic term for G_4 take the form

$$S_{\text{kin}} = \int_{M_{11}} \left[2\pi (2\pi \ell_P)^{-9} R * 1 - \frac{1}{2} (2\pi)^{-1} (2\pi \ell_P)^{-3} G_4 \wedge *G_4 \right],$$

where ℓ_P is the 11d Planck length. The action enters the path integral via e^{iS} and is defined mod 2π .

computing the effective action for the modes in (2.2) the term $C_3 X_8$ plays no role.⁴ The kinetic term for G_4 yields standard kinetic terms for the 5d gauge fields. The Chern-Simons coupling $C_3 G_4 G_4$ yields a set of topological terms in the 5d effective action. They are most conveniently written in terms of a gauge-invariant 6-form,

$$S_{\text{top}} = 2\pi \int_{\mathcal{M}_5} I_5^{(0)}, \quad dI_5^{(0)} = I_6, \quad (2.5)$$

where the 6-form I_6 is given by

$$I_6 = \frac{1}{(2\pi)^2} \left[-N_\alpha \gamma_4 \wedge F_2^\alpha + \frac{1}{2} N_\alpha \mathcal{K}^{\alpha}_{uv} H_3^u \wedge H_3^v \right] \\ + \frac{1}{(2\pi)^3} \left[-\frac{1}{6} \mathcal{K}_{\alpha\beta\gamma} F_2^\alpha \wedge F_2^\beta \wedge F_2^\gamma + \mathcal{K}_{xu\alpha} f_1^x \wedge F_2^\alpha \wedge H_3^u + \frac{1}{2} \mathcal{K}_{xy} \gamma_4 \wedge f_1^x \wedge f_1^y \right]. \quad (2.6)$$

The quantities $\mathcal{K}^{\alpha uv}$, $\mathcal{K}_{\alpha\beta\gamma}$, $\mathcal{K}_{xu\alpha}$, $\mathcal{K}_{xy}$ are integer intersection numbers which can be defined in terms of the closed forms on M_6 as

$$\mathcal{K}^{\alpha}_{uv} = \int_{M_6} \Omega_4^\alpha \wedge \lambda_{1u} \wedge \lambda_{1v}, \quad \mathcal{K}_{\alpha\beta\gamma} = \int_{M_6} \omega_{2\alpha} \wedge \omega_{2\beta} \wedge \omega_{3\gamma}, \\ \mathcal{K}_{xu\alpha} = \int_{M_6} \Lambda_{3x} \wedge \lambda_{1u} \wedge \omega_{2\alpha}, \quad \mathcal{K}_{xy} = \int_{M_6} \Lambda_{3x} \wedge \Lambda_{3y}. \quad (2.7)$$

These intersection numbers depend only on the cohomology classes of the internal forms, and not on the specific representatives used to write down G_4 in (2.3).

The first two terms in (2.6) are the sought-for topological mass terms in the 5d supergravity effective action. In contrast to the other topological couplings in (2.6), they are quadratic in the external gauge fields. We stress that the topological mass terms are due to the background flux quanta N_α .

When $b^2(M_6) \geq 2$ we are free to consider a change of basis in the lattice $H^2(M_6, \mathbb{Z})_{\text{free}}$, which is accompanied by a change of basis in the external 1-form gauge fields. A new basis $A_1'^\alpha$ can always be found such that $A_1'^{\alpha=1}$ is the only 1-form gauge field with a topological mass term with c_3 . Since $A_1'^{\alpha=1}$ plays a special role compared to the vectors $A_1'^{\alpha \neq 1}$, we introduce the notation

$$A_1'^\alpha = (\mathcal{A}_1, \widehat{\mathcal{A}}_1^\alpha), \quad \hat{\alpha} = 2, 3, \dots, n. \quad (2.8)$$

With this notation we have

$$-N_\alpha F_2^\alpha \wedge \gamma_4 = -k d\mathcal{A}_1 \wedge \gamma_4, \quad k = \text{gcd}(N_\alpha), \quad (2.9)$$

while the vectors $\widehat{\mathcal{A}}_1^\alpha$ do not enter the topological mass terms. Further information about the new basis $A_1'^\alpha$ is collected in appendix A.

⁴Even after the isometry gauge fields are turned on, the term $C_3 X_8$ does not yield topological mass terms (i.e. topological terms quadratic in the external fields) for the cases of interest in this work.

$g \geq 2$	$p \neq 0, q \neq 0, p \neq q$	$U(1)_1 \times U(1)_2$	$\mathcal{N} = 1$
	$p = 0$ or $q = 0$	$SU(2)_1 \times U(1)_2$ or $U(1)_1 \times SU(2)_2$	$\mathcal{N} = 2$ MN
	$p = q$	$SU(2) \times U(1)$	$\mathcal{N} = 1$ MN
$g = 0$	$ p - q > 2$	$U(1)_1 \times U(1)_2 \times SU(2)_\Sigma$	$\mathcal{N} = 1$
$g = 1$	$p \neq 0$	$U(1)_1 \times U(1)_2$	$\mathcal{N} = 1$

Table 1. Summary of the values of p, q that yield smooth AdS_5 solutions in M-theory. Recall $p + q = 2(g - 1)$. In the third column we list the isometries of the internal space M_6 . In the last column, MN stands for Maldacena-Nuñez and refers to the solutions of [37].

2.2 Applications to wrapped M5-branes

In this section we specialize the results of the previous section to two classes of AdS_5 solutions that are particularly relevant in connection to 4d $\mathcal{N} = 1$ SCFTs engineered with M5-branes wrapped on a Riemann surface. More precisely, we consider:

- M5-branes wrapped on a Riemann surface without punctures, which correspond to the solutions of [14, 15], referred to as BBBW.
- M5-branes probing a $\mathbb{Z}_2$ singularity and wrapped on a Riemann surface without punctures [27], which correspond to a class of solutions of [28], referred to as GMSW.

2.2.1 M5-branes wrapped on a Riemann surface

The BBBW solutions [14, 15] describe the near-horizon geometry of a stack of M5-branes wrapped on a genus- g Riemann surface Σ_g with a non-trivial topological twist preserving 4d $\mathcal{N} \geq 1$ superconformal symmetry. The internal space M_6 is topologically an S^4 bundle over Σ_g . Its topology is encoded in two integer numbers p, q satisfying

$$p + q = -\chi(\Sigma_g) = 2(g - 1) . \quad (2.10)$$

We can regard $S^4 \hookrightarrow M_6 \rightarrow \Sigma_g$ as the unit-sphere bundle associated to a real rank-5 vector bundle $\mathbb{R}^5 \hookrightarrow \mathcal{N} \rightarrow \Sigma_g$. The bundle $\mathcal{N}$ is identified with the normal bundle to the M5-brane stack. It splits as $\mathcal{N} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{N}_0$, where $\mathcal{L}_1, \mathcal{L}_2$ are complex line bundles over Σ_g , and $\mathcal{N}_0$ is a trivial real rank-1 vector bundle. The integers p, q are the Chern numbers of the complex line bundles $\mathcal{L}_1, \mathcal{L}_2$, respectively.

In table 1 we summarize the choices of g, p, q for which a smooth AdS_5 M-theory solution exists, and for each case we list the isometries of the internal space M_6 . Some comments are in order. In all cases, M_6 admits at least a $U(1)_1 \times U(1)_2$ isometry, which is the subgroup of the $SO(5)$ isometry of the S^4 fiber that is preserved by the fibration over Σ_g for any choice of p, q . When $g \geq 2, p = 0$, $U(1)_1$ enhances to $SU(2)_1$. Supersymmetry enhances to $\mathcal{N} = 2$ and the isometry group $SU(2)_1 \times U(1)_2$ is identified with the R-symmetry of the SCFT. This setup is the $\mathcal{N} = 2$ Maldacena-Nuñez (MN) solution [37]. Similar remarks apply to $g \geq 2, q = 0$. In the case $g \geq 2, p = q$, the difference of the generators of $U(1)_1$ and $U(1)_2$ enhances to $SU(2)$, which is identified with an enhanced flavor symmetry of the SCFT side. This is the $\mathcal{N} = 1$ MN solution [37]. When $g = 0$,

the Riemann surface is a round sphere S^2 . The space M_6 admits an additional $\text{SO}(3)_\Sigma$ isometry, originating from the isometry of S^2 . Finally, we would like to emphasize that the case $g = 1$, $p = q = 0$, which corresponds to 4d $\mathcal{N} = 4$ SYM theory, is not included in table 1, because there is no smooth AdS_5 M-theory solution with internal space $S^4 \times T^2$ without any twisting. The $\mathcal{N} = 4$ SYM theory is best studied holographically via the standard $AdS_5 \times S^5$ solution in type IIB string theory.

The number of external p -form fields entering the topological terms (2.6) in the 5d effective action is determined by the Betti numbers of M_6 . The latter do not depend on the twist parameters p, q and are given by

$$b^0(M_6) = b^2(M_6) = b^4(M_6) = b^6(M_6) = 1, \quad b^1(M_6) = b^5(M_6) = 2g, \quad b^3(M_6) = 0. \quad (2.11)$$

This claim is verified in appendix B.1, where we also construct the associated closed forms with integral periods. The fact that $b^4(M_6) = 1$ is consistent with the fact that BBBW solutions have only one flux parameter,

$$N_{\alpha=1} = N, \quad (2.12)$$

which is the number of M5-branes in the stack. We notice that $b^1(M_6) = 2g$ stems from the fact that the $2g$ harmonic 1-forms on Σ_g can be pulled back to M_6 , yielding closed but not exact 1-forms, whose de Rham classes account for the entire 1-cohomology of M_6 . The 5d p -form gauge fields originating from the expansion of C_3 are

$$c_3, \quad A_1^{\alpha=1} = A_1, \quad B_2^u, \quad u = 1, \dots, 2g, \quad (2.13)$$

while we do not find any 0-form gauge potential.

Making use of the closed forms of appendix B.1, we can compute explicitly the intersection numbers (2.7). The only non-zero intersection pairing is $\mathcal{K}^{\alpha=1}_{uv}$, which can be written as

$$\mathcal{K}^{\alpha=1}_{uv} = \Omega_{uv} = \mathcal{C}_{1u}^\Sigma \cdot \mathcal{C}_{1v}^\Sigma, \quad u, v = 1, \dots, 2g. \quad (2.14)$$

In the previous expression $\mathcal{C}_{1u}$ denotes a basis of integral 1-homology on Σ_g , and Ω_{uv} is the intersection pairing, which is antisymmetric and non-degenerate. The 6-form I_6 encoding the topological couplings of the 5d action as in (2.6) is given by

$$I_6 = \frac{1}{(2\pi)^2} \left[-N \gamma_4 \wedge F_2 + \frac{1}{2} N \Omega_{uv} H_3^u \wedge H_3^v \right]. \quad (2.15)$$

It is useful to choose a basis $\mathcal{C}_{1u}^\Sigma$ of integral 1-homology on Σ_g that is based on the standard A and B cycles on the Riemann surface. Correspondingly, we write

$$\mathcal{C}_{1u}^\Sigma = (\mathcal{C}_i, \tilde{\mathcal{C}}^i), \quad \begin{aligned} \mathcal{C}_i \cdot \tilde{\mathcal{C}}^j &= -\tilde{\mathcal{C}}^j \cdot \mathcal{C}_i = \delta_i^j, \\ \mathcal{C}_i \cdot \mathcal{C}_j &= \tilde{\mathcal{C}}^i \cdot \tilde{\mathcal{C}}^j = 0, \end{aligned} \quad i, j = 1, \dots, g, \quad (2.16)$$

In other words, the intersection pairing Ω_{uv} in this basis takes the standard form

$$\Omega_{uv} = \begin{pmatrix} 0 & \delta_i^j \\ -\delta_i^j & 0 \end{pmatrix}. \quad (2.17)$$

The group of linear transformations of the lattice $H_1(\Sigma_g, \mathbb{Z})$ that preserve this form of Ω_{uv} is $\text{Sp}(2g; \mathbb{Z})$. (In our notation, $\text{Sp}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})$.) The choice of basis (2.16) implies that the index u on the 2-form gauge fields B_2^u is split into two sets of g values,

$$B_2^u = (B_{2i}, \tilde{B}_2^i), \quad i = 1, \dots, g. \quad (2.18)$$

In this basis, the 6-form I_6 reads

$$I_6 = \frac{1}{(2\pi)^2} \left[-N \gamma_4 \wedge F_2 - N \tilde{H}_3^i \wedge H_{3i} \right]. \quad (2.19)$$

2.2.2 M5-branes probing a $\mathbb{Z}_2$ singularity and wrapped on a Riemann surface

Let us now consider a class of solutions first discussed in [28]. The space M_6 is topologically an S^2 bundle over the product of two Riemann surfaces. If one of the Riemann surfaces is a torus, the setup is best studied by dualizing the M-theory solution to a type IIB string theory solution. We thus focus on the case where both Riemann surfaces are non-flat. There is no smooth solutions if both Riemann surfaces are negatively curved. We are therefore left with one sphere and one Riemann surface Σ_g with $g = 0$ or $g \geq 2$. The line element has the form

$$ds^2(M_6) = f_\varphi(\mu) ds^2(S_\varphi^2) + f_\Sigma(\mu) ds^2(\Sigma_g) + f_\mu(\mu) d\mu^2 + f_\psi(\mu) D\psi^2. \quad (2.20)$$

In the previous expression, ψ is an angular coordinate with period 2π , while μ is a coordinate on an interval, $\mu \in [\mu_S, \mu_N]$. The quantity $ds^2(S_\varphi^2) = d\theta^2 + \sin^2 \theta d\varphi^2$ is the standard line element on a unit-radius two-sphere, while $ds^2(\Sigma_g)$ denotes the line element on a Riemann surface of constant curvature $\kappa = \pm 1$, respectively. The functions f_φ, f_Σ are strictly positive on the entire μ interval. The function f_μ has poles at $\mu = \mu_{N,S}$, while f_ψ has zeros at $\mu = \mu_{N,S}$; as a result, the μ and ψ coordinate describe a two-dimensional space S_ψ^2 which is topologically a 2-sphere, with isometry group $U(1)_\psi$. The circle S_ψ^1 shrinks smoothly at $\mu = \mu_{N,S}$. Finally, the fibration of S_ψ^2 over the base $S_\varphi^2 \times \Sigma_g$ is encoded in

$$dD\psi = -2V_\varphi - \chi V_\Sigma, \quad \chi = 2 - 2g, \quad (2.21)$$

where V_φ, V_Σ are the volume forms on S_φ^2, Σ_g , respectively, normalized according to

$$\int_{S_\varphi^2} V_\varphi = 2\pi, \quad \int_{\Sigma_g} V_\Sigma = 2\pi. \quad (2.22)$$

More details on these geometries can be found in appendix B.2.

To highlight the interpretation of M_6 in terms of wrapped M5-branes, it is convenient to present M_6 as

$$M_4 \hookrightarrow M_6 \rightarrow \Sigma_g, \quad (2.23)$$

where the space M_4 consists of the μ, ψ directions and the 2-sphere S_φ^2 . It is depicted schematically in figure 1. The space M_4 can be identified with the resolution of the quotient $S^4/\mathbb{Z}_2$ [27]. Notice that S_φ^2 does not shrink at $\mu = \mu_{N,S}$ and defines two 2-cycles in M_4 . The latter are identified with the resolution cycles originating from the blow-up of the

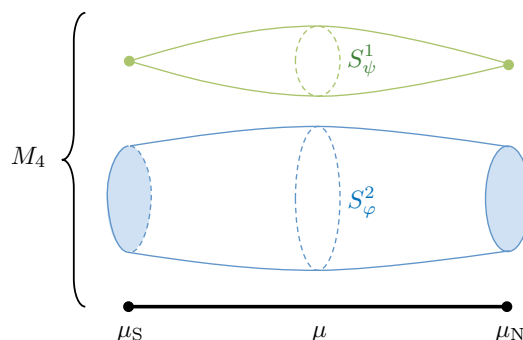


Figure 1. Schematic depiction of the space M_4 comprised by the 2-sphere S^2_φ , the circle S^1_ψ , and the μ interval. The space M_4 is the blow-up resolution of $S^4/\mathbb{Z}_2$. The blow-up $\mathbb{P}^1$'s are identified with S^2_φ at $\mu = \mu_N$ and $\mu = \mu_S$.

singularities of $S^4/\mathbb{Z}_2$.⁵ This motivates the interpretation of M_6 as near horizon geometry of a stack of M5-branes probing a $\mathbb{Z}_2$ singularity and wrapped on a Riemann surface.

The Betti numbers of M_6 are

$$\begin{aligned} b^0(M_6) &= b^6(M_6) = 1, & b^2(M_6) &= b^4(M_6) = 3, \\ b^1(M_6) &= b^5(M_6) = 2g, & b^3(M_6) &= 4g. \end{aligned} \quad (2.24)$$

This is verified in appendix B.2, where we also construct the closed forms needed to represent all cohomology classes of M_6 . In accordance with $b^4(M_6) = 3$, this class of GMSW solutions has three independent flux parameters. They can be taken to be

$$N_\alpha = (N, N_+, N_-), \quad \alpha = 1, 2, 3, \quad (2.25)$$

where N is the flux through M_4 (at a generic point on Σ_g) and is identified with the number of M5-branes in the stack, while $N_\pm = \frac{1}{2}(N_N \pm N_S)$ encode the fluxes through the 2-cycles in M_4 at $\mu = \mu_{N,S}$ combined with Σ_g . A more detailed discussion can be found in appendix B.2. With reference to (2.25), the three 1-form gauge fields originating from expansion of C_3 onto 2-cohomology classes are denoted

$$A_1^\alpha = (A_1, A_1^+, A_1^-), \quad F_2^\alpha = (F_2, F_2^+, F_2^-). \quad (2.26)$$

The 1-cohomology classes of M_6 are labeled by the same index $u = 1, \dots, 2g$ that labels the non-trivial 1-cycles on the Riemann surface. As in the previous section, we can choose a canonical basis of 1-cycles on Σ_g , and split the index u into two sets of g values. Accordingly, we have a total of $2g$ 2-form gauge fields, which we can arrange into two groups of g each, and similarly for their field strengths,

$$B_2^u = (B_{2i}, \tilde{B}_2^i), \quad H_3^u = (H_{3i}, \tilde{H}_3^i), \quad i = 1, \dots, g. \quad (2.27)$$

⁵The $\mathbb{Z}_2$ action is $(y^1, y^2, y^3, y^4, y^5) \mapsto (-y^1, -y^2, -y^3, -y^4, y^5)$ in terms the Cartesian coordinates $y^{1,2,3,4,5}$ of $\mathbb{R}^5 \supset S^4$. This action has two fixed points on S^4 at $y^5 = \pm 1$. Near each fixed point the space looks like $\mathbb{R}^4/\mathbb{Z}_2$ and can be resolved by a 2-center ALE Taub-NUT geometry.

multiplicity	fields	top. mass terms	5d bulk gauge symm.
$b^2(M_6) = 3$	A_1^α	$\frac{1}{2\pi} N_\alpha A_1^\alpha \wedge d\gamma_3$	$U(1)^2$ 0-form symm.
1	γ_3		$\mathbb{Z}_k$ 0-form symm.
$b^1(M_6) = 2g$	$B_{2i}, \tilde{B}_2^i$	$\frac{1}{2\pi} N \tilde{B}_2^i \wedge dB_{2i}$	$(\mathbb{Z}_N \times \mathbb{Z}_N)^g$ 1-form symm.
$b^3(M_6) = 4g$	$a_{0i}^\pm, \tilde{a}_0^{i\pm}$	—	5d axions

Table 2. Summary of p -form gauge fields in 5d supergravity obtained from expansion of C_3 onto cohomology classes in M_6 for M5-branes at a $\mathbb{Z}_2$ singularity wrapped on a genus- g Riemann surface. We have defined $k = \gcd(N_\alpha)$.

By a similar token, we organize the $4g$ 0-form gauge fields associated to 3-cohomology class of M_6 into four groups of g elements. Within each group, we label 0-form fields with the same index i as in (2.27),

$$a_0^x = (a_{0i}^+, \tilde{a}_0^{i+}, a_{0i}^-, \tilde{a}_0^{i-}), \quad f_1^x = (f_{1i}^+, \tilde{f}_1^{i+}, f_{1i}^-, \tilde{f}_1^{i-}), \quad i = 1, \dots, g. \quad (2.28)$$

Having introduced our choice of bases in cohomology and our notation, we can present the expression for I_6 . It reads (suppressing wedge products)

$$\begin{aligned} I_6 = & \frac{1}{(2\pi)^2} \left[- (N F_2 + N_+ F_2^+ + N_- F_2^-) \gamma_4 - N \tilde{H}_3^i H_{3i} \right] \\ & + \frac{1}{(2\pi)^3} \left[- \frac{1}{6} \chi (F_2^+)^3 - \frac{1}{2} \chi F_2^+ (F_2^-)^2 + F_2 F_2^+ F_2^- - \gamma_4 (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) \right. \\ & \left. + F_2^+ (f_{1i}^+ \tilde{H}_3^i - \tilde{f}_1^{i+} H_{3i}) + F_2^- (f_{1i}^- \tilde{H}_3^i - \tilde{f}_1^{i-} H_{3i}) \right]. \end{aligned} \quad (2.29)$$

The topological mass terms are collected in the first line. We summarize the p -forms fields and their topological mass terms in table 2. The bulk gauge groups in the last column are explained in greater detail in the next section.

3 BF theory in the bulk and holographic interpretation

In this section we analyze the 5d dynamics of the p -form gauge fields originating from the expansion of the M-theory 3-form C_3 . We make contact with well-known aspects of BF theories [24, 25] and argue that the 5d theory contains gauge fields with discrete gauge groups. When the 5d spacetime has a boundary, the theory has to be supplemented by suitable boundary conditions and boundary terms, which we partially review. Moreover, we describe the singleton modes that propagate on the boundary of spacetime. Finally, we discuss the holographic correspondence between discrete gauge fields in five dimensions and global discrete p -form symmetries in the 4d boundary theory, as well as the holographic interpretation of the singleton modes of the 5d bulk theory. Most of the ideas presented in this section are modeled on results that have appeared in the literature. Our main goal here is to collect useful observations to set the stage for the 't Hooft anomaly discussion of section 5.

3.1 Low-energy dynamics in five dimensions

The relevant couplings in the effective action for the p -form gauge fields coming from the M-theory 3-form are the kinetic terms and the topological terms (2.5). At very low-energies, the dynamics is governed by the topological terms that are quadratic in the p -form gauge fields, as these terms in the 5d action contain only one derivative. These topological terms are encoded in the first line of the formal 6-form I_6 in (2.6).

We have already argued that the basis of 1-form gauge fields in (2.8) is the best suited for discussing topological mass terms involving c_3 . We also notice that, in all setups described in section 2.2, the term $\frac{1}{2} N_\alpha \mathcal{K}_{uv}^{\alpha} H_3^u \wedge H_3^v$ takes the simple form $-N \tilde{H}_3^i \wedge H_{3i}$, see (2.19) and (2.29). For these reasons, for the remainder of this section we consider the 5d topological theory defined by the action

$$S = \int_{\mathcal{M}_5} \left[-\frac{1}{2\pi} k c_3 \wedge d\mathcal{A}_1 - \frac{1}{2\pi} N \tilde{B}_2^i \wedge dB_{2i} \right]. \quad (3.1)$$

In the previous expression $\mathcal{M}_5$ denotes external spacetime. In writing the action (3.1) we have chosen a specific antiderivative $I_5^{(0)}$ of the formal 6-form I_6 . If $\mathcal{M}_5$ has no boundary, this choice does not matter. The case $\partial\mathcal{M}_5 \neq \emptyset$ is discussed below. Recall that $i = 1, \dots, g$ and $k = \gcd(N_\alpha)$ (if $b^2(M_6) = 1$, we define $k = N_{\alpha=1}$). The action (3.1) describes a collection of decoupled standard BF theories. We refer the reader to e.g. [24, 25] for background material on BF theories and their relation to the Stückelberg mechanism.

The fact that the 1-form gauge fields $\hat{\mathcal{A}}_1^\alpha$ do not enter (3.1) means that their dynamics is governed by the kinetic terms and the cubic topological couplings in (2.5). As a result, the 1-form gauge fields $\hat{\mathcal{A}}_1^\alpha$ are standard U(1) gauge fields. In contrast, the dynamics of $\mathcal{A}_1$, c_3 , B_{2i} , and $\tilde{B}_2^i$ is governed by (3.1) and therefore:

- $\mathcal{A}_1$ describes a 1-form gauge field with gauge group $\mathbb{Z}_k$.
- c_3 describes a 3-form gauge field with gauge group $\mathbb{Z}_k$.
- B_{2i} , $\tilde{B}_2^i$ describe 2-form gauge fields with gauge group $\mathbb{Z}_N$.

The field $\mathcal{A}_1$ is a continuum description of a discrete gauge field because it is a flat connection and its holonomies are restricted in $\mathbb{Z}_k \subset \text{U}(1)$. Similar remarks apply to c_3 , B_{2i} , $\tilde{B}_2^i$: for arbitrary cycles $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_2$ in 5d spacetime $\mathcal{M}_5$,

$$\begin{aligned} \exp\left(i \int_{\mathcal{C}_1} \mathcal{A}_1\right) &\in \mathbb{Z}_k \subset \text{U}(1), & \exp\left(i \int_{\mathcal{C}_3} c_3\right) &\in \mathbb{Z}_k \subset \text{U}(1), \\ \exp\left(i \int_{\mathcal{C}_2} B_{2i}\right) &\in \mathbb{Z}_N \subset \text{U}(1), & \exp\left(i \int_{\mathcal{C}_2} \tilde{B}_2^i\right) &\in \mathbb{Z}_N \subset \text{U}(1). \end{aligned} \quad (3.2)$$

We stress that, since all these p -form gauge fields are flat on-shell, the holonomies written above only depend on the homology classes of $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_2$, and not on the specific representatives. Let us also emphasize that this description of discrete gauge fields in terms of local p -forms and their holonomies is convenient for our purposes, but in more general situations an approach based on cocycles is preferred [31, 32].

3.1.1 Boundary terms and boundary conditions

For applications to holography we have to consider 5d spacetimes with a conformal boundary. In this case, the bulk action (3.1) has to be supplemented with suitable boundary conditions and possibly additional boundary terms, in order to ensure a well-defined variational problem. In this section we describe some sets of boundary conditions that will be relevant below in the holographic discussion.

Topological boundary conditions. Let us first discuss boundary conditions for the $c_3, \mathcal{A}_1$ BF theory. A simple choice is to assign Dirichlet boundary conditions on $\mathcal{A}_1$, with free boundary conditions for c_3 . The variational problem is well-posed because the relevant terms in the on-shell variation of the action (3.1) are

$$\delta S = \int_{\partial \mathcal{M}_5} \frac{1}{2\pi} k c_3 \wedge \delta \mathcal{A}_1 + \dots \quad (3.3)$$

Let us stress that imposing Dirichlet boundary conditions for both $\mathcal{A}_1$ and c_3 would be inconsistent, since the variational problem defined by the bulk BF action (3.1) is first-order. If desired, the roles of $\mathcal{A}_1$ and c_3 can be exchanged. By adding the boundary term $-\frac{k}{2\pi} \int_{\partial \mathcal{M}_5} c_3 \wedge \mathcal{A}_1$ to (3.1), we can rewrite the relevant terms in the total action as

$$S' = \int_{\mathcal{M}_5} \left[-\frac{1}{2\pi} k \mathcal{A}_1 \wedge dc_3 \right] + \dots \quad (3.4)$$

In this case we impose Dirichlet boundary conditions on c_3 , with free boundary conditions for $\mathcal{A}_1$. The boundary conditions described so far are topological, since they are invariant under orientation-preserving diffeomorphisms of $\partial \mathcal{M}_5$ and do not require the choice of a boundary metric. (See [38] for a classification of topological boundary conditions in Abelian 3d Chern-Simons theory.)

If the integer k can be factorized as $k = mm'$, we can also consider a generalization of the above topological boundary conditions, along the lines of [1, 26]. Let us stress that, since $\mathcal{A}_1$ is a discrete 1-form gauge field, assigning Dirichlet boundary conditions for $\mathcal{A}_1$ means specifying its $\mathbb{Z}_k$ holonomies around 1-cycles in the boundary $\partial \mathcal{M}_5$. We can partially relax the boundary conditions on $\mathcal{A}_1$ as follows. To a given 1-cycle $\mathcal{C}_1$ in $\partial \mathcal{M}_5$ we no longer associate an element $x \in \mathbb{Z}_k$, but rather a coset $[x]_{\mathbb{Z}_m} \in \mathbb{Z}_k/\mathbb{Z}_m$. The holonomy $\exp(i \int_{\mathcal{C}_1} \mathcal{A}_1)$ is free to take any value $y \in [x]_{\mathbb{Z}_m}$, which is the same as $y = x \bmod m'$.⁶ Following the terminology of [26], we say that $\mathcal{A}_1$ is free in $\mathbb{Z}_k$ modulo $\mathbb{Z}_{m'}$. It is interesting to notice that, since $\mathbb{Z}_k/\mathbb{Z}_m \cong \mathbb{Z}_{m'}$, the data encoded in the boundary conditions for $\mathcal{A}_1$ is the same data that define a background $\mathbb{Z}_{m'}$ 1-form gauge field on the boundary.

In order to have a well-defined variational problem, we must partially restrict the field c_3 . Its boundary conditions are no longer free. To a 3-cycle $\mathcal{C}_3$, we assign a coset $[x]_{\mathbb{Z}_{m'}}$ and the holonomy $\exp(i \int_{\mathcal{C}_3} c_3)$ can take any value $y \in [x]_{\mathbb{Z}_{m'}}$, which is the same as $y = x \bmod m'$.

⁶For example, if $m = 3$, $m' = 4$, $k = 12$, the subgroup $\mathbb{Z}_3 \subset \mathbb{Z}_{12}$ consists of $\{0, 4, 8\}$. The elements of the quotient $\mathbb{Z}_{12}/\mathbb{Z}_3$ are the cosets $[0]_{\mathbb{Z}_3} = \{0, 4, 8\} \subset \mathbb{Z}_{12}$, $[1]_{\mathbb{Z}_3} = \{1, 5, 9\} \subset \mathbb{Z}_{12}$, $[2]_{\mathbb{Z}_3} = \{2, 6, 10\} \subset \mathbb{Z}_{12}$, and $[3]_{\mathbb{Z}_3} = \{3, 7, 11\} \subset \mathbb{Z}_{12}$. The boundary conditions for $\mathcal{A}_1$ select one coset, for example $[1]_{\mathbb{Z}_3}$, leaving the holonomy of $\mathcal{A}_1$ free to take any value $y = 1 \bmod 4$, namely $y = 1, 5$, or 9 .

m . In short, we say that c_3 is free in $\mathbb{Z}_k$ modulo $\mathbb{Z}_m$. Specifying the boundary condition for c_3 is the same as choosing a background $\mathbb{Z}_m$ 3-form gauge field on the boundary.

Let us now comment on topological boundary conditions for the $\tilde{B}_2^i, B_{2i}$ BF theory. For a given label $i = 1, \dots, g$, we may assign Dirichlet boundary conditions for $\tilde{B}_2^i$ and free boundary conditions for B_{2i} , or *vice versa*. If N can be factorized as $N = nn'$, we can also consider boundary conditions in which B_{2i} is free in $\mathbb{Z}_N$ modulo $\mathbb{Z}_{n'}$, while $\tilde{B}_2^i$ is free in $\mathbb{Z}_N$ modulo $\mathbb{Z}_n$ (in the same terminology explained above.)

The full set of boundary conditions for the $\tilde{B}_2^i, B_{2i}$ BF theory, however, is richer. Indeed, we can select suitable linear combinations $(B'_{2i}, \tilde{B}_2^i)$ of the original 2-forms $(B_{2i}, \tilde{B}_2^i)$, and impose that B'_{2i} be free in $\mathbb{Z}_N$ modulo $\mathbb{Z}_{n'}$, and $\tilde{B}_2^i$ be free in $\mathbb{Z}_N$ modulo $\mathbb{Z}_n$. Moreover, the duality group $\text{Sp}(2g, \mathbb{Z})$ acts on the set of topological boundary conditions. We leave the problem of classifying topological boundary conditions for the $\tilde{B}_2^i, B_{2i}$ BF theory to future work.

The role of kinetic terms. Let us close this section by emphasizing that the discussion of boundary conditions is qualitatively different if the kinetic terms are included in the analysis. This is because, if the kinetic terms are retained, the variational problem is a second-order problem. It is therefore possible, for instance, to impose Dirichlet boundary conditions on all fields. This point is discussed in [39, 40] in the context of 3d and 5d topological theories. We expect similar features in our 5d BF system.

3.1.2 Singleton modes propagating on the boundary

When a topological 5d BF theory with a coupling between a p -form gauge field and a $(4-p)$ -form gauge field is considered in a spacetime with a boundary, there is a massless $(p-1)$ -form gauge field propagating along the boundary. (Equivalently, the massless mode on the boundary can be thought of as a $(3-p)$ -form gauge field.) These massless boundary modes are usually referred to as singletons. For a justification of the previous claims and of the following statements, see e.g. [24, 25]. For the case at hand, the singleton modes are:

- One 0-form gauge field in $\partial\mathcal{M}_5$.
- $2g$ 1-form gauge fields in $\partial\mathcal{M}_5$.

All these gauge fields are standard $U(1)$ gauge fields, as opposed to discrete gauge fields.

While the Hilbert space of the singleton fields is insensitive to the choice of boundary terms and boundary conditions, its dynamics (i.e. the Hamiltonian on the Hilbert space) is different for different boundary terms and boundary conditions. In this work, we refrain from a detailed analysis of the singleton dynamics. We will be mainly interested in counting singletons and discussing their holographic duals. A thorough analysis of the singleton sector would require to take into account the kinetic terms, as in [40, 41]. We leave such investigation for the future.

3.2 Holographic interpretation

Let us now turn to a discussion of the holographic interpretation of the features of the bulk BF theory listed in the previous section.

3.2.1 Global discrete symmetries in four dimensions

We have argued above that the 1-form gauge fields $\mathcal{A}_1^{\hat{a}}$ are standard U(1) gauge fields in five dimensions. As a result, they are dual to global U(1) 0-form symmetries in the interacting CFT living on the boundary.

In contrast, the holographic interpretation of the discrete gauge fields $\mathcal{A}_1$, c_3 , B_{2i} , and $\tilde{B}_2^i$ is more subtle. We describe it in the purely topological BF theory, neglecting kinetic terms.

Holography of the topological BF theory. The holographic interpretation of the 5d bulk BF theory (3.1) depends on the choice of boundary conditions. In other words, different boundary conditions correspond to different dual CFTs, which may have different global symmetries. This is a standard phenomenon in the paradigmatic example of $AdS_5 \times S^5$ in type IIB [42] and has recently been studied in the context of ABJM theories [26].

Firstly, let us focus on the $\mathcal{A}_1$, c_3 system. The holographic interpretation of the topological boundary conditions discussed above is as follows.

- (a) Dirichlet boundary conditions for $\mathcal{A}_1$ and free boundary conditions for c_3 :
The dual interacting CFT admits a global $\mathbb{Z}_k$ 0-form symmetry. Specifying the boundary condition for $\mathcal{A}_1$ is the same as fixing a configuration for the 4d background 1-form gauge field that couples to this global symmetry.
- (b) Dirichlet boundary conditions for c_3 and free boundary conditions for $\mathcal{A}_1$:
The dual interacting CFT admits a global $\mathbb{Z}_k$ 2-form symmetry. Specifying the boundary condition for c_3 is the same as fixing a configuration for the 4d background 3-form gauge field that couples to this global symmetry.
- (c) The case $k = mm'$ with $\mathcal{A}_1$ free in $\mathbb{Z}_k$ modulo $\mathbb{Z}_{m'}$ and c_3 free in $\mathbb{Z}_k$ modulo $\mathbb{Z}_m$:
The dual interacting CFT admits both a global $\mathbb{Z}_{m'}$ 0-form symmetry and a global $\mathbb{Z}_m$ 2-form symmetry. Specifying the boundary conditions for $\mathcal{A}_1$ and c_3 is the same as fixing a configuration for the 4d background 1-form and 3-form gauge fields that couple to these global symmetry.

Case (c) is intermediate between cases (a) and (b). In case (c), there is a mixed 't Hooft anomaly between the $\mathbb{Z}_{m'}$ 0-form symmetry and the $\mathbb{Z}_m$ 2-form symmetry. This 't Hooft anomaly is encoded in the 6-form $I_6 = -k \frac{dc_3}{2\pi} \wedge \frac{d\mathcal{A}_1}{2\pi}$, which is related by descent to the BF coupling in the 5d bulk action.⁷

Let us also observe that cases (b) and (c) can be obtained from case (a) via gauging. More precisely, suppose $k = mm'$. The CFT of case (a) has a global $\mathbb{Z}_k$ 0-form symmetry. We may gauge a subgroup $\mathbb{Z}_m \subset \mathbb{Z}_k$ of this global symmetry. The gauging is performed by

⁷Roughly speaking, terms in I_6 involving two or more Dirichlet fields are interpreted as 't Hooft anomalies. If we choose Dirichlet boundary conditions for $\mathcal{A}_1$, the field c_3 has free boundary conditions. The 6-form $I_6 = -k \frac{dc_3}{2\pi} \wedge \frac{d\mathcal{A}_1}{2\pi}$ does not encode a 't Hooft anomaly, and indeed we only have a global $\mathbb{Z}_k$ 0-form symmetry. When we let $\mathcal{A}_1$ free in $\mathbb{Z}_k$ modulo $\mathbb{Z}_{m'}$ and c_3 free in $\mathbb{Z}_k$ modulo $\mathbb{Z}_m$, both fields $\mathcal{A}_1$ and c_3 are “partially Dirichlet”. As a result, $I_6 = -k \frac{dc_3}{2\pi} \wedge \frac{d\mathcal{A}_1}{2\pi}$ now encodes the mixed 't Hooft anomaly between the 0-form and 2-form global symmetries.

path-integrating over the background 1-form gauge field that couples to the $\mathbb{Z}_m$ subgroup. This is the same as modifying the boundary conditions for $\mathcal{A}_1$: we go from Dirichlet boundary conditions, to having $\mathcal{A}_1$ free in $\mathbb{Z}_k$ modulo $\mathbb{Z}_{m'}$. After gauging, the residual global 0-form symmetry is $\mathbb{Z}_{m'}$. There is also an emergent global $\mathbb{Z}_m$ 2-form symmetry. We recognize the features of the CFT of case (c). Selecting $m = k$, $m' = 1$ we recover case (b).

The B_{2i} , $\tilde{B}_2^i$ system can be analyzed in a similar way. As recalled in section 3.1.1, the full set of allowed topological boundary conditions is rich, and their classification is left for future work. To illustrate the relation between boundary conditions and global discrete symmetries, we consider a simple class of boundary conditions, in which we can treat each label $i = 1, \dots, g$ independently. One may thus consider the following three scenarios.

- (a') Dirichlet boundary conditions for B_{2i} and free boundary conditions for $\tilde{B}_2^i$:
The dual interacting CFT admits a global $(\mathbb{Z}_N)^g$ 1-form symmetry of “electric type”. Specifying the boundary condition for B_{2i} is the same as fixing a configuration for the 4d background 2-form gauge fields that couple to this global symmetry.
- (b') Dirichlet boundary conditions for $\tilde{B}_2^i$ and free boundary conditions for B_{2i} :
The dual interacting CFT admits a global $(\mathbb{Z}_N)^g$ 1-form symmetry of “magnetic type”. Specifying the boundary condition for $\tilde{B}_2^i$ is the same as fixing a configuration for the 4d background 2-form gauge fields that couple to this global symmetry.
- (c') The case $N = nn'$ with B_{2i} free in $\mathbb{Z}_N$ modulo $\mathbb{Z}_{n'}$ and $\tilde{B}_2^i$ free in $\mathbb{Z}_N$ modulo $\mathbb{Z}_n$:
The dual interacting CFT admits both a global $(\mathbb{Z}_{n'})^g$ 1-form symmetry of “electric type” and a global $(\mathbb{Z}_n)^g$ 1-form symmetry of “magnetic type”. Specifying the boundary conditions for B_{2i} and $\tilde{B}_2^i$ is the same as fixing configurations for the 4d background 2-forms that couple to these global symmetries.

As before, the case (c') is intermediate between (a') and (b'), and in case (c') there is a mixed 't Hooft anomaly between the $(\mathbb{Z}_{n'})^g$ and $(\mathbb{Z}_n)^g$ 1-form symmetries. This 't Hooft anomaly is encoded in the 6-form $I_6 = N \frac{d\tilde{B}_2^i}{2\pi} \wedge \frac{dB_{2i}}{2\pi}$.

3.2.2 Singleton modes as Goldstone modes

According to the usual holographic dictionary, the supergravity theory in the bulk of 5d spacetime $\mathcal{M}_5$ is dual to an interacting CFT living on $\partial\mathcal{M}_5$. The gravity theory in five dimensions has additional singleton modes, that only propagate on the conformal boundary of 5d spacetime. These modes do not gravitate. They are holographically dual to additional, decoupled free fields in four dimensions.

BF singletons as Goldstone modes. In section 3.1.2 we have identified a subset of the singleton modes for the 5d gravitational theories of interest in this work. More precisely, we have identified the singleton modes associated to the 5d BF theory (3.1). For these singleton modes we can offer an interpretation in terms of Goldstone's theorem, as follows.

The singleton mode associated to the BF coupling $k c_3 \wedge d\mathcal{A}_1$ is a 4d 0-form gauge field, i.e. an axion. The BF coupling $k c_3 \wedge d\mathcal{A}_1$ can be related by dualization to a Stückelberg

coupling between the vector $\mathcal{A}_1$ and the axion dual to c_3 . (Useful background material can be found in [24, 25].) The 5d theory describes a U(1) 0-form gauge symmetry Higgsed down to $\mathbb{Z}_k$. According to the usual holographic dictionary, the U(1) 0-form gauge symmetry in five dimensions is dual to a U(1) 0-form global symmetry in four dimensions. The boundary value of the 5d gauge field $\mathcal{A}_1$ is identified with the Noether 1-form current J_1 for the global U(1) 0-form symmetry in four dimensions. Since the U(1) 0-form gauge symmetry is spontaneously broken in five dimensions, the dual U(1) 0-form global symmetry in four dimensions is also spontaneously broken. As a result, we have a 4d massless Goldstone scalar Φ , related to the current J_1 by the schematic relation $J_1 \sim d\Phi$. The Goldstone mode Φ enjoys a global shift symmetry $\Phi \rightarrow \Phi + \text{const.}$ It is an axion and its interactions are derivative interactions. In the deep IR, Φ decouples from the rest of the 4d theory. We identify it with the holographic dual of the singleton mode from the BF coupling $k c_3 \wedge d\mathcal{A}_1$.

Similar remarks apply to the singleton modes associated to the BF coupling $N \tilde{B}_2^i \wedge B_{2i}$. A general statement is as follows:

A D -dimensional BF coupling $B_{D-1-p} \wedge dA_p$ between a p -form gauge field A_p and a $(D-1-p)$ -form gauge field B_{D-1-p} yields a singleton mode which is a massless U(1) $(p-1)$ -form gauge field in $(D-1)$ dimensions. It is identified with the Goldstone mode originating from spontaneous breaking of a global $(p-1)$ -form symmetry in the $(D-1)$ -dimensional dual field theory.

In the case $D = 2p + 1$ we can formulate a similar statement regarding Chern-Simons couplings.

A $(2p+1)$ -dimensional Chern-Simons coupling $A_p \wedge dA_p$ for a p -form gauge field A_p yields a singleton mode which is a massless *chiral* U(1) $(p-1)$ -form gauge field in $2p$ dimensions. It is identified with the Goldstone mode originating from spontaneous breaking of a global $(p-1)$ -form symmetry in the $2p$ -dimensional dual field theory.

Here a chiral $(p-1)$ -form gauge field is by definition a gauge field whose p -form field strength obeys a self-duality constraint of the form $*_{2p} F_p = \pm F_p$ or $*_{2p} F_p = \pm i F_p$, depending on the dimension and signature of spacetime.

4 Extended operators and discrete symmetries

In this section we review the extended operators of the 5d topological BF theory with action (3.1). We identify the 11d origin of these operators in terms of wrapped M2-branes. We also consider the interplay between these operators and the topological boundary conditions for the BF system (3.1) and infer what extended operators are expected in the dual 4d field theories.

4.1 Extended operators in the BF bulk theory

A natural set of gauge-invariant observables in the 5d theory (3.1) is given by the holonomies of the gauge fields c_3 , $\mathcal{A}_1$, $\tilde{B}_2^i$, B_{2i} on cycles in 5d spacetime,

$$\begin{aligned} W_c(\mathcal{C}_3^{\text{ext}}, n) &= \exp \left[i n \int_{\mathcal{C}_3^{\text{ext}}} c_3 \right], & W_{\mathcal{A}}(\mathcal{C}_1^{\text{ext}}, n) &= \exp \left[i n \int_{\mathcal{C}_1^{\text{ext}}} \mathcal{A}_1 \right], \\ W_{B, \tilde{B}}(\mathcal{C}_2^{\text{ext}}, n, \tilde{n}) &= \exp \left[i \int_{\mathcal{C}_2^{\text{ext}}} \left(\tilde{n}^i B_{2i} - n_i \tilde{B}_2^i \right) \right], \end{aligned} \quad (4.1)$$

where n , n_i , and $\tilde{n}^i$ are integers and $\mathcal{C}_1^{\text{ext}}$, $\mathcal{C}_2^{\text{ext}}$, $\mathcal{C}_3^{\text{ext}}$ and 1-, 2-, 3-cycles in 5d spacetime. The superscript ‘ext’ stands for external and is inserted to avoid possible confusions with cycles in the internal geometry M_6 . We remind the reader that p -form gauge fields are normalized to have periods quantized in units of 2π . In the topological BF theory, the operator $W_c(\mathcal{C}_3^{\text{ext}}, n)$ describes a 3d defect in 5d spacetime extended along $\mathcal{C}_3^{\text{ext}}$ with electric charge n under c_3 . In a similar way, $W_{\mathcal{A}}(\mathcal{C}_1^{\text{ext}}, n)$ represents a 1d defect extended along $\mathcal{C}_1^{\text{ext}}$ with charge n under $\mathcal{A}_1$, while $W_{B, \tilde{B}}(\mathcal{C}_2^{\text{ext}}, n, \tilde{n})$ describes a 2d defect along $\mathcal{C}_2^{\text{ext}}$ with charges n_i , $\tilde{n}^i$ under B_{2i} , $\tilde{B}_2^i$. The operators (4.1) will be referred to as electric operators.

If a defect charged under $\mathcal{A}_1$ is transported around a defect charged under c_3 , it acquires a non-trivial $\mathbb{Z}_k$ Aharonov-Bohm phase. The latter is encoded in the correlator

$$\langle W_c(\mathcal{C}_3^{\text{ext}}, n) W_{\mathcal{A}}(\mathcal{C}_1^{\text{ext}}, n') \rangle \sim \exp \left[i \frac{n n'}{k} L(\mathcal{C}_3^{\text{ext}}, \mathcal{C}_1^{\text{ext}}) \right], \quad (4.2)$$

where $L(\mathcal{C}_3^{\text{ext}}, \mathcal{C}_1^{\text{ext}})$ is the integer linking number of $\mathcal{C}_1^{\text{ext}}$ and $\mathcal{C}_3^{\text{ext}}$ in the ambient 5d spacetime. By a similar token, the Aharonov-Bohm phases of defects charged under B_{2i} , $\tilde{B}_{2i}$ are captured by the correlators

$$\langle W_{B, \tilde{B}}(\mathcal{C}_2^{\text{ext}}, n, \tilde{n}) W_{B, \tilde{B}}(\mathcal{C}_2^{\text{ext}'}, n', \tilde{n}') \rangle \sim \exp \left[i \frac{n_i \tilde{n}^i - \tilde{n}^i n'_i}{N} L(\mathcal{C}_2^{\text{ext}}, \mathcal{C}_2^{\text{ext}'}) \right]. \quad (4.3)$$

The derivation of (4.2) and (4.3) can be found e.g. in [24, 25].

In addition to the electric operators in (4.1), the 5d topological theory also admits “mixed” electric-magnetic operators. If we consider a 2-cycle $\mathcal{B}_2^{\text{ext}}$ in external spacetime, we can define a ’t Hooft operator for $\mathcal{A}_1$ supported on $\mathcal{B}_2^{\text{ext}}$. This is done in the usual way. We remove a small tubular neighborhood of $\mathcal{B}_2^{\text{ext}}$ from 5d spacetime. The boundary of the tubular neighborhood is an S^2 bundle over $\mathcal{B}_2^{\text{ext}}$. The ’t Hooft operator on $\mathcal{B}_2^{\text{ext}}$ is defined by performing the path integral over $\mathcal{A}_1$ with the boundary condition $\frac{1}{2\pi} \int_{S^2} \mathcal{F}_2 = 1$. Because of the $k c_3 \wedge d\mathcal{A}_1$ coupling in the action, the ’t Hooft operator is not gauge invariant. It must be supplemented with a charge- k Wilson operator for c_3 on a 3-chain $\mathcal{C}_3^{\text{ext}}$ such that $\partial \mathcal{C}_3^{\text{ext}} = \mathcal{B}_2^{\text{ext}}$ [42–44].⁸ In a completely analogous fashion, one can consider a ’t Hooft

⁸After removing a small tubular neighborhood U of $\mathcal{B}_2^{\text{ext}}$, the gauge variation of the 5d action reads

$$\delta S = -\frac{1}{2\pi} k \int_{\mathcal{M}_5 \setminus U} \delta c_3 \wedge \mathcal{F}_2 = -\frac{1}{2\pi} k \int_{\mathcal{M}_5 \setminus U} d\Lambda_2 \wedge \mathcal{F}_2 = -k \int_{\mathcal{B}_2^{\text{ext}}} \Lambda_2,$$

where we have used $\frac{1}{2\pi} \int_{S^2} \mathcal{F}_2 = 1$. The above expression shows that the gauge variation of the ’t Hooft operator for $\mathcal{A}_1$ can be cancelled by k Wilson operators for c_3 .

operator for c_3 supported on a 0-cycle $\mathcal{B}_0^{\text{ext}}$ (a collection of points taken with signs). To preserve gauge invariance, this must be supplemented with a charge- k Wilson operator for $\mathcal{A}_1$ supported on a 1-chain $\mathcal{C}_1^{\text{ext}}$ with $\partial\mathcal{C}_1^{\text{ext}} = \mathcal{B}_0^{\text{ext}}$. Finally, analogous mixed electric-magnetic operators exist for the $(\tilde{B}_2^i, B_{2i})$ system.

M-theory origin of extended operators. The purely electric operators $W_c(\mathcal{C}_3^e, n)$ are realized by a stack of n M2-brane probes sitting at a point in the internal space M_6 , and extending along $\mathcal{C}_3^e$ in the external spacetime directions. By a similar token, the operators $W_{\mathcal{A}}(\mathcal{C}_1^e, n)$ are a stack of n probe M2-branes wrapping a 2-cycle in M_6 . More precisely, the 2-cycle is $m_\alpha \mathcal{C}_2^\alpha = \mathcal{C}_2'^{\alpha=1}$, where the integers m_α are defined in (A.4), and in the second step we refer to the primed basis of 2-cycles defined by (A.2). Finally, the operators $W_{B, \tilde{B}}(\mathcal{C}_2^e, n, \tilde{n})$ originate from probe M2-branes wrapping a 1-cycle in M_6 . The charges $\tilde{n}^i, n_i$ are identified with the integers that define this 1-cycle, with respect to a fixed basis of 1-cycles in M_6 . It can also be verified that the 5d Aharonov-Bohm phases encoded in the correlators (4.2), (4.3) can be reproduced from an 11d perspective, using the $C_3 G_4 G_4$ coupling in the M-theory low-energy effective action.

The mixed electric-magnetic operators are realized using probe configurations with M2-branes ending on M5-branes. As an example, let us consider a charge-1 't Hooft operator for $\mathcal{A}_1$ on $\mathcal{B}_2^{\text{ext}}$ together with a charge- k Wilson operator for c_3 on $\mathcal{C}_3^{\text{ext}}$, with $\partial\mathcal{C}_3^{\text{ext}} = \mathcal{B}_2^{\text{ext}}$. This 5d operator is realized by one M5-brane wrapping a 4-cycle on M_6 and extending along $\mathcal{B}_2^{\text{ext}}$. With reference to the change of basis discussed in appendix A, we can characterize this 4-cycle as $\mathcal{C}_4'^{\alpha=1}$. Since there are k units of G_4 -flux threading the 4-cycle $\mathcal{C}_4'^{\alpha=1}$, there is a tadpole in the worldvolume theory of the probe M5-brane. This is canceled by adding k M2-branes ending on the M5-brane. The M2-branes sit at a point on $\mathcal{C}_4'^{\alpha=1} \subset M_6$ and are extended along $\mathcal{C}_3^{\text{ext}}$ in the external directions. In a similar way, one can describe the 11d origin of all other mixed electric-magnetic operators.

In our discussion so far we have not taken supersymmetry into consideration. One could determine the BPS conditions for probe M2-branes and M5-branes by analyzing κ -symmetry on their worldvolumes.

4.2 Extended operators in the dual field theory

The operators of the 5d topological theory discussed in the previous section can yield operators in the dual boundary field theory. The choice of boundary conditions for $\mathcal{A}_1, c_3, B_{2i}, \tilde{B}_2^i$ determines whether a 5d operator is allowed to end on the boundary $\partial\mathcal{M}_5$ or not. A similar discussion has recently appeared in [26] in the context of AdS_4/CFT_3 .

Let us discuss operators constructed with $\mathcal{A}_1, c_3$. We make contact to the cases (a), (b), (c) discussed in section 3.2.1. In case (a) the operators $\mathcal{W}_{\mathcal{A}}(\mathcal{C}_1^e, n)$ defined in (4.1) are allowed to end on $\partial\mathcal{M}_5$, while the operators $\mathcal{W}_c(\mathcal{C}_3^e, n)$ are forbidden from ending on $\partial\mathcal{M}_5$.⁹ The 5d operator $\mathcal{W}_{\mathcal{A}}(\mathcal{C}_1^e, 1)$ yields a local operator $\mathcal{O}$ on $\partial\mathcal{M}_5$ that has unit charge under the global $\mathbb{Z}_k$ 0-form symmetry of the field theory. In a similar way, in case (b) it is the operators $\mathcal{W}_c(\mathcal{C}_3^e, n)$ that can end on the boundary, and $\mathcal{W}_c(\mathcal{C}_3^e, 1)$ yields a surface operator

⁹More precisely, $\mathcal{W}_c(\mathcal{C}_3^e, n)$ can end on $\partial\mathcal{M}_5$ only if n is a multiple of k , in which case the operator $\mathcal{W}_c(\mathcal{C}_3^e, n)$ is trivial.

on $\partial\mathcal{M}_5$ with unit charge under the global $\mathbb{Z}_k$ 2-form symmetry of the field theory. In case (c), with factorization $k = m m'$, the operator $\mathcal{W}_{\mathcal{A}}(\mathcal{C}_1^e, n)$ is allowed to end on $\partial\mathcal{M}_5$ if the charge n is a multiple of m , while $\mathcal{W}_c(\mathcal{C}_3^e, n)$ can end on the boundary if n is a multiple of m' . In the boundary field theory we obtain both local operators and surface operators, compatibly with the global $\mathbb{Z}_{m'}$ 0-form and $\mathbb{Z}_m$ 2-form symmetry.

The mixed electric-magnetic operators of the 5d topological bulk theory act as baryon vertices [45] from the point of view of the dual field theory. For example, in case (a) we can consider the operator $\mathcal{O}^k$ on the boundary and connect it to a point in the bulk, where a charge-1 't Hooft operator for c_3 is supported (a “monopole event”). The arguments of [44] show that in this case $\mathcal{O}^k$ acquires a VEV. Notice that this phenomenon does not break the global $\mathbb{Z}_k$ 0-form symmetry. Analogous remarks apply to cases (b) and (c).

Let us now turn to the operators $W_{B,\tilde{B}}(\mathcal{C}_2^e, n, \tilde{n})$ in (4.1). In section 3.2.1 we have defined the cases (a'), (b'), (c'). Notice that we can choose any of these three options independently for each label $i = 1, \dots, g$. For the sake of simplicity, let us discuss the situation in which we choose case (a') for all $i = 1, \dots, g$. Other choices of topological boundary conditions for the $(\tilde{B}_2^i, B_{2i})$ system can be discussed in a similar way.

If we select case (a') for all $i = 1, \dots, g$, the operator $W_{B,\tilde{B}}(\mathcal{C}_2^e, n, \tilde{n})$ is allowed to end on $\partial\mathcal{M}_5$ if its $\tilde{B}_2^i$ charges n_i are all equal to zero. If this condition is met, we get a line operator in the dual field theory. It has charges $\tilde{n}^i$ under the global “electric” $(\mathbb{Z}_N)^g$ 1-form symmetry of the QFT. If $\tilde{n}^i = N \tilde{s}^i$ for some integers $\tilde{s}^i$, we can connect the line operator on the boundary to a mixed electric-magnetic operator (baryon vertex) in the bulk. More precisely, the line operator on $\mathcal{M}_5$ is connected by a 2d worksheet to a line $\mathcal{B}_1^{\text{ext}}$ in the bulk, which supports a 't Hooft operator for $\tilde{H}_3^i$ with charges $\tilde{s}^i$.¹⁰ According to the analysis of [44], the presence of the baryon vertex in the 5d bulk implies condensation of the line operator with charges $\tilde{n}^i = N \tilde{s}^i$.¹¹ This condensation does not trigger a spontaneous breaking of the $(\mathbb{Z}_N)^g$ 1-form symmetry.

The analysis of extended operators and boundary conditions in the $B_i, \tilde{B}^i$ BF theory can be used as a tool to access allowed line operators in 4d SCFTs from wrapped M5-branes. The goal is a classification that generalizes the results of [46] beyond Lagrangian gauge theories. We expect a rich variety of line operators and a non-trivial action of the duality group $\text{Sp}(2g, \mathbb{Z})$ on them. Notice that this strategy can be applied to both $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories. We plan to study this problem in greater detail in future work.

5 't Hooft anomalies from inflow

In this section we compute the inflow anomaly polynomial for the M-theory setups of interest in this work. We include all p -form gauge fields originating from expansion of C_3 onto cohomology classes of M_6 , as well as background fields for isometries of M_6 , and an arbitrary background metric. A systematic method for performing this computation was developed in [22]. One main novelty here is the interpretation of the terms involving $\mathcal{A}_1$,

¹⁰More explicitly, we consider a small tubular neighborhood of $\mathcal{B}_1^{\text{ext}}$ in $\mathcal{M}_5$. Its boundary is an S^3 bundle over $\mathcal{B}_1^{\text{ext}}$. We impose the boundary condition $\frac{1}{2\pi} \int_{S^3} \tilde{H}_3^i = \tilde{s}^i$.

¹¹A line operator is said to be condensed if it obeys a perimeter law [1].

c_3 , B_{2i} , $\tilde{B}_2^i$, which encode 't Hooft anomalies for discrete global symmetries. Moreover, in the case of M5-branes probing a $\mathbb{Z}_2$ singularity, we include background 0-form gauge fields and we discuss their interpretation in terms of anomalies in the space of coupling constants.

5.1 Inflow anomaly polynomial

Let us outline the recipe for the computation of the inflow anomaly polynomial. We refer to [22] for further explanations. The input data is the internal geometry M_6 and the background G_4 -flux configuration $\overline{G}_4$. In the notation of (2.3), the latter is $\overline{G}_4/(2\pi) = N_\alpha \Omega_4^\alpha$. Our goal is to compute the 6-form inflow anomaly polynomial I_6^{inflow} . As per usual descent formalism, I_6^{inflow} is a closed, gauge-invariant 6-form that is defined in a fiducial spacetime $\mathcal{M}_6$, which is taken to be Euclidean and six-dimensional.

In order to compute 't Hooft anomalies for symmetries associated to isometries of M_6 , we have to consider a fibration of M_6 over the fiducial spacetime $\mathcal{M}_6$. The relevant space is therefore an auxiliary 12-manifold M_{12} ,

$$M_6 \hookrightarrow M_{12} \rightarrow \mathcal{M}_6 . \quad (5.1)$$

The desired 6-form I_6^{inflow} is computed by fiber integration along M_6 of a globally defined 12-form $\mathcal{I}_{12}$ on M_{12} ,

$$I_6^{\text{inflow}} = \int_{M_6} \mathcal{I}_{12} . \quad (5.2)$$

The 12-form $\mathcal{I}_{12}$ is constructed from the class $X_8(TM_{12})$ (see (2.4) for the definition of X_8 in terms of Pontryagin classes) and from a 4-form E_4 on M_{12} , according to

$$\mathcal{I}_{12} = -\frac{1}{6} E_4 \wedge E_4 \wedge E_4 - E_4 \wedge X_8 . \quad (5.3)$$

The expression for E_4 is discussed below. It is argued in [22] that the 6-form (5.2) is equal to minus the 't Hooft anomalies of the full 4d theory living on the M5-branes stack. In the cases of interest in this paper, the 4d theory consists of an interacting SCFT, together with free decoupled modes. We may then write

$$I_6^{\text{inflow}} + I_6^{\text{SCFT}} + I_6^{\text{decoupl}} = 0 . \quad (5.4)$$

We comment further on I_6^{decoupl} in section 5.2.

Let us now turn to a description of the 4-form E_4 . It is a globally defined, closed 4-form on M_{12} with integral periods, which can be written as

$$E_4 = N_\alpha (\Omega_4^\alpha)^{\text{eq}} + \frac{F_2^\alpha}{2\pi} \wedge (\omega_{2\alpha})^{\text{eq}} + \frac{f_1^x}{2\pi} \wedge (\Lambda_{3x})^{\text{eq}} + \frac{H_3^u}{2\pi} \wedge (\lambda_{1u})^{\text{eq}} + \frac{\gamma_4}{2\pi} . \quad (5.5)$$

In the previous expression the forms F_2^α , f_1^x , H_3^u , γ_4 are closed forms on the base $\mathcal{M}_6$ of the fibration (5.1), pulled back to the total space M_{12} (the pullback is implicit in our notation). Exactly as in (2.3), we interpret F_2^α , f_1^x , H_3^u , γ_4 as the field strengths of p -form background gauge fields on $\mathcal{M}_6$, so that the periods of F_2^α , f_1^x , H_3^u , γ_4 are quantized in units of 2π .

The forms $(\Omega_4^\alpha)^{\text{eq}}, (\omega_{2\alpha})^{\text{eq}}, (\Lambda_{3x})^{\text{eq}}, (\lambda_{1u})^{\text{eq}}$ are globally defined, closed forms on M_{12} with integral periods.¹² They can be regarded as a gauge-invariant and closed extension of the forms $\Omega_4^\alpha, \omega_{2\alpha}, \Lambda_{3x}, \lambda_{1u}$ on the fiber M_6 . Indeed, if the fibration (5.1) is replaced by a direct product and all external gauge fields related to isometries of M_6 are turned off, the forms $(\Omega_4^\alpha)^{\text{eq}}, (\omega_{2\alpha})^{\text{eq}}, (\Lambda_{3x})^{\text{eq}}, (\lambda_{1u})^{\text{eq}}$ reduce to $\Omega_4^\alpha, \omega_{2\alpha}, \Lambda_{3x}, \lambda_{1u}$.

We discuss the construction of the forms $(\Omega_4^\alpha)^{\text{eq}}, (\omega_{2\alpha})^{\text{eq}}, (\Lambda_{3x})^{\text{eq}}, (\lambda_{1u})^{\text{eq}}$ in appendix C. We would like to emphasize here, however, that they are not uniquely determined by the forms $\Omega_4^\alpha, \omega_{2\alpha}, \Lambda_{3x}, \lambda_{1u}$ on M_6 . Different realizations of $(\Omega_4^\alpha)^{\text{eq}}$ differ by a closed (but not necessarily exact) 4-form on M_{12} , and similarly for the other forms. We show in appendix C that the ambiguities related to a specific choice of $(\Omega_4^\alpha)^{\text{eq}}, (\omega_{2\alpha})^{\text{eq}}, (\Lambda_{3x})^{\text{eq}}, (\lambda_{1u})^{\text{eq}}$ in E_4 can always be undone by adding exact pieces to E_4 (which do not alter the integral (5.2)) and/or performing a field redefinition of the external field strengths $F_2^\alpha, H_3^u, \gamma_4$. We also verify that the necessary field redefinitions preserve the lattice of periods of the field strengths. We conclude that the inflow anomaly polynomial I_6^{inflow} is unambiguously defined, up to a choice of basis in the space of external p -form gauge fields on $\mathcal{M}_6$.

After these preliminary remarks, we can discuss anomaly inflow for the wrapped M5-brane setups of interest in this work.

5.1.1 M5-branes wrapped on a Riemann surface

In section 2.2.1 we have summarized the choices of g, p, q that lead to a smooth supersymmetric AdS_5 M-theory solution, see table 1. In all cases, M_6 admits at least a $U(1)_1 \times U(1)_2$ isometry, associated to angular directions ϕ_1, ϕ_2 . We couple these isometries to external Abelian gauge fields A^{ϕ_1}, A^{ϕ_2} . We introduce the notation

$$c_1^{\phi_1} = \frac{dA^{\phi_1}}{2\pi} = c_1(U(1)_1), \quad c_1^{\phi_2} = \frac{dA^{\phi_2}}{2\pi} = c_1(U(1)_2) \quad (5.6)$$

for the first Chern classes of these background connections. When $g = 0$, the space M_6 admits an additional $SO(3)_\Sigma \cong SU(2)_\Sigma$ isometry. We can couple this isometry to a triplet of external gauge fields. We use the notation $c_2^\Sigma = c_2(SU(2)_\Sigma)$ for the second Chern class of these $SU(2)_\Sigma$ background gauge fields. appendix B.1 contains a more detailed discussion of the gauging of isometries of M_6 .

The derivation of the inflow anomaly polynomial is reported in appendix B.1. The result reads (suppressing wedge products)

$$\begin{aligned} I_6^{\text{inflow}} = & -\frac{2}{3} \left(N^3 - \frac{1}{4} N \right) \left[p c_1^{\phi_1} (c_1^{\phi_2})^2 + q c_1^{\phi_2} (c_1^{\phi_1})^2 \right] - \frac{1}{6} N \left[p (c_1^{\phi_1})^3 + q (c_1^{\phi_2})^3 \right] \\ & + \frac{1}{24} N \left[p c_1^{\phi_1} + q c_1^{\phi_2} \right] p_1(T) + \frac{1}{6} \left[(N^3 q^2 - N) p c_1^{\phi_1} + (N^3 p^2 - N) q c_1^{\phi_2} \right] c_2^\Sigma \\ & + \frac{1}{(2\pi)^2} \left[-N \gamma_4 F_2 - N \tilde{H}_3^i H_{3i} \right]. \end{aligned} \quad (5.7)$$

¹²The label “eq” stands for equivariant, because the forms $(\Omega_4^\alpha)^{\text{eq}}, (\omega_{2\alpha})^{\text{eq}}, (\Lambda_{3x})^{\text{eq}}, (\lambda_{1u})^{\text{eq}}$ on M_{12} can be regarded as representatives of classes in the G -equivariant cohomology of M_6 , where the group G is the isometry group of M_6 [22].

This result includes the terms originating from $-E_4 X_8$ in $\mathcal{I}_{12}$. In (5.7), $p_1(T)$ denotes the first Pontryagin class of the background metric on spacetime. Notice that the coefficients of the BF terms are both equal to N here. It is understood that the terms with c_2^Σ are only present if $g = 0$. We stress that (5.7) does not contain mixed 't Hooft anomalies between the symmetries related to isometries of M_6 and the symmetries associated to cohomology classes on M_6 .¹³

Interpretation. If we focus only on continuous symmetries, the background fields F_2 , γ_4 , $\tilde{H}_3^i$, H_3 are set to zero. This follows from the tadpole condition on $E_4^2 + 2 X_8$ discussed in [22]. Alternatively, we notice that I_6^{inflow} in (5.7) can be regarded as collecting all topological terms in the 5d AdS_5 effective action. Enforcing the tadpole condition is equivalent to using the 5d EOMs for A_1 , c_3 , $\tilde{B}_2^i$, B_{2i} that come from this topological action: all these fields are flat on-shell. After integrating out F_2 , γ_4 , $\tilde{H}_3^i$, H_3 , the first two lines of (5.7) reproduce known results [15, 22] and their interpretation is standard: they encode 't Hooft anomalies for the symmetries $U(1)_1$, $U(1)_2$, $SU(2)_\Sigma$ and Poincaré symmetry.

The terms $\gamma_4 F_2$ and $\tilde{H}_3^i H_{3i}$ on the last line of (5.7) are a proxy for 't Hooft anomalies involving discrete global symmetries. More precisely, the global symmetry on the boundary SCFT depends on the choice of boundary conditions for $A_1 \equiv \mathcal{A}_1$, c_3 , B_{2i} , $\tilde{B}_i^2$. The terms $\gamma_4 F_2$ and $\tilde{H}_3^i H_{3i}$ each encode a mixed 't Hooft anomaly if we choose boundary conditions of type (c), (c') in the terminology of section 3.2.1.

5.1.2 M5-branes probing a $\mathbb{Z}_2$ singularity and wrapped on a Riemann surface

If we consider a higher-genus Riemann surface, the internal space M_6 has isometry group $U(1)_\psi \times SU(2)_\varphi$. In the case $g = 0$ we have an additional $SO(3)_\Sigma \cong SU(2)_\Sigma$ isometry. We introduce the compact notation

$$c_1^\psi = c_1(U(1)_\psi), \quad c_2^\varphi = c_2(SU(2)_\varphi), \quad c_2^\Sigma = c_2(SU(2)_\Sigma) \quad (5.8)$$

for the Chern classes of the background gauge fields for $U(1)_\psi$, $SU(2)_\varphi$, $SU(2)_\Sigma$.

The computation of the inflow anomaly polynomial can be found in appendix B.2. Let us write the result as

$$I_6^{\text{inflow}} = I_6^{\text{inflow},1} + I_6^{\text{inflow},2} + I_6^{\text{inflow},3}, \quad (5.9)$$

where $I_6^{\text{inflow},1}$ encodes the anomalies involving exclusively the symmetries associated to isometries and Poincaré symmetry, $I_6^{\text{inflow},2}$ collects all terms that only involve p -form gauge fields originating from expansion of C_3 , and $I_6^{\text{inflow},3}$ contains all other terms. Explicitly,

$$\begin{aligned} I_6^{\text{inflow},1} = & \left(\frac{1}{3} \chi N^3 + N^2 N_- \right) c_1^\psi c_2^\varphi - \frac{1}{3} N_- (c_1^\psi)^3 + \frac{1}{12} N_- c_1^\psi p_1(T) - \frac{1}{3} \chi N c_1^\psi c_2^\varphi \\ & + \left[\frac{1}{3} N_-^3 - N^2 N_- - \frac{2}{3} N^3 + \frac{2}{3} (N + N_-) \right] c_1^\psi c_2^\Sigma. \end{aligned} \quad (5.10)$$

¹³The anomaly polynomial is sensitive to the choice of the forms $(\Omega_4^\alpha)^{\text{eq}}$, $(\omega_{2\alpha})^{\text{eq}}$, $(\Lambda_{3x})^{\text{eq}}$, $(\lambda_{1u})^{\text{eq}}$. There exists a choice such the result takes the form (5.7). As we shall see, in the GMSW setup such a choice is not possible.

$$\begin{aligned}
 I_6^{\text{inflow},2} = & \frac{1}{(2\pi)^2} \left[- (N F_2 + N_+ F_2^+ + N_- F_2^-) \gamma_4 - N \tilde{H}_3^i H_{3i} \right] \\
 & + \frac{1}{(2\pi)^3} \left[- \frac{1}{6} \chi (F_2^+)^3 - \frac{1}{2} \chi F_2^+ (F_2^-)^2 + F_2 F_2^+ F_2^- - \gamma_4 \left(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^- \right) \right. \\
 & \left. + F_2^+ \left(f_{1i}^+ \tilde{H}_3^i - \tilde{f}_1^{i+} H_{3i} \right) + F_2^- \left(f_{1i}^- \tilde{H}_3^i - \tilde{f}_1^{i-} H_{3i} \right) \right],
 \end{aligned} \tag{5.11}$$

$$\begin{aligned}
 I_6^{\text{inflow},3} = & - \frac{1}{(2\pi)^2} N_- c_1^\psi (F_2^+)^2 + \frac{1}{2\pi} \left(N N_- + \frac{1}{2} N^2 \chi - \frac{1}{2} \chi \right) (c_1^\psi)^2 F_2^+ \\
 & - \frac{1}{2\pi} N \left(N_- F_2^+ + N_+ F_2^- \right) c_2^\varphi + \frac{1}{24} \chi \frac{F_2^+}{2\pi} p_1(T) \\
 & + \frac{1}{2\pi} (N + N_-) \left[N_+ F_2^- + N_- F_2^+ - N_+ F_2 \right] c_2^\Sigma \\
 & + \frac{1}{(2\pi)^2} \left[N (c_1^\psi)^2 - N c_2^\varphi - 2 c_1^\psi \frac{F_2^+}{2\pi} \right] \left(f_{1i}^+ \tilde{f}_1^{i+} + f_{1i}^- \tilde{f}_1^{i-} \right) \\
 & - \frac{1}{(2\pi)^3} c_1^\psi F_2^- \left(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^- \right) - \frac{1}{(2\pi)^2} N c_1^\psi \left(f_{1i}^+ \tilde{H}_3^i - \tilde{f}_1^{i+} H_{3i} \right).
 \end{aligned} \tag{5.12}$$

It should be stressed that, in presenting I_6^{inflow} , we have implicitly chosen a basis of external p -form fields originating from expansion of C_3 . We are free to consider field redefinitions that shift these p -form fields with terms constructed with the background connections for the isometries of M_6 . Unlike the BBBW case, however, there is no such redefinition that can set to zero all mixed terms between symmetries originating from isometries, and symmetries originating from C_3 .

Perturbative anomalies. Let us extract physical information about perturbative anomalies for continuous symmetries from the inflow anomaly polynomial (5.9). The gauge fields B_{2i} , $\tilde{B}_2^i$, c_3 , together with one linear combination of the three vectors A_1 , $A_1^\pm$, are topologically massive gauge fields in five dimensions and therefore cannot be interpreted as background gauge fields for continuous symmetries in the 4d field theory. If we are only interested in studying local aspects of 't Hooft anomalies for continuous global symmetries, we have to eliminate the topologically massive fields from the anomaly polynomial. This is done enforcing the tadpole constraints on $E_4^2 + 2 X_8$ discussed in [22]. Equivalently, we impose the equations of motion for c_3 , B_{2i} , $\tilde{B}_2^i$ in the 5d topological theory defined by I_6^{inflow} . If we do so, we obtain the relations

$$\begin{aligned}
 0 &= \frac{1}{2\pi} \left(N F_2 + N_+ F_2^+ + N_- F_2^- \right) + \frac{1}{(2\pi)^2} \left(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^- \right), \\
 0 &= \frac{1}{2\pi} N H_{3i} + \frac{1}{(2\pi)^2} \left(F_2^+ f_{1i}^+ + F_2^- f_{1i}^- \right) - \frac{1}{2\pi} N c_1^\psi f_{1i}^+, \\
 0 &= \frac{1}{2\pi} N \tilde{H}_3^i + \frac{1}{(2\pi)^2} \left(F_2^+ \tilde{f}_1^{i+} + F_2^- \tilde{f}_1^{i-} \right) - \frac{1}{2\pi} N c_1^\psi \tilde{f}_1^{i+}.
 \end{aligned} \tag{5.13}$$

We may solve these relations for F_2 , H_{3i} , $\tilde{H}_3^i$. After plugging the corresponding expressions back into I_6^{inflow} , the field γ_4 drops away and we are left with a polynomial in c_1^ψ , c_2^φ , c_2^Σ , $p_1(T)$, $F_2^\pm$, $f_{1i}^\pm$, $\tilde{f}_1^{i\pm}$. This polynomial encodes the sought-for perturbative anomalies and

reads

$$\begin{aligned}
 I_6^{\text{inflow,pert}} = & \left(\frac{1}{3} \chi N^3 + N^2 N_- \right) c_1^\psi c_2^\varphi - \frac{1}{3} N_- (c_1^\psi)^3 + \frac{1}{12} N_- c_1^\psi p_1(T) - \frac{1}{3} \chi N c_1^\psi c_2^\varphi \\
 & + \left[\frac{1}{3} N_-^3 - N^2 N_- - \frac{2}{3} N^3 + \frac{2}{3} (N + N_-) \right] c_1^\psi c_2^\Sigma \\
 & + \frac{1}{(2\pi)^3} \left[-\frac{\chi}{6} (F_2^+)^3 - \left(\frac{N_-}{N} + \frac{\chi}{2} \right) F_2^+ (F_2^-)^2 - \frac{N_+}{N} F_2^- (F_2^+)^2 \right] \\
 & + \frac{1}{(2\pi)^4} \left[-\frac{1}{N} (F_2^-)^2 f_{1i}^- \tilde{f}_1^{i-} - \frac{1}{N} (F_2^+) f_{1i}^+ \tilde{f}_1^{i+} - \frac{2}{N} F_2^+ F_2^- (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) \right] \\
 & - \frac{1}{(2\pi)^2} N_- c_1^\psi (F_2^+)^2 - \frac{1}{(2\pi)^3} 2 c_1^\psi F_2^+ f_{1i}^- \tilde{f}_1^{i-} + \frac{1}{(2\pi)^2} N (c_1^\psi)^2 f_{1i}^- \tilde{f}_1^{i-} \\
 & + \frac{1}{2\pi} \left(N N_- + \frac{\chi}{2} N^2 - \frac{\chi}{2} \right) (c_1^\psi)^2 F_2^+ - \frac{1}{2\pi} N (N_+ F_2^- + N_- F_2^+) c_2^\varphi \\
 & - \frac{1}{(2\pi)^2} N (f_{1i}^+ \tilde{f}_1^{i+} + f_{1i}^- \tilde{f}_1^{i-}) c_2^\varphi + \frac{1}{2\pi} \left(N N_- + N_-^2 + N_+^2 + \frac{N_- N_+^2}{N} \right) F_2^+ c_2^\Sigma \\
 & + \frac{1}{2\pi} \left(N N_+ + 2 N_+ N_- + \frac{N_+ N_-^2}{N} \right) F_2^- c_2^\Sigma + \frac{1}{2\pi} \frac{\chi}{24} F_2^+ p_1(T) \\
 & + \frac{1}{(2\pi)^2} \left(N_+ + \frac{N_- N_+}{N} \right) c_2^\Sigma (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) . \tag{5.14}
 \end{aligned}$$

The above expression extends the results of [27] with the inclusion of the terms involving $f_{1i}^\pm, \tilde{f}_1^{i\pm}$. Notice the appearance of $1/N$ factors in the 't Hooft anomaly coefficients. They originate from solving the relations (5.13). Physically, they come from integrating out topologically massive modes. The perturbative anomaly polynomial (5.14) can be used to compute central charges via a -maximization; at leading order in N , one finds a perfect match with the dual supergravity computation based on the GMSW solutions [27].

Aside: topologically massive fields and Green-Schwarz terms in six dimensions.

A variant of the mechanism that generates $1/N$ terms in (5.14) by integrating out topologically massive fields is at play in six dimensions. More precisely, let us consider a stack of N M5-branes probing a $\Gamma_{\text{ADE}} \subset \text{SU}(2)$ singularity. The internal geometry is S^4/Γ_{ADE} . Upon resolution of the orbifold singularities at the north and south poles of S^4 , we get a smooth internal space M_4 . At each pole we have a collection of resolution 2-cycles. Expansion of the M-theory 3-form C_3 in cohomology of M_4 yields an external 3-form gauge field c_3 and a collection of 1-form gauge fields, associated to the resolution 2-cycles at the north and south poles. In the limit in which the resolution cycles are shrunk to zero size we have an $G_\Gamma^{\text{N}} \times G_\Gamma^{\text{S}}$ non-Abelian gauge symmetry in the 7d low-energy effective action, where G_Γ is the ADE Lie group associated to Γ_{ADE} . The topological couplings of the 7d effective action are conveniently encoded in a gauge-invariant 8-form, which contains the terms

$$-\frac{1}{2} N \frac{\gamma_4^2}{(2\pi)^2} - \frac{1}{4} \frac{\gamma_4}{2\pi} \left[\frac{\text{tr}(F^{\text{N}})^2}{(2\pi)^2} - \frac{\text{tr}(F^{\text{S}})^2}{(2\pi)^2} \right] . \tag{5.15}$$

The 4-form γ_4 is the field strength of the 3-form gauge field c_3 , while $F^{\text{N,S}}$ is the field strength of the gauge group $G_\Gamma^{\text{N,S}}$. The 3-form gauge field c_3 is topologically massive by

virtue of the 7d Chern-Simons coupling encoded in the term γ_4^2 in (5.15). In order to study the perturbative anomalies for continuous global symmetries of the system, we have to integrate out this massive field. Eliminating c_3 via its classical equation of motion is the same as enforcing the tadpole constraint of [22]. The terms (5.15) are traded for

$$\frac{1}{32N} \left[\frac{\text{tr}(F^N)^2}{(2\pi)^2} - \frac{\text{tr}(F^S)^2}{(2\pi)^2} \right]^2. \quad (5.16)$$

In the analysis of [29] this term is interpreted as a Green-Schwarz term related to the center-of-mass mode of the M5-branes, see also the recent field-theoretic analysis of [47]. Our analysis of topological mass terms in supergravity reveals how this term is automatically accounted for in inflow via integrating out massive modes.

Remarks on the background 0-form gauge fields $a_{0i}^\pm, \tilde{a}_0^{i\pm}$. The terms in I_6^{inflow} with $f_{1i}^\pm = da_{0i}^\pm, \tilde{f}_1^{i\pm} = d\tilde{a}_0^{i\pm}$ should be interpreted along the lines of [2] as 't Hooft anomalies in the space of coupling constants. We can think of $a_{0i}^\pm, \tilde{a}_0^{i\pm}$ as background fields for global “(−1)-form symmetries” in the 4d field theory. (We refer the reader to [2] for a careful discussion of the merits and limitations of the notion of “(−1)-form symmetry”.)

Recall that $a_{0i}^\pm, \tilde{a}_0^{i\pm}$ originate from expansion of C_3 onto 3-cycles in the internal space M_6 . We can offer an interpretation of $a_{0i}^\pm, \tilde{a}_0^{i\pm}$ in terms of the picture of M5-branes probing a $\mathbb{Z}_2$ singularity. The 6d SCFT on the worldvolume of the M5-branes has an $\text{SU}(2)_N \times \text{SU}(2)_S$ global symmetry. This theory is reduced on Σ_g with a non-zero flavor flux, which breaks $\text{SU}(2)_N \times \text{SU}(2)_S$ to the Cartan subgroup $\text{U}(1)_N \times \text{U}(1)_S$. The 6d background 1-form gauge fields for this 0-form symmetry can be dimensionally reduced along 1-cycles in Σ_g to yield 0-form gauge fields in four dimensions. Since Σ_g has $2g$ 1-cycles, this reduction generates a total of $2 \times 2g = 4g$ 0-form gauge fields in four dimensions, which matches the total number of $a_{0i}^\pm, \tilde{a}_0^{i\pm}$ fields.

The operators in the 4d SCFT coupled to $a_{0i}^\pm, \tilde{a}_0^{i\pm}$ are exactly marginal operators. In a schematic semi-Lagrangian language, the deformation of the SCFT associated to $a_{0i}^\pm, \tilde{a}_0^{i\pm}$ takes the form

$$\Delta \mathcal{S} = \int_{\mathcal{M}_4} \frac{*41}{2\pi} \left[a_{0i}^+ \tilde{\mathcal{O}}^{i+} - \tilde{a}_0^{i+} \mathcal{O}_i^+ + a_{0i}^- \tilde{\mathcal{O}}^{i-} - \tilde{a}_0^{i-} \mathcal{O}_i^- \right], \quad \Delta(\mathcal{O}_i^\pm) = \Delta(\tilde{\mathcal{O}}^{i\pm}) = 4. \quad (5.17)$$

In light of the discussion of the previous paragraph, we can regard $\mathcal{O}_i^\pm, \tilde{\mathcal{O}}^{i\pm}$ as coming from the dimensional reduction on Σ_g of the 6d 1-form conserved current operators associated to the 6d $\text{U}(1)_N \times \text{U}(1)_S$ 0-form symmetry. Schematically,

$$*_6 J_{1,6d}^\pm \sim (*_4 \mathcal{O}_i^\pm) \wedge \tilde{\lambda}^i - (*_4 \tilde{\mathcal{O}}^{i\pm}) \wedge \lambda_i, \quad (5.18)$$

where $\lambda_i, \tilde{\lambda}^i$ are closed 1-forms on Σ_g as in section 2.1. We also notice that $a_{0i}^\pm, \tilde{a}_0^{i\pm}$ are compact scalars with period 2π . This indicates that the spacetime integrals of the associated operators $\mathcal{O}_i^\pm, \tilde{\mathcal{O}}^{i\pm}$ satisfy a quantization condition of the form

$$\int_{\mathcal{M}_4} \mathcal{O}_i^\pm *_4 1 \in \mathbb{Z}, \quad \int_{\mathcal{M}_4} \tilde{\mathcal{O}}^{i\pm} *_4 1 \in \mathbb{Z}. \quad (5.19)$$

Intuitively, the operators $\mathcal{O}_i^\pm$, $\tilde{\mathcal{O}}^{i\pm}$ are analogous to $\text{tr}(FF)$ in gauge theory, and $a_{0i}^\pm$, $\tilde{a}_0^{i\pm}$ are analogous to θ angles.

Let us stress that $a_{0i}^\pm$, $\tilde{a}_0^{i\pm}$ are distinct from the axionic couplings that originate from the complex structure moduli of the Riemann surface Σ_g . The geometric origin of the latter resides in a deformation of the metric on Σ_g . 't Hooft anomalies associated to these coupling constants have been analyzed in [48].

5.1.3 Anomalies for discrete symmetries

The inflow anomaly polynomial balances against the total 't Hooft anomalies of interacting and decoupled modes in the 4d field theory, see (5.4). It should be stressed that the separation into interacting and decoupling modes does not necessary correspond to a simple factorization of the partition function in field theory. From the perspective of the dual gravity theory, the singleton sector in a string theory/M-theory compactification decouples from the rest of the dynamics of quantum gravity, but in general the full quantum gravity partition function does not simply factorize into a contribution from the singleton sector times a contribution from interacting modes. Rather, as argued in [40], one expects the total string theory/M-theory partition function to be of the schematic form

$$Z_{\text{tot}} \sim \sum_{\beta} Z^{\beta} Z_{\beta}^{\text{singleton}}. \quad (5.20)$$

In the previous expression, the discrete label β enumerates the relevant topological sectors of string theory/M-theory in the background under consideration. The quantities $Z_{\beta}^{\text{singleton}}$ encode the contribution of singleton modes, while Z^{β} encode the contributions of all other interacting bulk modes. The holographic duals of Z^{β} are the conformal blocks of an interacting 4d CFT, while the holographic duals of $Z_{\beta}^{\text{singleton}}$ are the conformal blocks of a free 4d theory.

As demonstrated in [40], the correct strategy to compute $Z_{\beta}^{\text{singleton}}$ on the gravity side is to consider both kinetic terms and topological terms in the 5d supergravity effective action. In this approach the Hamiltonian in the singleton sector is unambiguously determined. (In contrast, in the purely topological BF theory without kinetic terms, the Hamiltonian can be modified by adding boundary terms.) For the setups of interest in this work, one needs to consider the BF couplings (3.1) supplemented with standard kinetic terms.

In the setups with wrapped M5-branes studied in this work, the total worldvolume theory has a partition function of the form (5.20) with more than one term on the r.h.s. . Indeed, the different summands labeled by β correspond to inequivalent choices of boundary conditions for the fields entering the BF couplings (3.1). In the total worldvolume theory, the fields $\mathcal{A}_1$, c_3 , B_{2i} , $\tilde{B}_2^i$ are associated to global continuous U(1) 0-, 2-, and 1-form symmetries. These U(1) symmetries are spontaneously broken. The breaking pattern is different for the various β summands in (5.20). Indeed, we know that interacting theories associated to different choices of boundary conditions have different global discrete symmetries. For example, with reference to the terminology of section 3.2.1, the U(1) 0-form symmetry associated to $\mathcal{A}_1$ is broken to $\mathbb{Z}_k$ in case (a), is broken to nothing in case (b), and is broken to $Z_{m'}$ in case (c).

The inflow anomaly polynomial I_6^{inflow} in (5.7) or (5.9) is interpreted as minus the anomaly polynomial of the total worldvolume theory (5.20). Since this theory has continuous symmetries, we can describe its anomalies using the language of differential forms. If we ignore the specific breaking pattern of the $U(1)$'s to discrete symmetries, all interacting SCFTs with partition functions Z^β have the same perturbative 't Hooft anomalies for their unbroken continuous symmetries. For wrapped M5-branes these anomalies are the first two lines of (5.7), while for wrapped M5-branes at a $\mathbb{Z}_2$ singularity the perturbative anomalies are collected in (5.14).

Extracting the anomalies for discrete symmetries of a given interacting SCFT Z^β is more challenging. We expect that the language of differential cohomology should give us the proper mathematical framework to discuss these anomalies. In appendix D we provide a brief review of the aspects of differential cohomology that are relevant for this work. We use the notation $\check{H}^\ell(\mathcal{M}_4)$ to denote the ℓ -th differential cohomology group of external spacetime. An element of $\check{H}^\ell(\mathcal{M}_4)$ models an $(\ell - 1)$ -form $U(1)$ gauge field.

As a first case, let us consider the BBBW setup and assign Dirichlet boundary conditions to $\mathcal{A}_1$ and free boundary conditions to c_3 , case (a) in the terminology of section 3.2.1. We can dualize the 3-form gauge field c_3 to a 0-form gauge field ϕ_0 . The effect of the dualization is to convert the original BF theory (3.1) (supplemented by standard kinetic terms) into a Stückelberg theory written in terms of the combination $D\phi_0 = d\phi_0 - k\mathcal{A}_1$. In the deep IR, the 1-form gauge field $\mathcal{A}_1$ and the 0-form gauge field $d\phi_0$ are subject to the constraint $D\phi_0 = 0$, or

$$k\mathcal{A}_1 = d\phi_0. \quad (5.21)$$

The gauge field $k\mathcal{A}_1$ is pure gauge, because it is given in terms a *globally defined* closed 1-form $d\phi_0$ with periods that are quantized in units of 2π . Crucially, this does not mean that $\mathcal{A}_1$ is trivial. Instead, $\mathcal{A}_1$ is a flat gauge field that is allowed to have non-trivial holonomies that are k -th roots of unity. These features show that the pair $(\mathcal{A}_1, \phi_0)$ subject to the constraint (5.21) describes a background 1-form $\mathbb{Z}_k$ gauge field, as in [25, 32].¹⁴

In the process of dualizing c_3 to ϕ_0 , the BF term $k\mathcal{A}_1 \wedge dc_3$ is removed. As a result, the anomaly polynomial (5.7) does not contain c_3 nor $\mathcal{A}_1$. This is consistent with the global symmetries of the theory in case (a): we have a global 0-form $\mathbb{Z}_k$ symmetry from $\mathcal{A}_1$, but no global symmetry from c_3 , and thus no mixed anomaly between the two. Moreover, since (5.7) lacks mixed terms between $\mathcal{F}_2$, γ_4 , and the other field strengths, there are no mixed anomalies between the 0-form $\mathbb{Z}_k$ symmetry and other symmetries.

Similar remarks apply to case (b), in which we assign Dirichlet boundary conditions to c_3 . In this situation we dualize $\mathcal{A}_1$ to ϕ_2 , and we impose the constraint

$$k c_3 = d\phi_2. \quad (5.22)$$

¹⁴If G is a finite group, giving a connection on a principal G -bundle over $\mathcal{M}_4$ is the same as specifying an element of $\text{Hom}(\pi_1(\mathcal{M}_4), G)$. For the case at hand $G = \mathbb{Z}_k$ is Abelian, and therefore (by Hurewicz theorem) we can equivalently consider $\text{Hom}(H_1(\mathcal{M}_4), \mathbb{Z}_k)$. The pair $(\mathcal{A}_1, \phi_0)$ determines indeed an element of $\text{Hom}(H_1(\mathcal{M}_4), \mathbb{Z}_k)$, because the holonomies of $\mathcal{A}_1$ for any 1-cycle are in $\mathbb{Z}_k \subset U(1)$, and only depend on the homology class of the 1-cycle because $\mathcal{A}_1$ is flat. If external spacetime $\mathcal{M}_4$ has no torsion in homology, we also have $\text{Hom}(H_1(\mathcal{M}_4), \mathbb{Z}_k) \cong H^1(\mathcal{M}_4, \mathbb{Z}_k)$ from the universal coefficient theorem.

Thus, the pair (c_2, ϕ_2) models a 3-form gauge field for a $\mathbb{Z}_k$ symmetry. The $B_{2i}, \tilde{B}_2^i$ system is studied in a similar way.

The setup with wrapped M5-branes probing a $\mathbb{Z}_2$ singularity is considerably richer. Let us focus on the fields $\mathcal{A}_1, c_3$, and let us impose Dirichlet boundary conditions on $\mathcal{A}_1$. In the total anomaly polynomial (5.9) we can collect all terms with a γ_4 factor,

$$-\frac{\gamma_4}{2\pi} \left[k \frac{\mathcal{F}_2}{2\pi} + \frac{f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-}{(2\pi)^2} \right], \quad (5.23)$$

where we have recalled that $N_\alpha F_2^\alpha = k \mathcal{F}_2$. The dualization of c_3 yields a 0-form gauge field ϕ_0 as before. The analog of the constraint (5.21) reads now

$$k \mathcal{A}_1 + A_1^{f\tilde{f}} = d\phi_0. \quad (5.24)$$

In the previous expression, $A_1^{f\tilde{f}}$ denotes the 1-form gauge field whose field strength satisfies

$$\frac{dA_1^{f\tilde{f}}}{2\pi} = \frac{f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-}{(2\pi)^2}. \quad (5.25)$$

More precisely, the 0-form gauge fields $a_{0i}^\pm, \tilde{a}_0^{i\pm}$ can be modeled by elements of the differential cohomology group $\check{H}^1(\mathcal{M}_4)$. In differential cohomology a well-defined notion of product exists, which maps $\check{H}^1(\mathcal{M}_4) \times \check{H}^1(\mathcal{M}_4)$ to $\check{H}^2(\mathcal{M}_4)$. In other words, to a pair of 0-form gauge fields one can associate a 1-form gauge field, see appendix D for further details. It is in this sense that $A_1^{f\tilde{f}}$ is constructed from $a_{0i}^\pm, \tilde{a}_0^{i\pm}$. The relation (5.24) should be interpreted as a relation between elements of $\check{H}^2(\mathcal{M}_4)$. If we take the field strength of both sides, we get an equation for differential 2-forms,

$$k \mathcal{F}_2 + \frac{f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-}{2\pi} = 0. \quad (5.26)$$

This relation is one of the equations of motions of the topological theory defined by (5.9), or equivalently one of the tadpole constraints on $E_4^2 + 2 X_8$.

The 2-form equation (5.26) can be integrated on any 2-cycle in spacetime. Since $\mathcal{F}_2$ has periods that are quantized in units of 2π , we learn that $dA_1^{f\tilde{f}}$ has periods that are quantized in units of $2\pi k$. This indicates that we can introduce a new 1-form gauge field $\mathbf{A}_1$ defined as

$$\mathbf{A}_1 = \mathcal{A}_1 + \frac{1}{k} A_1^{f\tilde{f}}, \quad (5.27)$$

and that $d\mathbf{A}_1$ has periods that are quantized in units of 2π . It should therefore be possible to model $\mathbf{A}_1$ with an element of $\check{H}^2(\mathcal{M}_4)$. The new gauge field $\mathbf{A}_1$ satisfies

$$k \mathbf{A}_1 = d\phi_0. \quad (5.28)$$

Therefore, the pair $(\mathbf{A}_1, \phi_0)$ describes a background gauge field for a $\mathbb{Z}_k$ 0-form symmetry.

We can now go back to the anomaly polynomial (5.9). Dualization of γ_4 has removed all terms with a γ_4 factor. There are several other terms, however, that contain $\mathcal{A}_1$. We

rewrite these terms using (5.27) to trade $\mathcal{A}_1$ for $\mathbf{A}_1$ and $A_1^{f\tilde{f}}$. After this rewriting, we find terms with $d\mathbf{A}_1$, for example the term

$$I_6^{\text{inflow}} \supset n \frac{d\mathbf{A}_1}{2\pi} \wedge c_2^\varphi, \quad (5.29)$$

where n is an integer 't Hooft anomaly coefficient. The 2-form $d\mathbf{A}_1$ is zero: how should (5.29) be interpreted? We regard the 6-form I_6^{inflow} as the field strength of a U(1) 5-form gauge field, modeled by an element of $\check{H}^6(\mathcal{M}_6)$. The wedge product in (5.29) is reinterpreted as the product in differential cohomology. The second Chern class c_2^φ admits a natural extension in differential cohomology and defines an element of $\check{H}^4(\mathcal{M}_4)$. A more detailed discussion of this point can be found in appendix D. The 1-form gauge field $\mathbf{A}_1$ is thought of as an element of $\check{H}^2(\mathcal{M}_4)$. Their product is thus an element in $\check{H}^6(\mathcal{M}_6)$. Even though the field strength of this element of $\check{H}^6(\mathcal{M}_6)$ is zero (because $\mathbf{A}_1$ is flat), this object is still non-trivial. It encodes a non-zero 't Hooft anomaly between the discrete $\mathbb{Z}_k$ 0-form symmetry and the $\text{SU}(2)_\varphi$ symmetry.

The ideas outlined in the previous paragraphs can also be applied to the $B_{2i}, \tilde{B}_2^i$ system. For example, if we assign Dirichlet boundary conditions to B_{2i} (for each label i), we have to collect all terms in (5.9) with $\tilde{H}_3^i$ and dualize $\tilde{B}_2^i$ to a 1-form gauge field ϕ_{1i} . We obtain a Stückelberg-like system that enforces a constraint of the form

$$N B_{2i} + B_{2i}^{\text{comp}} = d\phi_{1i}. \quad (5.30)$$

The quantity B_{2i}^{comp} is a composite 2-form gauge field, whose field strength satisfies

$$dB_{2i}^{\text{comp}} = \frac{F_2^+ f_{1i}^+ + F_2^- f_{1i}^-}{2\pi} - N c_1^\psi f_{1i}^+. \quad (5.31)$$

As in the case of $A_1^{f\tilde{f}}$, the object dB_{2i}^{comp} is best thought of as a sum of products in differential cohomology of 1-form and 0-form gauge fields. The relation (5.30) is interpreted as an equation in $\check{H}^3(\mathcal{M}_4)$. Taking the field strength of both sides we get an equation for 3-forms, which is the second equation of motion in (5.13). The periods of dB_{2i}^{comp} are quantized in units of $2\pi N$, thus it makes sense to consider $1/N B_{2i}^{\text{comp}}$. Reasoning as above, a new 2-form gauge field $\mathbf{B}_{2i}$ can be introduced, in terms of which (5.30) takes a simpler form,

$$\mathbf{B}_{2i} = B_{2i} + \frac{1}{N} B_{2i}^{\text{comp}}, \quad N \mathbf{B}_{2i} = d\phi_{1i}. \quad (5.32)$$

The dualization of $\tilde{B}_2^i$ to ϕ_{1i} has removed all terms with $\tilde{H}_3^i$ from (5.9). There are other terms containing B_{2i} , however. We rewrite such terms trading B_{2i} for $\mathbf{B}_{2i}$. As before, the terms that contain $d\mathbf{B}_{2i}$ encode mixed 't Hooft anomalies between the “electric” $(\mathbb{Z}_N)^g$ 1-form symmetry and other symmetries of the field theory.

In appendix E we present a case study for a detailed analysis of the anomalies, along the lines explained in the previous paragraphs. In particular, we give the full anomaly polynomial in the case in which we assign Dirichlet boundary conditions to $\mathcal{A}_1$ and B_{2i} . We find a rich variety of mixed anomalies involving the $\mathbb{Z}_k$ 0-form symmetry, the $(\mathbb{Z}_N)^g$

1-form symmetry, and the continuous symmetries of the system, see (E.18)–(E.21). The terms in the anomaly polynomial that involve $d\mathbf{A}_1$ and $d\mathbf{B}_{2i}$ have the following structure,

$$\begin{aligned}
 I_6^{\text{inflow}} \supset & a_1 \frac{(d\mathbf{A}_1)^3}{(2\pi)^3} + a_2 c_1^\psi \frac{(d\mathbf{A}_1)^2}{(2\pi)^2} + a_{3,\hat{\alpha}} \frac{\mathcal{F}_2^{\hat{\alpha}} (d\mathbf{A}_1)^2}{(2\pi)^3} + a_4 \frac{(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) (d\mathbf{A}_1)^2}{(2\pi)^3} \\
 & + a_5 \frac{f_{1i}^+ \tilde{f}_1^{i+} (d\mathbf{A}_1)^2}{(2\pi)^3} + a_6 c_2^\varphi \frac{d\mathbf{A}_1}{2\pi} + a_7 c_2^\Sigma \frac{d\mathbf{A}_1}{2\pi} + a_8 (c_1^\psi)^2 \frac{d\mathbf{A}_1}{2\pi} + a_9 p_1(T) \frac{d\mathbf{A}_1}{2\pi} \\
 & + a_{10,\hat{\alpha}} c_1^\psi \frac{\mathcal{F}_2^{\hat{\alpha}} d\mathbf{A}_1}{(2\pi)^2} + a_{11,\hat{\alpha}\hat{\beta}} \frac{\mathcal{F}_2^{\hat{\alpha}} \mathcal{F}_2^{\hat{\beta}} d\mathbf{A}_1}{(2\pi)^3} + a_{12} c_1^\psi \frac{f_{1i}^- \tilde{f}_1^{i-} d\mathbf{A}_1}{(2\pi)^3} + a_{13,\hat{\alpha}} \frac{f_{1i}^+ \tilde{f}_1^{i+} \mathcal{F}_2^{\hat{\alpha}} d\mathbf{A}_1}{(2\pi)^4} \\
 & + a_{14} c_1^\psi \frac{(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) d\mathbf{A}_1}{(2\pi)^3} + a_{15,\hat{\alpha}} \frac{(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) \mathcal{F}_2^{\hat{\alpha}} d\mathbf{A}_1}{(2\pi)^3} \\
 & + a_{16} \frac{(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-)^2 d\mathbf{A}_1}{(2\pi)^5} + a_{17} \frac{f_{1ij}^- \tilde{f}_1^{j-} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) d\mathbf{A}_1}{(2\pi)^5} \\
 & + a_{18,\hat{\alpha}} \frac{\mathcal{F}_2^{\hat{\alpha}} \tilde{f}_1^{i+} d\mathbf{B}_{2i}}{(2\pi)^3} + a_{19,\hat{\alpha}} \frac{\mathcal{F}_2^{\hat{\alpha}} \tilde{f}_1^{i-} d\mathbf{B}_{2i}}{(2\pi)^3} + a_{20} c_1^\psi \frac{\tilde{f}_1^{i+} d\mathbf{B}_{2i}}{(2\pi)^2} \\
 & + a_{21} \frac{(f_{1j}^+ \tilde{f}_1^{j-} - \tilde{f}_1^{j+} f_{1j}^-) \tilde{f}_1^{i+} d\mathbf{B}_{2i}}{(2\pi)^4} + a_{22} \frac{d\mathbf{A}_1 d\mathbf{B}_{2i} \tilde{f}_1^{i+}}{(2\pi)^3} .
 \end{aligned} \tag{5.33}$$

In the previous expression we have made use of the notation introduced in (2.8), in which the index $\hat{\alpha}$ refers to the continuous $U(1)^2$ symmetry associated to two out of the three vectors coming from expansion of C_3 onto cohomology classes. The explicit expressions of the anomaly coefficients $a_1, \dots, a_{22}$ can be read off from (E.18)–(E.21). Among the various terms in (5.33) we notice in particular: terms that are cubic and quadratic in $d\mathbf{A}_1$; the term $p_1(T) d\mathbf{A}_1$ describing a mixed discrete-gravitational anomaly; the last term in (5.33) which mixes the two discrete symmetries with a continuous axionic “ (-1) -form symmetry”.

In closing this section, let us comment on boundary conditions of type (c) or (c'). Intuitively speaking, in case (c) only a part of the field c_3 should be dualized to ϕ_0 , and a part of $\mathcal{A}_1$ should be dualized to ϕ_2 . More precisely, we expect a difficulty in using a Lagrangian formalism to describe this case, analogous for instance to the difficulties that one encounters in formulating a 4d $U(1)$ gauge theory with both the electric and magnetic photon in the Lagrangian. Even though we are not able to describe the dualization from BF form to Stückelberg form with the same level of detail as in cases (a) and (b), we can still give an interpretation of (5.9) in terms of 't Hooft anomalies for discrete symmetries. As already anticipated in section 3.2.1, the term $k \mathcal{F}_2 \gamma_4$ describes a mixed anomaly between the $\mathbb{Z}_{m'}$ 0-form symmetry and the $\mathbb{Z}_m$ 2-form symmetry. By a similar token, all terms involving $\mathcal{F}_2$ signal non-zero 't Hooft anomalies between the $\mathbb{Z}_{m'}$ 0-form symmetry and the other symmetries in the system, and similarly for terms with γ_4 . Analogous remarks apply to B_{2i} , $\tilde{B}_2^i$. As in the case studied in appendix E, we find a rich structure of mixed 't Hooft anomalies.

5.2 Singletons and 't Hooft anomalies

A better understanding of the decoupled sector of the 4d field theory is crucial to obtain a detailed prediction for the anomalies of the interacting CFTs of interest. The holographic

dictionary suggests a general strategy to extract information about the decoupling modes on the field theory side from the gravity side: one has to study the singleton modes that propagate on the conformal boundary of $\mathcal{M}_5 \times_w M_6$.

Let us consider the setup with a stack of M5-branes wrapped on a Riemann surface. In this case we know the decoupling modes on the field theory side. They are obtained from dimensional reduction on Σ_g of a free 6d $\mathcal{N} = (2, 0)$ tensor multiplet, which corresponds to the center-of-mass degrees of freedom of the M5-brane stack. Our goal is to compare the set of decoupled fields with singleton modes in $\mathcal{M}_5 \times_w M_6$, where M_6 is the internal space of BBBW solutions.

The dimensional reduction of a free 6d $\mathcal{N} = (2, 0)$ tensor multiplet on a genus- g Riemann surface with twist parameters p, q is discussed in appendix F. Recall that, for any values of p, q , the internal space has a $U(1)_1 \times U(1)_2$ isometry. We find the following 4d $\mathcal{N} = 1$ multiplets:

$$\begin{aligned}
 g \text{ vector multiplets:} & \quad A_\mu(0, 0), \lambda(1, 1); \\
 \text{one chiral multiplet:} & \quad \Phi(0, 0), b_0(0, 0), \psi(-1, -1); \\
 h^0(K^{\frac{p}{p+q}}) \text{ chiral multiplets:} & \quad Q(2, 0), \Lambda(1, -1); \\
 h^0(K^{\frac{q}{p+q}}) \text{ chiral multiplets:} & \quad \hat{Q}(0, 2), \hat{\Lambda}(-1, 1).
 \end{aligned} \tag{5.34}$$

In the above expressions, K is the canonical bundle of Σ_g (we are assuming $g \neq 1$, see appendix F for the case $g = 1$). The scalars Φ, b_0 are real, while $Q, \hat{Q}$ are complex. The spinors $\lambda, \psi, \Lambda, \hat{\Lambda}$ are Weyl spinors of positive chirality. For each field, we have included its $U(1)_1 \times U(1)_2$ charges. The combination $U(1)_1 + U(1)_2$ is an R-symmetry, while $U(1)_1 - U(1)_2$ is a flavor symmetry.

The multiplicities and the $U(1)_1 \times U(1)_2$ charges of the free fields listed in (5.34) are such that their combined 't Hooft anomalies match exactly with the dimensional reduction on Σ_g of the 8-form anomaly polynomial of a free 6d $\mathcal{N} = (2, 0)$ tensor multiplet, as expected. Notice that the anomalies only depend on the difference $h^0(K^{\frac{p}{p+q}}) - h^0(K^{\frac{q}{p+q}})$, which is fixed by the Riemann-Roch theorem, see appendix F.

How are the free fields in (5.34) identified with singleton modes on $\mathcal{M}_5 \times_w M_6$? The g vectors A_μ (we omit the degeneracy label) are identified with the g singleton 1-form gauge field associated to the BF coupling $N \tilde{B}_2^i \wedge dB_{2i}$. In a similar way, the real scalar b_0 is identified with the singleton 0-form gauge field associated to the BF coupling $k \mathcal{A}_1 \wedge dc_3$. The origin of the other scalar modes and of the fermions is different. These fields are identified with suitable Kaluza-Klein modes of 11d supergravity on $\mathcal{M}_5 \times_w M_6$, whose internal wavefunction is such that they are pure gauge in the bulk of $\mathcal{M}_5$, but propagate on the conformal boundary $\partial\mathcal{M}_5$. We may refer to these modes as Kaluza-Klein singletons. They are well-understood for the $AdS_5 \times S^5$ solution in type IIB supergravity [49].

As we can see from (5.34), the Kaluza-Klein singletons Φ, λ, ψ sit in supermultiplets that contain the BF singletons. The existence and charges of these Kaluza-Klein singletons can be easily determined by counting BF singletons (which are neutral under $U(1)_1 \times U(1)_2$) and using 4d $\mathcal{N} = 1$ supersymmetry. In contrast, the chiral multiplets (Q, Λ) and $(\hat{Q}, \hat{\Lambda})$ do not contain BF singletons. It follows that to verify the existence, charges, and multiplicities

of these Kaluza-Klein singletons we cannot rely on a simple counting of BF terms, and we rather have to perform a direct analysis of the Kaluza-Klein spectrum.

The situation is different if we specialize to $g \geq 2$, $q = 0$, i.e. the $\mathcal{N} = 2$ Maldacena-Nuñez solution. The $U(1)_1$ isometry enhances to $SU(2)_1$ and the internal space M_6 contains a round S^2 . The free fields in (5.34) can be reorganized into 4d $\mathcal{N} = 2$ multiplets,

$$\begin{aligned} g \text{ vector multiplets:} & \quad A_\mu [1_0], \quad Q [1_2], \quad \lambda' [2_1] ; \\ \text{one hypermultiplet:} & \quad b_0 [1_0], \quad \hat{Q}' [3_0], \quad \psi' [2_{-1}] . \end{aligned} \quad (5.35)$$

For each field we have indicated its $SU(2)_1$ representation and $U(1)_2$ charge. The triplet of real scalars $\hat{Q}'$ comes from combining the complex scalar $\hat{Q}$ and the real scalar Φ in (5.34), while the fermion λ' comes from λ , Λ , and the fermion ψ' comes from ψ , $\hat{\Lambda}$. It is clear from (5.35) that in this 4d $\mathcal{N} = 2$ setup all Kaluza-Klein singletons are related to BF singletons by supersymmetry. Thus, the existence, charges, and multiplicities of the Kaluza-Klein singletons can be easily inferred from counting BF singletons and exploiting 4d $\mathcal{N} = 2$ supersymmetry.

If we consider the setup with wrapped M5-branes probing a $\mathbb{Z}_2$ singularity, the task at hand is to identify singleton modes for GMSW solutions on the gravity side. A subset of these modes is easily identified: a real 0-form BF singleton b_0 and a set of g real 1-form BF singletons A_μ . These fields are neutral under the $SU(2)_\varphi$ flavor symmetry and the $U(1)_\psi$ isometry, which is an R-symmetry.¹⁵ Exploiting 4d $\mathcal{N} = 1$ supersymmetry, we predict the following multiplets of singleton modes in GMSW,

$$\begin{aligned} g \text{ vector multiplets:} & \quad A_\mu [1_0], \quad \lambda [1_1] ; \\ \text{one chiral multiplet:} & \quad b_0 [1_0], \quad \Phi [1_0], \quad \psi [1]_{-1} . \end{aligned} \quad (5.36)$$

We have indicated the $SU(2)_\varphi$ representation and the $U(1)_\psi$ charge. (All these fields are neutral under the baryonic $U(1)^2$ symmetry.) In analogy with (5.34), we expect additional chiral multiplets of Kaluza-Klein singletons, whose charges and multiplicities cannot be inferred from the BF terms alone. To identify these chiral multiplets, we need to perform a more detailed study of the Kaluza-Klein spectrum of GMSW solutions. We plan to address this problem in future work.

It should be stressed, however, that we do not expect Kaluza-Klein singletons to exhaust the entire set of singleton modes for these geometries. This expectation is based on analogy with D3-brane setups in type II string theory. As pointed out in [24], if we consider type IIB supergravity on $AdS_5 \times T^{1,1}$, we only see one vector BF singleton (coming from the term $B_2 \wedge dC_2$, where B_2 is the NSNS 2-form and C_2 is the RR 2-form). On the other hand, the worldvolume theory on the D3-branes is a $U(N) \times U(N)$ quiver theory (with superpotential) in which the two $U(1)$'s in $U(N) \times U(N)$ decouple in the IR. The overall $U(1)$ is identified with the BF vector, but the relative $U(1)$ does not appear to have an obvious singleton interpretation within the supergravity approximation. The geometry $AdS_5 \times T^{1,1}$ can be regarded as originating from blow-up of a $\mathbb{Z}_2$ orbifold singularity. This feature is qualitatively similar to our interpretation of the smooth GMSW solutions

¹⁵The *superconformal* R-symmetry is the linear combination of $U(1)_\psi$ and the baryonic $U(1)^2$ symmetry fixed by a -maximization [50].

in terms of a blow-up of the $\mathbb{C}^2/\mathbb{Z}_2$ singularities at the north and south poles of S^4 . For this reason, we expect that, in order to capture all decoupling modes on the worldvolume theory of the M5-brane stack, one has to analyze singleton modes beyond the supergravity approximation, including stringy modes. This program could give an exact answer for the decoupled modes, which, combined with inflow, would yield the exact anomaly of the interacting SCFT, including $\mathcal{O}(1)$ terms.

6 Outlook

The problems studied in this work suggest several directions for future research. For instance, a systematic analysis of topological boundary conditions for the BF theory $N \tilde{B}_2^i \wedge dB_{2i}$, including the action of the duality group $\mathrm{Sp}(2g, \mathbb{Z})$, has not been performed. Such a study has the potential of furnishing an organizing principle for the classification of line operators in 4d SCFTs engineered with M5-branes, with either $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry.

Another problem that deserves further analysis is the computation of the partition function vector of the singleton modes in a setup with wrapped M5-branes. The full partition function from the gravity side is expected to take the form (5.20), where the vector Z^β encodes the contribution of interacting bulk modes (and is dual to the partition function vector of an interacting SCFT), while $Z_\beta^{\text{singleton}}$ is the partition function vector of singleton modes. The latter is computable following the methods of [40, 41]. The action of $\mathrm{Sp}(2g, \mathbb{Z})$ on the conformal blocks Z^β of the interacting SCFT can be determined from its action on $Z_\beta^{\text{singleton}}$. One may then explore the interplay between the duality group and 't Hooft anomalies for various global symmetries.

We have observed that BF couplings in the bulk 5d topological theory account for a set of singleton modes on the gravity side. On the other hand, additional singleton modes are present, which do not originate from BF terms. In general $\mathcal{N} = 1$ setups, supersymmetry is not sufficient to determine all singleton modes starting from BF singleton modes. It would be beneficial to perform a systematic study of singleton modes in string/M-theory compactifications, especially in setups with lower amounts of supersymmetry. On the basis of the holographic dictionary, it is expected that singleton modes on the gravity side should account for all modes that decouple in the IR on the field theory side. A detailed knowledge of decoupling modes can provide access to precision holography, allowing for example for a computation of exact anomalies, beyond the large- N limit, including $\mathcal{O}(1)$ terms. The role of singleton modes in holographic flows is also worth analyzing further.

It is natural to wonder how the results of this paper would be modified by the inclusion of punctures on the Riemann surface. In order to address this question in a more systematic way, a better understanding of punctures for 4d $\mathcal{N} = 1$ theories engineered with M5-branes would be useful. With regards to $\mathcal{N} = 1$ regular punctures for 6d $(2, 0)$ theories of type A_{N-1} on a Riemann surface, our expectation is that there should be no mixed anomalies between the continuous 0-form flavor symmetries at the punctures and the discrete and higher-form symmetries of the system. On the other hand, we anticipate a much richer structure in setups with M5-branes probing a $\mathbb{Z}_2$ singularity and wrapped on the Riemann surface with punctures.

Finally, it would be useful to extend the discussion of symmetries and anomalies in geometric engineering, including other possible sources of internal discrete symmetries (such as discrete isometries of the internal space or torsion cycles [33, 34]), as well as spacetime discrete symmetries (such as parity or time reversal).

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A A change of basis

In this appendix we consider an internal space M_6 with $n := b^2(M_6) \geq 2$. The lattice $H^2(M_6, \mathbb{Z})_{\text{free}}$ is preserved by the action of $\text{SL}(n, \mathbb{Z})$. In terms of the closed 2-forms $\omega_{2\alpha}$, we can consider a change of basis of the form

$$\omega'_{2\alpha} = \omega_{2\beta} (M^{-1})^\beta{}_\alpha, \quad M \in \text{SL}(n, \mathbb{Z}). \quad (\text{A.1})$$

This linear transformation is accompanied by transformations on the closed 4-forms Ω_4^α , as well as on the basis $\mathcal{C}_2^\alpha$ of 2-cycles in M_6 , and the basis $\mathcal{C}_4^\alpha$ of 4-cycles in M_6 . In order to preserve the relations $\int_{M_6} \omega_{2\alpha} \wedge \Omega_4^\beta = \delta_\alpha^\beta$, $\int_{\mathcal{C}_2^\alpha} \omega_{2\beta} = \delta_\beta^\alpha$, and $\int_{\mathcal{C}_4^\alpha} \Omega_4^\beta = \delta_\alpha^\beta$, we must set

$$\Omega_4'^\alpha = M^\alpha{}_\beta \Omega_4^\beta, \quad \mathcal{C}_2'^\alpha = M^\alpha{}_\beta \mathcal{C}_2^\beta, \quad \mathcal{C}_4'^\alpha = \mathcal{C}_4^\beta (M^{-1})^\beta{}_\alpha. \quad (\text{A.2})$$

The ansatz (2.3) for G_4 contains the terms $A_1^\alpha \wedge \omega_{2\alpha}$ and $N_\alpha \Omega_4^\alpha$. As a result, the linear transformation (A.1) induces a linear transformation for the external 1-form gauge fields and flux quanta,

$$A_1'^\alpha = M^\alpha{}_\beta A_1^\beta, \quad N'_\alpha = N_\beta (M^{-1})^\beta{}_\alpha. \quad (\text{A.3})$$

To proceed, we define the integers k and m_α via the relations

$$k = \text{gcd}(N_1, \dots, N_n), \quad N_\alpha = k m_\alpha. \quad (\text{A.4})$$

It can then be shown that a matrix $M \in \text{SL}(n, \mathbb{Z})$ exists, such that

$$A_1'^{\alpha=1} = m_\beta A_1^\beta. \quad (\text{A.5})$$

This can be argued as follows. The integers $\{m_\alpha\}_{\alpha=1}^n$ are relatively prime. There must exist labels $\alpha_1, \alpha_2 \in \{1, \dots, n\}$, $\alpha_1 \neq \alpha_2$, such that m_{α_1} and m_{α_2} are relatively prime. After reordering $\{m_\alpha\}_{\alpha=1}^n$ if necessary, we can take $\alpha_1 = 1$, $\alpha_2 = 2$. We may then consider the following matrix,

$$M^\alpha{}_\beta = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \dots & m_{n-1} & m_n \\ r & s & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad \det M = s m_1 - r m_2. \quad (\text{A.6})$$

Since m_1 and m_2 are relatively prime, there exist integers r and s satisfying $s m_1 - r m_2 = 1$. This follows from Bézout's identity in elementary number theory. Thus $M \in \text{SL}(n, \mathbb{Z})$ and clearly $A_1'^{\alpha=1} = m_\beta A_1^\beta$.

In the new basis N'_α of flux quanta, the component $N'_{\alpha=1} = k$ is the only non-zero component, and we have

$$N_\alpha dA_1^\alpha = k dA_1'^{\alpha=1} . \quad (\text{A.7})$$

This is why this basis is best suited to study the topological terms in the 5d action.

B Cohomology classes and gauging of isometries

In this appendix we study non-trivial cohomology classes of the internal space M_6 for BBBW solutions and GMSW solutions. To compute the full inflow anomaly polynomial, we need to activate background gauge fields for the isometries of M_6 . After turning on these gauge fields, the relevant spacetime is denoted M_{12} and is of the form

$$M_6 \hookrightarrow M_{12} \rightarrow \mathcal{M}_6 , \quad (\text{B.1})$$

where $\mathcal{M}_6$ denotes external spacetime. For the purposes of computing anomalies in the descent formalism, we take X_6 to be Euclidean and six-dimensional. We discuss representatives for cohomology classes in M_6 and their counterparts in M_{12} . We suppress wedge products throughout this appendix.

B.1 BBBW solutions

B.1.1 Cohomology classes in M_6

In the BBBW solutions, the internal space M_6 is topologically an S^4 fibration over a genus- g Riemann surface Σ_g . We refer to [15] for the full expression of the metric on M_6 . For the purposes of this work, we can use the following simplified line element on M_6 , which captures the topology and isometries of the metric in [15],

$$ds^2(M_6) = ds^2(\Sigma_g) + d\mu_0^2 + d\mu_1^2 + d\mu_2^2 + \mu_1^2 D\phi_1^2 + \mu_2^2 D\phi_2^2 , \quad (\text{B.2})$$

$$dD\phi_1 = -p V_\Sigma , \quad dD\phi_2 = -q V_\Sigma , \quad \int_{\Sigma_g} V_\Sigma = 2\pi , \quad p + q = -\chi = 2(g-1) .$$

The angles ϕ_1, ϕ_2 have periodicity 2π and $ds^2(\Sigma_g)$ denotes the metric on Σ_g with constant curvature $\kappa \in \{0, 1, -1\}$. We have also introduced three constrained coordinates μ_0, μ_1, μ_2 , satisfying

$$\mu_0^2 + \mu_1^2 + \mu_2^2 = 1 , \quad -1 \leq \mu_0 \leq 1 , \quad 0 \leq \mu_1 \leq 1 , \quad 0 \leq \mu_2 \leq 1 . \quad (\text{B.3})$$

We use B_2 to denote the 2d space parametrized by μ_0, μ_1, μ_2 . The total space of the $S_{\phi_1}^1 \times S_{\phi_2}^1$ fibration over B_2 is an S^4 . We refer to the points $\mu_0 = \pm 1$ as the north and south poles of S^4 , respectively. The space B_2 is depicted schematically in figure 2.

The space M_6 admits one integral 4-homology class. A representative is obtained by considering the 4-cycle $\mathcal{C}^4$ that is obtained by taking the S^4 fiber on top of a generic point

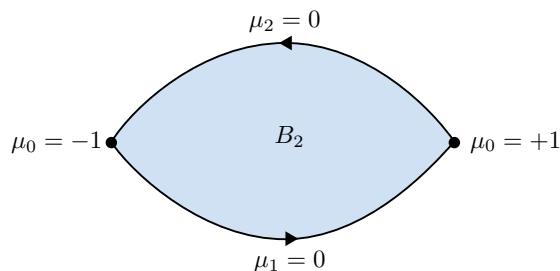


Figure 2. Schematic depiction of the space B_2 described by the constrained coordinates (B.3) with line element $d\mu_0^2 + d\mu_1^2 + d\mu_2^2$. We have also indicated the orientation of ∂B_2 used throughout this appendix.

on the Riemann surface. Correspondingly, there is one non-trivial 4-cohomology class on M_6 . We represent it by a closed but not exact 4-form ω_4 , which integrates to 1 on the $\mathcal{C}^4$ cycle,

$$\int_{\mathcal{C}^4} \Omega_4 = 1. \quad (\text{B.4})$$

The closed 4-form ω_4 is parametrized as follows,

$$\Omega_4 = h_2 \frac{D\phi_1}{2\pi} \frac{D\phi_2}{2\pi} + h_1^1 \frac{D\phi_1}{2\pi} \frac{dD\phi_2}{2\pi} + h_1^2 \frac{D\phi_2}{2\pi} \frac{dD\phi_1}{2\pi}. \quad (\text{B.5})$$

In the above expression, h_2 is a 2-form on B_2 , while $h_1^{1,2}$ are 1-forms on B_2 . Closure of ω_4 requires

$$h_2 = dh_1^1 = -dh_1^2. \quad (\text{B.6})$$

It follows that $d(h_1^1 + h_1^2) = 0$ and therefore (since B_2 has trivial cohomology) there exists a 0-form h_0 such that

$$h_1^1 + h_1^2 = -dh_0. \quad (\text{B.7})$$

Without loss of generality, we can take h_0 to satisfy

$$(h_0)^N = -(h_0)^S, \quad (\text{B.8})$$

where the superscript N, S means evaluation at $\mu_0 = \pm 1$, respectively. The 1-forms h_1^1 , h_1^2 satisfy additional requirements that ensure the regularity of ω_4 . In particular, we must demand that h_1^1 be zero if restricted to the locus $\{\mu_1 = 0\} \subset B_2$, and similarly for h_1^2 ,

$$h_1^1|_{\mu_1=0} = 0, \quad h_1^2|_{\mu_2=0} = 0. \quad (\text{B.9})$$

This is due to the fact that $S_{\phi_1}^1$ shrinks along $\mu_1 = 0$, and similarly for $S_{\phi_2}^1$. We also have to impose (B.4),

$$1 = \int_{B_2} h_2 = \int_{\partial B_2} h_1^1 = \int_{\{\mu_1=0\}} h_1^1 - \int_{\{\mu_2=0\}} h_1^2 = \int_{\{\mu_2=0\}} dh_0 = (h_0)^N - (h_0)^S. \quad (\text{B.10})$$

In the first step we have integrated over ϕ_1 , ϕ_2 . In the second step we used (B.6). The boundary ∂B_2 consists of the arcs $\{\mu_1 = 0\}$ and $\{\mu_2 = 0\}$, with a relative minus due to

orientation, see figure 2. We conclude recalling (B.9) and (B.7). We learn that $(h_0)^N - (h_0)^S = 1$, which, combined with (B.8), yields

$$(h_0)^N = \frac{1}{2}, \quad (h_0)^S = -\frac{1}{2}. \quad (\text{B.11})$$

Poincaré duality implies that the space M_6 admits one non-trivial 2-cohomology class. It can be represented by a closed but not exact 2-form ω_2 , normalized in such a way that

$$\int_{M_6} \Omega_4 \omega_2 = 1. \quad (\text{B.12})$$

We parametrize ω_2 as

$$\omega_2 = d \left[H_0^1 \frac{D\phi_1}{2\pi} + H_0^2 \frac{D\phi_2}{2\pi} \right] = dH_0^1 \frac{D\phi_1}{2\pi} + dH_0^2 \frac{D\phi_2}{2\pi} - (p H_0^1 + q H_0^2) \frac{V_\Sigma}{2\pi}, \quad (\text{B.13})$$

where H_0^1, H_0^2 are 0-forms on B_2 . We might have included an additional term $\Delta\omega_2 = (\text{const}) V_\Sigma$, but such term can always be reabsorbed by redefining H_0^1 or H_0^2 . (We do not consider the case $p = 0 = q$ because it does not correspond to a smooth M-theory solution.) The 0-forms H_0^1, H_0^2 satisfy additional requirements that ensure regularity of ω_2 . In particular, regularity the term $dH_0^1 \wedge D\phi_1$ implies that the function H_0^1 restricted to $\{\mu_1 = 0\}$ must be a constant. Similarly, H_0^2 restricted to $\{\mu_2 = 0\}$ must be a constant,

$$d(H_0^1|_{\mu_1=0}) = 0, \quad d(H_0^2|_{\mu_2=0}) = 0. \quad (\text{B.14})$$

Since we can connect the north and south poles with either the arc $\{\mu_1 = 0\}$ or $\{\mu_2 = 0\}$, we conclude that the values of H_0^1, H_0^2 at the poles are equal,

$$(H_0^1)^N = (H_0^1)^S, \quad (H_0^2)^N = (H_0^2)^S. \quad (\text{B.15})$$

Next, let us consider the integral in (B.12). It can be computed with manipulations similar to those in (B.10). The result is

$$\int_{M_6} \Omega_4 \omega_2 = - \left[p (H_0^1)^N + q (H_0^2)^N \right] \left[(h_0)^N - (h_0)^S \right]. \quad (\text{B.16})$$

Using (B.11), we conclude that, in order to have (B.12), the values $(H_0^1)^N, (H_0^2)^N$ must satisfy

$$p (H_0^1)^N + q (H_0^2)^N = -1. \quad (\text{B.17})$$

Next, let us discuss 1-cohomology classes on M_6 . The Riemann surface admits $2g$ independent non-trivial 1-cohomology classes, which are represented by closed but not exact 1-forms λ_{1u} , $u = 1, \dots, 2g$. These 1-forms can be pulled back to the total space M_6 , yielding 1-forms that we still denote λ_{1u} and that are still closed. It can be checked that they are not exact in M_6 , and that they furnish representatives for all 1-cohomology classes of M_6 . The associated 1-cycles on M_6 are realized by taking a 1-cycle on the Riemann surface, at the point $\mu_0 = 1$, where both $S_{\phi_1}^1$ and $S_{\phi_2}^1$ shrink. (Choosing $\mu_0 = -1$ yields 1-cycles in M_6 that are homologous to those at $\mu_0 = 1$.) Finally, the space M_6 admits no 3-cohomology class and no 3-cycles.

B.1.2 Inclusion of background gauge fields for isometries

The background gauge fields for the $U(1)_1 \times U(1)_2$ isometry are denoted A^{ϕ_1}, A^{ϕ_2} , with field strengths $F^{\phi_1} = dA^{\phi_1}$, $F^{\phi_2} = dA^{\phi_2}$. After turning on A^{ϕ_1}, A^{ϕ_2} we introduce the 1-forms

$$\tilde{D}\phi_i = D\phi_i - 2A^{\phi_i}, \quad d\tilde{D}\phi_1 = -pV_\Sigma - 2F^{\phi_1}, \quad d\tilde{D}\phi_2 = -qV_\Sigma - 2F^{\phi_2}. \quad (\text{B.18})$$

The periods of the field strengths F^{ϕ_1}, F^{ϕ_2} are quantized in units of 2π and we have¹⁶

$$c_1^{\phi_1} = c_1(U(1)_1) = \frac{F^{\phi_1}}{2\pi}, \quad c_1^{\phi_2} = c_1(U(1)_2) = \frac{F^{\phi_2}}{2\pi}. \quad (\text{B.19})$$

In the case $g = 0$, the Riemann surface is a round 2-sphere and the space M_6 admits an additional $SO(3)_\Sigma \cong SU(2)_\Sigma$ isometry. We find it convenient to describe the 2-sphere as the locus $y^a y_a = 1$ in $\mathbb{R}^3$, where $a = 1, 2, 3$ is a vector index of $SO(3)_\Sigma$. One can verify that the following 1-forms on M_6 are dual to Killing vectors in the metric (B.2),

$$\epsilon_{abc} y^b dy^c + \frac{1}{2} p y_a \mu_1^2 D\phi_1 + \frac{1}{2} q y_a \mu_2^2 D\phi_2. \quad (\text{B.20})$$

In other words, the $SO(3)_\Sigma$ isometry of the Riemann surface extends to an isometry of the total space M_6 for any value of p, q . We couple the $SO(3)_\Sigma$ isometry to a triplet A^a of external gauge fields. Our conventions are

$$Dy^a = dy^a + \frac{1}{2} \epsilon^{abc} A_b y_c, \quad F^a = dA^a + \frac{1}{2} \epsilon^{abc} A_b A_c. \quad (\text{B.21})$$

After turning on A^a , the volume form V_Σ on the Riemann surface in (B.18) must be replaced with the global angular form of $SO(3)_\Sigma$,

$$\frac{V_\Sigma}{2\pi} \rightarrow e_2^\Sigma = \frac{1}{8\pi} [\epsilon_{abc} Dy^a Dy^b y^c - 2F_a y^a]. \quad (\text{B.22})$$

The 2-form e_2^Σ is the closed and $SO(3)_\Sigma$ -invariant completion of $V_\Sigma/(2\pi)$. Integrals of powers of e_2^Σ on S^2 are computed via the Bott-Cattaneo formula [51],

$$\int_{S^2} (e_2^\Sigma)^{2s+1} = 2^{-2s} [p_1(SO(3)_\Sigma)]^s, \quad \int_{S^2} (e_2^\Sigma)^{2s} = 0, \quad s = 0, 1, 2, \dots \quad (\text{B.23})$$

and in particular $\int_{S^2} e_2^\Sigma = 1$ in our normalization. In writing the inflow anomaly polynomial below, we find it convenient to replace $p_1(SO(3)_\Sigma)$ with the second Chern class of $SU(2)_\Sigma$, according to

$$p_1(SO(3)_\Sigma) = -4c_2(SU(2)_\Sigma) \equiv -4c_2^\Sigma. \quad (\text{B.24})$$

After activating the gauge fields for isometries of M_6 we are effectively considering the auxiliary 12 space M_{12} in (B.1). We have to discuss how the closed forms $\omega_4, \omega_2, \lambda_{1u}$ extend to closed forms $(\omega_4)^{\text{eq}}, (\omega_2)^{\text{eq}}, (\lambda_{1u})^{\text{eq}}$ on M_{12} . We start by noting that the 1-forms λ_{1u} are unaffected by the gauging of the isometry $U(1)_1 \times U(1)_2$. Since they are

¹⁶The normalization of A_{ϕ_1}, A_{ϕ_2} in (B.18) can be checked by matching anomaly inflow with the known anomaly polynomial of the 4d SCFT.

only present for $g \neq 0$, the $\text{SO}(3)_\Sigma$ isometry plays no role.¹⁷ We conclude that we do not need to modify λ_{1u} in any way after gauging the isometries of M_6 ,

$$(\lambda_{1u})^{\text{eq}} = \lambda_{1u} . \quad (\text{B.25})$$

Next, let us consider the 2-form $(\omega_2)^{\text{eq}}$. It can be written as

$$(\omega_2)^{\text{eq}} = d \left[H_0^1 \frac{\tilde{D}\phi_1}{2\pi} + H_0^2 \frac{\tilde{D}\phi_2}{2\pi} \right] + 2 (H_0^1)^N \frac{F^{\phi_1}}{2\pi} + 2 (H_0^2)^N \frac{F^{\phi_2}}{2\pi} . \quad (\text{B.26})$$

This is manifestly closed and gauge-invariant. It is also globally defined, and reduces to ω_2 if all background gauge fields for isometries of M_6 are turned off. The 2-form $(\omega_2)^{\text{eq}}$ in M_{12} should have integral periods. In particular, we may consider a 2-cycle in external spacetime, sitting at $\mu_0 = \pm 1$ and a generic point on the Riemann surface in M_6 . We have defined $(\omega_2)^{\text{eq}}$ in such a way that its integral over such cycles is zero. Indeed, the relevant terms are

$$(\omega_2)^{\text{eq}} = 2 \left[(H_0^1)^N - H_0^1 \right] \frac{F^{\phi_1}}{2\pi} + 2 \left[(H_0^2)^N - H_0^2 \right] \frac{F^{\phi_2}}{2\pi} + \dots \quad (\text{B.27})$$

and the relation (B.15) implies that the prefactors of F^{ϕ_i} vanish both at $\mu_0 = 1$ and $\mu_0 = -1$.

Finally, let us turn to $(\omega_4)^{\text{eq}}$. We parametrize it as

$$(\Omega_4)^{\text{eq}} = h_2 \frac{\tilde{D}\phi_1}{2\pi} \frac{\tilde{D}\phi_2}{2\pi} + h_1 \frac{\tilde{D}\phi_1}{2\pi} \frac{d\tilde{D}\phi_2}{2\pi} + h_1^2 \frac{\tilde{D}\phi_2}{2\pi} \frac{d\tilde{D}\phi_1}{2\pi} - h_0 \frac{d\tilde{D}\phi_1}{2\pi} \frac{d\tilde{D}\phi_2}{2\pi} . \quad (\text{B.28})$$

Closure of $(\omega_4)^{\text{eq}}$ follows from the relations (B.6), (B.7). Moreover $(\omega_4)^{\text{eq}}$ is globally defined and has integral periods in M_{12} .

We conclude with two remarks. First, in the case $g = 0$ the background gauge fields for $\text{SO}(3)_\Sigma$ are implicitly included in (B.26) and (B.28) inside the 1-forms $\tilde{D}\phi_i$. Second, the forms $(\omega_2)^{\text{eq}}$, $(\omega_4)^{\text{eq}}$ are not the only possible choices of a closed and gauge-invariant completion of ω_2 , ω_4 . As argued in appendix C, however, any other choice leads to equivalent results for the inflow anomaly polynomial.

B.1.3 Computation of the inflow anomaly polynomial

Our first task is the computation of $-\frac{1}{6} \int_{M_6} E_4^3$, where E_4 is given by

$$E_4 = N (\Omega_4)^{\text{eq}} + \frac{F_2}{2\pi} (\omega_2)^{\text{eq}} + \frac{H_3^u}{2\pi} (\lambda_{1u})^{\text{eq}} + \frac{\gamma_4}{2\pi} . \quad (\text{B.29})$$

To compute the integral $\int_{M_6} E_4^3$, we first collect terms with exactly one $\tilde{D}\phi_1$ and one $\tilde{D}\phi_2$ factor. The integral over the Riemann surface for $g \neq 0$ is straightforward; in the case $g = 0$, we perform it with the help of the Bott-Cattaneo formula (B.23). We are left with an integral over B_2 . It is performed in a similar way as in (B.10). More precisely, the

¹⁷One might wonder if, in the case $g = 1$, the isometries of the T^2 base extend to isometries of M_6 . We have checked that, contrary to the $g = 0$ case, for $p \neq 0$ one cannot find globally-defined Killing vector fields on M_6 that reduce to the Killing vectors on the base T^2 .

integrand 2-form is cast as a total derivative of a 1-form on B_2 . Applying Stokes' theorem, we reduce the problem to an integral over the two arcs $\{\mu_1 = 0\}$, $\{\mu_2 = 0\}$, with a relative minus sign. In the computation, we make use of (B.6), (B.7), (B.9), (B.11), (B.14), (B.15), and (B.17). The result reads

$$-\frac{1}{6} \int_{M_6} E_4^3 = -\frac{2}{3} N^3 \left[p c_1^{\phi_1} (c_1^{\phi_2})^2 + q c_1^{\phi_2} (c_1^{\phi_1})^2 \right] + \frac{1}{6} \left[N^3 q^2 p c_1^{\phi_1} + N^3 p^2 q c_1^{\phi_2} \right] c_2^\Sigma + \frac{1}{(2\pi)^2} \left[-N \gamma_4 F_2 - N \tilde{H}_3^i H_{3i} \right]. \quad (\text{B.30})$$

We have used the relation (B.24) and we have written final expression in the notation introduced in the main text.

To proceed, we need the 8-form class X_8 , defined in (2.4). The computation of X_8 for BBBW setups is reviewed e.g. in [22]. One finds

$$-\int_{M_6} E_4 X_8 = \frac{1}{6} N \left[p c_1^{\phi_1} (c_1^{\phi_2})^2 + q c_1^{\phi_2} (c_1^{\phi_1})^2 \right] - \frac{1}{6} N \left[p (c_1^{\phi_1})^3 + q (c_1^{\phi_2})^3 \right] + \frac{1}{24} N \left[p c_1^{\phi_1} + q c_1^{\phi_2} \right] p_1(T) - \frac{1}{6} N \left[p c_1^{\phi_1} + q c_1^{\phi_2} \right] c_2^\Sigma. \quad (\text{B.31})$$

Combining (B.30) and (B.31) we get the result (5.7) quoted in the main text.

B.2 GMSW solutions

B.2.1 Cohomology classes in M_6

The exact line element on M_6 is recorded in [28]. For the purposes of this appendix, we can use the schematic line element in (2.20) without a detailed knowledge of the f functions.

We can define the following 4-cycles in M_6 ,

$$\begin{aligned} \mathcal{C}_N^4 &: S_\varphi^2 \times \Sigma_g \text{ at } \mu = \mu_N, \\ \mathcal{C}_S^4 &: S_\varphi^2 \times \Sigma_g \text{ at } \mu = \mu_S, \\ \mathcal{C}_C^4 &: S_\varphi^2 \times S_\psi^2 \text{ at a point on } \Sigma_g, \\ \mathcal{C}_\Sigma^4 &: \Sigma_g \times S_\psi^2 \text{ at a point on } S_\varphi^2. \end{aligned} \quad (\text{B.32})$$

We recall that S_ψ^2 is the two-dimensional space spanned by the angle ψ and the μ interval, with topology of a two-sphere and isometry $U(1)_\psi$. The 4-cycles $\mathcal{C}_N^4$, $\mathcal{C}_S^4$, $\mathcal{C}_C^4$, $\mathcal{C}_\Sigma^4$ define elements in the integral 4-homology of M_6 . They are not all independent, however, since the following relation holds in homology,

$$\mathcal{C}_N^4 - \mathcal{C}_S^4 + \chi \mathcal{C}_C^4 + 2 \mathcal{C}_\Sigma^4 = 0. \quad (\text{B.33})$$

The above can be verified by checking that the linear combination of 4-cycles on the l.h.s. yields integral zero when paired with an arbitrary closed 4-form on M_6 . The relation (B.33) implies that $\mathcal{C}_N^4 - \mathcal{C}_S^4$ represents an *even* 4-homology class. (The Euler characteristic χ is always an even integer.) The class $\mathcal{C}_N^4 + \mathcal{C}_S^4 = (\mathcal{C}_N^4 - \mathcal{C}_S^4) + 2 \mathcal{C}_S^4$ is therefore also an even class. This observation allows us to choose the following basis of integral 4-homology,

$$\mathcal{C}_{\alpha=1}^4 = \mathcal{C}_C^4, \quad \mathcal{C}_{\alpha=2}^4 = \frac{1}{2} (\mathcal{C}_N^4 + \mathcal{C}_S^4), \quad \mathcal{C}_{\alpha=3}^4 = \frac{1}{2} (\mathcal{C}_N^4 - \mathcal{C}_S^4). \quad (\text{B.34})$$

We can check that $\mathcal{C}_N^4$, $\mathcal{C}_S^4$, $\mathcal{C}_C^4$, and $\mathcal{C}_\Sigma^4$ can all be written as linear combinations of $\mathcal{C}_\alpha^4$ with *integer* coefficients.

To each 4-homology class $\mathcal{C}_\alpha^4$ there is a corresponding 4-cohomology class, which can be represented by a closed but not exact 4-form Ω_4^α . The 4-forms Ω_4^α are parametrized as

$$\Omega_4^\alpha = \left(dU_\varphi^\alpha \frac{V_\varphi}{2\pi} + dU_\Sigma^\alpha \frac{V_\Sigma}{2\pi} \right) \frac{D\psi}{2\pi} + \left(C^\alpha - 2U_\Sigma^\alpha - \chi U_\varphi^\alpha \right) \frac{V_\varphi}{2\pi} \frac{V_\Sigma}{2\pi}, \quad (\text{B.35})$$

where C^α is constant and U_φ , U_Σ are functions of μ only, satisfying

$$U_\varphi^\alpha(\mu_N) + U_\varphi^\alpha(\mu_S) = 0, \quad U_\Sigma^\alpha(\mu_N) + U_\Sigma^\alpha(\mu_S) = 0. \quad (\text{B.36})$$

We demand the standard pairing $\int_{\mathcal{C}_\alpha^4} \Omega_4^\beta = \delta_\alpha^\beta$ between 4-homology and 4-cohomology classes. This fixes the values of $U_\varphi(\mu_N)$, $U_\Sigma(\mu_N)$, C^α ,

	$U_\varphi^\alpha(\mu_N)$	$U_\Sigma^\alpha(\mu_N)$	C^α
$\alpha = 1$	1/2	$-\chi/4$	0
$\alpha = 2$	0	0	1
$\alpha = 3$	0	$-1/2$	0

(B.37)

To each 4-homology class $\mathcal{C}_\alpha^4$ there is also a Poincaré dual 2-cohomology class, which we represent by a closed but not exact 2-form $\omega_{2\alpha}$. The parametrization of $\omega_{2\alpha}$ reads

$$\omega_{2\alpha} = dH_\alpha \frac{D\psi}{2\pi} + \left(t_\alpha^\varphi - 2H_\alpha \right) \frac{V_\varphi}{2\pi} + \left(t_\alpha^\Sigma - \chi H_\alpha \right) \frac{V_\Sigma}{2\pi}, \quad (\text{B.38})$$

where t_α^φ , t_α^Σ are constants and H_α is a function of μ only, satisfying

$$H_\alpha(\mu_N) + H_\alpha(\mu_S) = 0. \quad (\text{B.39})$$

We impose the relation $\int_{M_6} \Omega_4^\alpha \omega_{2\beta} = \delta_\beta^\alpha$ to fix the values of $H_\alpha(\mu_N)$, t_α^φ , and t_α^Σ ,

	$H_\alpha(\mu_N)$	t_α^φ	t_α^Σ
$\alpha = 1$	0	0	1
$\alpha = 2$	1/2	0	0
$\alpha = 3$	0	-1	$-\chi/2$

(B.40)

We can define the following five 2-cycles inside M_6 ,

$$\begin{aligned} \mathcal{C}_2^{\varphi,N} : & \quad S_\varphi^2 \text{ at } \mu = \mu_N \text{ and at a point on } \Sigma_g, \\ \mathcal{C}_2^{\varphi,S} : & \quad S_\varphi^2 \text{ at } \mu = \mu_S \text{ and at a point on } \Sigma_g, \\ \mathcal{C}_2^{\Sigma,N} : & \quad \Sigma_g \text{ at } \mu = \mu_N \text{ and at a point on } S_\varphi^2, \\ \mathcal{C}_2^{\Sigma,S} : & \quad \Sigma_g \text{ at } \mu = \mu_S \text{ and at a point on } S_\varphi^2, \\ \mathcal{C}_2^{\text{fiber}} : & \quad \text{the } S_\psi^2 \text{ fiber at a point on the base } S_\varphi^2 \times \Sigma_g. \end{aligned} \quad (\text{B.41})$$

They all correspond to elements in the integral 2-homology of M_6 , but they satisfy two linear relations in homology,

$$\begin{aligned}\mathcal{C}_2^{\varphi,N} - \mathcal{C}_2^{\varphi,S} + 2\mathcal{C}_2^{\text{fiber}} &= 0, \\ \mathcal{C}_2^{\Sigma,N} - \mathcal{C}_2^{\Sigma,S} + \chi\mathcal{C}_2^{\text{fiber}} &= 0.\end{aligned}\tag{B.42}$$

The previous relations can be checked by pairing the combinations of 2-cycles with an arbitrary closed 2-form on M_6 . It is convenient to use a basis of integral 2-homology $\mathcal{C}_2^\alpha$ such that $\int_{\mathcal{C}_2^\alpha} \omega_{2\beta} = \delta_\beta^\alpha$. One can check that such a basis is given by

$$\begin{aligned}\mathcal{C}_2^{\alpha=1} &= \mathcal{C}_2^{\Sigma,N} - \frac{\chi}{2}\mathcal{C}_2^{\varphi,N}, \\ \mathcal{C}_2^{\alpha=1} &= \mathcal{C}_2^{\text{fiber}}, \\ \mathcal{C}_2^{\alpha=3} &= -\mathcal{C}_2^{\text{fiber}} - \mathcal{C}_2^{\varphi,N}.\end{aligned}\tag{B.43}$$

Moreover, we verify that all the five 2-cycles defined in (B.41) can be written as *integral* linear combinations of the basis 2-cycles $\mathcal{C}_2^\alpha$.

Let us now discuss 3-cycles in M_6 . We can define several 3-cycles in terms of the 1-cycles on the Riemann surface. Let Ω^{uv} denote the inverse of the intersection pairing Ω_{uv} , with the convention $\Omega_{uv}\Omega^{vw} = -\delta_u^w$. Since Ω_{uv} is integral and unimodular, so is Ω^{uv} . From $\int_{\Sigma_g} \lambda_{1u} \lambda_{1v} = \Omega_{uv} = \mathcal{C}_{1u}^\Sigma \cdot \mathcal{C}_{1v}^\Sigma$, we see that a basis of 1-cycles $\mathcal{C}_1^{\Sigma u}$ with $\int_{\mathcal{C}_1^{\Sigma u}} \lambda_{1v} = \delta_v^u$ is given by $\mathcal{C}_1^{\Sigma u} = \Omega^{uv} \mathcal{C}_{1v}^\Sigma$. We use these 1-cycles on the Riemann surface to construct 3-cycles in M_6 ,

$$\begin{aligned}\mathcal{C}_3^{Nu} : & \quad S_\varphi^2 \times \mathcal{C}_1^{\Sigma u} \text{ at } \mu = \mu_N, \\ \mathcal{C}_3^{Su} : & \quad S_\varphi^2 \times \mathcal{C}_1^{\Sigma u} \text{ at } \mu = \mu_S, \\ \mathcal{C}_3^{\psi u} : & \quad S_\psi^2 \text{ fibered over } \mathcal{C}_1^{\Sigma u} \text{ at a point on } S_\varphi^2.\end{aligned}\tag{B.44}$$

These 3-cycles represent integral 3-homology classes, subject to the relations

$$\mathcal{C}_3^{Nu} - \mathcal{C}_3^{Su} + 2\mathcal{C}_3^{\psi u} = 0.\tag{B.45}$$

In total, we have $4g$ independent 3-homology classes. The above relation implies that $\mathcal{C}_3^{Nu} - \mathcal{C}_3^{Su}$ is an even 3-homology class. The class $\mathcal{C}_3^{Nu} + \mathcal{C}_3^{Su} = (\mathcal{C}_3^{Nu} - \mathcal{C}_3^{Su}) + 2\mathcal{C}_3^{Su}$ is also even. It follows that the following classes are integral,

$$\mathcal{C}_3^{u\pm} = \frac{1}{2}(\mathcal{C}_3^{Nu} \pm \mathcal{C}_3^{Su}).\tag{B.46}$$

We use them as a basis of 3-homology,

$$\mathcal{C}_3^x = (\mathcal{C}_3^{u+}, \mathcal{C}_3^{u-}).\tag{B.47}$$

Notice that $\mathcal{C}_3^{Nu}$, $\mathcal{C}_4^{Su}$, and $\mathcal{C}_3^{\psi u}$ can all be written as linear combinations of $\mathcal{C}_3^x$ with *integer* coefficients.

The 3-cohomology classes dual to the 3-homology classes $\mathcal{C}_3^x$ can be represented by closed but not exact 3-forms Λ_{3x} . We parametrize them as follows,

$$\Lambda_{3x} = (\Lambda_{3u+}, \Lambda_{3u-}), \quad \Lambda_{3u\pm} = \left(d\mathcal{S}_{\pm} \frac{D\psi}{2\pi} - 2\mathcal{S}_{\pm} \frac{V_{\varphi}}{2\pi} \right) \lambda_{1u}, \quad (\text{B.48})$$

where λ_{1u} are the harmonic 1-forms on the Riemann surface, pulled back to M_6 , and $\mathcal{S}_{\pm}$ are functions of μ only. If we demand $\int_{\mathcal{C}_3^x} \Lambda_{3y} = \delta_y^x$ we derive

$$\mathcal{S}_+(\mu_N) = -\frac{1}{2}, \quad \mathcal{S}_+(\mu_S) = -\frac{1}{2}, \quad \mathcal{S}_-(\mu_N) = -\frac{1}{2}, \quad \mathcal{S}_-(\mu_S) = +\frac{1}{2}. \quad (\text{B.49})$$

The pairing $\mathcal{K}_{xy}$ defined in (2.7) takes the form

$$\mathcal{K}_{xy} = \begin{pmatrix} \mathcal{K}_{u+,v+} & \mathcal{K}_{u+,v-} \\ \mathcal{K}_{u-,v+} & \mathcal{K}_{u-,v-} \end{pmatrix} = \begin{pmatrix} 0 & -\Omega_{uv} \\ -\Omega_{uv} & 0 \end{pmatrix}, \quad (\text{B.50})$$

and is antisymmetric and unimodular. Finally, we may compute the intersection numbers $\mathcal{K}_{xv\alpha}$ in (2.7). The only non-zero components of $\mathcal{K}_{xv\alpha}$ are

$$\mathcal{K}_{u+,v,\alpha=2} = \Omega_{uv}, \quad \mathcal{K}_{u-,v,\alpha=3} = \Omega_{uv}. \quad (\text{B.51})$$

B.2.2 Inclusion of background gauge fields for isometries

For $g \geq 2$, the isometry group of M_6 is $U(1)_{\psi} \times SO(3)_{\varphi}$. To describe the isometries of $ds^2(S_{\varphi}^2) = d\theta^2 + \sin^2 \theta d\varphi^2$ it is convenient to introduce three constrained coordinates

$$\tilde{y}^{\tilde{a}} \tilde{y}_{\tilde{a}} = 1, \quad \tilde{y}^{\tilde{a}} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad \tilde{a} = 1, 2, 3. \quad (\text{B.52})$$

In the previous expression, $\tilde{a}$ is a vector index of $SO(3)_{\varphi}$. The gauging of $U(1)_{\psi}$ is performed by introducing a background connection A^{ψ} , while for $SO(3)_{\varphi}$ we introduce a triplet $\tilde{A}^{\tilde{a}}$ of connections. Our conventions for the gauging of the 1-forms $d\tilde{y}^{\tilde{a}}$ on S_{φ}^2 are

$$D\tilde{y}^{\tilde{a}} = d\tilde{y}^{\tilde{a}} + \frac{1}{2} \epsilon^{\tilde{a}\tilde{b}\tilde{c}} \tilde{A}_{\tilde{b}} \tilde{y}_{\tilde{c}}, \quad \tilde{F}^{\tilde{a}} = d\tilde{A}^{\tilde{a}} + \frac{1}{2} \epsilon^{\tilde{a}\tilde{b}\tilde{c}} \tilde{A}_{\tilde{b}} \tilde{A}_{\tilde{c}}. \quad (\text{B.53})$$

It is worth commenting on how the $SO(3)_{\varphi}$ isometry of S_{φ}^2 , considered in isolation, extends to an isometry of the total space M_6 . We have verified that the 1-forms

$$f_{\varphi} \epsilon_{\tilde{a}\tilde{b}\tilde{c}} \tilde{y}^{\tilde{b}} D\tilde{y}^{\tilde{c}} + f_{\psi} \tilde{y}_{\tilde{a}} D\psi \quad (\text{B.54})$$

are globally defined on M_6 and dual to Killing vectors in the line element (2.20).

In the case $g = 0$, the space M_6 admits an additional $SO(3)_{\Sigma}$ isometry, originating from the isometries of the Riemann surface. As in section B.1.2, we describe the Riemann surface in terms of three constrained coordinates y^a , where $a = 1, 2, 3$ is an index of $SO(3)_{\Sigma}$ (not to be confused with the $\tilde{a}$ indices of $SO(3)_{\varphi}$). The gauge fields of $SO(3)_{\Sigma}$ are denoted A^a . The gauging of dy^a is performed as in (B.21). The $SO(3)_{\Sigma}$ isometry of the Riemann surface extends to an isometry of M_6 because the following 1-forms are dual to Killing vectors in the metric (2.20),

$$f_{\Sigma} \epsilon_{abc} y^b Dy^c + f_{\psi} y_a D\psi. \quad (\text{B.55})$$

After turning on A^ψ , $\tilde{A}^{\tilde{a}}$, and (for $g = 0$) A^a , the 1-form $D\psi$ is replaced by its gauged version $\tilde{D}\psi$, which satisfies (cfr. with the ungauged version (2.21))

$$\frac{d\tilde{D}\psi}{2\pi} = -2e_2^\varphi - \chi e_2^\Sigma + 2\frac{F^\psi}{2\pi}. \quad (\text{B.56})$$

In the previous expression $F^\psi = dA^\psi$. The 2-form e_2^φ is the global angular form of $\text{SO}(3)_\varphi$,

$$e_2^\varphi = \frac{1}{8\pi} \left[\epsilon_{\tilde{a}\tilde{b}\tilde{c}} D\tilde{y}^{\tilde{a}} D\tilde{y}^{\tilde{b}} \tilde{y}^{\tilde{c}} - 2\tilde{F}_{\tilde{a}} \tilde{y}^{\tilde{a}} \right]. \quad (\text{B.57})$$

It is the closed and gauge-invariant completion of $V_\varphi/(2\pi)$ and satisfies Bott-Cattaneo identities analogous to (B.23). The 2-form e_2^Σ is understood in different way depending on $g \geq 2$ or $g = 0$: for $g \geq 2$ it is simply proportional to the volume form V_Σ , while for $g = 0$ it is the global angular form of $\text{SO}(3)_\Sigma$,

$$\text{for } g \geq 2: \quad e_2^\Sigma = \frac{V_\Sigma}{2\pi}, \quad \text{for } g = 0: \quad e_2^\Sigma = \frac{1}{8\pi} \left[\epsilon_{abc} Dy^a Dy^b y^c - 2F_a y^a \right]. \quad (\text{B.58})$$

The anomaly polynomial will be written in terms of the second Chern classes of $\text{SU}(2)_\Sigma$ and $\text{SU}(2)_\varphi$. They are related to the first Pontryagin classes of $\text{SO}(3)_\Sigma$ and $\text{SO}(3)_\varphi$ by (B.24) and the analogous relation for $p_1(\text{SO}(3)_\varphi)$.

Let us now turn to a discussion of the extension of the closed 4-forms Ω_4^α on M_6 to closed 4-forms $(\Omega_4^\alpha)^{\text{eq}}$ on M_{12} . We define

$$(\Omega_4^\alpha)^{\text{eq}} = d \left[\left(U_\varphi^\alpha e_2^\varphi + U_\Sigma^\alpha e_2^\Sigma \right) \frac{\tilde{D}\psi}{2\pi} \right] + C^\alpha e_2^\varphi e_2^\Sigma, \quad (\text{B.59})$$

and we verify that $(\Omega_4^\alpha)^{\text{eq}}$ are globally defined, closed 4-forms on M_{12} with integral periods. By a similar token, the 2-forms $\omega_{2\alpha}$ on M_6 extend to the 2-forms $(\omega_{2\alpha})^{\text{eq}}$, defined as

$$(\omega_{2\alpha})^{\text{eq}} = d \left[H_\alpha \frac{\tilde{D}\psi}{2\pi} \right] + t_\alpha^\varphi e_2^\varphi + t_\alpha^\Sigma e_2^\Sigma. \quad (\text{B.60})$$

The 1-forms λ_{1u} are unaffected by the gauging of the $\text{U}(1)_\psi \times \text{SU}(2)_\varphi$ isometry,

$$(\lambda_{1u})^{\text{eq}} = \lambda_{1u}. \quad (\text{B.61})$$

Finally, the 3-forms $\Lambda_{3u\pm}$ on M_6 extend to the following 3-forms on M_{12} ,

$$(\Lambda_{3u\pm})^{\text{eq}} = d \left[\mathcal{S}_\pm \frac{\tilde{D}\psi}{2\pi} \right] \lambda_{1u}. \quad (\text{B.62})$$

B.2.3 Computation of the inflow anomaly polynomial

Having defined the closed forms $(\Omega_4^\alpha)^{\text{eq}}$, $(\omega_{2\alpha})^{\text{eq}}$, $(\Lambda_{3u\pm})^{\text{eq}}$, and $(\lambda_{1u})^{\text{eq}}$, we have a fully explicit expression for the quantity E_4 . The other ingredient for the computation of the inflow anomaly polynomial is the 8-form X_8 . Following [27], we can compute it using the following relations among Pontryagin classes,

$$\begin{aligned} p_1(TM_{12}) &= p_1(T) + p_1(\text{SO}(3)_\varphi) + p_1(\text{SO}(3)_\Sigma) + \left[\frac{d\tilde{D}\psi}{2\pi} \right]^2, \\ p_2(TM_{12}) &= \left[p_1(T) + p_1(\text{SO}(3)_\varphi) + p_1(\text{SO}(3)_\Sigma) \right] \left[\frac{d\tilde{D}\psi}{2\pi} \right]^2. \end{aligned} \quad (\text{B.63})$$

In the previous expressions, $p_1(T)$ is the first Pontryagin class of external spacetime. The terms with $p_1(\text{SO}(3)_\Sigma)$ are understood to be present only in the case $g = 0$. By combining (B.63) and (B.56) we obtain an explicit expression for the 8-form X_8 .

The computation of the integrals $\int_{M_6} E_4^3$ and $\int_{M_6} E_4 X_8$ is straightforward. After collecting the terms with exactly one $\tilde{D}\psi$ factor, we can integrate over S_φ^2 with the help of the Bott-Cattaneo formula. The integral over Σ_g receives two contributions: terms with an odd power of e_2^Σ (treated with the Bott-Cattaneo formula), and terms with exactly two λ_{1u} factors (treated using $\int_{\Sigma_g} \lambda_{1u} \lambda_{1v} = \Omega_{uv}$). We are left with a one-dimensional integral over the μ interval, which is evaluated making use of (B.37), (B.40), and (B.49). The final result is recorded in the main text.

Let us conclude with a comment on the large N limit. We define this limit by letting the three flux quanta N_α scale in the same way, $N_\alpha \sim \mathcal{O}(N)$, with fixed ratios N_α/N_β for $\alpha \neq \beta$. We assign scaling N^0 to $p_1(T)$ and to the background fields associated to isometries of M_6 , while we assign scaling N^1 to all external gauge fields originating from expansion of C_3 onto cohomology classes of M_6 . In this way, the $\mathcal{O}(N^3)$ terms in I_6^{inflow} all originate from the E_4^3 term in $\mathcal{I}_{12}$, while the $\mathcal{O}(N)$ terms originate from $E_4 X_8$.

C Construction of E_4

In this appendix we discuss the construction and properties of the forms $(\Omega_4^\alpha)^{\text{eq}}$, $(\omega_{2\alpha})^{\text{eq}}$, $(\Lambda_{3x})^{\text{eq}}$, $(\lambda_{1u})^{\text{eq}}$ that enter the parametrization (5.5) of E_4 .

Suppose that M_6 admits a collection of Killing vectors k_I^m , with $m = 1, \dots, 6$ a curved tangent index on M_6 , and I labeling a basis of Killing vectors. The latter obey the Lie algebra

$$\mathcal{L}_I k_J \equiv \mathcal{L}_{k_I} k_J = [k_I, k_J] = f_{IJ}^K k_K, \quad (\text{C.1})$$

where $\mathcal{L}$ denotes Lie derivative. The fibration in (5.1) includes arbitrary background gauge fields associated to the isometries of the M_6 fiber. These gauge fields are 1-form gauge fields on the base $\mathcal{M}_6$. We refer to the operation of turning them on as gauging. In terms of local coordinates ξ^m on the M_6 fiber, the gauging is conveniently described by the replacement

$$d\xi^m \rightarrow D\xi^m = d\xi^m + k_I^m A^I, \quad (\text{C.2})$$

where A^I is the external gauge field associated to the Killing vector k_I^m . In our conventions, the field strength F^I of A^I reads (we suppress wedge products throughout this appendix)

$$F^I = dA^I - \frac{1}{2} f_{JK}^I A^J A^K. \quad (\text{C.3})$$

Let ω_q be a q -form on M_6 ,

$$\omega_q = \frac{1}{q!} \omega_{m_1 \dots m_q} d\xi^{m_1} \dots d\xi^{m_q}, \quad (\text{C.4})$$

where the components $\omega_{m_1 \dots m_q}$ depend only on the coordinates ξ^m on M_6 . We use the symbol $(\omega_q)^g$ for the gauged version of ω , obtained by means of the replacement (C.2),

$$(\omega_q)^g = \frac{1}{q!} \omega_{m_1 \dots m_q} D\xi^{m_1} \dots D\xi^{m_q}. \quad (\text{C.5})$$

A useful identity for $(\omega_q)^g$ is

$$d(\omega_q)^g + A^I (\mathcal{L}_I \omega_q)^g = (d\omega_q)^g + F^I (\iota_I \omega_q)^g, \quad (\text{C.6})$$

where ι_I denotes the interior product of the vector k_I^m with a differential form.

If we choose a metric on M_6 , we can select the harmonic representatives for the de Rham classes defined by $\Omega_4^\alpha, \omega_{2\alpha}, \Lambda_{3x}, \lambda_{1u}$. A harmonic form is automatically invariant under all isometries of M_6 .¹⁸ This means that (after adding suitable exact forms, if necessary) we can take the closed forms $\Omega_4^\alpha, \omega_{2\alpha}, \Lambda_{3x}, \lambda_{1u}$ to be invariant under all isometries of M_6 ,

$$\mathcal{L}_I \lambda_{1u} = 0, \quad \mathcal{L}_I \omega_{2\alpha} = 0, \quad \mathcal{L}_I \Lambda_{3x} = 0, \quad \mathcal{L}_I \Omega_4^\alpha = 0. \quad (\text{C.7})$$

It follows that the forms $\iota_I \lambda_{1u}, \iota_I \omega_{2\alpha}, \iota_I \Lambda_{3x}, \iota_I \Omega_4^\alpha$ are closed. We may then write

$$\begin{aligned} 2\pi \iota_I \lambda_{1u} &= c_{uI}, & 2\pi \iota_I \omega_{2\alpha} + d\omega_{0\alpha I} &= c_\alpha^u{}_I \lambda_{1u}, \\ 2\pi \iota_I \Lambda_{3x} + d\Lambda_{1xI} &= c_x^\alpha{}_I \omega_{2\alpha}, & 2\pi \iota_I \Omega_4^\alpha + d\Omega_{2I}^\alpha &= c^{\alpha x}{}_I \Lambda_{3x}, \end{aligned} \quad (\text{C.8})$$

where $c_{uI}, c_\alpha^u{}_I, c_x^\alpha{}_I, c^{\alpha x}{}_I$ are constants. For example, we have observed that $2\pi \iota_I \omega_{2\alpha}$ is a closed 1-form, and that the de Rham classes of λ_{1u} furnish a basis of $H^1(M_6, \mathbb{R})$. It follows there are suitable constants $c_\alpha^u{}_I$ such that the difference $2\pi \iota_I \omega_{2\alpha} - c_\alpha^u{}_I \lambda_{1u}$ is exact. Similar remarks apply to the other expressions in (C.8). The forms $\omega_{0\alpha I}, \Lambda_{1xI}, \Omega_{2I}^\alpha$ are only defined modulo addition of a closed form. Without loss of generality, they can be taken to satisfy

$$\mathcal{L}_I \omega_{0\alpha J} = f_{IJ}^K \omega_{0\alpha K}, \quad \mathcal{L}_I \Lambda_{1xJ} = f_{IJ}^K \Lambda_{1xK}, \quad \mathcal{L}_I \Omega_{2J}^\alpha = f_{IJ}^K \Omega_{2K}^\alpha. \quad (\text{C.9})$$

Symmetrizing in IJ and using (C.8) we derive

$$\begin{aligned} 0 &= 2\pi \mathcal{L}_{(I} \omega_{0\alpha|J)} = c_\alpha^u{}_{(I} c_{u|J)}, \\ 0 &= 2\pi \mathcal{L}_{(I} \Lambda_{1x|J)} = c_x^\alpha{}_{(I} \left[c_\alpha^u{}_J \lambda_{1u} - d\omega_{0\alpha|J)} \right] + 2\pi d\iota_{(I} \Lambda_{1x|J)}, \\ 0 &= 2\pi \mathcal{L}_{(I} \Omega_{2|J)}^\alpha = c^{\alpha x}{}_{(I} \left[c_x^\beta{}_J \omega_{2\beta} - d\Lambda_{1x|J)} \right] + 2\pi d\iota_{(I} \Omega_{2|J)}^\alpha. \end{aligned} \quad (\text{C.10})$$

If we integrate the second relation on a non-trivial 1-cycle in M_6 , only the term with λ_{1u} contributes. It follows that its coefficient must be zero. Similar remarks apply to the third line. We conclude that the constants c satisfy

$$c_\alpha^u{}_{(I} c_{u|J)} = 0, \quad c_x^\alpha{}_{(I} c_\alpha^u{}_J) = 0, \quad c^{\alpha x}{}_{(I} c_x^\beta{}_J) = 0. \quad (\text{C.11})$$

It follows from (C.10) that the forms $2\pi \iota_{(I} \Lambda_{1x|J)} - c_x^\alpha{}_{(I} \omega_{0\alpha|J)}$ and $2\pi \iota_{(I} \Omega_{2|J)}^\alpha - c^{\alpha x}{}_{(I} \Lambda_{1x|J)}$ are closed. We can therefore write

$$2\pi \iota_{(I} \Lambda_{1x|J)} = c_x^\alpha{}_{(I} \omega_{0\alpha|J)} + b_{xIJ}, \quad 2\pi \iota_{(I} \Omega_{2|J)}^\alpha + d\Omega_{0IJ}^\alpha = c^{\alpha x}{}_{(I} \Lambda_{1x|J)} + b^{\alpha u}{}_{IJ} \lambda_{1u}, \quad (\text{C.12})$$

where b_{xIJ} and $b^{\alpha u}{}_{IJ}$ are constants and Ω_{0IJ}^α are 0-forms, defined up to a constant.

¹⁸For example, if ω_2 is a harmonic 2-form, the fact that $\mathcal{L}_I \omega_2 = 0$ can be seen as follows. From $d\omega_2 = 0$ we derive $\mathcal{L}_I \omega_2 = d(\iota_I \omega_2)$. Making use of $\nabla_{(m} k_{I|n)} = 0$ and $\nabla^m \omega_{mn} = 0$, we verify $(\mathcal{L}_I \omega_2)_{mn} = \nabla^p (k_I \wedge \omega_2)_{pmn}$. We have thus established that the 2-form $\mathcal{L}_I \omega_2$ is both exact and co-exact. It follows that $\int_{M_{10-d}} (\mathcal{L}_I \omega_2) * (\mathcal{L}_I \omega_2) = 0$ (no sum over I), which in turn guarantees $\mathcal{L}_I \omega_2 = 0$.

We can now write the forms $(\Omega_4^\alpha)^{\text{eq}}$, $(\omega_{2\alpha})^{\text{eq}}$, $(\Lambda_{3x})^{\text{eq}}$, $(\lambda_{1u})^{\text{eq}}$. They are given by

$$\begin{aligned}(\Omega_4^\alpha)^{\text{eq}} &= (\Omega_4^\alpha)^{\text{g}} + \frac{F^I}{2\pi} (\Omega_{2I}^\alpha)^{\text{g}} + \frac{F^I}{2\pi} \frac{F^J}{2\pi} \Omega_{0IJ}^\alpha, \\(\Lambda_{3x})^{\text{eq}} &= (\Lambda_{3x})^{\text{g}} + \frac{F^I}{2\pi} (\Lambda_{1xI})^{\text{g}}, \\(\omega_{2\alpha})^{\text{eq}} &= (\omega_{2\alpha})^{\text{g}} + \frac{F^I}{2\pi} \omega_{0\alpha I}, \\(\lambda_{1u})^{\text{eq}} &= (\lambda_{1u})^{\text{g}}.\end{aligned}\tag{C.13}$$

Making use of the identity (C.6), the Bianchi identity for F^I , and the relations (C.8), (C.12) we compute

$$\begin{aligned}d(\lambda_{1u})^{\text{eq}} &= \frac{F^I}{2\pi} c_{uI}, \\d(\omega_{2\alpha})^{\text{eq}} &= \frac{F^I}{2\pi} c_\alpha{}^u{}_I (\lambda_{1u})^{\text{eq}}, \\d(\Lambda_{3x})^{\text{eq}} &= \frac{F^I}{2\pi} c_x{}^\alpha{}_I (\omega_{2\alpha})^{\text{eq}} + \frac{F^I}{2\pi} \frac{F^J}{2\pi} b_{xIJ}, \\d(\Omega_4^\alpha)^{\text{eq}} &= \frac{F^I}{2\pi} c^{\alpha x}{}_I (\Lambda_{3x})^{\text{eq}} + \frac{F^I}{2\pi} \frac{F^J}{2\pi} b^{\alpha u}{}_{IJ} (\lambda_{1u})^{\text{eq}}.\end{aligned}\tag{C.14}$$

For the spaces M_6 of interest in this work, all c and b constant vanish, and we verify closure of $(\Omega_4^\alpha)^{\text{eq}}$, $(\omega_{2\alpha})^{\text{eq}}$, $(\Lambda_{3x})^{\text{eq}}$, $(\lambda_{1u})^{\text{eq}}$, as anticipated in the main text. If we were to study a setup with non-zero c or b constants, we could still make use of (5.5), but we would have to modify the Bianchi identities for the external field strengths,

$$\begin{aligned}df_1^x &= -N_\alpha c^{\alpha x}{}_I F^I, \\dF_2^\alpha &= c_x{}^\alpha{}_I f_1^x F^I, \\dH_3^u &= -c_\alpha{}^u{}_I F_2^\alpha F^I - N_\alpha b^{\alpha u}{}_{IJ} F^I F^J, \\d\gamma_4 &= c_{uI} H_3^u F^I + b_{xIJ} f_1^x F^I F^J.\end{aligned}\tag{C.15}$$

We leave further investigation of this case to future work.

We noticed above that the forms Ω_{2I}^α , Λ_{1xI} , $\omega_{0\alpha I}$ are only defined up to addition of a closed form. We can parametrize this ambiguity by writing

$$\begin{aligned}\Omega_{2I}^{\alpha'} &= \Omega_{2I}^\alpha + d\mathcal{Y}_{1I}^\alpha + \nu^{\alpha\beta}{}_I \omega_{2\beta}, \\\Lambda_{1xI}' &= \Lambda_{1xI} + d\mathcal{Y}_{0xI} + \nu_x{}^u{}_I \lambda_{1u}, \\\omega_{0\alpha I}' &= \omega_{0\alpha I} + \nu_{\alpha I}\end{aligned}\tag{C.16}$$

where the ν parameters are constant, and the $\mathcal{Y}$ forms can be taken to satisfy relations analogous to (C.9). Since we have a new Ω_{2I}^α , we have to determine a new Ω_{0IJ}^α , by solving the second relation in (C.12). For simplicity, we only consider the situation in which the c and b constants are zero. We can then write

$$\Omega_{0IJ}^{\alpha'} = \Omega_{0IJ}^\alpha + 2\pi \iota_{(I} \mathcal{Y}_{1|J)}^\alpha + \nu^{\alpha\beta}{}_{(I} \omega_{0\alpha|J)} + \tau_{IJ}^\alpha,\tag{C.17}$$

where τ_{IJ}^α are arbitrary constants. If we insert the primed objects into the expression of E_4 , we obtain a new realization of E_4 , denoted E'_4 ,

$$\begin{aligned} E'_4 = & N_\alpha (\Omega_4^\alpha)^{\text{eq}} + \left[\frac{F_2^\alpha}{2\pi} + N_\beta \nu^{\beta\alpha}{}_I \frac{F^I}{2\pi} \right] (\omega_{2\alpha})^{\text{eq}} + \frac{f_1^x}{2\pi} (\Lambda_{3x})^{\text{eq}} \\ & + \left[\frac{H_3^u}{2\pi} + \nu_x{}^u{}_I \frac{f_1^x}{2\pi} \frac{F^I}{2\pi} \right] (\lambda_{1u})^{\text{eq}} + \left[\frac{\gamma_4}{2\pi} + N_\alpha \tau_{IJ}^\alpha \frac{F^I}{2\pi} \frac{F^J}{2\pi} + \nu_{\alpha I} \frac{F_2^\alpha}{2\pi} \frac{F^I}{2\pi} \right] \\ & + d \left[\frac{F^I}{2\pi} N_\alpha (\mathcal{Y}_{1I}^\alpha)^{\text{g}} - \frac{F^I}{2\pi} \frac{f_1^x}{2\pi} \mathcal{Y}_{0xI} \right]. \end{aligned} \quad (\text{C.18})$$

The last line collects the total derivative of a globally defined 3-form on M_{12} . Adding an exact piece to E_4 has no effect on the computation of I_6^{inflow} . We see that, up to this immaterial total derivative, E'_4 has the same form as E_4 , if we perform a redefinition of the external gauge fields,

$$\begin{aligned} \frac{F_2^{\alpha'}}{2\pi} &= \frac{F_2^\alpha}{2\pi} + N_\beta \nu^{\beta\alpha}{}_I \frac{F^I}{2\pi}, & \frac{H_3^{u'}}{2\pi} &= \frac{H_3^u}{2\pi} + \nu_x{}^u{}_I \frac{f_1^x}{2\pi} \frac{F^I}{2\pi}, \\ \frac{\gamma_4'}{2\pi} &= \frac{\gamma_4}{2\pi} + N_\alpha \tau_{IJ}^\alpha \frac{F^I}{2\pi} \frac{F^J}{2\pi} + \nu_{\alpha I} \frac{F_2^\alpha}{2\pi} \frac{F^I}{2\pi}. \end{aligned} \quad (\text{C.19})$$

Let us stress that the constants $\nu^{\beta\alpha}{}_I$, $\nu_x{}^u{}_I$, $\nu_{\alpha I}$, τ_{IJ}^α are not completely arbitrary: they must be chosen in such a way that E'_4 has integral periods. Let us assume that the normalization of the Killing vectors in (C.2) has been chosen in such a way that F^I has periods that are quantized in units of 2π (here we are assuming an Abelian isometry group for simplicity). The ν and τ constants have to be chosen in such a way that

$$N_\beta \nu^{\beta\alpha}{}_I \in \mathbb{Z}, \quad \nu_x{}^u{}_I \in \mathbb{Z}, \quad N_\alpha \tau_{IJ}^\alpha \in \mathbb{Z}, \quad \nu_{\alpha I} \in \mathbb{Z}. \quad (\text{C.20})$$

It then follows that the redefinition (C.19) preserves the lattice of periods of the external gauge fields.¹⁹

D Aspects of differential cohomology

In this appendix we give a brief review of some basic aspects of differential cohomology. We follow a presentation based on Cheeger-Simons differential characters [52]. Introductions aimed at physicists can be found e.g. in [2, 30].

Cheeger-Simons differential characters. A degree- ℓ Cheeger-Simons differential character χ on a manifold $\mathcal{M}$ is a group homomorphism $\chi \in \text{Hom}(Z_{\ell-1}(\mathcal{M}), \text{U}(1))$ with the following property: there exists a globally defined ℓ -form F_χ such that

$$\chi(\partial B_\ell) = \exp \left[2\pi i \int_{B_\ell} F_\chi \right], \quad \text{for all } B_\ell \in C_\ell(\mathcal{M}). \quad (\text{D.1})$$

¹⁹We notice that the field redefinitions for H_3^u and γ_4 are non-linear. For example, the quantity $\frac{f_1^x}{2\pi} \frac{F^I}{2\pi}$ can be regarded as the 3-form field strength of a “composite” 2-forms gauge field constructed from a_0^x and A^I . This notion of product of a p -form gauge field and a q -form gauge field to yield a $(p+q+1)$ -gauge field can be made mathematically precise in the framework of differential cohomology, see appendix D.

The notation $C_\ell(\mathcal{M})$ stands for the group of ℓ -chains in $\mathcal{M}$, while $Z_{\ell-1}(\mathcal{M})$ denotes the group of $(\ell-1)$ -cycles. (Chains and cycles are understood in the context of smooth singular homology.) One can verify from the definition of χ that the ℓ -form F_χ is uniquely determined, is closed, and has integral periods. The set of degree- ℓ Cheeger-Simons differential characters has a natural Abelian group structure. We find it convenient to adopt an additive notation, and write

$$(\chi_1 + \chi_2)(\Sigma_{\ell-1}) := \chi_1(\Sigma_{\ell-1}) \chi_2(\Sigma_{\ell-1}), \quad \Sigma_{\ell-1} \in Z_{\ell-1}(\mathcal{M}). \quad (\text{D.2})$$

In this notation, $\chi = 0$ means that χ associates $1 \in \text{U}(1)$ to every $\Sigma_{\ell-1} \in Z_{\ell-1}(\mathcal{M})$. The group of degree- ℓ Cheeger-Simons differential characters is denoted $\check{H}^\ell(\mathcal{M})$. (Contrary to ordinary cohomology groups, $\check{H}^\ell(\mathcal{M})$ is usually infinite-dimensional.)

The mathematical object χ models an $(\ell-1)$ -form $\text{U}(1)$ gauge field, or more precisely, the equivalence class of an $(\ell-1)$ -form $\text{U}(1)$ gauge field up to gauge transformations. To see this, we interpret the map $\chi : Z_{\ell-1}(\mathcal{M}) \rightarrow \text{U}(1)$ as the map that to each $(\ell-1)$ -cycle in spacetime $\mathcal{M}$ assigns the holonomy of the gauge field on that cycle. The globally-defined, closed ℓ -form F_χ with integral periods is identified with the field strength of the $(\ell-1)$ -form gauge field. (Notice that, in the main text, field strengths are normalized to have periods that are quantized in units of 2π .) The equation (D.1) encodes the expected physical relation between the holonomy of a gauge field along a boundary of a chain, and the flux of its field strength through that chain.

A differential character $\chi \in \check{H}^\ell(\mathcal{M})$ determines uniquely an element $a_\chi \in H^\ell(\mathcal{M}, \mathbb{Z})$, called the characteristic class of χ .²⁰ The characteristic class a_χ and the field strength F_χ satisfy the following compatibility condition,

$$[F_\chi]_{\text{dR}} = \varrho(a_\chi). \quad (\text{D.3})$$

The notation $[F_\chi]_{\text{dR}} \in H^\ell(\mathcal{M}, \mathbb{R})$ stands for the de Rham class of the closed form F_χ , while ϱ is the natural map

$$\varrho : H^\ell(\mathcal{M}, \mathbb{Z}) \rightarrow H^\ell(\mathcal{M}, \mathbb{R}). \quad (\text{D.4})$$

The relation (D.3) might erroneously suggest that all interesting information about a_χ is already contained in the field strength F_χ . Crucially, however, the map ϱ forgets torsion: a_χ is determined by F_χ only up to torsion elements in integer cohomology, i.e. up to an element of $\text{Tor } H^\ell(\mathcal{M}, \mathbb{Z})$. This additional data encoded in a_χ (and missed by F_χ) is particularly important if the spacetime manifold $\mathcal{M}$ has torsion in homology, as already emphasized for instance in [31].

A differential character χ is called topologically trivial if $a_\chi = 0$. It can be proven that $a_\chi = 0$ if and only χ can be written in terms of a globally defined $(\ell-1)$ -form A as

$$\chi(\Sigma_{\ell-1}) = \exp \left[2\pi i \int_{\Sigma_{\ell-1}} A \right], \quad \Sigma_{\ell-1} \in Z_{\ell-1}(\mathcal{M}). \quad (\text{D.5})$$

²⁰This can be seen as follows. Every group homomorphism $\chi : Z_{\ell-1}(\mathcal{M}) \rightarrow \text{U}(1)$ admits a (non-unique) lift, i.e. a group homomorphism $T : C_{\ell-1}(\mathcal{M}) \rightarrow \mathbb{R}$, such that $\chi = \exp(2\pi i T)$. From (D.1) one shows that $\delta T = F_\chi - c$ for some $c \in Z^\ell(\mathcal{M}, \mathbb{Z})$ (the group of integer cocycles on $\mathcal{M}$). While T and c are not uniquely determined, the cohomology class $a_\chi := [c] \in H^\ell(\mathcal{M}, \mathbb{Z})$ is uniquely fixed by χ .

In this case, $F_\chi = dA$ and (D.1) follows from Stokes' theorem. Moreover, (D.3) is satisfied because $[F_\chi]_{\text{dR}} = 0$ (since $F_\chi = dA$ and A is globally defined).

A differential character χ is called flat if $F_\chi = 0$. It can be proven that flat characters are identified with elements of the (ordinary) cohomology group $H^{\ell-1}(\mathcal{M}, \text{U}(1))$. Interestingly, there exist flat but topologically non-trivial characters. Indeed, $H^{\ell-1}(\mathcal{M}, \text{U}(1))$ is a compact Abelian group that generically has more than one connected component. The connected component of the identity consists of characters that are both flat and topologically trivial (we may refer to them as Wilson lines). The connected components of $H^{\ell-1}(\mathcal{M}, \text{U}(1))$ are labeled by $\text{Tor } H^\ell(\mathcal{M}, \mathbb{Z})$. This fits with the fact that a_χ for a flat character χ is an element of $\text{Tor } H^\ell(\mathcal{M}, \mathbb{Z})$ (this follows from (D.3) and $F_\chi = 0$).

Let us emphasize that $\chi \in \check{H}^\ell(\mathcal{M})$ contains more information than its field strength F_χ and its characteristic class a_χ . In fact, F_χ and a_χ are unaffected if we shift χ by a Wilson line.

The language of differential characters offers a uniform way to describe $\text{U}(1)$ p -form gauge fields, including 0-form fields. In fact, one can prove that $\check{H}^1(\mathcal{M})$ is the same as the group of smooth maps from $\mathcal{M}$ to S^1 . This mathematical fact fits with the physics picture of a 0-form gauge field as a circle-valued scalar field.

Product in differential cohomology. There is a notion of product in differential cohomology compatible with the grading by the degree ℓ ,

$$\star : \check{H}^{\ell_1}(\mathcal{M}) \times \check{H}^{\ell_2}(\mathcal{M}) \rightarrow H^{\ell_1+\ell_2}(\mathcal{M}) . \quad (\text{D.6})$$

With reference to the additive notation of (D.2), the product $\star$ is distributive,

$$(\chi_1 + \chi_2) \star \chi_3 = \chi_1 \star \chi_3 + \chi_2 \star \chi_3 , \quad \chi_1, \chi_2 \in \check{H}^{\ell_1}(\mathcal{M}) , \quad \chi_3 \in \check{H}^{\ell_2}(\mathcal{M}) . \quad (\text{D.7})$$

The product is graded commutative, like the wedge product of differential forms,

$$\chi_1 \star \chi_2 = (-)^{\ell_1 \ell_2} \chi_2 \star \chi_1 , \quad \chi_1 \in \check{H}^{\ell_1}(\mathcal{M}) , \quad \chi_2 \in \check{H}^{\ell_2}(\mathcal{M}) . \quad (\text{D.8})$$

The field strength and characteristic class of the character $\chi_1 \star \chi_2$ are determined by those of χ_1, χ_2 via

$$\chi_3 := \chi_1 \star \chi_2 , \quad F_{\chi_3} = F_{\chi_1} \wedge F_{\chi_2} , \quad a_{\chi_3} = a_{\chi_1} \smile a_{\chi_2} , \quad (\text{D.9})$$

where in the last relation $\smile$ denotes the cup product in integer cohomology. If χ_1 is topologically trivial, then χ_3 is topologically trivial, for any χ_2 . Indeed, if χ_1 is determined by the (ℓ_1-1) -form A_1 , then χ_3 is determined by the $(\ell_1+\ell_2-1)$ -form $A_1 \wedge F_{\chi_2}$. By a similar token, if χ_1 is flat, so is χ_3 , for any χ_2 . If we regard the flat character χ_1 as an element of $H^{\ell_1-1}(\mathcal{M}, \text{U}(1))$, and the flat character χ_3 as an element of $H^{\ell_1+\ell_2-1}(\mathcal{M}, \text{U}(1))$, then we can write $\chi_3 = \chi_1 \smile a_{\chi_2}$, where $\smile : H^{\ell_1-1}(\mathcal{M}, \text{U}(1)) \times H^{\ell_2}(\mathcal{M}, \mathbb{Z}) \rightarrow H^{\ell_1+\ell_2-1}(\mathcal{M}, \text{U}(1))$ is a well-defined cup product in cohomology.

Cheeger-Simons characters and $\mathbb{Z}_k$ gauge fields. In section 5.1.3 we have encountered a constrained 1-form gauge field $\mathcal{A}_1$, subject to (5.21). If we describe the (gauge-equivalence class of the) 1-form gauge field $\mathcal{A}_1$ with a differential character $\chi \in \check{H}^2(\mathcal{M})$, we have the correspondence

$$k \mathcal{A}_1 = d\phi_0 \quad \leftrightarrow \quad k \chi := \underbrace{\chi + \cdots + \chi}_{k \text{ times}} = 0 \in \check{H}^2(\mathcal{M}) . \quad (\text{D.10})$$

Indeed, we have argued that $k \mathcal{A}_1 = d\phi_0$ means that $k \mathcal{A}_1$ is pure gauge. Since differential characters are gauge-equivalence classes of gauge fields, $k \mathcal{A}_1$ is described by the zero character $0 \in \check{H}^\ell(\mathcal{M})$. The equation $k \chi = 0$ implies

$$F_\chi = 0, \quad k a_\chi = 0, \quad \chi(\Sigma_1) \in \mathbb{Z}_k \subset \text{U}(1) \quad \text{for all } \Sigma_1 \in Z_1(\mathcal{M}) . \quad (\text{D.11})$$

Crucially, $k a_\chi = 0$ does not imply $a_\chi = 0$, but merely that a_χ is a k -torsion element in integer cohomology. Even if $a_\chi = 0$ (which is the case if $\mathcal{M}$ has no torsion in homology), the character χ can be non-zero: it is a Wilson line with holonomies in $\mathbb{Z}_k \subset \text{U}(1)$ determined by a globally-defined closed 1-form. In physics terms, we may simply write $\mathcal{A}_1 = \frac{1}{k} d\phi_0$ [25].

There exist other realizations of the differential cohomology groups $\check{H}^\ell(\mathcal{M})$, for instance in terms of Hopkins-Singer cocycles [53] or Deligne-Beilinson cocycles, see e.g. [32, 54] for a review. Loosely speaking, in these formalisms one can model not only the gauge-equivalence class of a gauge field, but the gauge field itself. In these mathematical frameworks we can give a precise definition to $\mathcal{A}_1$ and $d\phi_0$ separately, and impose the relation $k \mathcal{A}_1 = d\phi_0$. This approach is taken in [32] using Deligne-Beilinson cocycles.

Cheeger-Simons characters and characteristic classes. The notions of Chern classes, Pontryagin classes, Euler classes admit a natural generalization in the framework of differential cohomology. For definiteness, let us focus on Chern classes; analogous remarks hold for other characteristic classes. Our exposition follows [55].

Let $\mathcal{V}$ be a complex rank- n vector bundle over $\mathcal{M}$, with structure group $\text{U}(n)$, equipped with a hermitian fiber metric and a connection ∇ compatible with the fiber metric. The curvature of ∇ is the 2-form F_Δ on $\mathcal{M}$. In our conventions, F_Δ is antihermitian. The Chern forms $c_k(\nabla)$ are defined via

$$\det\left(\mathbb{I} + \frac{i F_\Delta}{2\pi}\right) = 1 + c_1(\nabla) + c_2(\nabla) + \cdots, \quad c_k(\nabla) \in \Omega_{\mathbb{Z}}^{2k}(\mathcal{M}) . \quad (\text{D.12})$$

The $2k$ -form $c_k(\nabla)$ is closed and has integral periods. If we choose a different connection ∇' on the same vector bundle, the form $c_k(\nabla')$ is generically different from $c_k(\nabla)$, but they differ by an exact piece. Their de Rham classes are the same, allowing us to define

$$c_k^{\mathbb{R}}(\mathcal{V}) = [c_k(\nabla)]_{\text{dR}} \in H^{2k}(\mathcal{M}, \mathbb{R}) . \quad (\text{D.13})$$

The superscript $\mathbb{R}$ on $c_k^{\mathbb{R}}(\mathcal{V})$ is inserted to emphasize that it is an object in the *real* cohomology of $\mathcal{M}$. It is known, however, that $c_k^{\mathbb{R}}(\mathcal{V})$ admits an integral refinement: an *integer* cohomology class $c_k(\mathcal{V})$ can be defined, such that

$$c_k^{\mathbb{R}}(\mathcal{V}) = \varrho(c_k(\mathcal{V})), \quad c_k(\mathcal{V}) \in H^{2k}(\mathcal{M}, \mathbb{Z}), \quad (\text{D.14})$$

where ϱ is the map (D.4). The integer class $c_k(\mathcal{V})$ contains more information than the real class $c_k^{\mathbb{R}}(\mathcal{V})$. For example, if the bundle $\mathcal{V}$ can be equipped with a flat connection, $c_k^{\mathbb{R}}(\mathcal{V}) = 0$ but $c_k(\mathcal{V})$ can be a non-trivial element in $\text{Tor } H^{2k}(\mathcal{M}, \mathbb{Z})$.

By definition, a differential refinement of the k -th Chern class is a map that sends a pair $(\mathcal{V}, \nabla)$ to an element $\check{c}_k(\nabla) \in \check{H}^{2k}(\mathcal{M})$, satisfying the following properties:

- (i) The field strength of the differential character $\check{c}_k(\nabla)$ is the Chern form $c_k(\nabla) \in \Omega_{\mathbb{Z}}^{2k}(\mathcal{M})$.
- (ii) The characteristic class of the differential character $\check{c}_k(\nabla)$ is the integral Chern class $c_k(\mathcal{V}) \in H^{2k}(\mathcal{M}, \mathbb{Z})$.
- (iii) For every smooth map $f : \mathcal{M}' \rightarrow \mathcal{M}$, one has $f^*\check{c}_k(\nabla) = \check{c}_k(f^*\nabla)$.

In the last point, $f^*\check{c}_k(\nabla) \in \check{H}^{2k}(\mathcal{M}')$ is the pullback from $\mathcal{M}$ to $\mathcal{M}'$ of the differential character $\check{c}_k(\nabla)$, while $\check{c}_k(f^*\nabla)$ denotes the element of $\check{H}^{2k}(\mathcal{M}')$ that is associated to the pullback vector bundle $f^*\mathcal{V}$ equipped with the pullback connection $f^*\nabla$. It can be proven that Chern classes admit a unique differential refinement. Similar theorems hold for Pontryagin classes and Euler classes.

Notice that the differential refinement $\check{c}_k(\nabla)$ retains information about the specific choice of connection ∇ . In more physical terms, $\check{c}_k(\nabla)$ has information about the specific $U(n)$ background gauge field configuration, whereas the integral Chern class $c_k(\mathcal{V})$ only depends on the topology of the bundle $\mathcal{V}$.

E Case study: wrapped M5-branes at a $\mathbb{Z}_2$ singularity

In this appendix we consider the total anomaly polynomial (5.9) for wrapped M5-branes at a $\mathbb{Z}_2$ singularity and we extract physical information about 't Hooft anomalies for discrete symmetries. More precisely, we consider the case in which we assign Dirichlet boundary conditions to $\mathcal{A}_1$ and B_{2i} (for each label $i = 1, \dots, g$). The interacting SCFT has therefore a global $\mathbb{Z}_k$ 0-form symmetry and an “electric” global $(\mathbb{Z}_N)^g$ 1-form symmetry.

Our strategy is as follows:

1. Perform an $SL(3, \mathbb{Z})$ transformation on the 1-form gauge fields $A_1^\alpha = (A_1, A_1^+, A_1^-)$ to single out the linear combination that enters the BF coupling with γ_4 , as described in appendix A. The new basis is denoted $(\mathcal{A}_1, \mathcal{A}_1^+, \mathcal{A}_1^-)$.
2. Collect all terms with γ_4 and $\tilde{H}_3^i$ and define new gauge fields $\mathbf{A}_1, \mathbf{B}_{2i}$ in such a way that these terms take the form

$$I_6^{\text{inflow}} \supset -k \frac{\gamma_4}{2\pi} \frac{d\mathbf{A}_1}{2\pi} - N \frac{\tilde{H}_3^i}{2\pi} \frac{d\mathbf{B}_{2i}}{2\pi} . \quad (\text{E.1})$$

As explained in section 5.1.3, we can then dualize c_3 and $\tilde{B}_2^i$ to ϕ_0, ϕ_{1i} , respectively. We get a Stückelberg-type theory for the pairs $(\mathbf{A}_1, \phi_0), (\mathbf{B}_{2i}, \phi_{1i})$ with constraints

$$k \mathbf{A}_1 = d\phi_0 , \quad N \mathbf{B}_{2i} = d\phi_{1i} . \quad (\text{E.2})$$

Thus $\mathbf{A}_1$ is the background gauge field for the global $\mathbb{Z}_k$ 0-form symmetry and $\mathbf{B}_{2i}$ is the background gauge field for the global $(\mathbb{Z}_N)^g$ 1-form symmetry.

3. Remove all terms with γ_4 and $\tilde{H}_3^i$ from I_6^{inflow} , and write the rest of I_6^{inflow} in terms of $\mathcal{A}_1^\pm$, $\mathbf{A}_1$, $\mathbf{B}_{2i}$, $a_{0i}^\pm$, $\tilde{a}_0^{i\pm}$, $p_1(T)$, and the background gauge fields for isometries of M_6 .

Let us address each step in turn.

Given the flux quanta $N_\alpha = (N, N_+, N_-)$ we define the integers $k, m, m_\pm$ via

$$k = \gcd(N, N_+, N_-), \quad N = k m, \quad N_\pm = k m_\pm. \quad (\text{E.3})$$

As a simplifying technical assumption, we suppose that m and m_+ are relatively prime. (Other cases are studied in a similar way.) It follows that integers r, s exist such that

$$m s - m_+ r = 1. \quad (\text{E.4})$$

The integers r, s are not uniquely determined by this equation. We suppose that a choice for r, s is made and kept fixed throughout. The change of basis of the 1-form gauge fields can be written as

$$\begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_1^+ \\ \mathcal{A}_1^- \end{pmatrix} = M^{-1} \begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_1^+ \\ \mathcal{A}_1^- \end{pmatrix}, \quad M = \begin{pmatrix} m & m_+ & m_- \\ r & s & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (\text{E.5})$$

The field strengths of $\mathcal{A}_1, \mathcal{A}_1^\pm$ are denoted $\mathcal{F}_2, \mathcal{F}_2^\pm$.

Next, we examine the terms in (5.9) with γ_4 and $\tilde{H}_3^i$. We find

$$\begin{aligned} I_6^{\text{inflow}} \supset & -\frac{\gamma_4}{2\pi} \left[k \frac{\mathcal{F}_2}{2\pi} + \frac{f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-}{(2\pi)^2} \right] \\ & - \frac{\tilde{H}_3^i}{2\pi} \left[N \frac{H_{3i}}{2\pi} + \left(\frac{m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^- - r \mathcal{F}_2}{2\pi} - N c_1^\psi \right) \frac{f_{1i}^+}{2\pi} + \frac{\mathcal{F}_2^-}{2\pi} \frac{f_{1i}^-}{2\pi} \right]. \end{aligned} \quad (\text{E.6})$$

This means that the new gauge fields $\mathbf{A}_1$ are defined by $\mathbf{B}_{2i}$ satisfy

$$\begin{aligned} k \frac{d\mathbf{A}_1}{2\pi} &= k \frac{\mathcal{F}_2}{2\pi} + \frac{f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-}{(2\pi)^2}, \\ N \frac{d\mathbf{B}_{2i}}{2\pi} &= N \frac{H_{3i}}{2\pi} + \left(\frac{m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^- - r \mathcal{F}_2}{2\pi} - N c_1^\psi \right) \frac{f_{1i}^+}{2\pi} + \frac{\mathcal{F}_2^-}{2\pi} \frac{f_{1i}^-}{2\pi}. \end{aligned} \quad (\text{E.7})$$

The final step is to remove the terms in the anomaly polynomial with $\gamma_4, \tilde{H}_3^i$, and write the rest using (E.7) to trade $\mathcal{F}_2, H_{3i}$ for $d\mathbf{A}_1, d\mathbf{B}_{2i}$. The result is quite lengthy: we present it as the sum of several contributions, listed as follows.

- *Terms containing only fields for isometries of M_6 and Poincaré symmetry:*

$$\begin{aligned} I_6^{\text{inflow}} \supset & -\frac{1}{3} N_- (c_1^\psi)^3 + \frac{1}{12} N_- c_1^\psi p_1(T) + \left(N^2 N_- + \frac{\chi}{3} N^3 - \frac{\chi}{3} N \right) c_1^\psi c_2^\varphi \\ & + \left(-\frac{2}{3} N^3 - N^2 N_- + \frac{1}{3} N_-^3 + \frac{2}{3} N + \frac{2}{3} N_- \right) c_1^\psi c_2^\Sigma. \end{aligned} \quad (\text{E.8})$$

- *Terms with three factors $\mathcal{F}_2^\pm$:*

$$I_6^{\text{inflow}} \supset \frac{1}{(2\pi)^3} \left[-\frac{\chi}{6} m^3 (\mathcal{F}_2^+)^3 - \frac{1}{2} m (2m_+ + m m_- r \chi) \mathcal{F}_2^- (\mathcal{F}_2^+)^2 \right. \\ \left. - \frac{1}{6} m_- r (6m_- s + 3\chi + m_-^2 r^2 \chi) (\mathcal{F}_2^-)^3 \right. \\ \left. - \frac{1}{2} (2m_- (m_+ r + m s) + m \chi + m m_-^2 r^2 \chi) \mathcal{F}_2^+ (\mathcal{F}_2^-)^2 \right]. \quad (\text{E.9})$$

- *Terms with two factors $\mathcal{F}_2^\pm$ and two factors $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$I_6^{\text{inflow}} \supset \frac{1}{(2\pi)^4} \left[-\frac{1}{k m} (\mathcal{F}_2^-)^2 f_{1i}^- \tilde{f}_1^{i-} - \frac{1}{k m} (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-)^2 f_{1i}^+ \tilde{f}_1^{i+} \right. \\ \left. - \frac{1}{2 k m} \left\{ m^3 r \chi (\mathcal{F}_2^+)^2 + 2m (1 + m_+ r + m s + m m_- r^2 \chi) \mathcal{F}_2^+ \mathcal{F}_2^- \right. \right. \\ \left. \left. + r (m_- (2 + 4m s) + m \chi + m m_-^2 r^2 \chi) (\mathcal{F}_2^-)^2 \right\} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) \right]. \quad (\text{E.10})$$

- *Terms with one factor $\mathcal{F}_2^\pm$ and four factors $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$I_6^{\text{inflow}} \supset \frac{1}{(2\pi)^5} \left[-\frac{2r}{k^2 m} (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-) f_{1j}^+ \tilde{f}_1^{j+} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) \right. \\ \left. - \frac{r}{2 k^2 m} (m^2 r \chi \mathcal{F}_2^+ + (2 + 2m s + m m_- r^2 \chi) \mathcal{F}_2^-) (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-)^2 \right]. \quad (\text{E.11})$$

- *Terms with six factors $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$I_6^{\text{inflow}} \supset \frac{1}{(2\pi)^6} \left[-\frac{r^2}{k^3 m} f_{1j}^+ \tilde{f}_1^{j+} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-)^2 - \frac{r^3 \chi}{6 k^3} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-)^3 \right]. \quad (\text{E.12})$$

- *Terms with one factor c_1^ψ and $\mathcal{F}_2^\pm$ and/or $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$I_6^{\text{inflow}} \supset -\frac{1}{(2\pi)^2} k m_- (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-)^2 c_1^\psi \\ - \frac{1}{(2\pi)^3} 2 (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-) f_{1j}^- \tilde{f}_1^{j-} c_1^\psi \\ - \frac{1}{(2\pi)^3} 2 m_- r (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-) (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) c_1^\psi \\ + \frac{1}{(2\pi)^4} \left[-\frac{2r}{k} f_{1j}^- \tilde{f}_1^{j-} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) c_1^\psi \right. \\ \left. - \frac{m_- r^2}{k} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-)^2 \right] c_1^\psi. \quad (\text{E.13})$$

- *Terms with two factors c_1^ψ and $\mathcal{F}_2^\pm$ and/or $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$\begin{aligned}
 I_6^{\text{inflow}} \supset & \frac{1}{2\pi} \frac{1}{2} (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-) \left[k^2 m (2m_- + m \chi) - \chi \right] (c_1^\psi)^2 \\
 & + \frac{1}{(2\pi)^2} \frac{r}{2k} \left[k^2 m (2m_- + m \chi) - \chi \right] \left(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^- \right) (c_1^\psi)^2 \\
 & + \frac{1}{(2\pi)^2} k m f_{1j}^- \tilde{f}_1^{j-} (c_1^\psi)^2 .
 \end{aligned} \tag{E.14}$$

- *Terms with one factor c_2^φ and $\mathcal{F}_2^\pm$ and/or $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$\begin{aligned}
 I_6^{\text{inflow}} \supset & \frac{1}{2\pi} \left[-k^2 m^2 m_- \mathcal{F}_2^+ - k^2 m (m_+ + m_-^2 r) \mathcal{F}_2^- \right] c_2^\varphi \\
 & + \frac{1}{(2\pi)^2} \left[-k m m_- r \left(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^- \right) - k m (f_{1i}^+ \tilde{f}_1^{i+} + f_{1i}^- \tilde{f}_1^{i-}) \right] c_2^\varphi
 \end{aligned} \tag{E.15}$$

- *Terms with one factor c_2^Σ and $\mathcal{F}_2^\pm$ and/or $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$\begin{aligned}
 I_6^{\text{inflow}} \supset & \frac{1}{2\pi} k^2 (m + m_-) \left[(m m_- + m_+^2) \mathcal{F}_2^+ + (m_+ + m_-^2 r + m_- m_+ s) \mathcal{F}_2^- \right] c_2^\Sigma \\
 & + \frac{1}{(2\pi)^2} k (m + m_-) (m_- r + m_+ s) \left(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^- \right) c_2^\Sigma .
 \end{aligned} \tag{E.16}$$

- *Terms with one factor $p_1(T)$ and $\mathcal{F}_2^\pm$ and/or $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$I_6^{\text{inflow}} \supset \frac{1}{2\pi} \frac{\chi}{24} (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-) p_1(T) + \frac{1}{(2\pi)^2} \frac{\chi r}{24k} \left(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^- \right) p_1(T) \tag{E.17}$$

- *Terms cubic and quadratic in $d\mathbf{A}_1$:*

$$\begin{aligned}
 I_6^{\text{inflow}} \supset & \frac{1}{(2\pi)^3} \frac{r^3 \chi}{6} (d\mathbf{A}_1)^3 - \frac{1}{(2\pi)^2} k m_- r^2 c_1^\psi (d\mathbf{A}_1)^2 \\
 & + \frac{1}{(2\pi)^3} \left[-\frac{\chi}{2} m r^2 \mathcal{F}_2^+ - \frac{r}{2} (2s + m_- r^2 \chi) \mathcal{F}_2^- \right] (d\mathbf{A}_1)^2 \\
 & + \frac{1}{(2\pi)^4} \left[-\frac{r^3 \chi}{2k} \left(f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^- \right) - \frac{r^2}{k m} f_{1i}^+ \tilde{f}_1^{i+} \right] (d\mathbf{A}_1)^2 .
 \end{aligned} \tag{E.18}$$

- *Terms linear in $d\mathbf{A}_1$, without $f_{1i}^\pm, \tilde{f}_1^{i\pm}$ or $d\mathbf{B}_{2i}$:*

$$\begin{aligned}
 I_6^{\text{inflow}} \supset & + \frac{1}{2\pi} \left[k^2 m m_- r c_2^\varphi - k^2 (m + m_-) (m_- r + m_+ s) c_2^\Sigma - \frac{r \chi}{24} p_1(T) \right. \\
 & + \frac{r}{2} (\chi - k^2 m (2m_- + m \chi)) (c_1^\psi)^2 \left. \right] d\mathbf{A}_1 \\
 & + \frac{1}{(2\pi)^3} \left[\frac{m^2 r \chi}{2} (\mathcal{F}_2^+)^2 + \frac{r}{2} (4m_- s + \chi + m_-^2 r^2 \chi) (\mathcal{F}_2^-)^2 \right.
 \end{aligned}$$

$$\begin{aligned}
 & + (m_+ r + m(s + m_- r^2 \chi)) \mathcal{F}_2^+ \mathcal{F}_2^- \Big] d\mathbf{A}_1 \\
 & + \frac{1}{(2\pi)^2} 2k m_- r c_1^\psi (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-) d\mathbf{A}_1 .
 \end{aligned} \tag{E.19}$$

- *Terms linear in $d\mathbf{A}_1$ with two or four factors $f_{1i}^\pm, \tilde{f}_1^{i\pm}$:*

$$\begin{aligned}
 I_6^{\text{inflow}} \supset & \frac{1}{(2\pi)^3} 2r c_1^\psi f_{1i}^- \tilde{f}_1^{i-} d\mathbf{A}_1 + \frac{1}{(2\pi)^3} 2m_- r^2 c_1^\psi (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) d\mathbf{A}_1 \\
 & + \frac{1}{(2\pi)^4} \frac{2r}{k m} f_{1i}^+ \tilde{f}_1^{i+} (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-) d\mathbf{A}_1 \\
 & + \frac{1}{(2\pi)^4} \frac{r}{k m} \left[m^2 r \chi \mathcal{F}_2^+ + (1 + 2m s + m m_- r^2 \chi) \mathcal{F}_2^- \right] \\
 & \quad \times (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) d\mathbf{A}_1 \\
 & + \frac{1}{(2\pi)^5} \frac{r^3 \chi}{2 k^2} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-)^2 d\mathbf{A}_1 \\
 & + \frac{1}{(2\pi)^5} \frac{2r^2}{k^2 m} f_{1i}^+ \tilde{f}_1^{i+} (f_{1i}^+ \tilde{f}_1^{i-} - \tilde{f}_1^{i+} f_{1i}^-) d\mathbf{A}_1 .
 \end{aligned} \tag{E.20}$$

- *Terms with $d\mathbf{B}_{2i}$:*

$$\begin{aligned}
 I_6^{\text{inflow}} \supset & \frac{1}{(2\pi)^3} \mathcal{F}_2^- d\mathbf{B}_{2i} \tilde{f}_1^{i-} + \frac{1}{(2\pi)^3} (m \mathcal{F}_2^+ + m_- r \mathcal{F}_2^-) d\mathbf{B}_{2i} \tilde{f}_1^{i+} \\
 & - \frac{1}{(2\pi)^2} k m c_1^\psi d\mathbf{B}_{2i} \tilde{f}_1^{i+} + \frac{1}{(2\pi)^4} \frac{r}{k} (f_{1j}^+ \tilde{f}_1^{j-} - \tilde{f}_1^{j+} f_{1j}^-) d\mathbf{B}_{2i} \tilde{f}_1^{i+} \\
 & - \frac{1}{(2\pi)^3} r d\mathbf{A}_1 d\mathbf{B}_{2i} \tilde{f}_1^{i+} .
 \end{aligned} \tag{E.21}$$

We have a rich variety of 't Hooft anomalies involving the $\mathbb{Z}_k$ 0-form symmetry and the $(\mathbb{Z}_N)^g$ 1-form symmetry.

F Free tensor multiplet reduction on Σ_g with topological twist

A free 6d $\mathcal{N} = (2, 0)$ tensor multiplet consists of: a chiral 2-form $b_{\mu\nu}$ which is a singlet of $\text{SO}(5)_R$; a symplectic Majorana-Weyl fermion χ in the representation $\mathbf{4}$ of $\text{USp}(4)_R \cong \text{SO}(5)_R$; five real scalar fields $\phi^1, \dots, \phi^5$ in the vector representation of $\text{SO}(5)_R$. In this appendix we study the reduction of this multiplet on a genus- g Riemann surface Σ_g with a non-zero $\text{SO}(5)_R$ background connection. The latter is encoded in the twist parameters p, q defined in section 2.2.1 and satisfying $p + q = -\chi$.

Since the chiral 2-form $b_{\mu\nu}$ is a singlet of $\text{SO}(5)_R$, it is unaffected by the topological twist. Its reduction on a genus- g Riemann surface yields the following massless fields: g real 4d vectors and one real 4d scalar b_0 . The reduction of the 6d fermion χ and the 6d scalars $\phi^1, \dots, \phi^5$, on the other hand, is sensitive to the twist parameters. We collect all massless 4d fields, their origins, and their multiplicities in table 3.

6d origin	U(1) ₁	U(1) ₂	U(1) _Σ	U(1) _Σ '	4d field	multiplicity
χ	+1	+1	$+\frac{1}{2}$	1	λ_α	g
	-1	-1	$+\frac{1}{2}$	0	ψ_α	1
	+1	-1	$+\frac{1}{2}$	$\frac{p}{p+q}$	Λ_α	$h^0(K^{\frac{p}{p+q}})$
	-1	+1	$+\frac{1}{2}$	$\frac{q}{p+q}$	$\hat{\Lambda}_\alpha$	$h^0(K^{\frac{q}{p+q}})$
$\phi^1 + i\phi^2$	+2	0	0	$\frac{p}{p+q}$	q	$h^0(K^{\frac{p}{p+q}})$
$\phi^3 + i\phi^4$	0	+2	0	$\frac{q}{p+q}$	$\hat{q}$	$h^0(K^{\frac{q}{p+q}})$
ϕ^5	0	0	0	0	Φ	1
$b_{\mu\nu}$	0	0	± 1	± 1	A_μ	g
	0	0	0	0	b_0	1

Table 3. Massless 4d fields originating from dimensional reduction of a 6d $\mathcal{N} = (2, 0)$ free tensor multiplet on a genus- g Riemann surface ($g \neq 1$) with twist parameters p, q satisfying $p+q = 2(g-1)$.

As we can see from the charges of $\phi^1 + i\phi^2$ and $\phi^3 + i\phi^4$, the subgroup $U(1)_1 \subset SO(5)_R$ is identified with rotations in the 12 plane, and $U(1)_2 \subset SO(5)_R$ is identified with rotations in the 34 plane. They are both normalized in such a way that their minimal charge is ± 1 . The notation $U(1)_\Sigma$ refers to local frame rotations on the Riemann surface. Its normalization is such that a chiral spinor on Σ_g has $U(1)_\Sigma$ charge $\pm \frac{1}{2}$. The symbol $U(1)_\Sigma'$ stands for the twisted local frame rotations on the Riemann surface. More precisely,

$$t'_\Sigma = t_\Sigma + \frac{p}{2(p+q)} t_1 + \frac{q}{2(p+q)} t_2, \quad (\text{F.1})$$

where $t'_\Sigma, t_\Sigma, t_1, t_2$ are the generators of $U(1)_\Sigma', U(1)_\Sigma, U(1)_1, U(1)_2$, respectively. Our discussion applies to $g \neq 1$. The case $g = 1$ is discussed at the end of this appendix.

In table 3, the 4d fields $\lambda_\alpha, \psi_\alpha, \Lambda_\alpha, \hat{\Lambda}_\alpha$ are Weyl spinors of positive chirality, q and $\hat{q}$ are complex scalars, Φ and b_0 are real scalars, and A_μ are real vectors. For each 4d field, the $U(1)_\Sigma'$ charge determines the bundle of which the corresponding internal wavefunctions must be a section. Massless fields originate from covariantly constant sections, or equivalently holomorphic sections. The symbol K stands for the canonical bundle on Σ_g .

The fields listed in table 3 are organized into the following multiplets of 4d $\mathcal{N} = 1$ supersymmetry:

- (A_μ, λ_α) : a collection of g vector multiplets;
- (Φ, b_0, ψ_α) : one chiral multiplet with $U(1)_1 \times U(1)_2$ charges $(0, 0)$;
- (q, Λ_α) : a collection of $h^0(K^{\frac{p}{p+q}})$ chiral multiplets with $U(1)_1 \times U(1)_2$ charges $(2, 0)$;
- $(\hat{q}, \hat{\Lambda}_\alpha)$: a collection of $h^0(K^{\frac{q}{p+q}})$ chiral multiplets with $U(1)_1 \times U(1)_2$ charges $(0, 2)$.

If an integer m divides $2(g-1)$, it is possible to define an m -th root $K^{\frac{1}{m}}$ of the canonical bundle, but the root is not unique. Since $p+q = 2(g-1)$, the bundles $K^{\frac{p}{p+q}}, K^{\frac{q}{p+q}}$ can be defined, but we would require more data to fully specify them. (For example,

for $p = q$ the additional data is a choice of spin structure on Σ_g .) Even though we are not able to determine the multiplicities $h^0(K^{\frac{p}{p+q}})$ and $h^0(K^{\frac{q}{p+q}})$ without further input, the Riemann-Roch theorem implies the relation

$$h^0(K^{\frac{p}{p+q}}) - h^0(K^{\frac{q}{p+q}}) = \frac{1}{2}(p - q). \quad (\text{F.2})$$

Notice that, since $p + q$ is an even integer, so is $p - q$, so the r.h.s. is an integer. To justify (F.2), we notice that the Riemann-Roch theorem can be stated as

$$h^0(\mathcal{L}) - h^0(\mathcal{L}^{-1} \otimes K) = \deg(\mathcal{L}) + 1 - g, \quad (\text{F.3})$$

where $\mathcal{L}$ is a line bundle on Σ_g . If we set $\mathcal{L} = K^{\frac{p}{p+q}}$, then we have $\mathcal{L}^{-1} \otimes K = K^{\frac{q}{p+q}}$. Moreover, $\deg(K) = 2(g - 1)$ gives $\deg(\mathcal{L}) = 2(g - 1)\frac{p}{p+q} = p$, and (F.3) implies (F.2).

Interestingly, the 't Hooft anomaly polynomial of the 4d fields listed in table 3 (with those $U(1)_1, U(1)_2$ charge assignments) only depends on the difference $h^0(K^{\frac{p}{p+q}}) - h^0(K^{\frac{q}{p+q}})$. We can thus make use of (F.2) and verify that the anomaly polynomial computed from table 3 matches exactly with the integration over Σ_g of the 8-form anomaly polynomial of a free 6d $\mathcal{N} = (2, 0)$ tensor multiplet.

We also notice that if we set $q = 0, p = -\chi$, we get a number $h^0(K^{\frac{p}{p+q}}) = h^0(K) = g$ of chiral multiplets with $U(1)_1 \times U(1)_2$ charges $(2, 0)$, and a number $h^0(K^{\frac{q}{p+q}}) = h^0(K^0) = 1$ of chiral multiplets with $U(1)_1 \times U(1)_2$ charges $(0, 2)$. The 4d multiplets (A_μ, λ_α) and (q, Λ_α) fit into $g \mathcal{N} = 2$ vector multiplets, while (Φ, b_0, ψ_α) and $(\hat{q}, \hat{\Lambda}_\alpha)$ fit into one $\mathcal{N} = 2$ hypermultiplet. It should be stressed that, because of its charge assignments, the contribution of this hypermultiplet to the 4d 't Hooft anomalies is equal to -1 times the contribution of a vector multiplet.

We may also consider the case $p = q = -\chi/2$. The chiral multiplets $(q, \Lambda_\alpha), (\hat{q}, \hat{\Lambda}_\alpha)$ have the same multiplicity and fit into a doublet of the enhanced flavor symmetry $SU(2)_F$. In contrast, the chiral multiplet (Φ, b_0, ψ_α) is a singlet of $SU(2)_F$. (The Cartan of $SU(2)_F$ is proportional to the difference $t_1 - t_2$.)

Finally, let us comment on the case $g = 1, p \neq 0$. The Riemann surface is flat and its canonical bundle is trivial. The total covariant derivative on T^2 (in local flat coordinates) has no spin connection term but includes the terms originating from the background $U(1)_1 \times U(1)_2 \subset SO(5)_R$ fields. It takes the schematic form $D_m = \partial_m + p A_m(t_1 - t_2)$, where A_m is a local antiderivative of the volume form on T^2 . With reference to table 3, the modes of χ with $t_1 = t_2 = \pm 1$ are unaffected by the $U(1)_1 \times U(1)_2$ background. To get massless modes in four dimensions, we take their internal wavefunction to be a covariantly constant spinor on T^2 . Since T^2 is flat, a covariantly constant spinor is constant, yielding a multiplicity 1 for both λ_α and ψ_α . (We select periodic boundary conditions on both 1-cycles of T^2 .) As before, the fermion λ_α combines with A_μ in one vector multiplet, and ψ_α combines with b_0 and Φ in one chiral multiplet with $U(1)_1 \times U(1)_2$ charges $(0, 0)$. The mode of χ with $(t_1, t_2) = (+1, -1)$, denoted Λ_α in table 3, is affected by the topological twist, and behaves as a section of $\mathcal{L}^p$, where $\mathcal{L}$ is a degree-one line bundle on T^2 . The same holds true for the scalar $\phi^1 + i\phi^2$. The fields $\Lambda_\alpha, \phi^1 + i\phi^2$ fit into chiral multiplets with $U(1)_1 \times U(1)_2$ charges $(2, 0)$ and multiplicity $h^0(\mathcal{L}^p)$. In a similar way, $\hat{\Lambda}_\alpha$ and $\phi^1 - i\phi^2$ fit into chiral multiplets

with $U(1)_1 \times U(1)_2$ charges $(0, 2)$ and multiplicity $h^0(\mathcal{L}^{-p})$. The difference between $h^0(\mathcal{L}^p)$ and $h^0(\mathcal{L}^{-p})$ can be computed using (F.3), with the replacement $\mathcal{L} \rightarrow \mathcal{L}^p$, to give

$$h^0(\mathcal{L}^p) - h^0(\mathcal{L}^{-p}) = p. \quad (\text{F.4})$$

As in the $g \neq 1$ case, this relation can be used to verify that the 't Hooft anomalies of the 4d fields match with the result obtained by integration over the Riemann surface of the 8-form anomaly polynomial of a free 6d $\mathcal{N} = (2, 0)$ tensor multiplet.

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