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A predator–prey model with Crowley–Martin functional response: A nonautonomous study

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Abstract

We investigate a nonautonomous predator–prey model system with a Crowley–Martin functional response. We perform rigorous mathematical analysis and obtain conditions for (a) global attractivity and permanence in the form of integrals which improve the traditional conditions obtained by using bounds of involved parameters; and (b) the existence of periodic solutions applying continuation theorem from coincidence degree theory which has stronger results than using Brouwer fixed point theorem. Our result also indicates that the global attractivity of periodic solution is positively affected by the predator's density dependent death rate. We employ partial rank correlation coefficient method to focus on how the output of the model system analysis is influenced by variations in a particular parameter disregarding the uncertainty over the remaining parameters. We discuss the relations between results (permanence and global attractivity) for autonomous and nonautonomous systems to get insights on the effects of time-dependent parameters.

Recommendations for Resource Managers:

- The natural environment fluctuates because of several factors, for example, mating habits, food supplies, seasonal effects of weathers, harvesting, death rates, birth rates, and other important population rates. The temporal fluctuations in physical environment (periodicity) plays a major role in community and population dynamics along with the impacts of population densities.
- Periodic system may suppress the permanence of its corresponding autonomous system with parameters being the averages of periodic parameters.
- As the human needs crosses a threshold level, then we require to observe the sustainability of resources of the associated exploited system. Therefore, the concept of stability and permanence become our main concern in an exploited model system (system with harvesting).
- The mutual interference at high prey density may leave negative effect on the permanence of the system.
- In harvested system, permanence becomes an important issue because if we harvest too many individuals then species may be driven to extinction. Interestingly, in many biological/agricultural systems, harvesting (due to fishing in marine system, hunting or disease) of a particular species/crop can only be more beneficial at certain times (e.g., the time and stage of harvest of a particular crop play greater role in its production and hence the particular crop is many times harvested at its physiological maturity or at harvest maturity).

KEYWORDS

almost periodic solution, coincidence degree, functional response, global attractivity, periodic solution, sensitivity analysis

1 | INTRODUCTION

The predator–prey model system has been one of the most important topics in biological systems (Berryman, 1992; Lotka, 1956). The predator–prey relationships play significant roles in determining the stability and persistence for a large number of species in ecosystems (Allesina & Tang, 2012). The survival of species depends on how efficiently they eat/take their food resource (like, prey for predators because prey serves as food resource and energy for associated predator). Thus the respective predator directly influences the associated ecosystem including the prey population via direct interactions. Such direct interactions between prey and predator have been mathematically formulated via different functional responses. The functional response (response function) is an important feature of the prey–predator interactions (Berryman, 1992). The understanding of the role of functional responses helps to get more biological insight into the predator–prey dynamics.

Functional responses describe how predator and prey interact in their ecosystems. The direct interactions between predator and prey have been modeled via linear (L-V type functional response; Berryman, 1992; Lotka, 1956), Holling type (III, II, and I) response functions (Holling, 1959; Xiao & Ruan, 2001), and ratio-dependent functional responses (Arditi & Ginzburg, 1989; Banerjee & Petrovskii, 2011). Recently, several authors have explored the dynamics of prey–predator system with a ratio-dependent response function (see, e.g., Arditi & Ginzburg, 1989; Banerjee & Petrovskii, 2011; Chen & Cao, 2008, and references therein). However, not only the direct interactions between prey and predator influence the dynamics of associated model system but also direct interactions among predators affect the overall dynamics of respective predator–prey system via modifying the functional response based on spatial factors (Cosner et al., 1999; Crowley & Martin, 1989). Beddington (1975) and DeAngelis et al. (1975) separately derived a response function that acclimate interference between predator (direct interactions in predators). Here the assumption is that individuals not only assign time to forage and process prey but also use some time fetching in encounters with predator (Beddington, 1975; DeAngelis et al., 1975; Tripathi et al., 2015). Thus, the expected consequence is that feeding rate of predator becomes free from the density of predator at the high density of prey.

However, empirical results suggest that the predator feeding rate is decreased with respect to the higher density of predator even when the density of its prey is high (Collazo et al., 2010; Skalski & Gilliam, 2001; Zimmermann et al., 2015). This concept was modeled mathematically by Crowley–Martin (1989) (hereafter the CM model system; Crowley & Martin, 1989). Also in Skalski and Gilliam (2001), a statistical inference from 19 prey–predator systems ensures that three predator-dependent response functions (viz., Hassel–Varley, Beddington–DeAngelis, and Crowley–Martin) give a better explanation of predator's feeding over the range of prey–predator richness. We would like to point out that the Crowley–Martin response function is akin to the Beddington–DeAngelis response function but it includes one more term explaining mutual interferences of predators at the high density of its prey (Parshad et al., 2017; Tripathi et al., 2020, 2016).

Thus incorporating the above idea, the per capita feeding rate for a particular predator (y) in CMFR is given by, $\eta(x, y) = \frac{ax}{1 + bx + cy + bcxy}$. Here x denotes the density of prey. The three parameters c, b, and a have similar interpretations as in Beddingtion–DeAngelis type functional response (Beddington, 1975; DeAngelis et al., 1975; Skalski & Gilliam, 2001). A notable difference between the Crowley–Martin and Beddington–DeAngelis functional responses is: Beddington–DeAngelis predicts that the impacts of predator's interference on feeding rate is much less under the conditions of high abundance of prey while Crowley–Martin considers the interference effects on feeding rate (Hassell, 1971). The limiting value of $\eta(x, y)$ depends on x

only, as $y \to 0$ (almost no interference amongst predators) and $\eta(x, y) \to 0$ when $y \to \infty$, (showing maximum interference among predators).

In theoretical ecology, there are many work on Lotka–Volterra systems in the constant environment. However in real life, the constant environment is a rare case (see, e.g., Chesson, 2003; Cushing, 1977; Fan & Kuang, 2004; Lin & Chen, 2009; Tripathi, 2016; Tripathi & Abbas, 2016). The natural environment fluctuates because of several factors, for example, mating habits, food supplies, seasonal effects of weathers, harvesting, death rates, birth rates, and other important population rates (Fan & Kuang, 2004). In an experiment on a host–parasite system, Utida (1957) has given suggestions an explanation for oscillatory data (Cushing, 1977). Moreover, in Utida (1953), cyclic fluctuations of population have also been demonstrated by taking 25 generations of interactions between populations of *Heterospilus prosopidi* (a larval parasite) and *azuki bean weevil*. This indicates that the physical environment plays a major role in community and population dynamics along with the impacts of population densities. Though the past studies reveal the fact that the temporal fluctuations in physical environment (temporal inhomogeneity in the model parameters) are key drivers of population fluctuations, yet only few theoretical attempts are found to forecast the characteristics of the consequential population fluctuation (Chesson, 2003; Fan & Kuang, 2004). Thus, there is a need to study ecosystems in the temporal inhomogeneous environment.

If we consider the temporal inhomogeneity of environment, a model system becomes nonautonomous (Fan & Kuang, 2004; Fan et al., 2003; Li & Takeuchi, 2015; Tripathi & Abbas, 2016). For nonautonomous model systems, researchers consider periodic and almost periodic coefficients. One can also find several important studies on neural networks with time-dependent parameters (nonautonomous neural networks). Recently, Yang et al. (2018) investigated the discontinuous nonautonomous networks and associated exponential synchronization control. Other significant studies related to important nonautonomous model systems on similar topics can be found in Duan et al. (2018, 2017), Huang and Bingwen (2019), Huang and Zhang (2019), and Huang et al. (2016, 2019). However, in ecology, the nonautonomous phenomenon occurs mainly due to seasonal variations, which make the population to grow periodically or almost periodically. More precisely, the model systems are also considered with time-varying parameters if the relevant environmental factors fluctuate periodically with time (Abbas et al., 2012; Rinaldi et al., 1993; Tripathi, 2016). This paper concerns complex delayed neural networks with discontinuous activations. Permanence, almost periodic and periodic solutions of Lotka–Volterra systems have been discussed by several authors (see Chen, 2006; Chen & Shi, 2006; Fan & Kuang, 2004). In particular, Li and Takeuchi (2015) established the existence of periodic solutions of a prey–predator system with a Beddington–DeAngelis response function. Recently, Tripathi and Abbas in Tripathi and Abbas (2015) discussed a nonautonomous model system with a modified Leslie–Gower response function. The global attractivity and permanence of a Lotka–Volterra competitive system was investigated in Chen (2006).

In this paper, we consider the following nonautonomous predator–prey model system with a CMFR and density-dependent death rates in both predator and prey:

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left(-b(t)x(t) + a(t) - \frac{c(t)y(t)}{a_2(t)x(t) + a_1(t) + a_3(t)y(t) + a_4(t)y(t)x(t)}\right), \\ \frac{dy(t)}{dt} &= y(t)\left(-e(t)y(t) - d(t) + \frac{f(t)x(t)}{a_2(t)x(t) + a_1(t) + a_3(t)y(t) + a_4(t)x(t)y(t)}\right), \end{aligned} \tag{1}$$

where $y(t)$ and $x(t)$ represent predator and prey densities at time t, respectively. Here we assume that $f(t), e(t), d(t), c(t), b(t), a(t), a_i(t)$ $(i = 1, 2, 3, 4)$ are continuous and bounded functions by positive constants with the following ecological interpretations: $f(t)$ (the coefficient of conversion from prey to predator); $e(t)$ (the predator population decreases due to competition among the predators); $d(t)$ (in the absence of prey, the predator population decreases); $c(t)$ (predator populations feed upon the prey population); $b(t)$ (due to competition amongst the preys, the prey population decreases); $a(t)$ (in the absence of predators, the prey population increases); $a_1(t)$ (measures the half saturation of prey species); $a_2(t)$ (measures the handling time); $a_3(t)$ (coefficient of interference among predators); $a_4(t)$ (the coefficient of interference among predators at the high density of prey).

The main goal of this study is to present the complete dynamics and to establish the conditions of existence of a unique global attractive almost periodic (periodic) solution of the model system (1) using a suitable Lyapunov functional and continuation theorem in degree theory. In present study, we have obtained the following important results and improvements:

- A nonautonomous prey–predator model system with a CMFR has been considered. All time-dependent parameter functions are considered bounded below and above by positive constants. Nonautonomous system has more reasonable biological interpretation than the corresponding autonomous system Tripathi et al. (2016).
- The conditions of extinction of both prey and predator and the global stability of boundary periodic solutions are given in both parametric and integral forms. The conditions in integral forms reflect the effects of the long-term predation behaviors on the number of species. The results have more reasonable biological interpretation rather than those for the corresponding autonomous system.
- The permanence conditions of the considered model are more flexible than usual conditions obtained by using supremum and infimum of the time-dependent model parameters. Moreover, flexible conditions involving integrals have been obtained rather than conditions obtained using lower and upper bounds of model parameter. Thus the persistence results of the present study improves the conditions from traditional methods (e.g., refer Fan & Kuang, 2004; Fan et al., 2003; Li & Takeuchi, 2015).
- Numerical examples show that periodic system may suppress the permanence of its corresponding autonomous system with parameters being the averages of periodic parameters.

The remaining part of manuscript is organized as follows. We establish permanence, boundedness, and global asymptotic stability of the considered model system in Section 2. Sufficient conditions for the global asymptotic stability and existence of a periodic solution have been discussed in Section 3. In Section 4, the existence of a unique almost periodic solution have been established. In Section 5, to support our analytical findings, numerical examples are demonstrated. Following numerical evaluations, we have performed the sensitivity analysis in Section 6. A brief discussion followed by ecological implications and future scope are given in the final section. Some preliminary results along with some conventional proofs have been presented in the appendix.

2 | A GENERAL NONAUTONOMOUS CASE: POSITIVITY, PERMANENCE, AND GLOBAL ATTRACTIVITY

Here, we establish the positive invariance, boundedness, permanence, and global asymptotic stability. Let $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2: y \geq 0, x \geq 0\}$. Suppose $g(t)$ be a bounded and continuous function on $\mathbb{R}$ and

g_L and g_M denote $\inf_{t\in\mathbb{R}} g(t)$ and $\sup_{t\in R} g(t)$, respectively. Based on the biological context of the proposed model system (1), we assume that its coefficients satisfy the following conditions:

$$\min\{a_L, b_L, c_L, d_L, e_L, f_L, \min_{i=1,2,3,4}\{a_{i_L}\}\} > 0$$

and

$$\max\{a_M, b_M, c_M, d_M, e_M, f_M, \max_{i=1,2,3,4}\{a_{i_M}\}\} < \infty.$$

Thus, we could have the following positive invariance of the model system (1).

Lemma 1. *For the model system* (1), *the positive cone is positively invariant.*

Proof. Refer to the appendix. □

Now define the following notations:

$$\begin{aligned} M_1^\epsilon &:= \frac{a_M}{b_L} + \epsilon, \quad m_1^\epsilon := \frac{1}{b_M}\left[a_L - \frac{c_M M_2}{a_{1_L} + a_{3_L} M_2}\right] - \epsilon, \\ M_2^\epsilon &:= \frac{1}{e_L}\left[-d_L + \frac{f_M M_1}{a_{1_L} + a_{2_L} M_1}\right] + \epsilon, \\ m_2^\epsilon &:= \frac{1}{e_M}\left[-d_M + \frac{f_L m_1}{a_{1_M} + a_{2_M} m_1 + a_{3_M} M_2 + a_{4_M} m_1 M_2}\right] - \epsilon. \end{aligned} \tag{2}$$

Then we obtain the subsequent theorem:

Theorem 1. *If time-dependent coefficients of the model system* (1) *satisfy*

$$d_L < \frac{f_M M_1^\epsilon}{a_{1_L} + a_{2_L} M_1^\epsilon}, \quad a_L > \frac{c_M M_2^\epsilon}{a_{1_L} + a_{3_L} M_2^\epsilon}, \quad d_M < \frac{f_L m_1^\epsilon}{a_{1_M} + a_{2_M} m_1^\epsilon + a_{3_M} M_2^\epsilon + a_{4_M} m_1^\epsilon M_2^\epsilon}, \tag{3}$$

then the set

$$\varkappa_\epsilon := \left\{(x, y) \in \mathbb{R}^2 : m_2^\epsilon \le y \le M_2^\epsilon, m_1^\epsilon \le x \le M_1^\epsilon\right\}, \tag{4}$$

is positively invariant with respect to the model system (1), *where* $\epsilon \ge 0$ *is sufficiently small so that* $m_1^\epsilon > 0$ *and* $m_2^\epsilon > 0$.

Using the Definition A1, we summarize the above theorem as the following result of the system (1) on permanence:

Theorem 2. *If the time-dependent coefficients of the model system* (1) *satisfy*

$$d_L < \frac{f_M M_1}{a_{1_L} + a_{2_L} M_1}, \quad a_L > \frac{c_M M_2}{a_{1_L} + a_{3_L} M_2}, \quad d_M < \frac{f_L m_1}{a_{1_M} + a_{2_M} m_1 + a_{3_M} M_2 + a_{4_M} m_1 M_2}, \tag{5}$$

then the system (1) is permanent.

Remark 1. The detailed proofs of Theorems 1 and 2 have been given in the appendix. If all the parameters in the model system (1) are constants (positive), then autonomous version of the model system (1) (also given in Tripathi et al., 2016) provides the following result on permanence:

$$d < \frac{fM}{a_1 + a_2 M}, \quad a > \frac{cL}{a_1 + a_3 L}, \quad d < \frac{fK}{a_1 + a_2 K + a_3 L + a_4 KL}, \tag{6}$$

where $\limsup_{t\to+\infty} y(t) \le L$, $\limsup_{t\to+\infty} x(t) \le M$, $\liminf_{t\to+\infty} y(t) \ge N$, $\liminf_{t\to+\infty} x(t) \ge K$. Note that these conditions are different to those conditions given in Tripathi et al. (2016).

Remark 2. All solutions of the model system (1) are eventually bounded (refer the Definition A2) under the conditions (5). One can also prove that the set $\varkappa_\epsilon \neq \phi$ that is there exists at least one positive bounded solution for the model system (1) (Definition A2). The proof follows similarly as in Du and Lv (2013).

Remark 3. For the same value of coefficient functions as in Example 1 (Section 5) with sufficiently small value of ϵ, the sufficient conditions of Theorem 1 would be well satisfied. Moreover, one can also compute the set $\varkappa_\epsilon$. Here for $\epsilon = 0$, (3) is same as (5). Hence the model system (1) is permanent if $\varkappa_\epsilon$ is positively invariant in model system (1). Here it is important to mention that permanence ensures for all the solutions to satisfy the property given in Definition A1.

Lemma 2. *If $q(t)$ and $p(t)$ are continuous functions defined on $\mathbb{R}$ and bounded by constants (positive) and $\frac{du(t)}{dt} \le (\ge) u(t)(-p(t)u(t) + q(t))$, $t \in [t_0, +\infty)$, then we obtain* $\limsup_{t\to+\infty} u(t) \le \sup_{t\in\mathbb{R}} \frac{q(t)}{p(t)}$, *if* $0 < u(t_0) \le \sup_{t\in\mathbb{R}} \frac{q(t)}{p(t)}$, *or* $\liminf_{t\to+\infty} u(t) \ge \inf_{t\in\mathbb{R}} \frac{q(t)}{p(t)}$, *if* $u(t_0) \ge \inf_{t\in\mathbb{R}} \frac{q(t)}{p(t)}$.

Define

$$\begin{aligned}
\tilde{M}_1 &= \sup_{t\in\mathbb{R}} \frac{a(t)}{b(t)}, \quad \tilde{M}_2 = \sup_{t\in\mathbb{R}} \frac{1}{e(t)} \left[-d(t) + \frac{\tilde{M}_1 f(t)}{a_2(t)\tilde{M}_1 + a_1(t)} \right], \\
\tilde{m}_1 &= \inf_{t\in\mathbb{R}} \frac{1}{b(t)} \left[a(t) - \frac{\tilde{M}_2 c(t)}{a_3(t)\tilde{M}_2 + a_1(t)} \right], \\
\tilde{m}_2 &= \inf_{t\in\mathbb{R}} \frac{1}{e(t)} \left[-d(t) + \frac{\tilde{m}_1 f(t)}{a_2(t)\tilde{m}_1 + a_1(t) + a_3(t)\tilde{M}_2 + a_4(t)\tilde{m}_1\tilde{M}_2} \right],
\end{aligned} \tag{7}$$

and

$$K := \{(x, y) \in \mathbb{R}^2 | \tilde{m}_2 \le y \le \tilde{M}_2, \tilde{m}_1 \le x \le \tilde{M}_1\}.$$

Now define the following conditions:

$$\begin{aligned}
&\inf_{t\in\mathbb{R}}\left[\frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1} - d(t)\right] > 0,\\
&\inf_{t\in\mathbb{R}}\left[-\frac{c(t)}{a_1(t) + a_3(t)\tilde{M}_2} + a(t)\right] > 0,\\
&\inf_{t\in\mathbb{R}}\left[\frac{f(t)\tilde{m}_1}{a_2(t)\tilde{m}_1 + a_1(t) + a_3(t)\tilde{M}_2 + a_4(t)\tilde{m}_1\tilde{M}_2} - d(t)\right] > 0,
\end{aligned} \tag{8}$$

which ensures that $\tilde{M}_1$ is greater than $\tilde{m}_1$ and $\tilde{m}_2$ is less than $\tilde{M}_2$.

Remark 4. The conditions given in Equation (8) are important for the model system (1) being permanent. The permanence of the model system (1) ensures the following ecological interpretations:

1. At any time t, the death rate $d(t)$ of predator y in the absence of prey is smaller than the benefits due to eating its prey x.
2. At any time t, the intrinsic growth rate $a(t)$ of prey x in the absence of predator y is large enough such that $\inf_{t\in\mathbb{R}}\left[a(t) - \frac{c(t)\tilde{M}_2}{a_1(t) + a_3(t)\tilde{M}_2}\right] > 0$.

Remark 5. By the comparison between conditions (5) and (8) and conditions (6) of the corresponding autonomous system, one can see that conditions (8) are more flexible than (5) and (6).

Theorem 3. *If the conditions* (8) *holds, then for model system* (1), *the set* K_ϵ *is positively invariant, and the model system* (1) *is permanent and the set* K_ϵ *is defined by*

$$K_\epsilon := \{(x, y) \in \mathbb{R}^2 | \tilde{m}_2 - \epsilon \le y \le \tilde{M}_2 + \epsilon, \tilde{m}_1 - \epsilon \le x \le \tilde{M}_1 + \epsilon\}$$

is an ultimate bounded region. Here ϵ *is a sufficiently small number such that* $\tilde{m}_1 - \epsilon > 0$ *and* $\tilde{m}_2 - \epsilon > 0$.

Theorem 4. *If the following condition holds*

$$\frac{f_M\tilde{M}_1}{a_{1_L} + a_{2_L}\tilde{M}_1} < d_L, \tag{9}$$

then the predator y goes extinct.

Theorem 5. *If the following condition holds*

$$a_M < \frac{c_L\tilde{m}_2}{a_{1_M} + a_{2_M}\tilde{M}_1 + a_{3_M}\tilde{m}_2 + a_{4_M}\tilde{m}_2\tilde{M}_1}, \tag{10}$$

then the prey x goes extinct.

Remark 6. If all the parameters in the model system (1) are time-independent, then the conditions for extinction of corresponding autonomous model system are given by

1. The predator y goes extinct if $\frac{fM}{a_1 + a_2 M} < d$.
2. The prey x goes extinct if $a < \frac{cN}{a_1 + a_2 M + a_3 N + a_4 NM}$.

Theorem 6.

(1) *If the following condition holds*

$$\int_0^{+\infty}\left(-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1}\right)dt = -\infty \tag{11}$$

then the predator y goes extinct.

(2) *If*

$$\int_0^{+\infty}\left(a(t) - \frac{\tilde{m}_2 c(t)}{a_2(t)\tilde{M}_1 + a_1(t) + a_3(t)\tilde{m}_2 + a_4(t)\tilde{M}_1\tilde{m}_2}\right)dt = -\infty, \tag{12}$$

holds then the prey x become extinct.

Remark 7. Conditions (11) and (12) have more reasonable biological interpretations than conditions with infimum and supremum of parameter functions given in (9) and (10) and those for corresponding autonomous model system.

Remark 8. Condition (11) shows that if for a long period of time the benefit of predator y from predating its prey x is less than the death rate of predator y, the predator y goes to extinction in the system (1). Condition (12) indicates that if long term effects of the predation behavior to prey x is larger than its intrinsic growth rate, the prey x goes extinct in the model system (1). Conditions (11) and (12) also show that $a(t) - \frac{c(t)\tilde{m}_2}{a_1(t) + a_2(t)\tilde{M}_1 + a_3(t)\tilde{m}_2 + a_4(t)\tilde{M}_1\tilde{m}_2}$ and $-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1}$, are allowed to change their signs.

For the boundary solution of model system (1), in the absence of predator, the model system (1) becomes (Riccatti equation):

$$\frac{dx(t)}{dt} = x(t)(-b(t)x(t) + a(t)) \tag{13}$$

Obviously, $x(t) = 0$ is a solution of Equation (13). Moreover, solution $\tilde{x}(t)$ such that $\tilde{x}(0) = x_0(\neq 0)$ is given by

$$\tilde{x}(t) = \left(\int_0^t b(s)\exp\left\{-\int_0^t a(\tau)d\tau\right\}ds + \frac{1}{x_0}\exp\left\{-\int_0^t a(s)ds\right\}\right)^{-1} \tag{14}$$

Hence, the existence of the boundary solution is guaranteed.

Theorem 7. *Let $Y(t) = (x_1(t), y_1(t))$ be a positive bounded solution of system (1). If conditions in Equation (8) and*

$$\begin{aligned}
&\inf_{t\in\mathbb{R}}\left\{-\frac{\left[a_2(t)M_2^{\epsilon} + a_4(t)M_2^{\epsilon 2}\right]c(t)}{\left(a_2(t)m_1^{\epsilon} + a_1(t) + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right.\\
&\left.-\frac{\left[a_1(t) + a_3(t)M_2^{\epsilon}\right]f(t)}{\left(a_1(t) + a_2(t)m_1^{\epsilon} + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2} + b(t)\right\} > 0,\\
&\inf_{t\in\mathbb{R}}\left\{-\frac{\left[a_1(t) + a_2(t)M_1^{\epsilon}\right]c(t)}{\left(a_2(t)m_1^{\epsilon} + a_1(t) + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right.\\
&\left.+e(t) + \frac{\left[a_3(t)m_1^{\epsilon} + a_4(t)m_1^{\epsilon 2}\right]f(t)}{\left(a_1(t) + a_2(t)M_1^{\epsilon} + a_3(t)M_2^{\epsilon} + a_4(t)M_1^{\epsilon}M_2^{\epsilon}\right)^2}\right\} > 0,
\end{aligned} \tag{15}$$

hold, then $(x_1(t), y_1(t))$ is globally attractive.

Proof. Suppose $X(t) = (x(t), y(t))$ be any positive bounded solution (refer the Definition A2) of model system (1). Hence there exists a positive number T such that $(x_1(t), y_1(t)), (x(t), y(t)) \in K_{\epsilon}$, for all $t \geq t_0 + T$. Define $\zeta(t, x(t), y(t)) = (a_1(t) + a_3(t)y(t) + a_4(t)x(t)y(t) + a_2(t)x(t))(a_1(t) + a_2(t)x_1(t) + a_4(t)x_1(t)y_1(t) + a_3(t)y_1(t))$.

Let $S_1(t) = |\ln x(t) - \ln x_1(t)|$.

For $S_1(t)$, the upper right Dini derivative Chen and Jinde (2003) is given by

$$\begin{aligned}
D^+S_1(t) &= \operatorname{sgn}(x(t) - x_1(t))\left(-\frac{\dot{x}_1(t)}{x_1(t)} + \frac{\dot{x}(t)}{x(t)}\right)\\
&= \operatorname{sgn}(x(t) - x_1(t))\left[-b(t)(x(t) - x_1(t)) - c(t)\left(-\frac{y_1(t)}{\zeta(t, x_1(t), y_1(t))}\right.\right.\\
&\quad\left.\left.+\frac{y(t)}{\zeta(t, x(t), y(t))}\right)\right]\\
&= -b(t)|x(t) - x_1(t)|\\
&\quad -\operatorname{sgn}(x(t) - x_1(t))c(t)\left(\frac{-y_1(t)\zeta(t, x(t), y(t)) + y(t)\zeta(t, x_1(t), y_1(t))}{(\zeta(t, x(t), y(t)))(\zeta(t, x_1(t), y_1(t)))}\right)\\
&\leq -b(t)|x(t) - x_1(t)|\\
&\quad + c(t)\left(\frac{y_1(t)(a_4(t)y(t) + a_2(t))}{\zeta(t, x_1(t), y_1(t)).\ \zeta(t, x(t), y(t))}|x(t) - x_1(t)|\right.\\
&\quad\left.+\frac{(a_2(t)x_1(t) + a_1(t))}{\zeta(t, x_1(t), y_1(t)).\ \zeta(t, x(t), y(t))}|y(t) - y_1(t)|\right).
\end{aligned}$$

Moreover, consider $S_2(t) = |\ln y(t) - \ln y_1(t)|$. For $S_2(t)$, we have

$$\begin{aligned}
D^+S_2(t) &= \operatorname{sgn}(y(t) - y_1(t))\left(-\frac{\dot{y}_1(t)}{y_1(t)} + \frac{\dot{y}(t)}{y(t)}\right) \\
&= \operatorname{sgn}(y(t) - y_1(t))\left[-e(t)(y(t) - y_1(t)) + f(t)\left(-\frac{x_1(t)}{\zeta(t, x_1(t), y_1(t))}\right.\right. \\
&\quad \left.\left. + \frac{x(t)}{\zeta(t, x(t), y(t))}\right)\right] \\
&= -e(t)|y(t) - y_1(t)| \\
&\quad + \operatorname{sgn}(y(t) - y_1(t)) f(t)\left(\frac{-x_1(t)\zeta(t, x(t), y(t)) + x(t)\zeta(t, x_1(t), y_1(t))}{(\zeta(t, x(t), y(t)))(\zeta(t, x_1(t), y_1(t)))}\right) \\
&\le -e(t)|y(t) - y_1(t)| \\
&\quad + f(t)\left(\frac{a_1(t) + a_3(t)y_1(t)}{\zeta(t, x_1(t), y_1(t)).\ \zeta(t, x(t), y(t))}|x(t) - x_1(t)|\right. \\
&\quad \left. - \frac{x_1(t)(a_3(t) + a_4(t)x_1(t))}{\zeta(t, x_1(t), y_1(t)).\ \zeta(t, x(t), y(t))}|y(t) - y_1(t)|\right).
\end{aligned}$$

Combining the two functions $S_i(t)$, for $i = 1, 2$, we find $S(t) = S_1(t) + S_2(t)$. For $t \ge t_0$, we have

$$\begin{aligned}
D^+S(t) &= D^+S_2(t) + D^+S_1(t) \\
&\le -\left[-\frac{c(t)\left[a_4(t)M_2^{\epsilon 2} + a_2(t)M_2^{\epsilon}\right]}{\left(a_2(t)m_1^{\epsilon} + a_1(t) + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2} + b(t)\right. \\
&\quad \left. - \frac{f(t)\left[a_1(t) + a_3(t)M_2^{\epsilon}\right]}{\left(a_1(t) + a_2(t)m_1^{\epsilon} + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right]|-\bar{x}_2(t) + \bar{x}_1(t)| \\
&\quad - \left[-\frac{c(t)\left[a_1(t) + a_2(t)M_1^{\epsilon}\right]}{\left(a_2(t)m_1^{\epsilon} + a_1(t) + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right. \\
&\quad \left. + e(t) + \frac{f(t)\left[a_3(t)m_1^{\epsilon} + a_4(t)m_1^{\epsilon 2}\right]}{\left(a_1(t) + a_2(t)M_1^{\epsilon} + a_3(t)M_2^{\epsilon} + a_4(t)M_1^{\epsilon}M_2^{\epsilon}\right)^2}\right]|\bar{y}_1(t) - \bar{y}_2(t)|.
\end{aligned}$$

Equation (15) implies the existence of a positive constant ρ defined as follows:

$$\begin{aligned}
\rho = \min\Bigg\{\inf_{t\in\mathbb{R}}\Bigg\{&-\frac{c(t)\left[a_2(t)M_2^{\epsilon} + a_4(t)M_2^{\epsilon 2}\right]}{\left(a_2(t)m_1^{\epsilon} + a_1(t) + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2} + b(t) \\
&- \frac{f(t)\left[a_1(t) + a_3(t)M_2^{\epsilon}\right]}{\left(a_2(t)m_1^{\epsilon} + a_1(t) + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\Bigg\}
\end{aligned}$$

and

$$\inf_{t\in\mathbb{R}}\left\{-\frac{\left[a_2(t)M_1^{\epsilon}+a_1(t)\right]c(t)}{\left(a_1(t)+a_2(t)m_1^{\epsilon}+a_3(t)m_2^{\epsilon}+a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right.$$
$$\left.+e(t)+\frac{\left[a_3(t)m_1^{\epsilon}+a_4(t)m_1^{\epsilon 2}\right]f(t)}{\left(a_1(t)+a_2(t)M_1^{\epsilon}+a_4(t)M_1^{\epsilon}M_2^{\epsilon}+a_3(t)M_2^{\epsilon}\right)^2}\right\}\Bigg\} > 0.$$

Hence, we obtain that

$$D^+S(t) \le -\rho[|y(t)-y_1(t)|+|x(t)-x_1(t)|]. \tag{16}$$

Integrating (16), from t_0+T to t, we find

$$\rho\int_{t_0+T}^{t}[|x(s)-x_1(s)|+|y(s)-y_1(s)|]ds+S(t)<S(t_0+T)<+\infty,$$

which gives

$$\limsup_{t\to\infty}\int_{t_0+T}^{t}[|y(s)-y_1(s)|+|x(s)-x_1(s)|]ds<\frac{S(T+t_0)}{\rho}<+\infty.$$

Hence $|y(t)-y_1(t)|, |x(t)-x_1(t)| \in L^1([T+t_0,+\infty))$. Boundedness of $y(t), x(t), x_1(t), y_1(t)$ imply the boundedness of their derivatives for $t \ge t_0+T$ (see, model system (1)). Hence one can easily observe that $|y(t)-y_1(t)|$ and $|x(t)-x_1(t)|$ are uniformly continuous on $[T+t_0,+\infty)$. Thus using the Lemma A2, one can obtain

$$\lim_{t\to\infty}|y(t)-y_1(t)|=0, \quad \lim_{t\to\infty}|x(t)-x_1(t)|=0.$$

Therefore $Y(t)=(x_1(t), y_1(t))$ is globally attractive solution of the model system (1). □

Remark 9. One can also show that above property is satisfied by any two positive solutions (with positive initial conditions) that is we can establish the global asymptotic stability of the model system (1). For $e(t)=0$ and $a_4(t)=0$, the model system (1) becomes the nonautonomous Beddington–DeAngelis type prey–predator model discussed by Fan and Kuang (2004). In this case, the above discussion remain valid.

3 | PERIODIC CASES

Apart from general nonautonomous models, here, the parameters in the system (1) are taken as periodic functions as relevant environmental factors fluctuate periodically in time (Abbas et al., 2010; Cushing, 1977; Rinaldi et al., 1993). The periodicity of parameters may incorporate the periodicity of the environment. Periodicity of parameters is also reasonable assumption in the aspect of seasonal factors, for example, harvesting, hunting, availability of food.

There are several more mechanisms that causes periodic environment, for example, phytoplankton-zooplankton populations with primary class fish feeding on zooplankton throughout the summer and tree–insect pest systems regulated by migratory insectivores; variations of the habitat facilitate the escape/capture of the prey in some particular seasons; the relaxing time of the predator varies throughout the year, as populations characterized by some degree of diapause; periodic existence of a super predator abusing the predator population causes to the periodic variations of predator death rate; the caloric content of the prey fluctuates throughout the year, such as, in some plant-herbivore communities, the availability of energy to the predator for reproduction fluctuates consistently, excess in the prey mortality rate due to competition at high densities, and so forth (Rinaldi et al., 1993). In this section, we will discuss the existence of a periodic solution (positive) of the resulting periodic nonautonomous model system followed by the global attractivity of the solution using Lemmas A3 and A4. Here we assume that

$$\begin{aligned} d(t+\omega) = d(t), \quad c(t+\omega) = c(t), \quad a(t+\omega) = a(t), \quad b(t+\omega) = b(t), \quad e(t+\omega) \\ = e(t), \quad f(t+\omega) = f(t), \quad a_4(t+\omega) = a_4(t), \quad a_3(t+\omega) = a_3(t), \quad a_2(t+\omega) \\ = a_2(t), \quad a_1(t+\omega) = a_1(t), \end{aligned} \tag{17}$$

that is all the parameters of model system (1) are ω-periodic in time (t). Let $\hat{\psi} = \frac{1}{\omega}\int_0^\omega \psi(t)dt$ denotes the mean value of the periodic continuous function $\psi(t)$ with period ω.

There are three natural phenomena to understand the evolution of dynamics of the autonomous version of model system (1) under the periodic (almost-periodic) perturbation when model system (1) exhibits a limit cycle. Let T be the period of limit cycle and ω be the period of the periodic (almost-periodic) perturbations. If $T = \omega$, then limit cycle may develop into a positive harmonic periodic solution with period ω. If $T \neq \omega$ and rationally dependent, then the limit cycle may develop into a positive harmonic or subharmonic periodic solution with the period of the least common multiple (LCM) of T and ω. If $T \neq \omega$ and rationally independent, then the limit cycle may develop into an almost periodic solution. The periodic nature of solution can also be observed in planar piecewise linear systems of node saddle type Wang et al. (2019), in delayed Cai, Zuowei, Jianhua Huang, and Lihong Huang, Periodic orbit analysis for the delayed Filippov system Cai et al. (2018).

Theorem 8. *If the condition* (4) *of the Theorem* 2 *holds, then the model system* (1) *has at least one positive periodic solution* (*with period* ω), *say*, (x_1, y_1), *which lies in* κ_ϵ.

Proof. The Theorem 8 can easily be proved by using Brouwer fixed point theorem (refer the Lemma A3). □

Now in the next theorem, we use an alternative approach (continuation theorem) to prove the existence of a positive periodic solution.

Theorem 9. *If the following conditions holds:*

$$\hat{a} > \widehat{\left(\frac{c}{a_3}\right)}, \quad \exp\{-2\hat{a}\omega\}(\hat{f} - \hat{d}a_{2_M})\left[\left(\hat{a} - \widehat{\left(\frac{c}{a_3}\right)}\right)/\hat{b}\right] > \hat{d}a_{1_M}, \tag{18}$$

then there exists at least one ω-periodic positive solution for model system (1).

Proof. Let $y(t)=\exp\{v(t)\}, x(t)=\exp\{u(t)\}$, the model system (1) is rewritten as follows:

$$\begin{aligned}\frac{du(t)}{dt} &= -b(t)\exp\{u(t)\}+a(t)\\ &\quad-\frac{\exp\{v(t)\}c(t)}{a_2(t)\exp\{u(t)\}+a_1(t)+a_3(t)\exp\{v(t)\}+a_4(t)\exp\{u(t)\}\exp\{v(t)\}},\\ \frac{dv(t)}{dt} &= -e(t)\exp\{v(t)\}-d(t)\\ &\quad+\frac{\exp\{u(t)\}f(t)}{a_2(t)\exp\{u(t)\}+a_1(t)+a_3(t)\exp\{v(t)\}+a_4(t)\exp\{u(t)\}\exp\{v(t)\}}.\end{aligned}\tag{19}$$

Now we define the operators L, N, and projectors P and Q:

$$L:\ DomL\subset X\to X,\quad Lw=L\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}\dfrac{du}{dt}\\ \dfrac{dv}{dt}\end{bmatrix},$$

$$P,Q:\ X\to X,\quad Pw=Qw=\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\omega}\displaystyle\int_0^{\omega}u(t)dt\\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega}v(t)dt\end{bmatrix},$$

$$N:\ X\to X, Nw=N\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}N_1(t)\\ N_2(t)\end{bmatrix}$$

$$=\begin{bmatrix}-b(t)\exp\{u(t)\}+a(t)\\ -\dfrac{c(t)\exp\{v(t)\}}{a_2(t)\exp\{u(t)\}+a_1(t)+a_3(t)\exp\{v(t)\}+a_4(t)\exp\{v(t)\}\exp\{u(t)\}}\\ -d(t)-e(t)\exp\{v(t)\}\\ +\dfrac{f(t)\exp\{u(t)\}}{a_2(t)\exp\{u(t)\}+a_1(t)+a_3(t)\exp\{v(t)\}+a_4(t)\exp\{v(t)\}\exp\{u(t)\}}\end{bmatrix}.$$

Here, we have

$$KerL=\{(u,v)\in X\,|\,(u(t),v(t))=(p_1,p_2)\in\mathbb{R}^2\quad\text{for } t\in\mathbb{R}\},$$
$$ImL=\left\{(u,v)\in X\,|\int_0^{\omega}v(t)dt=0,\int_0^{\omega}u(t)dt=0\right\},$$

and codim Im $L=$ dim Ker $L=2$. L is a Fredholm mapping of index zero as Im L is closed in X. We observe that P is continuous projection such that Ker $L=$ Im P, Im $L=Im(I-Q)=$ Ker P. Moreover, the generalized inverse (to L), $K_P:$ Im $L\to$ Dom $L\cap KerP$ is given by

$$
\begin{aligned}
K_P(w) &= \int_0^t w(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t w(s)dsdt = K_P\begin{bmatrix} u \\ v \end{bmatrix} \\
&= \begin{bmatrix} \int_0^t u(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t u(s)dsdt \\ \int_0^t v(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t v(s)dsdt \end{bmatrix}.
\end{aligned}
$$

Here, $K_P(I - P)N$ and QN are continuous. For any open bounded set $\Omega \in X$, one can show that $K_P(I - P)N(\bar{\Omega})$ is compact. Also $PN(\bar{\Omega})$ is bounded. So N is L-compact on $\bar{\Omega}$.

For $Lx = \lambda Nx$, for each $\lambda \in (0, 1)$, we have

$$
\begin{aligned}
\frac{du(t)}{dt} &= \lambda\Bigg(-b(t)\exp\{u(t)\} + a(t) \\
&\quad - \frac{\exp\{v(t)\}c(t)}{a_2(t)\exp\{u(t)\} + a_1(t) + a_3(t)\exp\{v(t)\} + a_4(t)\exp\{v(t)\}\exp\{u(t)\}}\Bigg), \\
\frac{dv(t)}{dt} &= \lambda\Bigg(-d(t) - e(t)\exp\{v(t)\} \\
&\quad + \frac{\exp\{u(t)\}f(t)}{a_2(t)\exp\{u(t)\} + a_1(t) + a_3(t)\exp\{v(t)\} + a_4(t)\exp\{v(t)\}\exp\{u(t)\}}\Bigg).
\end{aligned} \tag{20}
$$

If $(u(t), v(t)) \in X$ be an arbitrary solution of the model system (20) for some $\lambda \in (0, 1)$, we obtain

$$
\begin{aligned}
\hat{a}\omega &= \int_0^\omega \Bigg(b(t)\exp\{u(t)\} \\
&\quad + \frac{\exp\{v(t)\}c(t)}{a_2(t)\exp\{u(t)\} + a_1(t) + a_3(t)\exp\{v(t)\} + a_4(t)\exp\{v(t)\}\exp\{u(t)\}}\Bigg)dt, \\
\hat{d}\omega &= -\int_0^\omega e(t)\exp\{v(t)\}dt \\
&\quad + \int_0^\omega \frac{\exp\{u(t)\}f(t)}{a_2(t)\exp\{u(t)\} + a_1(t) + a_3(t)\exp\{v(t)\} + a_4(t)\exp\{v(t)\}\exp\{u(t)\}}dt.
\end{aligned} \tag{21}
$$

From (20) and (21), we have

$$
\begin{aligned}
\int_0^\omega \left|\frac{du}{dt}\right| dt &\le \lambda\int_0^\omega a(t)dt + \lambda\int_0^\omega \Bigg(b(t)\exp\{u(t)\} \\
&\quad + \frac{\exp\{v(t)\}c(t)}{a_2(t)\exp\{u(t)\} + a_1(t) + a_3(t)\exp\{v(t)\} + a_4(t)\exp\{u(t)\}\exp\{v(t)\}}\Bigg)dt \\
&< 2\hat{a}\omega, \quad \int_0^\omega \left|\frac{dv}{dt}\right| dt \le \lambda\int_0^\omega d(t)dt - \int_0^\omega e(t)\exp\{v(t)\}dt \\
&\quad + \lambda\int_0^\omega \frac{\exp\{u(t)\}f(t)}{a_2(t)\exp\{u(t)\} + a_1(t) + a_3(t)\exp\{v(t)\} + a_4(t)\exp\{u(t)\}\exp\{v(t)\}}dt \\
&< 2\hat{d}\omega.
\end{aligned} \tag{22}
$$

As $(u(t), v(t)) \in X$, there exists $\mu_i, \nu_i \in [0, \omega]$, $(i = 1,2)$ such that

$$u(\nu_1) = \max_{t\in[0,\omega]} u(t), \quad u(\mu_1) = \min_{t\in[0,\omega]} u(t), \quad v(\nu_2) = \max_{t\in[0,\omega]} v(t), \quad v(\mu_2) = \min_{t\in[0,\omega]} v(t). \tag{23}$$

It follows from (21) and (23) that

$$\int_0^\omega b(t)\exp\{u(\mu_1)\}dt = \hat{b}\omega \exp\{u(\mu_1)\} \le \hat{a}\omega,$$

which gives $u(\mu_1) \le \ln\left(\frac{\hat{a}}{\hat{b}}\right)$, and hence we obtain

$$u(t) \le u(\mu_1) + \int_0^\omega \left|\frac{du}{dt}\right| dt \le \ln\left(\frac{\hat{a}}{\hat{b}}\right) + 2\hat{a}\omega := \Theta_1. \tag{24}$$

Moreover, from the first equation of (21) and (23), one can obtain that

$$\hat{a}\omega \le \int_0^\omega \left(b(t)\exp\{u(\nu_1)\} + \frac{c(t)}{a_3(t)}\right)dt = \widehat{\left(\frac{c}{a_3}\right)}\omega + \hat{b}\omega \ \exp\{u(\nu_1)\}.$$

Thus, we find $u(\nu_1) \ge \ln\left(\hat{a} - \widehat{\left(\frac{c}{a_3}\right)}\right)/\hat{b}$. Therefore, we have

$$u(t) \ge u(\nu_1) - \int_0^\omega \left|\frac{du}{dt}\right| dt \ge \ln\left\{\hat{a} - \widehat{\left(\frac{c}{a_3}\right)}\right\}/\hat{b} - 2\hat{a}\omega := \Theta_2. \tag{25}$$

Thus, (25) together with (24) implies that $\max_{t\in[0,\omega]}|u(t)| \le \max\{|\Theta_1|, |\Theta_2|\} := D_1$. Moreover, from the second Equation of (21) and (23), we get

$$\hat{d}\omega \le \int_0^\omega \frac{f(t)\exp\{\Theta_1\}}{a_{1_L} + a_{2_L}\exp\{\Theta_1\}} - \int_0^\omega e(t)\exp\{v(\mu_2)\}dt,$$
$$\exp\{v(\mu_2)\} \le \frac{\hat{f}\ \exp\{\Theta_1\} - \hat{d}(a_{1_L} + a_{2_L}\exp\{\Theta_1\})}{\hat{e}(a_{1_L} + a_{2_L}\exp\{\Theta_1\})}.$$

Hence,

$$\begin{aligned} v(t) &\le v(\mu_2) + \int_0^\omega \left|\frac{dv}{dt}\right| dt \\ &\le \ln\left\{\frac{\hat{f}\ \exp\{\Theta_1\} - \hat{d}(a_{1_L} + a_{2_L}\exp\{\Theta_1\})}{\hat{e}(a_{1_L} + a_{2_L}\exp\{\Theta_1\})}\right\} + 2\hat{d}\omega := \Theta_3. \end{aligned} \tag{26}$$

Again, from the second Equation of (21) and (23), we find

$$\hat{d}\omega \ge \int_0^\omega \frac{f(t)\exp\{\Theta_2\}}{a_{1_M} + a_{2_M}\exp\{\Theta_2\} + a_{4_M}\exp\{\Theta_2\}\exp\{v(\nu_2)\}} - \int_0^\omega e(t)\exp\{v(\nu_2)\}dt,$$

and it leads to

$$\hat{e}a_{4_M}\exp\Theta_2(\exp\{v(\mu_2)\})^2+\left[\hat{e}(a_{1_M}+a_{2_M}\exp\{\Theta_2\})+\hat{d}a_{4_M}\exp\{\Theta_2\}\right]\exp\{v(\nu_2)\}$$
$$+\hat{d}(a_{1_M}+a_{2_M}\exp\{\Theta_2\})-\hat{f}\exp\{\Theta_2\}\geq 0.$$

Here, we assume that $A=\hat{e}a_{4_M}\exp\{\Theta_2\}, B=\hat{e}(a_{1_M}+a_{2_M}\exp\{\Theta_2\})+\hat{d}a_{4_M}\exp\{\Theta_2\}$, $C=\hat{d}(a_{1_M}+a_{2_M}\exp\{\Theta_2\})-\hat{f}\exp\{\Theta_2\}$. As (18) implies that $C<0$, hence, we have

$$\exp\{v(\nu_2)\}\geq\frac{B+\sqrt{B^2-4AC}}{2A},$$

which in turn implies that

$$v(t)\geq v(\nu_2)-\int_0^\omega\left|\frac{dv}{dt}\right|\geq\ln\left\{\frac{B+\sqrt{B^2-4AC}}{2A}\right\}-2\hat{d}\omega:=\Theta_4. \tag{27}$$

Therefore, the Equations (26) and (27) together give that $\max_{t\in[0,\omega]}v(t)\leq\max\{|\Theta_3,\Theta_4|\}:=D_2$. Obviously, D_1 and D_2 are independent of λ. Define $D=D_1+D_2+D_3$, where $D_3>0$ is taken sufficiently large so that $D_3>|l_2|+|L_2|+|l_1|+|L_1|$.

For $\mu\in[0,1]$ and $(u,v)\in\mathbb{R}^2$, consider the following algebraic equations

$$\begin{aligned}
&-\hat{b}\exp\{u\}+\hat{a}-\frac{1}{\omega}\int_0^\omega\frac{\exp\{v\}\mu c(t)}{a_1(t)+a_3(t)\exp\{v\}+a_2(t)\exp\{u\}+a_4(t)\exp\{u\}\exp\{v\}}dt\\
&\quad=0,\\
&-\hat{e}\exp\{v\}-\hat{d}+\frac{1}{\omega}\int_0^\omega\frac{\exp\{u\}f(t)}{a_1(t)+a_3(t)\exp\{v\}+a_2(t)\exp\{u\}+a_4(t)\exp\{v\}\exp\{u\}}dt\\
&\quad=0.
\end{aligned}\tag{28}$$

One can easily show that any solution (u_1,v_1) of the above equations satisfies

$$l_1\leq u_1\leq L_1, l_2\leq v_1\leq L_2. \tag{29}$$

Define $\Omega=\{(u,v)^T\in X:D>\|(u,v)\|\}$. Then one can easily conclude that for each $(u,v)\in\partial\Omega\cap DomL,\lambda\in(0,1)$, each solution x of Equation $Lx=\lambda Nx$ is such that $x\notin\partial\Omega$, that is, Ω satisfies the condition (i) of the Lemma A4. Furthermore, $(u,v)\in\mathbb{R}^2$ with norm $\|(u,v)\|=|v|+|u|=D$. Now, from definition of D and Equation (29) we have $PNw=PN(u,v)^T\neq 0$, for if $PNw=PN(u,v)^T=0$, then $(u,v)^T$ is a constant solution of (28) with $\mu=1$. Hence, we have $\|(u,v)\|\leq D_1+D_2$, which is contradictory to $\|(u,v)^T\|=D$. Thus we have

$$PNw=PN\begin{bmatrix}u\\v\end{bmatrix}$$
$$=\begin{bmatrix}\hat{a}-\hat{b}\exp\{u\}\\ \quad-\dfrac{1}{\omega}\displaystyle\int_0^\omega\frac{c(t)\exp\{v\}}{a_1(t)+a_2(t)\exp\{u\}+a_3(t)\exp\{v\}+a_4(t)\exp\{u\}\exp\{v\}}\\ -\hat{d}+\dfrac{1}{\omega}\displaystyle\int_0^\omega\frac{f(t)\exp\{u\}}{a_1(t)+a_2(t)\exp\{u\}+a_3(t)\exp\{v\}+a_4(t)\exp\{u\}\exp\{v\}}\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}.$$

Thus the requirement of the condition (ii) of the Lemma A4 is accomplished.

Now we need to compute the Brouwer degree of the map PN. For this we define a homotopy and use its invariance property. Consider the homotopy

$$H_\mu w = H_\mu \begin{bmatrix} u \\ v \end{bmatrix} = \mu QN \begin{bmatrix} u \\ v \end{bmatrix} + (1-\mu) G \begin{bmatrix} u \\ v \end{bmatrix},$$

where $\mu \in [0, 1]$ and

$$G\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \hat{a} - \hat{b}\exp\{u\} \\ \hat{d} - \dfrac{1}{\omega}\displaystyle\int_0^\omega \dfrac{f(t)\exp\{u\}}{a_1(t) + a_2(t)\exp\{u\} + a_3(t)\exp\{v\} + a_4(t)\exp\{u\}\exp\{v\}} dt \end{bmatrix}. \tag{30}$$

From (29), we know that $H_\mu w = H_\mu(u, v) \neq (0, 0)^T$ on $\partial\Omega \cap KerL$. Note that $J = I$ as $ImP = KerL$. Hence due to invariance property of homotopy of topological degree (refer the Definition A8), we get

$$\begin{aligned} deg(PN, \Omega \cap Ker\, L, (0, 0)^T) &= deg(JPN(u, v)^T, \Omega \cap Ker\, L, (0, 0)^T) \\ &= deg(G, \Omega \cap Ker\, L, (0, 0)^T). \end{aligned}$$

Clearly, we have $G((u, v)^T) = 0$, that is,

$$-\hat{b}\exp\{u\} + \hat{a} = 0, \quad \hat{d} - \frac{1}{\omega}\int_0^\omega \frac{\exp\{u\} f(t)}{a_1(t) + a_3(t)\exp\{v\} + a_2(t)\exp\{u\} + a_4(t)\exp\{u\}\exp\{v\}} dt = 0,$$

has unique solution $\tilde{w} = (\tilde{u}, \tilde{v})^T \in \Omega \cap KerL$. $detM$ stands for determinant of a matrix M while $J_f(w)$ is Jacobian matrix of the function f at w. Then one can obtain that

$$deg(JPN(u, v)^T, \Omega \cap Ker\, L, (0, 0)^T) = sig(det\, J_G(\tilde{w})) \neq 0.$$

Thus we have verified all the requirements of Lemma A4 and hence the equation $Lx = Nx$, that is, Equation (19) has at least one ω-periodic solution in $DomL \cap \overline{\Omega}$, say, $(u_1(t), v_1(t))$. As $x_1(t) = \exp\{u_1(t)\}$, $y_1(t) = \exp\{v_1(t)\}$, hence $(x_1(t), y_1(t))$ is ω-periodic solution of model system (1). □

Remark 10. One can observe that Theorem 9 is weaker than Theorem 8 under certain parametric condition. Theorems 9 and 8 ensure for a periodic solution, while conditions (4) and (18) are different. Clearly if $d_M > 1$ and we replace m_1 by m_1^ϵ then from (4) one can obtain that $(f_L - d_M a_{2_M})\left(\left(a_L - \frac{c_M}{a_{3_L}}\right)/b_M - \epsilon\right) > d_M a_{1_M}$. Moreover, when $(f_L - d_M a_{2_M})$ $\left(\left(a_L - \frac{c_M}{a_{3_L}}\right)/b_M\right) > d_M a_L > (f_L - d_M a_{2_M})\left(\left(a_L - \frac{c_M}{a_{3_L}}\right)/b_M - \epsilon\right)$ then (4) implies (18). Thus if $d_M > 1$, $\epsilon \in \left((1 - \exp\{-2\hat{a}\omega\})\frac{a_L a_{3_L} - c_M}{b_M a_{3_L}}, \frac{a_L a_{3_L} - c_M}{b_M a_{3_L}}\right)$ and the condition (5) hold then (18) holds. This ensures the betterment of Theorem 9 over Theorem 8. In case of Beddington–DeAngelis type predator-prey model system, the condition $d_M > 1$ is relaxed as parameter a_4, the mutual interference in presence of high prey density, is not present.

3.1 | Dynamics of a boundary ω-periodic solution

If the conditions (17) hold, the solution $\tilde{x}_1(t)$ with $\tilde{x}_1(0) = \left(\exp\left\{\int_0^\omega a(s)ds\right\} - 1\right)\left(\int_0^\omega b(s)\exp\left\{\int_0^s a(\tau)d\tau\right\}ds\right)^{-1}$ is given by

$$\tilde{x}_1(t) = \left(\int_t^{t+\omega} b(s)\exp\left\{-\int_s^t a(\tau)d\tau\right\}ds\right)^{-1}\left(\exp\left\{\int_0^\omega a(s)ds\right\} - 1\right). \tag{31}$$

One can easily check that under the conditions (17), $\tilde{x}_1(t) = \tilde{x}_1(t+\omega)$, that is $\tilde{x}_1(t)$ is a unique ω-periodic solution. Hence under the conditions (17), the existence of boundary periodic solutions is also guaranteed.

Theorem 10. *The model system* (1) *has a ω-periodic solution* $(\tilde{x}_1(t), 0)$. *In addition, if*

$$d(t) - \frac{f(t)}{a_2(t)} - \frac{c(t)}{a_1(t)} > 0, \tag{32}$$

then $\lim_{t\to+\infty}|(x(t), y(t)) - (\tilde{x}_1(t), 0)| = 0$, *that is, all the solutions with* $x(t_0) > 0, y(t_0) > 0$ *are attracted by* $(\tilde{x}_1(t), 0)$.

Proof. It can be shown that $(\tilde{x}_1(t), 0)$ is a periodic solution of model system (1) with period ω. The global attractivity can be discussed with a suitable Lyapunov functional. □

Theorem 11. *If the following condition holds*

$$\int_0^\omega \left(\frac{\tilde{M}_1 f(t)}{a_1(t) + a_2(t)\tilde{M}_1} - d(t)\right)dt < 0, \tag{33}$$

then the solution $(\tilde{x}_1(t), 0)$ *is globally asymptotically stable for model system* (1).

Remark 11. Theorem 11 gives the following implications:

1. If all the parameters are constants (positive), then the boundary solution (periodic) $(\tilde{x}_1(t), 0)$ degenerate the axial equilibrium $\left(\frac{a}{b}, 0\right)$ of associated autonomous model system. Theorem 11 shows that $\left(\frac{a}{b}, 0\right)$ is globally asymptotically stable for corresponding autonomous model system if $d > \frac{fM}{a_1 + a_2 M}$.
2. Similar to condition (11), condition (33) has more reasonable biological interpretation than condition those expressed by infimum and supremum of parameter functions or by considering positive constants.

3. The condition (33) implies that the model system (1) is nonpermanent. But its corresponding autonomous model system with parameters being replaced by their averages in the periodic interval $[0, \omega]$. The corresponding autonomous model system may be permanent (refer to Example 7).

In the following table, we present some of the comparative results obtained by using continuation theorem (in coincidence degree theory) and Brouwer fixed point theorem.

4 | ALMOST-PERIODIC CASE

The idea of almost periodic functions was presented by Bohr in his wonderful paper published in Acta Mathematica (Bohr, 1947; Chen & Cao, 2008). Upon considering long-term dynamical behaviors, the periodic parameters often turn out to experience certain interruptions that may cause small perturbations, that is, parameters become periodic up to a small error. Thus, almost periodic oscillatory behavior is considered to be more accordant with reality. The predator–prey interactions in the real world are affected by many factors and undergo all kinds of perturbation, among which some are almost periodic for seasonal reasons. The model system with almost periodic coefficients is considered when the numerous components of environment are periodic but not necessarily with commensurate periods (e.g., mating habits, seasonal effects of weather, food supplies, and harvesting) Lin and Chen (2009), that is, when the periods of the components of environment are rationally independent. Thus, the assumption of almost periodicity makes the model system more realistic. For detailed study of almost periodic functions, its properties and certain applications, interested readers may refer to Huang et al. (2019), Qian and Yuhui (2020), Yoshizawa (2012), Zhang et al. (2020).

Here, we prove the existence of almost periodic solution of system (1), which generalizes the concept of periodicity. Here, we assume that $c(t)$, $d(t)$, $a(t)$, $b(t)$, $f(t)$, $e(t)$, $a_1(t)$, $a_2(t)$, $a_4(t)$, and $a_3(t)$ are almost periodic in t. Let

$$y(t) = \exp(\bar{y}(t)), \quad x(t) = \exp(\bar{x}(t)).$$

The system (1) becomes:

$$\begin{aligned}
\frac{d\bar{x}(t)}{dt} &= -b(t)\exp(\bar{x}(t)) + a(t) \\
&\quad - \frac{\exp(\bar{y}(t))c(t)}{a_2(t)\exp(\bar{x}(t)) + a_1(t) + a_3(t)\exp(\bar{y}(t)) + a_4(t)\exp(\bar{x}(t))\exp(\bar{y}(t))}, \\
\frac{d\bar{y}(t)}{dt} &= -e(t)\exp(\bar{y}(t)) - d(t) \\
&\quad + \frac{f(t)\exp(\bar{x}(t))}{a_2(t)\exp(\bar{x}(t)) + a_1(t) + a_3(t)\exp(\bar{y}(t)) + a_4(t)\exp(\bar{x}(t))\exp(\bar{y}(t))}.
\end{aligned} \tag{34}$$

From Theorem 1, one can easily prove.

Theorem 12. *If*

$$\begin{aligned}
&\inf_{t\in\mathbb{R}}\left[-d(t)+\frac{\tilde{M}_1 f(t)}{a_1(t)+a_2(t)\tilde{M}_1}\right]>0,\\
&\inf_{t\in\mathbb{R}}\left[-\frac{\tilde{M}_2 c(t)}{a_1(t)+a_3(t)\tilde{M}_2}+a(t)\right]>0,\\
&\inf_{t\in\mathbb{R}}\left[-d(t)+\frac{\tilde{m}_1 f(t)}{a_2(t)\tilde{m}_1+a_1(t)+a_3(t)\tilde{M}_2+a_4(t)\tilde{m}_1\tilde{M}_2}\right]>0,
\end{aligned} \tag{35}$$

then the following set:

$$K_\epsilon^* := \left\{(x,y)\in\mathbb{R}^2 \,|\, \ln\left(\tilde{m}_2^\epsilon\right)\le y\le \ln\left(\tilde{M}_2^\epsilon\right), \ln\left(\tilde{m}_1^\epsilon\right)\le x\le \ln(\tilde{M}_1^\epsilon)\right\}$$

is positively invariant for system (34), where $\tilde{m}_2, \tilde{m}_1, \tilde{M}_1, \tilde{M}_2$ are given in Section 2.

Now, consider the following ordinary differential equation

$$x' = f(t,x), \quad f(t,x)\in C(R\times D, R^n). \tag{36}$$

Here $f(t,x)$ is almost periodic in t, uniformly with respect to $x\in D$ and D is an open set in R^n.

To prove the existence of an almost-periodic solution for system (36), the following product system for (36) is considered:

$$y' = f(t,y), \quad x' = f(t,x). \tag{37}$$

Lemma 3 (Theorem 19.1 in Yoshizawa, 2012). *Consider a Lyapunov function $V(t,x,y)$ defined on $[0,+\infty)\times D\times D$ such that:*

1. $\beta(\|x-y\|)\ge V(t,x,y)\ge\alpha(\|x-y\|)$, *where $\beta(\gamma)$ and $\alpha(\gamma)$ are increasing, continuous and positive definite.*
2. $K(\|x_1-x_2\|+\|y_1-y_2\|)\ge|V(t,x_1,y_1)-V(t,x_2,y_2)|$, *where K is a positive constant.*
3. $-\mu V(|x-y|)\ge V'(t,x,y)$, *where μ is also a positive constant.*

Furthermore, let $S\subset D$ be a compact set and let the system (36) has a solution that remains in S for all $t\ge t_0\ge 0$. Then there exists a uniformly asymptotically stable unique almost-periodic solution in S for the system (36).

Theorem 13. *The model system (1) has a unique almost periodic solution provided conditions of Theorem 7 hold.*

Proof. Refer to appendix. □

5 | NUMERICAL SIMULATIONS

To demonstrate analytical findings graphically, we numerically simulate solutions of system (1). Numerical simulations also show that the periodic system may suppress the permanence of its corresponding autonomous system with parameters being the averages of the corresponding periodic parameters. For this, we consider the following examples:

Example 1 (Theorems 2 and 3). Consider $b(t) = 2 + \cos t, a(t) = 3.2, c(t) = 1.5, d(t) = \frac{1}{20} + \frac{1}{30}\cos t, e(t) = 3, f(t) = 1, a_1(t) = \frac{1}{5} + \frac{1}{10}\sin t, a_2(t) = 3 + \frac{1}{5}\sin t, a_3(t) = 2 + \cos t, a_4(t) = \frac{1}{20} + \frac{1}{30}\sin t$, then the system (1) becomes:

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\Bigg(-(2+\cos t)x(t) + 3.2\\ &\quad - \frac{1.5y(t)}{\left(3+\frac{1}{5}\sin t\right)x(t) + \frac{1}{5} + \frac{1}{10}\sin t + (2+\cos t)y(t) + \left(\frac{1}{20}+\frac{1}{30}\sin t\right)x(t)y(t)}\Bigg),\\ \frac{dy(t)}{dt} &= y(t)\Bigg(-\left(\frac{1}{20}+\frac{1}{30}\cos t\right) - 3y(t)\\ &\quad + \frac{x(t)}{\left(3+\frac{1}{5}\sin t\right)x(t) + \frac{1}{5} + \frac{1}{10}\sin t + (2+\cos t)y(t) + \left(\frac{1}{20}+\frac{1}{30}\sin t\right)x(t)y(t)}\Bigg).\end{aligned} \tag{38}$$

We compute lower and upper bounds of all the time dependent parameters. We have $a_L = a_M = 3.2, b_L = 1, b_M = 3, c_L = c_M = 1.5, d_L = \frac{1}{60}, d_M = \frac{1}{12}, e_L = e_M = 3, f_L = f_M = 1, a_{1_L} = \frac{1}{10}, a_{1_M} = \frac{3}{10}, a_{2_L} = \frac{14}{5}, a_{2_M} = \frac{16}{5}, a_{3_L} = a_{3_M} = 3, a_{4_L} = \frac{1}{60}, a_{4_M} = \frac{1}{12}, M_1 = 3.2, m_1 = 0.94, M_2 = 0.1122, m_2 = 0.059$. Furthermore, we have

$$\begin{aligned} d_L = 0.016 &< 0.34 = \frac{f_M M_1}{a_{1_L} + a_{2_L} M_1},\\ a_L = 3.2 &> 2.82 = \frac{c_M M_2}{a_{1_L} + a_{3_L} M_2},\\ d_M = 0.083 &< 0.26 = \frac{f_L m_1}{a_{1_M} + a_{2_M} m_1 + a_{3_M} M_2 + a_{4_M} m_1 M_2}.\end{aligned}$$

The parametric values considered in Example 1 satisfy condition (5). We also have $\tilde{M}_1 = 3.2, \tilde{M}_2 = 0.106, \tilde{m}_1 = 0.96, \tilde{m}_2 = 0.064$, and conditions (8) satisfied as follows:

$$\begin{aligned} &\inf_{t\in\mathbb{R}}\left[\frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1} - d(t)\right] = 0.23 > 0,\\ &\inf_{t\in\mathbb{R}}\left[-\frac{c(t)\tilde{M}_2}{a_1(t) + a_3(t)\tilde{M}_2} + a(t)\right] = 2.60 > 0,\\ &\inf_{t\in\mathbb{R}}\left[-d(t) + \frac{f(t)\tilde{m}_1}{a_1(t) + a_3(t)\tilde{M}_2 + a_2(t)\tilde{m}_1 + a_4(t)\tilde{m}_1\tilde{M}_2}\right] = 0.19 > 0.\end{aligned}$$

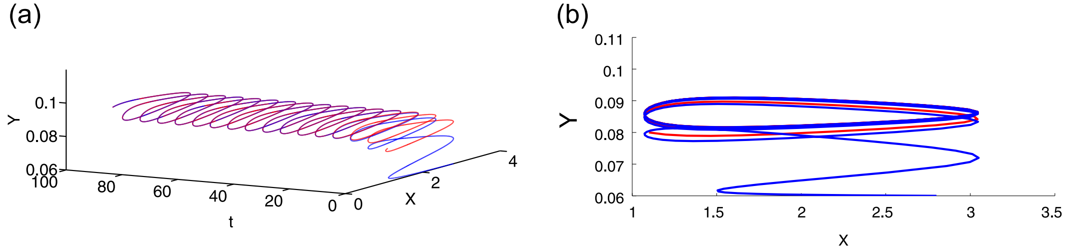

FIGURE 1 Phase diagram of the model system (38). Trajectories start with initial points (1.1, 0.08) and (2.8, 0.06). (a) The orbit of predator–prey-time. (b) The orbit of predator–prey

Moreover, the permanence of model system (38) is ensured by Theorems 2 and 3. Figures 1 and 2 also support the permanence of system (38). The integral curves are shown in Figure 2 and phase diagram has been shown in Figure 1.

Example 2 (Theorem 7). Let $b(t) = 25 + 0.1\cos t, a(t) = 100, c(t) = 0.5, d(t) = 1 + 0.5\sin t, f(t) = 12, e(t) = 1, a_1(t) = 0.14 + 0.05\sin t, a_2(t) = 1, a_4(t) = 0.3 + 0.2\cos t,$ $a_3(t) = 0.83 + 0.01\sin t$ then the model system (1) becomes:

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\Bigg(-(25 + 0.1\cos t)x(t) + 100\\ &\quad - \frac{0.5y(t)}{(0.83 + 0.01\sin t)y(t) + 0.1 + 0.05\sin t + x(t) + (0.3 + 0.2\cos t)x(t)y(t)}\Bigg),\\ \frac{dy(t)}{dt} &= y(t)\Bigg(-(1 + 0.5\sin t) - y(t)\\ &\quad + \frac{12x(t)}{(0.83 + 0.01\sin t)y(t) + 0.1 + 0.05\sin t + x(t) + (0.3 + 0.2\cos t)x(t)y(t)}\Bigg).\end{aligned} \tag{39}$$

By simple numerical computations one can obtain that $a_L = a_M = 100$, $b_L = 24.90$, $b_M = 25.10$, $c_L = c_M = 0.5$, $d_L = 0.5$, $d_M = 1.50$, $e_L = e_M = 1$, $f_L = f_M = 12$, $a_{1_L} = 0.09$, $a_{1_M} = 0.19$, $a_{2_L} = 1 = a_{2_M}$, $a_{3_L} = 0.07$, $a_{3_M} = 0.09$, $a_{4_L} = 0.1$, $a_{4_M} = 0.5$, $M_1^\epsilon = 4.01$, $M_2^\epsilon = 11.24$, $m_1^\epsilon = 3.73$, $m_2^\epsilon = 0.73$. And furthermore

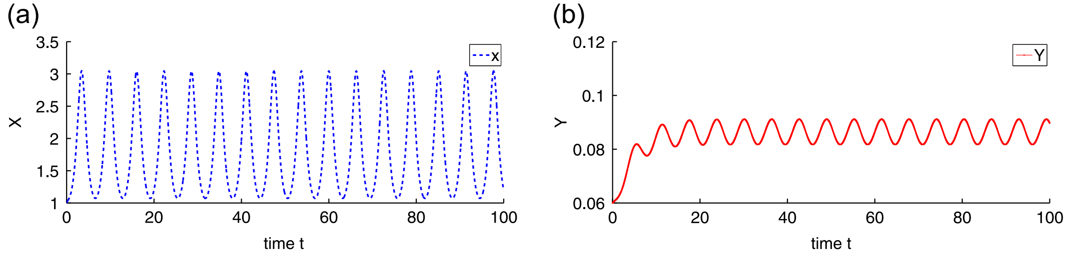

FIGURE 2 Time series of system (38). (a) The integral curve of prey. (b) The integral curve of predator

$$\inf_{t\in\mathbb{R}}\left[\frac{\tilde{M}_1 f(t)}{a_1(t)+a_2(t)\tilde{M}_1}-d(t)\right]=9.958>0,$$
$$\inf_{t\in\mathbb{R}}\left[-\frac{\tilde{M}_2 c(t)}{a_1(t)+a_3(t)\tilde{M}_2}+a(t)\right]=99.449>0,$$
$$\inf_{t\in\mathbb{R}}\left[\frac{\tilde{m}_1 f(t)}{a_1(t)+a_2(t)\tilde{m}_1+a_3(t)\tilde{M}_2+a_4(t)\tilde{m}_1\tilde{M}_2}-d(t)\right]=0.0736>0,$$

and

$$\inf_{t\in\mathbb{R}}\left\{-\frac{c(t)\left[a_4(t)M_2^{\epsilon 2}+a_2(t)M_2^{\epsilon}\right]}{\left(a_2(t)m_1^{\epsilon}+a_1(t)+a_3(t)m_2^{\epsilon}+a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}+b(t)\right.$$
$$\left.-\frac{f(t)\left[a_1(t)+a_3(t)M_2^{\epsilon}\right]}{\left(a_2(t)m_1^{\epsilon}+a_1(t)+a_3(t)m_2^{\epsilon}+a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right\}=16.970>0,$$
$$\inf_{t\in\mathbb{R}}\left\{-\frac{\left[a_1(t)+a_2(t)M_1^{\epsilon}\right]c(t)}{\left(a_1(t)+a_2(t)m_1^{\epsilon}+a_3(t)m_2^{\epsilon}+a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right.$$
$$\left.+e(t)+\frac{\left[a_3(t)m_1^{\epsilon}+a_4(t)m_1^{\epsilon 2}\right]f(t)}{\left(a_2(t)M_1^{\epsilon}+a_1(t)+a_3(t)M_2^{\epsilon}+a_4(t)M_1^{\epsilon}M_2^{\epsilon}\right)^2}\right\}=0.9909>0.$$

Thus the values of parameters considered in Example 2 satisfy conditions (8) and (15). Therefore, Theorem 7 ensures the global asymptotic stability (global attractivity) of a bounded positive solution of system (39). One can also refer Figure 3.

Example 3. Let $b(t)=2+\cos t$, $a(t)=0.5$, $c(t)=0.3$, $d(t)=0.01+0.03\cos t$, $e(t)=2$, $a_1(t)=0.1+0.01\sin t$, $f(t)=3.5+0.1\sin t$, $a_2(t)=4+0.1\sin t$, $a_4(t)=0.1+0.2\cos t$, $a_3(t)=2+\cos t$ and $\omega=2\pi$. The system (1) becomes:

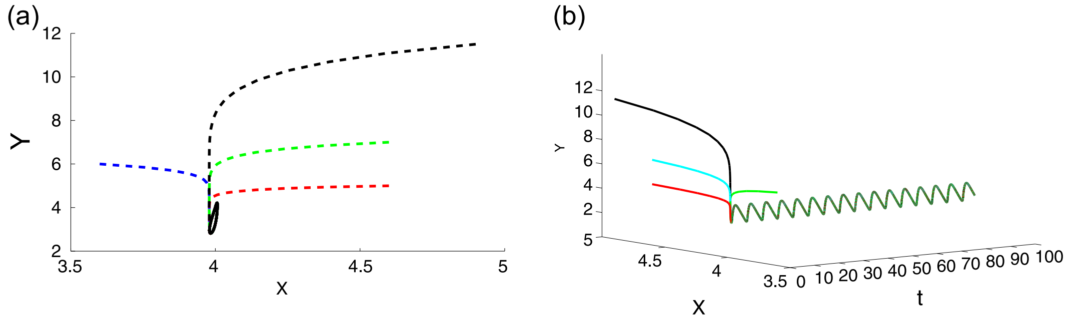

FIGURE 3 Phase portrait of the model system (39). Trajectories starting with the initial conditions (3.6, 6), (4.6, 5), (4.6, 7), and (4.9, 11.5). Different colored trajectories start from different initial conditions. (a) The orbit of predator–prey. (b) The orbit of predator–prey-time

$$\begin{aligned}
\frac{dx(t)}{dt} &= x(t)\Bigg(-(2+\cos t)x(t)+0.5\\
&\quad-\frac{0.3y(t)}{0.1+0.01\sin t+(2+\cos t)y(t)+(4+0.1\sin t)x(t)+(0.1+0.2\cos t)x(t)y(t)}\Bigg),\\
\frac{dy(t)}{dt} &= y(t)\Bigg(-2y(t)-(0.01+0.03\cos t)\\
&\quad+\frac{(0.1\sin t+3.5)x(t)}{0.1+0.01\sin t+(2+\cos t)y(t)+(4+0.1\sin t)x(t)+(0.1+0.2\cos t)x(t)y(t)}\Bigg).
\end{aligned}\tag{40}$$

One can compute that

$$\hat{a}=0.5, \hat{c}=0.3, \hat{b}=2, \hat{d}=0.01, \hat{f}=3.5, \hat{e}=2, \hat{a_1}=0.1, \hat{a_2}=4, \hat{a_3}=2, \hat{a_4}=0.1, a_{1_M}=0.11, a_{2_M}=4.1.$$

And furthermore

$$\hat{a}=0.5>\widehat{\left(\frac{c}{a_3}\right)}=0.15, \quad \exp\{-2\hat{a}\omega\}(\hat{f}-\hat{d}a_{2_M})\left[\left(\hat{a}-\widehat{\left(\frac{c}{a_3}\right)}\right)/\hat{b}\right]=0.0016>\hat{d}a_{1_M}=0.0011.$$

Hence the parametric values in the Example 3 satisfy condition (18). The model system (40) has at least one 2π-periodic solution (positive). Its phase diagram has been shown in Figure 8.

Finally, we consider the following example:

Example 4. Let $b(t)=2+0.1\cos t, a(t)=2, c(t)=1+0.1\cos t, e(t)=3+0.1\cos t, d(t)=1.5, f(t)=0.3, a_1(t)=1=a_2(t), a_3(t)=2.1+0.2\cos t, a_4(t)=0.1+0.1\cos t,$ and $\omega=2\pi$ then the system (1) becomes:

$$\begin{aligned}
\frac{dx(t)}{dt} &= x(t)\Bigg(-(2+0.1\cos t)x(t)+2\\
&\quad-\frac{y(t)(1+0.1\cos t)}{1+(2.1+0.2\cos t)y(t)+x(t)+(0.1+0.1\cos t)x(t)y(t)}\Bigg),\\
\frac{dy(t)}{dt} &= y(t)\Bigg(-(3+0.1\cos t)y(t)-1.2\\
&\quad+\frac{0.3x(t)}{1+(2.1+0.2\cos t)y(t)+x(t)+(0.1+0.1\cos t)x(t)y(t)}\Bigg).
\end{aligned}\tag{41}$$

By easy calculations, we have

$$\begin{aligned}
&d_M=d_L=1.5, c_L=0.99, c_M=1.1, f_M=0.3, a_{1_L}=a_{1_M}=1, a_{2_L}=a_{2_M}=1,\\
&d(t)-\frac{f(t)}{a_2(t)}-\frac{c(t)}{a_1(t)}>d_L-\frac{c_M}{a_{1_L}}-\frac{f_M}{a_{2_L}}=1.5-1.1-0.3=0.10.
\end{aligned}$$

Thus the condition (32) is well satisfied. Hence $\lim_{t\to+\infty}|(x(t), y(t)) - (x_1(t), 0)| = 0$. Figure 9 ensures the global asymptotic stability of (1.046, 0) of model system (41).

Moreover, for $a = 3.2, b = 2 + \cos t, c = 1.5, d = \frac{1}{20} + \frac{1}{30}\cos t, e = 3, f = 1, a_1 = \frac{1}{5} + \frac{1}{10}\sin t, a_2 = 3 + \frac{1}{5}\sin t, a_3 = 1.4 + \cos t, a_4 = \frac{1}{90} + \frac{1}{100}\sin t$, one can compute that $a_L = a_M = 3.2, b_L = 1, b_M = 3, c_L = c_M = 1.5, d_L = 0.02, d_M = 0.08, e_L = e_M = 3, f_L = f_M = 1, a_{1_L} = 0.14$, $a_{1_M} = 0.34, a_{2_L} = 2.76, a_{2_M} = 3.24, a_{3_L} = 0.4, a_{3_M} = 2.4, a_{4_L} = 0.001, a_{4_M} = 0.011$. Here $a_L a_{3_L} = 3.2 \times 0.4 = 1.28 < c_L = 1.5$. This confirms that the above values of parameters fail to satisfy permanence conditions (Theorem 2). However numerical evaluation of the system (1) for the above set of parametric values, leads to periodic coexistence scenario as presented in Figure 10. This result establishes the fact that the conditions for permanence of the system (1) (refer Section 2) are sufficient but not necessary.

Example 5 (Theorems 4 and 6). Let $b(t) = 2 + 0.1\cos t, a(t) = 2, c(t) = 0.2 + 0.1\sin t$, $d(t) = 1, e(t) = 3 + 0.1\cos t, f(t) = 0.2, a_2(t) = 1, a_1(t) = 0.2 + 0.1\sin t, a_4(t) = 1$, $a_3(t) = 5 + 0.1\sin t$, then system (1) becomes:

$$\begin{aligned}
\frac{dx(t)}{dt} &= x(t)\Bigg(-(2 + 0.1\cos t)x(t) + 2 \\
&\quad - \frac{y(t)(0.2 + 0.1\sin t)}{0.2 + 0.1\sin t + x(t) + x(t)y(t) + (5 + 0.1\sin t)y(t)}\Bigg), \quad \frac{dy(t)}{dt} \\
&= y(t)\Bigg(-(3 + 0.1\cos t)y(t) - 1 \\
&\quad + \frac{0.2x(t)}{0.1\sin t + 0.2 + x(t) + (5 + 0.1\sin t)y(t) + x(t)y(t)}\Bigg).
\end{aligned} \tag{42}$$

By simple numerical computations, one can obtain that $a_L = a_M = 2, b_L = 1.9, b_M = 2.1$, $c_L = 0.1, c_M = 0.3, d_L = d_M = 1, e_L = 2.9, e_M = 3.1, f_L = f_M = 0.2, a_{1_L} = 0.1, a_{1_M} = 0.3, a_{2_L} = a_{2_M} = 1, a_{3_L} = 4.9, a_{3_M} = 5.1, a_{4_L} = a_{4_M} = 1$. And furthermore

$$\frac{f_M \tilde{M}_1}{a_{1_L} + a_{2_L}\tilde{M}_1} = 0.18 < 1 = d_L.$$

Also $\int_0^{+\infty}\left(-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1}\right)dt = \int_0^{+\infty}\left(-1 + \frac{0.21}{1.25 + 0.1\sin t}\right)dt = -\infty$. Hence $\lim_{t\to+\infty} y(t) = 0$. Therefore, Theorems 4 and 6 ensure that predator y of model system (42) will extinct.

Example 6 (Theorems 5 and 6). Let $b(t) = 0.06, a(t) = 0.02 + \sin t, c(t) = 1.8 + 0.1\sin t, d(t) = 0.001, e(t) = 0.05 + 0.01\cos t, f(t) = 2.7 + 0.1\sin t, a_1(t) = 0.002 + 0.001\sin t$, $a_2(t) = 1, a_3(t) = 2 + \sin t, a_4(t) = 0.001$, then the model system (1) becomes:

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\Bigg(0.02 + \sin t - (0.06)x(t) \\ &\quad - \frac{y(t)(1.8 + 0.1\sin t)}{0.002 + (2 + \sin t)y(t) + 0.001\sin t + \sin t + x(t) + (0.001)x(t)y(t)}\Bigg), \\ \frac{dy(t)}{dt} &= y(t)\Bigg(-0.001 - (0.05 + 0.01\cos t)y(t) \\ &\quad + \frac{x(t)(2.7 + 0.1\sin t)}{0.002 + 0.001\sin t + (2 + \sin t)y(t) + \sin t + x(t) + (0.001)x(t)y(t)}\Bigg).\end{aligned} \tag{43}$$

By simple numerical computations, we obtain $a_L = 0.01, a_M = 0.03, b_L = 0.06$, $b_M = 0.06, c_L = 1.7, c_M = 1.9, d_L = d_M = 0.001, e_L = 0.04, e_M = 0.06, f_L = 2.6, f_M = 2.8, a_{1_L} = 0.001, a_{1_M} = 0.003, a_{2_L} = a_{2_M} = 1, a_{3_L} = 1, a_{3_M} = 3, a_{4_L} = a_{4_M} = 0.001$. And furthermore

$$a_M = 0.03 < \; =0.24 \frac{c_L m_2}{a_{1_M} + a_{2_M} M_1 + a_{3_M} m_2 + a_{4_M} m_2 M_1}.$$

Also $\int_0^{+\infty}\left(a(t) - \frac{c(t)m_2}{a_1(t) + a_2(t)M_1 + a_3(t)m_2 + a_4(t)M_1 m_2}\right)dt = -\infty$. Hence $\lim_{t\to+\infty} x(t) = 0$. Thus Theorems 5 and 6 ensure that prey x of model system (43) will extinct.

Example 7 (Theorem 11). Let $b(t) = 2 + 0.1\cos t, a(t) = 2, c(t) = 1 + 0.1\cos t$, $d(t) = 0.3, f(t) = 1.3 + \cos t, e(t) = 3 + 0.1\cos t, a_1(t) = 1, a_2(t) = 1.5 + \cos t, a_3(t) = 2.1 + 0.2\cos t, a_4(t) = 1 + 0.1\sin t$, and $\omega = 2\pi$ then system (1) becomes:

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\Bigg(2 - (2 + 0.1\cos t)x(t) \\ &\quad - \frac{(1 + 0.1\cos t)y(t)}{1 + (1.5 + \cos t)x(t) + (2.1 + 0.2\cos t)y(t) + (1 + 0.1\cos t)x(t)y(t)}\Bigg), \\ \frac{dy(t)}{dt} &= y(t)\Bigg(-0.3 - (3 + 0.1\cos t)y(t) \\ &\quad + \frac{(1.3 + \cos t)x(t)}{1 + (1.5 + \cos t)x(t) + (2.1 + 0.2\cos t)y(t) + (1 + 0.1\cos t)x(t)y(t)}\Bigg).\end{aligned} \tag{44}$$

By numerical calculations, we have $a_L = a_M = 2, b_L = 0.9, b_M = 2.1, c_L = 0.9, c_M = 1.1$, $d_L = d_M = 0.3, f_L = 0.3, f_M = 2.3, a_{1_L} = a_{1_M} = 1, a_{2_L} = 0.5, a_{2_M} = 2.5, a_{3_L} = 1.9, a_{3_M} = 2.3$, $a_{4_L} = 0.9, a_{4_M} = 1.1$, and

$$\int_0^{2\pi}\left(-0.3 + \frac{2.22(1.3 + \cos t)}{4.33 + 2.22\cos t}\right)dt = -1.50 < 0,$$

Therefore Theorem 11 ensures the globally asymptotically stability of boundary periodic solution $(\tilde{x}_1(t), 0)$ of model system (44).

The model system (44) is nonpermanent with periodic coefficients. Hence the corresponding autonomous model system (with its parameter values being the average of the corresponding periodic functions in system (44) is permanent. Average of the parameter values of model system (44) is given as

$$a(t) = 2, b(t) = 2, c(t) = 1, d(t) = 0.3, f(t) = 1.3, e(t) = 3, a_2(t) = 1.5, a_1(t) = 1, a_4(t) = 1, a_3(t) = 2.1,$$

and the corresponding autonomous model system is

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left(2 - 2x(t) - \frac{y(t)}{1 + 1.5x(t) + x(t)y(t) + 2.1y(t)}\right), \\ \frac{dy(t)}{dt} &= y(t)\left(-0.3 - 3y(t) + \frac{1.3x(t)}{1 + 2.1y(t) + 1.5x(t) + x(t)y(t)}\right). \end{aligned} \tag{45}$$

It may easily be verified that the conditions in Equation (6)

$$\begin{aligned} d &= 0.3 < 0.52 = \frac{fM}{a_1 + a_2M}, \\ a &= 2 > 0.06 = \frac{cL}{a_1 + a_3L}, \\ d &= 0.3 < 0.5 = \frac{fK}{a_1 + a_2K + a_3L + a_4KL}, \end{aligned}$$

are satisfied. Hence model system (45) is permanent.

It is very interesting that Example 7 shows that the nonautonomous model system may suppress the permanence of its corresponding autonomous model system.

6 | SENSITIVITY ANALYSIS

The outcomes of deterministic model systems are governed by the input parameters of model systems, which may show some uncertainty in their selection or determination. We employed a global sensitivity analysis to evaluate the impact of uncertainty and the sensitivity of the outputs of numerical simulations to variations in each parameter of the system (1) using the method of partial rank correlation coefficients (PRCC) and Latin hypercube sampling (LHS; Marino et al., 2008). The parameters with significant impact on the outcome of numerical simulations are determined by sensitivity analysis. To generate the LHS matrices, we assume that all the model parameters are uniformly distributed. Then 200 simulations of the model per LHS run were performed, using the baseline values are: Example 1 ⇒ Figure 11a,b, Example 2 ⇒ Figure 11c,d, Example 3 ⇒ Figure 11e,f, Example 4 ⇒ Figure 11g,h, Example 5 ⇒ Figure 11i,j, Example 6 ⇒ Figure 11k,l, Example 7 ⇒ Figure 11m,n and the ranges as 25% from the baseline values (in either direction). Notice that the PRCC values remain between −1 and 1. Negative (positive) values represent a negative (positive) correlation of the model outcome with its parameter. A negative (positive) correlation indicates that a negative (positive) change in the parameter will decrease (increase) the model output. Bigger absolute value of the PRCC represents the larger correlation of the parameter with the outcome. The PRCC values are represented by bar graphs in Figure 11a,c,e,g,i,k,m and its time evolution has been illustrated in Figure 11b,d,f,h,j,l,n.

7 | DISCUSSION

Variability in environment plays a critical role in shaping population dynamics. Predator–prey relationship is one of the basic links among populations which affect population dynamics and trophic structures. The classical predator–prey model has commonly been studied in an idiosyncratic fashion, without considering variability in the surrounding environment in which population grows and survives. In this paper, environmental variability is captured in the model parameters with time-dependent periodic and almost periodic functions. This approach makes the model being nonautonomous in nature.

We studied the a nonautonomous prey–predator system with a CMFR and density-dependent death rate. We provided global dynamics of the model system (1) systematically. The global qualitative behavior (e.g., permanence and global asymptotic stability) of the general nonautonomous model system (1) have been discussed. The conditions (15) and (5) provide the sufficient conditions for global asymptotic stability and permanence of the system (1), respectively (see, Figures 7 and 8). Using continuation theorem and Brouwer fixed-point theorem, we have also derived the sufficient conditions (5) and (18) for a positive periodic solution. A comparative study about the application of both the theorems for a positive periodic solution is presented in Table 1. Different numerical examples with numerical simulations are considered to agree with the analytical findings. To assess the role of sensitivity and uncertainty of the outputs of the numerical simulations with respect to variations in each parameter of the model system (1), we have also employed a global sensitivity analysis using PRCC and LHS. More precisely, the analysis of the considered system discloses the following conclusions:

(i) We have established practical persistence for the model system (1) (refer Theorem 1 and Figures 1 and 2) while the definition of permanence provides the theoretical persistence for the system. The condition (5) ensures that the mutual interference at high-prey density (a_4) leaves negative effect on the permanence of the system (1). When the value of mutual interference (a_4), crosses a specific value, the sufficient condition for permanence (5) violates. Moreover, we have also obtained more flexible permanence conditions (8) for the model system (1) rather than conditions obtained in Equation (5).

TABLE 1 Comparative results obtained by using continuation and Brouwer fixed point theorem

Brouwer fixed point theorem	Continuation theorem
Uses the supremum and infimum of the parameters (refer the proof of the Theorem 8)	Uses average values of the related parameters (refer the proof of the Theorem 9)
The condition (5) is same as permanence condition (5) Guarantees for a positive periodic solution under the condition (5)	The condition (18) is different from permanence condition (5) Ensures for a positive periodic solution under the condition (18)
Theorem 8 is stronger (i.e., provides better result)	Theorem 9 is weaker

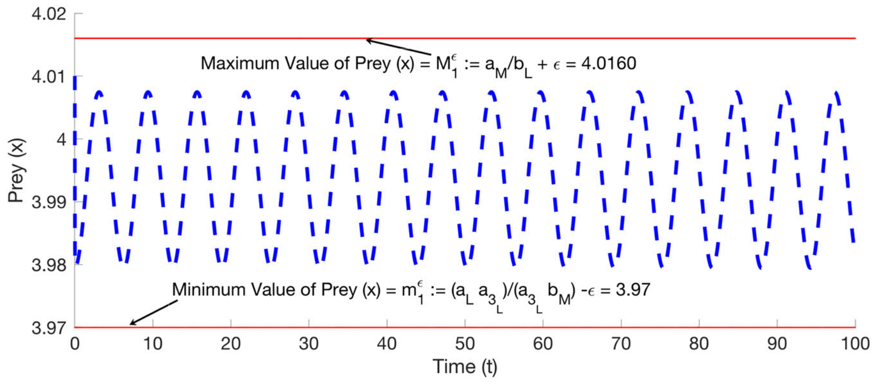

FIGURE 4 Solution curves for the prey in the model system (39). Trajectories starting through different initial conditions ([3.97, 2.5], [3.99, 4], and [4.009, 7]) and each of these ultimately are attracted by a single trajectory

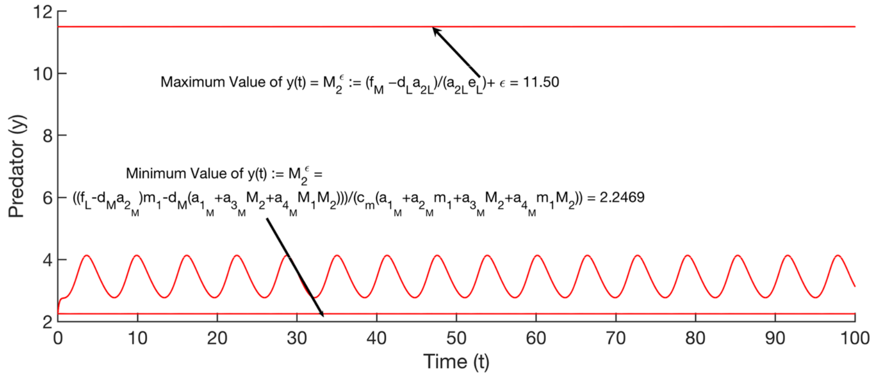

FIGURE 5 Solution curves for the predator in the model system (39) initiating from different initial conditions ([3.97, 2.5], [3.99, 4], and [4.009, 7]). All these curves approach to the same curve

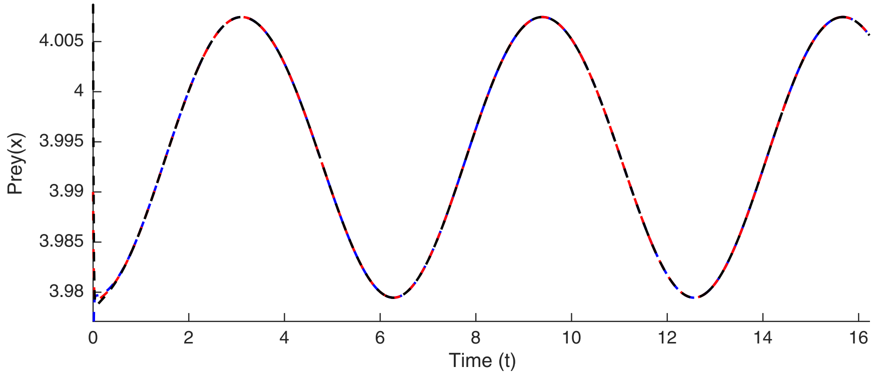

FIGURE 6 Solution curves for the prey in the model system (39). Trajectories initiating from the initial conditions ([3.97, 2.5], [3.99, 4], and [4.009, 7]) attracted by one a particular trajectory

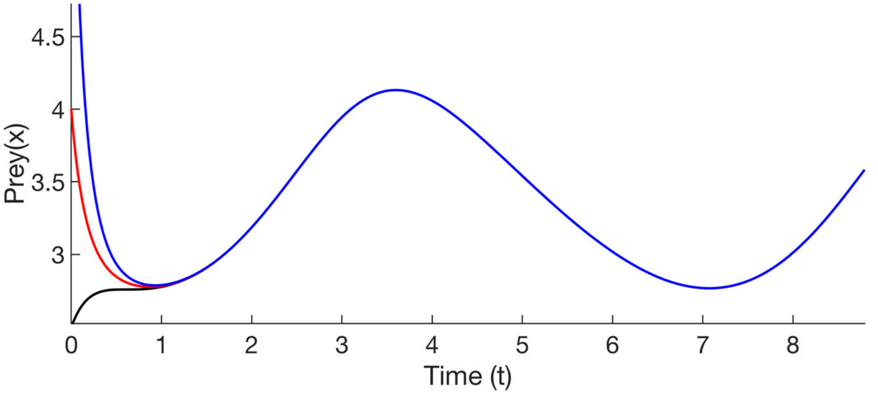

FIGURE 7 Solution curves for the prey in the model system (39) initiating from different initial conditions ([3.97, 2.5], [3.99, 4], and [4.009, 7]). All these curves approach to the same curve)

(ii) The conditions (11) and (12) of extinction of both prey and predator and global stability of boundary periodic solutions (refer Equation 33) have been obtained in both parametric and integral forms (refer the Theorems 6 and 11). The conditions involving integrals reflect the effects of the long-term predation behaviors on the number of species and provides reasonable biological interpretation rather than those for the corresponding autonomous system.

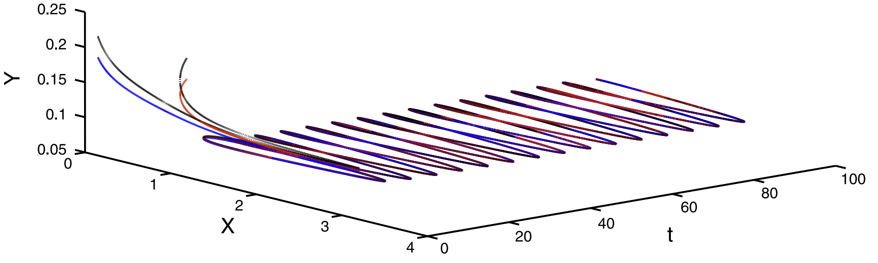

FIGURE 8 Positive periodic solution for the model system (40) starting with the initial conditions (0.15, 0.19), (0.15, 0.22), (1.19, 0.19), and (1.19, 0.22)

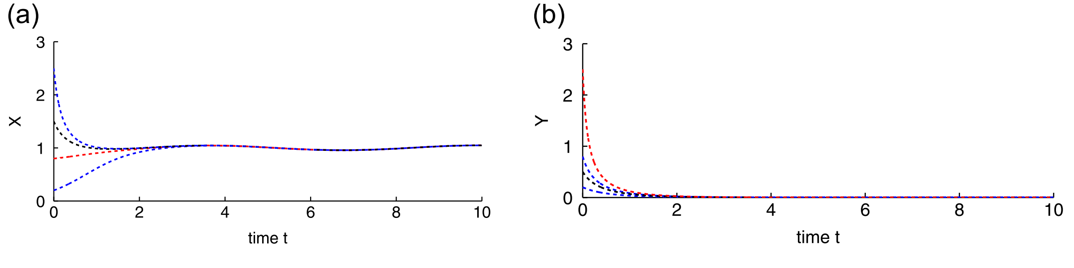

FIGURE 9 Time series of system (41). The extinction of predator-y. Solution curves starting with the initial conditions (0.2, 0.2), (0.8, 0.8), (1.5, 0.5), and (2.5, 2.5). (a) The integral curves of prey. (b) The integral curves of predator

These conditions also improves the usual conditions obtained using bounds of parameters (Fan & Kuang, 2004; Li & Takeuchi, 2015). Moreover the last condition in (5) (i.e., $d_M < \frac{f_L m_1}{a_{1_M} + a_{2_M} m_1 + a_{3_M} M_2 + a_{4_M} m_1 M_2}$ reduces to the permanence condition for the Beddington-DeAngelis type prey-predator model system with $a_4 = 0$. Thus the condition (5) is more general than the condition obtained in Li and Takeuchi (2015). We have also shown that the existence of a positively invariant set is sufficient for the permanence of the system.

(iii) We discuss the relations between results (permanence and global attractivity) for autonomous and nonautonomous systems to get insights on the effects of time dependent parameters. The boundary periodic solution $(\tilde{x}_1(t), 0)$ of nonautonomous model system (1) degenerate the boundary equilibrium $\left(\frac{a}{b}, 0\right)$ of the corresponding autonomous model system. For global stability of positive periodic solution of nonautonomous model system (1), the condition (33) (refer the Theorem 11) ensure the globally asymptotically stability of boundary equilibrium $\left(\frac{a}{b}, 0\right)$ for corresponding autonomous model system if $d < \frac{fM}{a_1 + a_2 M}$. One of the interesting findings is that the nonautonomous model system may suppress the permanence of its corresponding autonomous model system.

(iv) It is pretty clear that in both Figures 6 and 7, there is one trajectory that attracts other trajectories (prey and predator population initiating from different initial values) toward itself. This ensures the global stability of both the populations (prey and predator). Moreover, the conditions of both the Theorems 7 and 1 are well satisfied by the numerical Example 2, ensuring the existence of a global stable solution. The global attractivity condition (15) from the Theorem 7 ensures that the global attractivity of positive solution depends on both the density dependent death rate of predators and Crowley–Martin coefficient a_4. The density dependent death rate $e(t)$ leaves positive effect on the global attractivity of the positive solution, that is the predator density dependence death rate $e(t)$ shows stabilizing effect on the system. Theorem 9

(v) is weaker than Theorem 8. If $\epsilon \in \left((1 - \exp\{-2\hat{a}\omega\})\frac{a_L a_{3_L} - c_M}{b_M a_{3_L}}, \frac{a_L a_{3_L} - c_M}{b_M a_{3_L}}\right)$, $d_M > 1$ and (5) hold then (18) holds. This provides the existence range of periodic solution. Global stability of solution (boundary) and the predator species extinction and is discussed in the Theorem 10 (refer Figure 9).

(vi) We have also discussed more general case than the existence of periodic solution that is, we established the existence of a positive almost periodic solution (refer the Theorem 13). It is important to mention that this particular proof of existence of unique almost periodic solution do not make use of Arzela–Ascoli's Lemma (Rudin, 2008; Zhou & Shao, 2017).

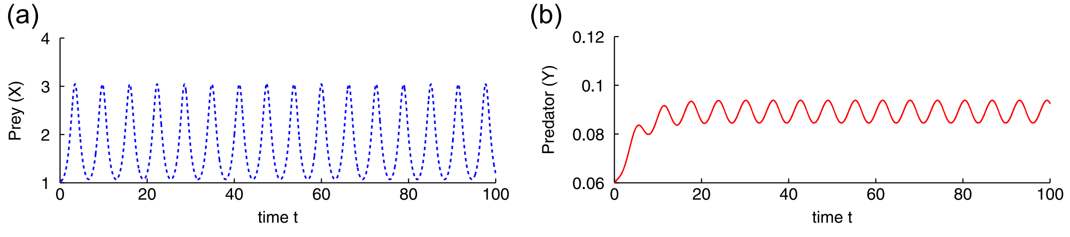

FIGURE 10 Permanence of the model system (1). Solution curves start with the initial condition (1, 0.06)

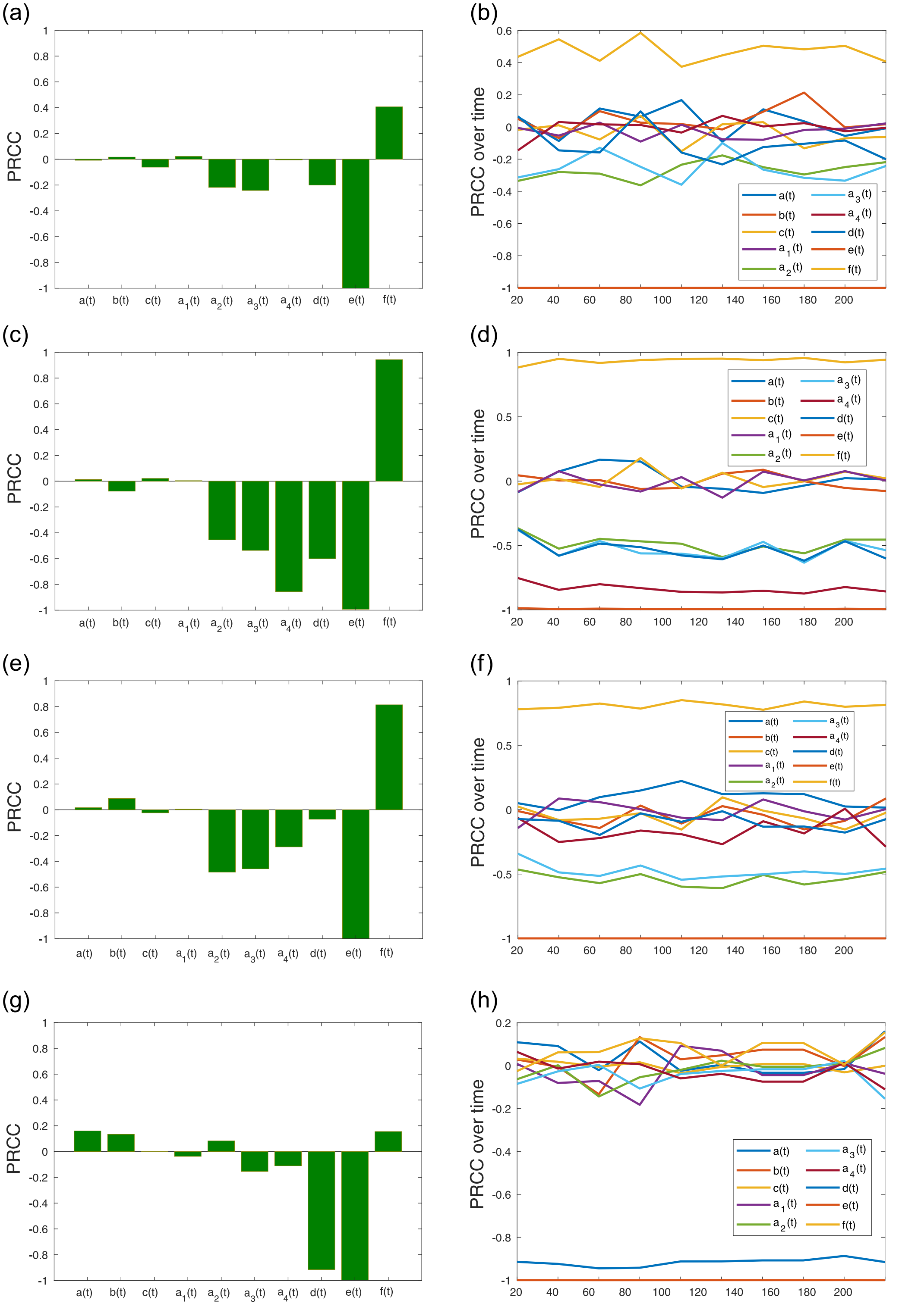

FIGURE 11 Bar graphs of PRCC of the parameters of the model system (1) and time course plots of the PRCCs of the parameters of the model system (1) at 10 different time points (days 20, 40, 60, 80, 100, 120, 140, 160, 180, and 200). Model parameters were sampled 1000 times. Baseline parameters are in the text. PRCC, partial rank correlation coefficient

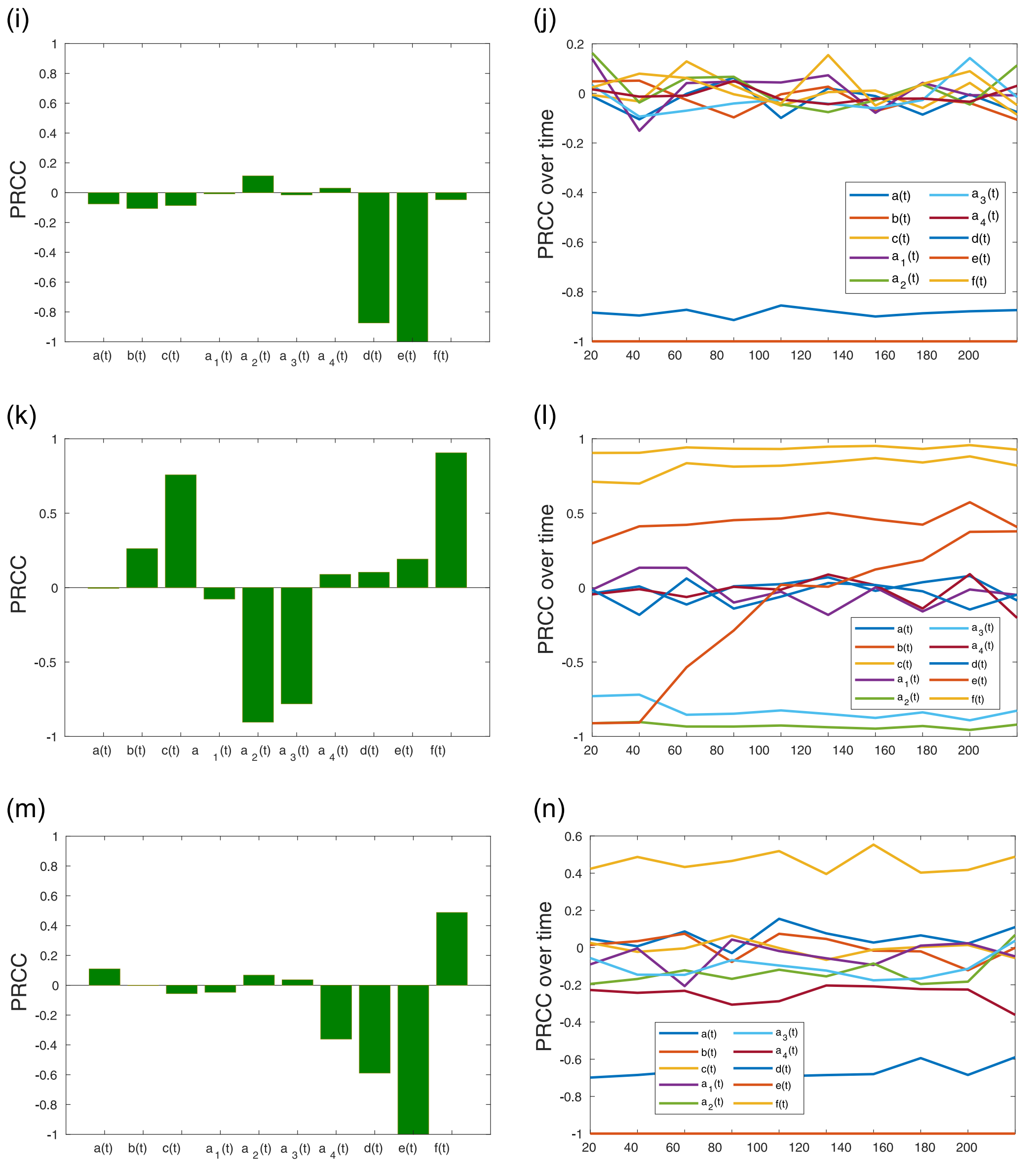

FIGURE 11 (Continued)

7.1 | Ecological implications and future scopes

Conditions of Theorem 2 are well satisfied for $\epsilon = 0$ in Theorem 1. Thus the positive invariance of K_ϵ ensures the permanence of the model system (1) (for a geometrical illustration, see Figures 4 and 5). Therefore, whenever, the population of prey lies within the particular range $\left[m_1^\epsilon, M_1^\epsilon\right]$ (see Theorems 1 and 2) and remain in the same range (as $t \to \infty$), then the prey population will remain in a region having positive distance from boundary and would always persist. The same explanation holds for predator population. The existence of a nonconstant globally attractive solution (refer the Theorem 7) describes the inevitability of prey and predator population regardless of their initial conditions (Figures 6 and 7). This particular result holds for the model system (1), however, in real scenario, for various kind of necessities (like, food resources, financial income, water, air and several other resources of modern time), our lives are dependent upon natural resources. As the human needs crosses a threshold level, then we require to observe the sustainability of resources of the associated exploited system (Arrow et al., 1995; Holling, 1973; Ludwig et al., 1997). Therefore, the concept of stability and permanence become our main concern in an exploited model system (system with harvesting).

In harvested system, permanence becomes an important issue because if we harvest too many individual then species may be driven to extinction. Interestingly, in many biological/agricultural systems, harvesting (due to fishing in marine system, hunting or disease) of a particular species/crop can only be more beneficial at certain times (for example, the time and stage of harvest of a particular crop play greater role in its production and hence the particular crop is many times harvested at its physiological maturity or at harvest maturity). The good examples of the periodic harvesting (seasonal harvesting) are fishing seasons, crop spraying for parasites or seasonal open hunting (Brauer & Sànchez, 2003). Moreover, if in a model system, the exploitation of a particular species crosses a threshold level, then the stability and resilience of the system may get disturbed. For such nonautonomous model systems with age selective harvesting (or, time-dependent harvesting, like periodic harvesting), establishing a globally attractive solution and analyzing the effect of harvesting (and the role of time variant parameters) on permanence and globally stability would be an interesting problem.

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CONFLICT OF INTERESTS

The authors declare that there are no conflict of interests.

AUTHOR CONTRIBUTIONS

Jai Prakash Tripathi (investigation, conceptualization, methodology, supervision, writing) developed and analyzed the mathematical model, Sarita Bugalia (computation, writing, editing), Vandana Tiwari (writing, reviewing, original draft preparation), Yun Kang (original draft preparation, supervision, writing, editing) helped in mathematical and numerical analysis of the model system. All authors contributed to the writing of the manuscript and have read and agreed to the published version of the manuscript.

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APPENDIX A

Proof of Lemma 1

Proof. From the prey equation of the model system (1) it follows that $x = 0$ is invariant. This implies that $x(t) > 0$ for all t if $x(0) > 0$. A similar argument using predator equation of the model system (1), shows that $y = 0$ is also an invariant set, so, $y(t) > 0$ for all t if $y(0) > 0$. Thus any trajectory starting in $\mathbb{R}_+^2$ can not cross the co-ordinate axes. Hence the result follows. □

Proof of Theorem 1

Proof. Using the bounds of the coefficients, from the first equation of the model system (1), we obtain

$$\frac{dx(t)}{dt} \leq x(t)(a_M - b_L x(t)).$$

Using Lemma A1, we have $\limsup_{t\to\infty} x(t) \le \frac{a_M}{b_L} := M_1$.

Thus for sufficiently small $\epsilon > 0$, there exist a positive real number T_1 such that $x(t) \le M_1 + \epsilon, \forall\, t \ge T_1$.

Further from second equation of the system (1), we find

$$\frac{dy(t)}{dt} \le y(t)\left(-d_L - e_L y(t) + \frac{f_M M_1}{a_{1_L} + a_{2_L} M_1}\right),$$

which implies that $\limsup_{t\to\infty} y(t) \le \frac{1}{e_L}\left[-d_L + \frac{f_M M_1}{a_{1_L} + a_{2_L} M_1}\right] := M_2$, provided $d_L < \frac{f_M M_1}{a_{1_L} + a_{2_L} M_1}$.

Hence for sufficiently small $\epsilon > 0, \exists$ a positive real number $T_3 \ge T_2 \ge 0$ such that $y(t) \le M_2 + \epsilon, \forall\, t \ge T_3$.

Again from the first Equation of (1), one can find that

$$\frac{dx(t)}{dt} \ge x(t)\left(a_L - b_M x(t) - \frac{c_M M_2}{a_{1_L} + a_{3_L} M_2}\right).$$

which implies that $\liminf_{t\to+\infty} x(t) \ge \frac{1}{b_M}\left[a_L - \frac{c_M M_2}{a_{1_L} + a_{3_L} M_2}\right] := m_1$, provided $a_L > \frac{c_M M_2}{a_{1_L} + a_{3_L} M_2}$.

So, again for arbitrary sufficiently small $\epsilon > 0$, there exists a positive real number $T_2 > T_1$ such that $x(t) \ge m_1 - \epsilon, \forall\, t \ge T_2$.

Moreover, using the lower and upper bounds of $x(t)$ and $y(t)$, from the second equation of (1), we obtain

$$\frac{dy(t)}{dt} \ge y(t)\left(-d_M - e_M y(t) + \frac{f_L m_1}{a_{1_M} + a_{2_M} m_1 + a_{3_M} M_2 + a_{4_M} m_1 M_2}\right),$$

which implies that $\liminf_{t\to+\infty} y(t) \ge \frac{1}{e_M}\left[-d_M + \frac{f_L m_1}{a_{1_M} + a_{2_M} m_1 + a_{3_M} M_2 + a_{4_M} m_1 M_2}\right] := m_2$, provided that $d_M < \frac{f_L m_1}{a_{1_M} + a_{2_M} m_1}$.

Hence again for $\epsilon > 0 \exists\, T_4 > T_5$ such that $y(t) \ge m_2 - \epsilon, \forall\, t \ge T_4$. Thus, any positive solution $(x(t), y(t))$ of system (1) satisfies $m_1 \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le M_1$, $m_2 \le \liminf_{t\to\infty} y(t) \le \limsup_{t\to\infty} y(t) \le M_2$, whenever (i) $d_L < \frac{f_M M_1}{a_{1_L} + a_{2_L} M_1}$, (ii) $a_L > \frac{c_M M_2}{a_{1_L} + a_{3_L} M_2}$, (iii) $d_M < \frac{f_L m_1}{a_{1_M} + a_{2_M} m_1 + a_{3_M} M_2 + a_{4_M} m_1 M_2}$. □

Proof of Theorem 3

Proof. First we prove that the set K is positively invariant for model system (1). Let $((x_1(t), y_1(t))$ be any solution of model system (1), with $(x_1(t_0), y_1(t_0)) \in K$. From the first equation of model system (1) and positivity of solutions of model system (1), we have

$$\frac{dx_1(t)}{dt} \le x_1(t)(a(t) - b(t)x_1(t)), \quad t \ge t_0, \tag{A1}$$

from Lemma 2 and $0 < x_1(t_0) \leq \tilde{M}_1$, implies that $x_1(t) \leq \tilde{M}_1$, for all $t \geq t_0$. From second equation of model system (1), we have

$$\frac{dy_1(t)}{dt} \leq y_1(t)\left(-d(t) - e(t)y_1(t) + \frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1}\right), \quad t \geq t_0, \tag{A2}$$

thus by Lemma 2 and $0 < y_1(t_0) \leq \tilde{M}_2$, implies that $y_1(t) \leq \tilde{M}_2$, for all $t \geq t_0$. It follows that

$$\frac{dx_1(t)}{dt} \geq x_1(t)\left(a(t) - b(t)x_1(t) - \frac{c(t)\tilde{M}_2}{a_1(t) + a_3(t)\tilde{M}_2}\right), \quad t \geq t_0, \tag{A3}$$

therefore from Lemma 2 and $x_1(t_0) \geq \tilde{m}_1 > 0$, implies that $x_1(t) \geq \tilde{m}_1$, for all $t \geq t_0$. Furthermore

$$\frac{dy_1(t)}{dt} \geq y_1(t)\left(-d(t) - e(t)y_1(t) + \frac{f(t)\tilde{m}_1}{a_1(t) + a_2(t)\tilde{m}_1}\right), \quad t \geq t_0, \tag{A4}$$

hence from Lemma 2 and $y_1(t_0) \geq \tilde{m}_2 > 0$, implies that $y_1(t) \geq \tilde{m}_2$, for all $t \geq t_0$. Therefore, K is positively invariant with respect to model system (1).

Assume that conditions (8) holds. We prove that model system (1) is permanent under the conditions (8). Precisely, we want to show that for any solution $((x_1(t), y_1(t))$ of model system (1) with $x_1(t_0) > 0, y_1(t_0) > 0$,

(i) $\limsup_{t\to+\infty} x_1(t) \leq \tilde{M}_1$,
(ii) $\limsup_{t\to+\infty} y_1(t) \leq \tilde{M}_2$ if $\inf_{t\in\mathbb{R}}\left[-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t)+a_2(t)\tilde{M}_1}\right] > 0$,
(iii) $\liminf_{t\to+\infty} x_1(t) \geq \tilde{m}_1$ if $\inf_{t\in\mathbb{R}}\left[-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t)+a_2(t)\tilde{M}_1}\right] > 0$ and $\inf_{t\in\mathbb{R}}\left[a(t) - \frac{c(t)\tilde{M}_2}{a_1(t)+a_3(t)\tilde{M}_2}\right] > 0$,
(iv) $\liminf_{t\to+\infty} y_1(t) \geq \tilde{m}_2$ if (8) holds.

From conditions (8), we can choose a sufficiently small $\epsilon > 0$ such that $\tilde{m}_1 - \epsilon > 0$ and

$$\inf_{t\in\mathbb{R}}\left[a(t) - \frac{c(t)(\tilde{M}_2+\epsilon)}{a_1(t) + a_3(t)(\tilde{M}_2+\epsilon)}\right] > 0,$$
$$\inf_{t\in\mathbb{R}}\left[-d(t) + \frac{f(t)(\tilde{m}_1-\epsilon)}{a_1(t) + a_2(t)(\tilde{m}_1-\epsilon) + a_3(t)(\tilde{M}_2+\epsilon) + a_4(t)(\tilde{m}_1-\epsilon)(\tilde{M}_2+\epsilon)}\right] > 0.$$

From Lemma 2 and inequality (A1) implies that $\limsup_{t\to+\infty} x_1(t) \leq \tilde{M}_1$. It follows that there exists $T_0 > t_0$ such that for $t > T_0$, $x_1(t) \leq \tilde{M}_1 + \epsilon$, for sufficiently small ϵ. From second equation of model system (1), implies that

$$\frac{dy_1(t)}{dt} \leq y_1(t)\left(-d(t) - e(t)y_1(t) + \frac{f(t)(\tilde{M}_1+\epsilon)}{a_1(t) + a_2(t)(\tilde{M}_1+\epsilon)}\right), \quad t \geq t_0.$$

Since $\inf_{t\in\mathbb{R}}\left[-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t)+a_2(t)\tilde{M}_1}\right] > 0$, from Lemma 2 we obtain

$$\limsup_{t\to+\infty} y_1(t) \le \sup_{t\in\mathbb{R}} \frac{1}{e(t)}\left[-d(t) + \frac{f(t)(\tilde{M}_1+\epsilon)}{a_1(t)+a_2(t)(\tilde{M}_1+\epsilon)}\right],$$
$$\le \sup_{t\in\mathbb{R}} \frac{1}{e(t)}\left[-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t)+a_2(t)\tilde{M}_1}\right] + \sup_{t\in\mathbb{R}} \frac{1}{e(t)}\left[\frac{f(t)\tilde{M}_1}{a_1(t)+a_2(t)\tilde{M}_1}\right]\epsilon.$$

Thus by boundedness of $\frac{f(t)\tilde{M}_1}{a_1(t)+a_2(t)\tilde{M}_1}$ and arbitrariness of ϵ, we obtain $\limsup_{t\to+\infty} y_1(t) \le \tilde{M}_2$.

It follows that there exists $T_1 > t_0$ such that for $t > T_1, y_1(t) \le \tilde{M}_2 + \epsilon$, for sufficiently small ϵ. And

$$\frac{dx_1(t)}{dt} \ge x_1(t)\left(a(t) - b(t)x_1(t) - \frac{c(t)(\tilde{M}_2+\epsilon)}{a_1(t)+a_3(t)(\tilde{M}_2+\epsilon)}\right).$$

Since $\inf_{t\in\mathbb{R}}\left[a(t) - \frac{c(t)\tilde{M}_2}{a_1(t)+a_3(t)\tilde{M}_2}\right] > 0$, again from Lemma 2, we obtain

$$\liminf_{t\to+\infty} x_1(t) \ge \inf_{t\in\mathbb{R}} \frac{1}{b(t)}\left[a(t) - \frac{c(t)(\tilde{M}_2+\epsilon)}{a_1(t)+a_3(t)(\tilde{M}_2+\epsilon)}\right]$$
$$\ge \inf_{t\in\mathbb{R}} \frac{1}{b(t)}\left[a(t) - \frac{c(t)\tilde{M}_2}{a_1(t)+a_3(t)\tilde{M}_2}\right] - \inf_{t\in\mathbb{R}} \frac{1}{b(t)}\left[\frac{c(t)}{a_1(t)+a_3(t)\tilde{M}_2}\right]\epsilon.$$

By arbitrariness of ϵ, we obtain $\liminf_{t\to+\infty} x_1(t) \ge \tilde{m}_1$. It follows that there exists $T_2 > t_0$ such that for $t > T_2, x_1(t) \ge \tilde{m}_1 - \epsilon$, for sufficiently small ϵ. Again

$$\frac{dy_1(t)}{dt} \ge y_1(t)\Bigg(-d(t) - e(t)y_1(t)$$
$$+ \frac{f(t)(\tilde{m}_1-\epsilon)}{a_1(t)+a_2(t)(\tilde{m}_1-\epsilon)+a_3(t)(\tilde{M}_2+\epsilon)+a_4(t)(\tilde{m}_1-\epsilon)(\tilde{M}_2+\epsilon)}\Bigg).$$

Since $\inf_{t\in\mathbb{R}}\frac{1}{e(t)}\left[-d(t) + \frac{f(t)\tilde{m}_1}{a_1(t)+a_2(t)\tilde{m}_1+a_3(t)\tilde{M}_2+a_4(t)\tilde{m}_1\tilde{M}_2}\right] > 0$, and from Lemma 2 we obtain

$$\liminf_{t\to+\infty} y_1(t) \ge \inf_{t\in\mathbb{R}} \frac{1}{e(t)}\left[-d(t) + \frac{f(t)(\tilde{m}_1-\epsilon)}{a_1(t)+a_2(t)(\tilde{m}_1-\epsilon)+a_3(t)(\tilde{M}_2+\epsilon)+a_4(t)(\tilde{m}_1-\epsilon)(\tilde{M}_2+\epsilon)}\right]$$
$$\ge \inf_{t\in\mathbb{R}} \frac{1}{e(t)}\left[-d(t) + \frac{f(t)\tilde{m}_1}{a_1(t)+a_2(t)\tilde{m}_1+a_3(t)\tilde{M}_2+a_4(t)\tilde{m}_1\tilde{M}_2}\right]$$
$$-\inf_{t\in\mathbb{R}} \frac{1}{e(t)}\left[\frac{f(t)}{a_1(t)+a_2(t)\tilde{m}_1+a_3(t)\tilde{M}_2+a_4(t)\tilde{m}_1\tilde{M}_2}\right]\epsilon.$$

By arbitrariness of ϵ, we obtain $\liminf_{t\to+\infty} y_1(t) \ge \tilde{m}_2$.

Hence we can conclude that K_ϵ is an ultimate bounded region of model system (1). Here ϵ is sufficiently small such that $\tilde{m}_1 - \epsilon > 0$ and $\tilde{m}_2 - \epsilon > 0$. □

Proof of Theorem 4

Proof. Let $((x_1(t), y_1(t))$ be any solution of model system (1), with $x_1(t_0) > 0, y_1(t_0) > 0$. From the second equation of model system (1), we obtain

$$\begin{aligned}\frac{dy_1(t)}{dt} &\le y_1(t)\left(-d(t) - e(t)y_1(t) + \frac{f(t)x_1(t)}{a_1(t) + a_2(t)x_1(t)}\right),\\ &\le y_1(t)\left(-d_L + \frac{f_M\tilde{M}_1}{a_{1_L} + a_{2_L}\tilde{M}_1}\right),\end{aligned}$$

it follows that, $y_1(t) \le y_1(0)e^{\left(-d_L + \frac{f_M\tilde{M}_1}{a_{1_L}+a_{2_L}\tilde{M}_1}\right)}$. By our assumption (9), we have $\lim_{t\to+\infty} y_1(t) = 0$. □

Proof of Theorem 5

Proof. Let $((x_1(t), y_1(t))$ be any solution of model system (1), with $x_1(t_0) > 0, y_1(t_0) > 0$. From Theorem 1, we know that $\limsup_{t\to+\infty} x_1(t) \le M_1$ and $\liminf_{t\to+\infty} y_1(t) \ge m_2$ and there exists a T_1 such that $x_1(t) \le M_1 + \epsilon$ and $y_1(t) \ge m_2 - \epsilon$, for sufficiently small $\epsilon > 0$ and for all $t \ge T_1$. From the first equation of model system (1), we obtain

$$\begin{aligned}\frac{dx_1}{dt} &\le x_1(t)\left(a(t) - b(t)x_1(t) - \frac{c(t)(m_2-\epsilon)}{a_1(t) + a_2(t)x_1(t) + a_3(t)(m_2-\epsilon) + a_4(t)(m_2-\epsilon)x_1(t)}\right),\\ &\le x_1(t)\left(a_M - \frac{c_L(m_2-\epsilon)}{a_{1_M} + a_{2_M}(M_1+\epsilon) + a_{3_M}(m_2-\epsilon) + a_{4_M}(m_2-\epsilon)(M_1+\epsilon)}\right),\end{aligned}$$

it follows that, $x_1(t) \le x_1(0)e^{\left(a_M - \frac{c_L(m_2-\epsilon)}{a_{1M}+a_{2M}(M_1+\epsilon)+a_{3M}(m_2-\epsilon)+a_{4M}(m_2-\epsilon)(M_1+\epsilon)}\right)}$. Letting $\epsilon \to 0$ and by assumption (10), we have $\lim_{t\to+\infty} x_1(t) = 0$. □

Proof of Theorem 6

Proof. Let $((x_1(t), y_1(t))$ be any solution of model system (1), with $x_1(t_0) > 0, y_1(t_0) > 0$.

1. Since we have $\limsup_{t\to\infty} x(t) \le \tilde{M}_1$. For sufficiently small $\epsilon > 0$, there exists a positive real number T_1 such that $x(t) \le \tilde{M}_1 + \epsilon, \forall\ t \ge T_1$. From the second equation of model system (1), we obtain

$$y'_1(t) \le y_1(t)\left(-d(t) + \frac{f(t)(\tilde{M}_1+\epsilon)}{a_1(t) + a_2(t)(\tilde{M}_1+\epsilon)}\right), \quad t \ge T_1.$$

   Let $q(t) = -d(t) + \frac{f(t)(\tilde{M}_1+\epsilon)}{a_1(t)+a_2(t)(\tilde{M}_1+\epsilon)}$. By our assumption, we know that $\int_{T_1}^{+\infty} q(t)dt = -\infty$. Thus from the above inequality, we obtain $y_1(t) \le y_1(T_1)e^{\int_{T_1}^{t} q(s)ds}$ and $\lim_{t\to+\infty} y_1(t) = 0$.

2. From Theorem 3, we know that $\limsup_{t\to+\infty} x(t) \le \tilde{M}_1$ and $\liminf_{t\to\infty} y(t) \ge \tilde{m}_2$. It follows that there exists a $T > T_1$ such that for $t > T$, $x(t) \le \tilde{M}_1 + \epsilon$, $y(t) \ge \tilde{m}_2 - \epsilon$, for sufficiently small $\epsilon > 0$. From the first equation of model system (1), we obtain

$$x'_1(t) \le x_1(t)\left(a(t) - \frac{c(t)(\tilde{m}_2 - \epsilon)}{a_1(t) + a_2(t)(\tilde{M}_1 + \epsilon) + a_3(t)(\tilde{m}_2 - \epsilon) + a_4(t)(\tilde{M}_1 + \epsilon)(\tilde{m}_2 - \epsilon)}\right), t \ge T.$$

Let $r(t) = \frac{c(t)(\tilde{m}_2 - \epsilon)}{a_1(t) + a_2(t)(\tilde{M}_1 + \epsilon) + a_3(t)(\tilde{m}_2 - \epsilon) + a_4(t)\tilde{M}_1 + \epsilon)(\tilde{m}_2 - \epsilon)}$. By our assumption we know that $\int_T^{+\infty} r(t)dt = -\infty$. Thus from the above inequality, we obtain $x_1(t) \le x_1(T)e^{\int_T^t r(s)ds}$ and $\lim_{t\to+\infty} x_1(t) = 0$. The proof is complete. □

Proof of Theorem 11

Proof. Let $(x_1(t), y_1(t))$ be any solution of model system (1) with $x_1(t_0) > 0, y_1(t_0) > 0$. From the first equation of model system (1) and Lemma 2, we have $\limsup_{t\to+\infty} x_1(t) \le \tilde{M}_1$. Obviously (33) implies that (11) holds. Since by Theorem 6, we know that $\lim_{t\to+\infty} y_1(t) = 0$. Therefore we need to prove that $\lim_{t\to+\infty}|x_1(t) - \tilde{x}_1(t)| = 0$. By (33), there exists $\epsilon > 0$ such that

$$\int_0^{\omega}\left(-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1} + (\rho_1 + \rho_2)\epsilon\right)dt = 0. \tag{A5}$$

Actually $\epsilon = -\frac{1}{(\rho_1 + \rho_2)\omega}\int_0^{\omega}(-d(t) + \frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1})dt$, where

$$\rho_1 = \sup_{t\in\mathbb{R}^+}|c(t) - e(t)|, \quad \rho_2 = \sup_{t\in\mathbb{R}^+}\frac{f(t)}{a_1(t) + a_2(t)\tilde{M}_1}.$$

Define a function

$$V(t) = |\ln(x_1(t)) - \ln(\tilde{x}_1(t))| + |\ln(y_1(t))|, \quad t > 0.$$

Since $\limsup_{t\to+\infty} x_1(t) \le \tilde{M}_1$ and $\lim_{t\to+\infty} y_1(t) = 0$, there exists a $T_0 > t_0$ such that for all $t > T_0$, $0 < x_1(t) \le \tilde{M}_1 + \epsilon$ and $y_1(t) < \epsilon$. Therefore calculating the right derivative of $V(t)$ along the solution of model system (1), we obtain

$$\begin{aligned} D^+V(t) \le& -b(t)|x_1(t) - \tilde{x}_1(t)| + \frac{c(t)y_1(t)}{a_1(t) + a_2(t)x_1(t) + a_3(t)y_1(t) + a_4(t)x_1(t)y_1(t)} \\ & -d(t) - e(t)y_1(t) + \frac{f(t)x_1(t)}{a_1(t) + a_2(t)x_1(t) + a_3(t)y_1(t) + a_4(t)x_1(t)y_1(t)} \\ \le& -b_L|x_1(t) - \tilde{x}_1(t)| + |c(t) - e(t)|\, y_1(t) - d(t) + \frac{f(t)(\tilde{M}_1 + \epsilon)}{a_1(t) + a_2(t)(\tilde{M}_1 + \epsilon)} \\ \le& -b_L|x_1(t) - \tilde{x}_1(t)| - d(t) + \frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1} + (\rho_1 + \rho_2)\epsilon, \quad t > T_0. \end{aligned}$$

integrating both sides from T_0 to t, we obtain by (A5)

$$V(t) + b_L \int_{T_0}^{t} |x_1(s) - \tilde{x}_1(s)|\, ds \le V(T_0) + \int_{T_0}^{t} \left(-d(s) + \frac{f(t)\tilde{M}_1}{a_1(t) + a_2(t)\tilde{M}_1} + (\rho_1 + \rho_2)\epsilon \right) ds \le +\infty, \quad t > T_0.$$

Which implies that $|x_1(t) - \tilde{x}_1(t)| \in L^1([T_0, +\infty))$.

The boundedness of $\tilde{x}_1(t)$ and $x_1(t)$ imply that both $\tilde{x}_1(t)$ and $x_1(t)$ have bounded derivatives for $t > T_0$. Then $|x_1(t) - \tilde{x}_1(t)|$ is uniformly continuous on $[T_0, +\infty)$. Hence by Lemma A2, we have $|x_1(t) - \tilde{x}_1(t)| = 0$. Which completes the proof. □

Proof of Theorem 13

Proof. For $(x, y) \in \mathbb{R}_+^2$, we define $\|(x, y)^T\| = x + y$. To prove that the model system (1) has a unique positive almost-periodic solution, which is uniformly asymptotically stable in K_ϵ, it is equivalent to show that model system (34) has a unique almost-periodic solution to be uniformly asymptotically stable in K_ϵ^*.

Consider the product system of (34)

$$\begin{aligned}
\frac{d\bar{x}_1(t)}{dt} &= a(t) - b(t)\exp(\bar{x}_1(t)) \\
&\quad - \frac{c(t)\exp(\bar{y}_1(t))}{a_1(t) + a_2(t)\exp(\bar{x}_1(t)) + a_3(t)\exp(\bar{y}_1(t)) + a_4(t)\exp(\bar{x}_1(t))\exp(\bar{y}_1(t))}, \\
\frac{d\bar{y}_1(t)}{dt} &= -d(t) - e(t)\exp(\bar{y}_1(t)) \\
&\quad + \frac{f(t)\exp(\bar{x}_1(t))}{a_1(t) + a_2(t)\exp(\bar{x}_1(t)) + a_3(t)\exp(\bar{y}_1(t)) + a_4(t)\exp(\bar{x}_1(t))\exp(\bar{y}_1(t))}, \\
\frac{d\bar{x}_2(t)}{dt} &= a(t) - b(t)\exp(\bar{x}_2(t)) \\
&\quad - \frac{c(t)\exp(\bar{y}_2(t))}{a_1(t) + a_2(t)\exp(\bar{x}_2(t)) + a_3(t)\exp(\bar{y}_2(t)) + a_4(t)\exp(\bar{x}_2(t))\exp(\bar{y}_2(t))}, \\
\frac{d\bar{y}_2(t)}{dt} &= -d(t) - e(t)\exp(\bar{y}_2(t)) \\
&\quad + \frac{f(t)\exp(\bar{x}_2(t))}{a_1(t) + a_2(t)\exp(\bar{x}_2(t)) + a_3(t)\exp(\bar{y}_2(t)) + a_4(t)\exp(\bar{x}_2(t))\exp(\bar{y}_2(t))}.
\end{aligned} \tag{A6}$$

Now we define a Lyapunov function on $[0, +\infty) \times K_\epsilon^* \times K_\epsilon^*$ as follows-

$$V(t, \bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2) = |\bar{x}_1(t) - \bar{x}_2(t)| + |\bar{y}_1(t) - \bar{y}_2(t)|.$$

Then condition 1 of Lemma 3 is satisfied for $\alpha(\gamma) = \beta(\gamma) = \gamma$ for $\gamma \ge 0$. Additionally

$$
\begin{aligned}
|V(t,\bar{x}_1,\bar{y}_1,\bar{x}_2,\bar{y}_2) - V(t,\bar{x}_1,\bar{y}_1,\bar{x}_2,\bar{y}_2)| &= (|\bar{x}_1(t) - \bar{x}_2(t)| + |\bar{y}_1(t) - \bar{y}_2(t)|) \\
&\quad - (|\bar{x}_3(t) - \bar{x}_4(t)| + |\bar{y}_3(t) - \bar{y}_4(t)|) \\
&\leq |\bar{x}_1(t) - \bar{x}_3(t)| + |\bar{y}_1(t) - \bar{y}_3(t)| + |\bar{x}_2(t) \\
&\quad - \bar{x}_4(t)| + |\bar{y}_2(t) - \bar{y}_4(t)| \\
&\leq \|(\bar{x}_1(t),\bar{y}_1(t)) - (\bar{x}_3(t),\bar{y}_3(t))\| \\
&\quad + \|(\bar{x}_2(t),\bar{y}_2(t)) - (\bar{x}_4(t),\bar{y}_4(t))\|,
\end{aligned}
\tag{A7}
$$

which shows that condition 2 of Lemma 3 is also satisfied.

Let $(\bar{x}_i(t),\bar{y}_i(t))^T$, $i = 1, 2$, be any two solutions of (34) defined on $[0,+\infty) \times K_\epsilon^* \times K_\epsilon^*$. Calculating the upper right derivative of $V(t)$ along the solutions of (34), we obtain

$$
D^+V(t) = A \times \operatorname{sgn}(\bar{x}_1(t) - \bar{x}_2(t)) + B \times \operatorname{sgn}(\bar{y}_1(t) - \bar{y}_2(t)) \tag{A8}
$$

where

$$
\begin{aligned}
A &= -b(t)(\exp(\bar{x}_1(t)) - \exp(\bar{x}_2(t))) \\
&\quad - \left(\frac{c(t)\exp(\bar{y}_1(t))}{a_1(t) + a_2(t)\exp(\bar{x}_1(t)) + a_3(t)\exp(\bar{y}_1(t)) + a_4(t)\exp(\bar{x}_1(t))\exp(\bar{y}_1(t))}\right. \\
&\quad \left. - \frac{c(t)\exp(\bar{y}_2(t))}{a_1(t) + a_2(t)\exp(\bar{x}_2(t)) + a_3(t)\exp(\bar{y}_2(t)) + a_4(t)\exp(\bar{x}_2(t))\exp(\bar{y}_2(t))}\right), \\
B &= -e(t)(\exp(\bar{y}_1(t)) - \exp(\bar{y}_2(t))) \\
&\quad + \left(\frac{f(t)\exp(\bar{x}_1(t))}{a_1(t) + a_2(t)\exp(\bar{x}_1(t)) + a_3(t)\exp(\bar{y}_1(t)) + a_4(t)\exp(\bar{x}_1(t))\exp(\bar{y}_1(t))}\right. \\
&\quad \left. - \frac{f(t)\exp(\bar{x}_2(t))}{a_1(t) + a_2(t)\exp(\bar{x}_2(t)) + a_3(t)\exp(\bar{y}_2(t)) + a_4(t)\exp(\bar{x}_2(t))\exp(\bar{y}_2(t))}\right).
\end{aligned}
$$

After some algebraic calculation, we obtain

$$
\begin{aligned}
A \times \operatorname{sgn}(\bar{x}_1(t) - \bar{x}_2(t)) \leq &\left(-b(t) + \frac{c(t)\left[a_2(t)M_2^\epsilon + a_4(t)M_2^{\epsilon 2}\right]}{\left(a_1(t) + a_2(t)m_1^\epsilon + a_3(t)m_2^\epsilon + a_4(t)m_1^\epsilon m_2^\epsilon\right)^2}\right) \\
&| \exp(\bar{x}_1(t)) - \exp(\bar{x}_2(t))| \\
&+ \frac{c(t)\left[a_1(t) + a_2(t)M_1^\epsilon\right]}{\left(a_1(t) + a_2(t)m_1^\epsilon + a_3(t)m_2^\epsilon + a_4(t)m_1^\epsilon m_2^\epsilon\right)^2} \; | \exp(\bar{y}_1(t)) \\
&- \exp(\bar{y}_2(t))|
\end{aligned}
$$

and

$$B \times \operatorname{sgn}(\bar{y}_1(t) - \bar{y}_2(t)) \leq \left(-e(t) - \frac{f(t)\left[a_3(t)m_1^{\epsilon} + a_4(t)m_1^{\epsilon 2}\right]}{\left(a_1(t) + a_2(t)M_1^{\epsilon} + a_3(t)M_2^{\epsilon} + a_4(t)M_1^{\epsilon}M_2^{\epsilon}\right)^2}\right)$$
$$| \exp(\bar{y}_1(t)) - \exp(\bar{y}_2(t))|$$
$$+ \frac{f(t)\left[a_1(t) + a_3(t)M_2^{\epsilon}\right]}{\left(a_1(t) + a_2(t)m_1^{\epsilon} + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2} \mid \exp(\bar{x}_1(t))$$
$$- \exp(\bar{x}_2(t))|$$

Note that

$$\begin{aligned} \exp(\bar{x}_1(t)) - \exp(\bar{x}_2(t)) &= \exp(\rho_1(t))(\bar{x}_1(t) - \bar{x}_2(t)), \\ \exp(\bar{y}_1(t)) - \exp(\bar{y}_2(t)) &= \exp(\rho_1(t))(\bar{y}_1(t) - \bar{y}_2(t)), \end{aligned} \tag{A9}$$

where $\rho_1(t)$ lies between $\bar{x}_1(t)$ and $\bar{x}_2(t)$ and $\rho_2(t)$ lies between $\bar{y}_1(t)$ and $\bar{y}_2(t)$. Hence, we obtain

$$\begin{aligned} D^{+}V(t) \leq &-\left[b(t) - \frac{c(t)\left[a_2(t)M_2^{\epsilon} + a_4(t)M_2^{\epsilon 2}\right]}{\left(a_1(t) + a_2(t)m_1^{\epsilon} + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right. \\ &\left. - \frac{f(t)\left[a_1(t) + a_3(t)M_2^{\epsilon}\right]}{\left(a_1(t) + a_2(t)m_1^{\epsilon} + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right] m_1^{\epsilon} \mid \bar{x}_1(t) - \bar{x}_2(t)| \\ &-\left[- \frac{c(t)\left[a_1(t) + a_2(t)M_1^{\epsilon}\right]}{\left(a_1(t) + a_2(t)m_1^{\epsilon} + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right. \\ &\left. + e(t) + \frac{f(t)\left[a_3(t)m_1^{\epsilon} + a_4(t)m_1^{\epsilon 2}\right]}{\left(a_1(t) + a_2(t)M_1^{\epsilon} + a_3(t)M_2^{\epsilon} + a_4(t)M_1^{\epsilon}M_2^{\epsilon}\right)^2}\right] m_2^{\epsilon} \mid \bar{y}_1(t) - \bar{y}_2(t)| \\ \leq & -\mu(|\bar{x}_1(t) - \bar{x}_2(t)| + |\bar{y}_1(t) - \bar{y}_2(t)|) \\ = & -\mu \; ||(\bar{x}_1(t), \bar{y}_1(t)) - (\bar{x}_2(t), \bar{y}_2(t))||, \end{aligned}$$

where

$$\begin{aligned} \mu = \min\left\{\inf_{t\in\mathbb{R}}\left\{\left[b(t) - \frac{c(t)\left[a_2(t)M_2^{\epsilon} + a_4(t)M_2^{\epsilon 2}\right]}{\left(a_1(t) + a_2(t)m_1^{\epsilon} + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right.\right.\right. \\ \left.\left. - \frac{f(t)\left[a_1(t) + a_3(t)M_2^{\epsilon}\right]}{\left(a_1(t) + a_2(t)m_1^{\epsilon} + a_3(t)m_2^{\epsilon} + a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right] m_1^{\epsilon}\right\} \end{aligned}$$

and

$$\inf_{t\in\mathbb{R}}\left\{\left[-\frac{c(t)\left[a_1(t)+a_2(t)M_1^{\epsilon}\right]}{\left(a_1(t)+a_2(t)m_1^{\epsilon}+a_3(t)m_2^{\epsilon}+a_4(t)m_1^{\epsilon}m_2^{\epsilon}\right)^2}\right.\right.$$

$$\left.\left.+\,e(t)+\frac{f(t)\left[a_3(t)m_1^{\epsilon}+a_4(t)m_1^{\epsilon 2}\right]}{\left(a_1(t)+a_2(t)M_1^{\epsilon}+a_3(t)M_2^{\epsilon}+a_4(t)M_1^{\epsilon}M_2^{\epsilon}\right)^2}\right]m_2^{\epsilon}\right\}>0.$$

Hence condition 3 of Lemma 3 is also satisfied. Therefore, by Theorem 12 and Lemma 3, it can be concluded that the model system (34) has a unique almost-periodic solution $(\bar{x}^*(t), \bar{y}^*(t))$ (say) in K_ϵ^*, which is uniformly asymptotically stable in K_ϵ^*. Hence the model system (1) has a unique positive almost-periodic solution $(\bar{x}^*(t), \bar{y}^*(t))$ in K_ϵ^*, which is uniformly asymptotically stable in K_ϵ^*. From Theorem 7, we have that $(\bar{x}^*(t), \bar{y}^*(t))$ is globally asymptotically stable, which completes the proof. □

Definitions and preliminaries

Here, to present sufficient conditions for the existence of a positive periodic and almost periodic solutions for the model system (1), some notations, definitions and lemmas have been introduced.

Lemma A1 (Comparison Lemma, Abbas et al., 2012). *If* $p > 0$, $q > 0$ *and* $\frac{du}{dt} \le (\ge) u(t)(q - pu(t))$, $u(0) > 0$, *then we have* $\limsup_{t\to+\infty} u(t) \le \frac{q}{p}$ $(\liminf_{t\to+\infty} u(t) \ge \frac{q}{p})$.

Definition A1 (Permanence and nonpermanence, Li & Takeuchi, 2015). The system (1) is said to be permanent if there exist positive real constants m and M with $0 < m \le M$ such that $min\{\liminf_{t\to\infty} x(t), \liminf_{t\to\infty} y(t)\} \ge m$, $max\{\limsup_{t\to\infty} x(t), \limsup_{t\to\infty} y(t)\} \le M$, for all the solutions $(x(t), y(t))$ of model system (1) with positive initial values. Model system (1) is said to be nonpermanent if there is a positive solution $(x(t), y(t))$ of (1) such that $min\{\limsup_{t\to\infty} x(t), \limsup_{t\to\infty} y(t)\} = 0$.

Definition A2 (Fan & Kuang, 2004). The solution set of the model system (1) is ultimately bounded if there exists a real constant $S > 0$ such that for every solution $(x(t), y(t))$ of (1), there exists a real constant $T > 0$ such that $\|(x(t), y(t))\| < S, \forall\, t \ge t_0 + T$, where S is independent of particular solution while T may depend on the solution. Here $\|.\|$ is the standard euclidian norm.

Definition A3 (Globally attractive solution, Fink & Dold, 1974). A bounded positive solution $X(t) = (\hat{x}(t), \hat{y}(t))$ of the model system (1) with $X(0) > 0$ is said to be globally attractive (globally asymptotically stable), if any other solution $Y(t) = (x(t), y(t))$ of the system (1) with $Y(0) > 0$, satisfies $\lim_{t\to+\infty}\|X(t) - Y(t)\| = 0$.

Lemma A2 (Barbalat, 1959; Kannan & Krueger, 2012). *Let* ζ *be a real number and* h *be a non-negative function defined on* $[\zeta, +\infty)$ *such that* h *is integrable on* $[\zeta, +\infty)$ *and is uniformly continuous on* $[\zeta, +\infty)$, *then* $\lim_{t\to+\infty} h(t) = 0$.

Lemma A3 (Brouwer fixed-point theorem Agarwal et al., 2001). *Let $\bar{\Upsilon}$ be a closed bounded convex subset of $\mathbb{R}^n$. Let ρ be a continuous operator that maps $\bar{\Upsilon}$ into itself. Then the operator ρ has at least one fixed point in $\bar{\Upsilon}$ that is there exists a point $\hat{x} \in \bar{\Upsilon}$ such that $\rho(\hat{x}) = \hat{x}$.*

Definition A4 (Almost periodic solution, Fink & Dold, 1974). A vectorial function $f : \mathbb{R}^{n+1} \to \mathbb{R}^m f(t, x)$, where f is an m-vector, t is a real scalar and x is an n-vector, is said to be almost periodic in t uniformly with respect to $x \in X \subset \mathbb{R}^n$, if $f(t, x)$ is continuous in $t \in \mathbb{R}$ and $x \in X$, and if for any $\epsilon > 0$, it is possible to find a constant $l(\epsilon) > 0$ such that in any interval of length $l(\epsilon)$ there exists a τ such that the inequality

$$\|f(t+\tau, x) - f(t, x)\| = \sum_{i=1}^{m} |f_i(t+\tau, x) - f_i(t, x)| < \epsilon$$

is satisfied for all $t \in \mathbb{R}$, $x \in X$. The number τ is called an ϵ-translation number of $f(t, x)$.

Definition A5 (Du & Lv, 2013). A real function $f : \mathbb{R} \to \mathbb{R}$ is said to be asymptotically almost periodic function if there exists an almost-periodic function $q(t)$ and a continuous real function $r(t)$ such that $f(t) = q(t) + r(t)$, $r(t) \to 0$ as $t \to \infty$.

Definition A6 (Gaines & Mawhin, 1977; Guo et al., 1995). Let Y and Z be two Banach spaces. Let $L : Dom(L) \subset Y \to Z$ is a linear map, and $N : Y \to Z$ be a continuous map. The operator L is called Fredholm operator of index 0 if dim (Ker L) = codim (Im L) $< +\infty$ and Im L is closed in Z. If L is a Fredholm operator of index zero and there exist continuous projections $P : Y \to Y$ and $Q : Z \to Z$ such that Im (P) = Ker (L), Ker (Q) = Im (L) = Im$(I - Q)$, it follows that L | Dom ($L \cup KerP$): $(I - P)X \to$ Im (L) is invertible. We denote the inverse of the map by K_P. If Υ is an open bounded subset, the mapping L is called L- compact on $\bar{\Upsilon}$ if $QN(\bar{\Upsilon})$ is bounded and $K_p(I - Q)N$: $\bar{\Upsilon} \to Y$ is compact. Since *Im* Q is isomorphic to *Ker* L, then there exists an isomorphism J : *Im* Q $\to$ *Ker* L. Here dim stands for dimension, codim for codimension and Dom is used for Domain.

Definition A7 (Deimling, 1989; Gaines & Mawhin, 1977). Let $\Upsilon \subset \mathbb{R}^n$ be an open and bounded set, $f \in C^1(\Upsilon, \mathbb{R}^n) \cap C(\bar{\Upsilon}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n / f(\partial\Upsilon \cup N_f)$, that is, y is a regular value of f. Here, $N_f = \{x \in \Upsilon: J_f(x) = 0\}$, the critical set of f and J_f is the Jacobian of f at x. Then the degree of f is denoted by deg $\{f, \Upsilon, y\}$, is defined by

$$deg\{f, \Upsilon, y\} = \sum_{x \in f^{-1}y} sgn \quad J_f(x). \tag{A10}$$

For more details about Degree Theory, the interested readers may refer Deimling (1989).

Definition A8 (Homotopy invariance, Flanders, 1963). Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $V(\Omega) = \{f \in C(\bar{\Omega}, \mathbb{R}^n)$: $0 \in f(\partial\Omega)\}$. Then the mapping $deg(.,\Omega)$: $V(\Omega) \to \mathbb{Z}$ is well defined. Moreover If $h : [0, 1] \times \bar{\Omega} \to \mathbb{R}^n$ is continuous and such that $0 \notin h(t, \partial\Omega)$ for all $t \in [0, 1]$ then $deg(h(t, .), \Omega)$ does not depend on t. If we "deform with continuity" a function $f \in V(\Omega)$ into another function $g \in \Omega$ then $deg(f, \Omega) = deg(g, \Omega)$, with the essential assumption that no zeros appear in $\partial\Omega$ throughout the homotopy.

Lemma A4 (Continuation theorem, Gaines & Mawhin, 1977). *Let L be a Fredholm mapping of index zero and N be a L-compact space on $\bar{\Upsilon}$. Furthermore assume*

(*i*) *for each $\lambda \in (0, 1)$, $x \in \partial\Upsilon \cup DomL$, $Lx \neq \lambda Nx$;*
(*ii*) *for each $x \in \partial\Upsilon \cup KerL$, $QNx \neq 0$ and $deg\{JQN, \Upsilon \cup KerL, 0\} \neq 0$.*

Then the operator equation $Lx = Nx$ has at least one solution in $\bar{\Upsilon} \cup DomL$.

Now we introduce the following function space with its norm:

$$X = \{w(t) = (u, v)^T \in C(\mathbb{R}, \mathbb{R}^2) | w(t + \omega) = w(t)\},$$

with norm

$$\|w\| = \max_{t\in[0,\omega]} \|w(t)\| = \max_{t\in[0,\omega]} \|u(t)\| + \max_{t\in[0,\omega]} \|v(t)\|,$$

for $(u, v) \in X$. Obviously, X is the Banach spaces when it endowed with the above norm $\|.\|$.