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QUADRATIC GROWTH AND STRONG METRIC SUBREGULARITY OF THE SUBDIFFERENTIAL VIA SUBGRADIENT GRAPHICAL DERIVATIVE*

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Abstract. This paper mainly studies the relationship between quadratic growth and strong metric subregularity of the subdifferential in finite dimensional settings by using the subgradient graphical derivative. We prove that the positive definiteness of the subgradient graphical derivative of an extended-real-valued lower semicontinuous proper function at a proximal stationary point is sufficient for the point to be a local minimizer at which the subdifferential is strongly subregular for 0. The latter was known to imply the quadratic growth. When the function is either subdifferentially continuous, prox-regular, twice epidifferentiable, or variationally convex, we show that the quadratic growth, the strong metric subregularity of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent. For $\mathcal{C}^2$-cone reducible constrained programs satisfying the metric subregularity constraint qualification, we obtain the same results for the sum of the objective function and the indicator function of the feasible set.

Key words. quadratic growth, strong local minimizer, second-order sufficient condition, subgradient graphical derivative, metric subregularity constraint qualification, conic programming

AMS subject classifications. 49J53, 90C31, 90C46

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1. Introduction.

Given a proper extended-real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we say f satisfies the *quadratic growth condition* (QGC) at a point $\bar{x} \in \mathbb{R}^n$ (such a point $\bar{x}$ is called a *strong local minimizer* of f) if there exist $\gamma > 0$ and modulus $\kappa > 0$ such that

$$f(x) - f(\bar{x}) \geq \frac{\kappa}{2}\|x - \bar{x}\|^2 \quad \text{for all} \quad x \in \mathbb{B}_\gamma(\bar{x}). \tag{1.1}$$

It is an important concept in optimization [3, 6, 7, 19, 24, 25, 27]. In the case of unconstrained $\mathcal{C}^2$-smooth optimization problems, strong local minimizers are fully characterized by the positive definiteness of the Hessian of the objective function at stationary points. When the problem is not smooth, several different types of second-order *directional derivatives* are useful for studying strong minimizers [3, 6, 7, 19, 24, 25, 27]. Especially, the *second subderivative* can be used to characterize strong local minimizers [24, Theorem 13.24].

In recent years, the connection between the quadratic growth condition and the strong metric subregularity of the subdifferential at local minimizers, which is

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interesting from a numerical optimization viewpoint [4, 12, 26], has been investigated in [1, 2, 11, 13]. Their relation dates back to the original work of Zhang and Treiman [28], where they studied functions with *upper-Lipschitz* inverse subdifferentials. Aragón Artacho and Geoffroy [1] developed the idea of Zhang and Treiman [28] by replacing the upper-Lipschitz property by some metric regularity properties of the subdifferential, but they focused on the case of convex functions. Among other things, they [1, Theorem 3.5] showed that when f is a proper lower semicontinuous convex function, the QGC holds at $\bar{x}$ if and only if the subdifferential ∂f is *strongly metrically subregular* at the minimizer $\bar{x}$ for 0 in the sense that there exist $\varepsilon > 0$ and modulus $\ell > 0$ such that

$$\ell d(0; \partial f(x)) \geq \|x - \bar{x}\| \quad \text{for all} \quad x \in \mathbb{B}_\varepsilon(\bar{x}), \tag{1.2}$$

where $d(0; \partial f(x))$ is the distance from 0 to the set of subgradients $\partial f(x)$. Without convexity, Drusvyatskiy, Mordukhovich, and Nghia [13, Corollary 3.5] proved that the latter property of the limiting subdifferential at a local minimizer implies the quadratic growth condition. Using some facts from semialgebraic geometry, Drusvyatskiy and Ioffe [11, Theorem 3.1] showed that the converse is also true for nonconvex semialgebraic functions. The question of whether this implication holds for other favorable classes of functions remains open.

Our main aim in the present paper is to find out an answer for the above open question. To this end, we utilize the *subgradient gradient derivative* defined as the graphical derivative acting on the subgradient mapping. Our choice is motivated by the Levy–Rockafellar criterion on strong metric subregularity via the graphical derivative, formulas for computing the subgradient graphical derivative [8, 16, 17, 18], and the results of Aragón Artacho and Geoffroy [1, 2] showing that the quadratic growth, the strong metric subregularity of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent if the function is lower semicontinuous convex.

The idea of using the subgradient graphical derivative to investigate strong local minimizers for nonconvex functions dates back to Eberhard and Wenczel [14]. In that paper, they [14, Theorem 71(2)] claimed that for lower semicontinous prox-bounded and proximally stable functions, the so-called *sufficient condition of the second kind* at a proximal stationary point (see Definition 3.3), which is weaker than the aforementioned positive definiteness of the subgradient graphical derivative, ensures the validity of the QGC at the reference point. However, our Example 3.4 shows that the claim fails even for convex functions. On the other hand, since there may exist stationary points that are not local minimizers if the function is nonconvex, in order to use [13, Corollary 3.5], we need a subgradient graphical derivative–related sufficient optimality condition. These lead us to the question of whether the positive definiteness of the subgradient graphical derivative at a stationary point implies that the point is a local minimizer.

Our main contributions and the organization of the paper are as follows. After recalling in section 2 the needed preliminary material from variational analysis [10, 20, 21, 24], in the first main part of our paper, section 3, we show that the positive definiteness of the subgradient graphical derivative at *proximal* stationary points is sufficient for the point to be a local minimizer at which the subdifferential is strongly subregular for 0. The latter was known to imply the quadratic growth by [13, Corollary 3.5]. Our proof is strongly based on [13, Corollary 3.3], the Levy–Rockafellar criterion, and a sum rule for the graphical derivative. For functions from the two favorable classes of *subdifferentially continuous, prox-regular, twice epidifferentiable functions* [24, Chapter 13] and *variationally convex functions* [22, 23], we show further that the quadratic growth, the strong metric subregularity of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent. Since functions from the just mentioned two classes are not necessarily semialgebraic and convex, our results complement the corresponding ones constructed in [1, 11, 13].

In the second main part of this paper, section 4, we examine $\mathcal{C}^2$-cone reducible constrained programs satisfying the metric subregularity constraint qualification. For the sum of the objective function and the indicator function of the feasible set of such a conic program, we prove that the quadratic growth, the strong metric subregularity of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent. To the best of our knowledge, this result is new even for nonlinear programs. We note that the constraint qualification also allows us to make the connection between the positive definiteness of the subgradient graphical derivative and some known second-order sufficient conditions (Remark 4.8). Furthermore, Example 4.10 shows that in the absence of the constraint qualification the quadratic growth does not imply the strong metric subregularity of the subdifferential.

Finally, in section 5, we summarize the obtained results of the paper and discuss the perspectives of this research direction.

2. Preliminaries.

In this section, we recall some basic notions and facts from variational analysis that will be used repeatedly in what follows; see [10, 20, 21, 24] for more details. Let Ω be a nonempty subset of the Euclidean space $\mathbb{R}^n$ and $\bar{x}$ be a point in Ω. The (Bouligand–Severi) *tangent/contingent cone* to the set Ω at $\bar{x}\in\Omega$ is known as

$$T_\Omega(\bar{x}) := \big\{v\in\mathbb{R}^n \,|\, \text{there exist } t_k\downarrow 0,\ v_k\to v \ \text{ with } \ \bar{x}+t_kv_k\in\Omega \text{ for all } k\in\mathbb{N}\big\}.$$

The polar cone of the tangent cone is the (Fréchet) *regular normal cone* to Ω at $\bar{x}$ defined by

$$\widehat{N}_\Omega(\bar{x}) := T_\Omega(\bar{x})^\circ. \tag{2.1}$$

Another normal cone construction used in our work is the (Mordukhovich) *limiting/basic normal cone* to Ω at $\bar{x}\in\Omega$ defined by

$$N_\Omega(\bar{x}) := \big\{v\in\mathbb{R}^n \,|\, \text{there exist } x_k \overset{\Omega}{\to} \bar{x},\ v_k\in\widehat{N}_\Omega(x_k) \text{ with } v_k\to v\big\}.$$

When $\bar{x}\notin\Omega$, we set $T_\Omega(\bar{x})=\emptyset$ and $N_\Omega(\bar{x})=\widehat{N}_\Omega(\bar{x})=\emptyset$ by convention. When the set Ω is convex, the above tangent cone and normal cones reduce to the tangent cone and normal cone in the sense of classical convex analysis.

Consider the set-valued mapping $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ with the domain $\operatorname{dom}F := \big\{x\in\mathbb{R}^n|\ F(x)\neq\emptyset\big\}$ and graph $\operatorname{gph}F := \big\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m|\, y\in F(x)\big\}$. Suppose that $(\bar{x},\bar{y})$ is an element of $\operatorname{gph}F$. The *graphical derivative* of F at $\bar{x}$ for $\bar{y}\in F(\bar{x})$ is the set-valued mapping $DF(\bar{x}|\bar{y}):\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ defined by

$$DF(\bar{x}|\bar{y})(w) := \big\{z\in\mathbb{R}^m \,|\, (w,z)\in T_{\operatorname{gph}F}(\bar{x},\bar{y})\big\} \ \text{ for } \ w\in\mathbb{R}^n, \tag{2.2}$$

which means $\operatorname{gph}DF(\bar{x}|\bar{y}) = T_{\operatorname{gph}F}(\bar{x},\bar{y})$; see, e.g., [10, 24]. Another generalized derivative for set-valued mappings used (infrequently) in our paper is the *regular* *coderivative* [20] $\widehat{D}^*F(\bar{x}|\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of F at $\bar{x}$ for $\bar{y} \in F(\bar{x})$ defined by

$$\widehat{D}^*F(\bar{x}|\bar{y})(z) := \big\{w \in \mathbb{R}^n \,|\, (w, -z) \in \widehat{N}_{\mathrm{gph}F}(\bar{x}, \bar{y})\big\} \quad \text{for all } z \in \mathbb{R}^m. \tag{2.3}$$

We note further that if $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a single-valued mapping differentiable at $\bar{x}$, then $D\Phi(\bar{x}|\Phi(\bar{x}))(w) = \{\nabla\Phi(\bar{x})w\}$ for any $w \in \mathbb{R}^n$, while $\widehat{D}^*\Phi(\bar{x}|\Phi(\bar{x}))(z) = \{\nabla\Phi(\bar{x})^*z\}$ for any $z \in \mathbb{R}^m$.

Following [10, section 3H], we say F is *metrically subregular* at $\bar{x} \in \mathrm{dom}\, F$ for $\bar{y} \in F(\bar{x})$ with modulus $\kappa > 0$ if there exists a neighborhood U of $\bar{x}$ such that

$$d(x; F^{-1}(\bar{y})) \le \kappa\, d(\bar{y}; F(x)) \quad \text{for all} \quad x \in U, \tag{2.4}$$

where $d(x; \Omega)$ represents the distance from a point $x \in \mathbb{R}^n$ to a set $\Omega \subset \mathbb{R}^n$. The infimum of all such κ is the modulus of metric subregularity, denoted by $\mathrm{subreg}\, F(\bar{x}|\bar{y})$. If additionally $\bar{x}$ is an isolated point to $F^{-1}(\bar{y})$, we say F is *strongly metrically subregular* at $\bar{x}$ for $\bar{y}$. It is known from [10, Theorem 4E.1] that F is strongly metrically subregular at $\bar{x}$ for $\bar{y}$ if and only if

$$DF(\bar{x}|\bar{y})^{-1}(0) = \{0\}. \tag{2.5}$$

Moreover, in the latter case, its modulus of (strong) metric subregularity is computed by

$$\mathrm{subreg}\, F(\bar{x}|\bar{y}) = \frac{1}{\inf\{\|z\| \,|\, z \in DF(\bar{x}|\bar{y})(w), \|w\| = 1\}}. \tag{2.6}$$

Assume that $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is an extended-real-valued lower semicontinuous (l.s.c.) proper function with $\bar{x} \in \mathrm{dom}\, f := \big\{x \in \mathbb{R}^n |\, f(x) < \infty\big\}$. The *limiting subdifferential* (known also as the Mordukhovich/basic subdifferential) of f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) := \big\{v \in \mathbb{R}^n \,|\, (v, -1) \in N_{\mathrm{epi}f}(\bar{x}, f(\bar{x}))\big\},$$

where $\mathrm{epi}\, f := \big\{(x, r) \in \mathbb{R}^n \times \mathbb{R} |\, r \ge f(x)\big\}$ is the epigraph of f. Another subdifferential construction used in this paper is the *proximal subdifferential* of f at $\bar{x}$ defined by

$$\partial_p f(\bar{x}) := \left\{v \in \mathbb{R}^n |\, \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|^2} > -\infty\right\}. \tag{2.7}$$

It is well known that

$$\partial f(\bar{x}) = \big\{v \in \mathbb{R}^n |\, \text{there exists } (x_k, v_k) \to (\bar{x}, v) \text{ with } v_k \in \partial_p f(x_k) \text{ and } f(x_k) \to f(\bar{x})\big\}, \tag{2.8}$$

which shows that $\partial_p f(\bar{x}) \subset \partial f(\bar{x})$.

Function f is said to be *prox-regular* at $\bar{x} \in \mathrm{dom}\, f$ for $\bar{v} \in \partial f(\bar{x})$ if there exist $r, \varepsilon > 0$ such that for all $x, u \in \mathbb{B}_\varepsilon(\bar{x})$ with $|f(u) - f(\bar{x})| < \varepsilon$ we have

$$f(x) \ge f(u) + \langle v, x - u\rangle - \frac{r}{2}\|x - u\|^2 \quad \text{for all} \quad v \in \partial f(u) \cap \mathbb{B}_\varepsilon(\bar{v}), \tag{2.9}$$

where $\mathbb{B}_\varepsilon(\bar{x}) := \{x |\, \|x - \bar{x}\| \le \varepsilon\}$ is the closed ball with center $\bar{x}$ and radius ε; see [24, Definition 13.27]. This clearly implies that $\partial f(u) \cap \mathbb{B}_\varepsilon(\bar{v}) \subset \partial_p f(x)$ whenever $\|u - \bar{x}\| < \varepsilon$ with $|f(u) - f(\bar{x})| < \varepsilon$. Moreover, f is said to be *subdifferentially continuous* at $\bar{x}$ for $\bar{v}$ if whenever $(x_k, v_k) \to (\bar{x}, \bar{v})$ and $v_k \in \partial f(x_k)$, one has $f(x_k) \to$

$f(\bar{x})$; see [24, Definition 13.28]. In the case that f is subdifferentially continuous at $\bar{x}$ for $\bar{v}$, the inequality "$|f(u) - f(\bar{x})| < \varepsilon$" in the definition of prox-regularity above could be omitted.

Recall from [24, Definition 13.3] that the *second subderivative* of f at $\bar{x}$ for $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ is given by

$$d^2 f(\bar{x}|v)(w) = \liminf_{\substack{\tau \searrow 0 \\ w' \longrightarrow w}} \Delta_\tau^2 f(\bar{x}|v)(w'), \tag{2.10}$$

where

$$\Delta_\tau^2 f(\bar{x}|v)(w') = \frac{f(\bar{x} + \tau w') - f(\bar{x}) - \tau\langle v, w'\rangle}{\frac{1}{2}\tau^2}.$$

Function f is said to be *twice epidifferentiable* at $\bar{x} \in \mathbb{R}^n$ for $v \in \mathbb{R}^n$ if for every $w \in \mathbb{R}^n$ and choice of $\tau_k \searrow 0$ there exists $w^k \to w$ such that

$$\frac{f(\bar{x} + \tau_k w^k) - f(\bar{x}) - \tau_k\langle v, w^k\rangle}{\frac{1}{2}\tau_k^2} \to d^2 f(\bar{x}|v)(w);$$

see, e.g., [24, Definition 13.6]. We note that *fully amenable functions*, including the maximum of finitely many C^2-functions, are important examples for subdifferentially continuous prox-regular and twice epidifferentiable l.s.c. proper functions [24, Corollary 13.15 and Proposition 13.32].

The main second-order structure used in this paper is the *subgradient graphical derivative* $D(\partial f)(\bar{x}|\bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$, which is defined from (2.2) by

$$D(\partial f)(\bar{x}|\bar{v})(w) := \big\{z \,|\, (w, z) \in T_{\mathrm{gph}\, \partial f}(\bar{x}, \bar{v})\big\} \quad \text{for all} \quad w \in \mathbb{R}^n. \tag{2.11}$$

In the case that f is twice epidifferentiable, prox-regular, and subdifferentially continuous at $\bar{x}$ for $\bar{v}$, it is known from [24, Theorem 13.40] that

$$D(\partial f)(\bar{x}|\bar{v}) = \partial h \quad \text{with} \quad h = \frac{1}{2} d^2 f(\bar{x}|\bar{v}), \tag{2.12}$$

which is an important formula in our study. When f is twice differentiable at $\bar{x}$, it is clear that $D(\partial f)(\bar{x}|\nabla f(\bar{x})) = \{\nabla^2 f(\bar{x})\}$.

3. Quadratic growth and strong metric subregularity of the subdifferentials of extended-real-valued functions. Given a function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and a point $\bar{x} \in \operatorname{dom} f$, $\bar{x}$ is said to be a strong local minimizer of f with modulus $\kappa > 0$ if there is a number $\gamma > 0$ such that the quadratic growth condition (QGC) (1.1) holds. We define the exact modulus for the QGC of f at $\bar{x}$ by

$$\mathrm{QG}\,(f; \bar{x}) := \sup\big\{\kappa > 0 \mid \bar{x} \text{ is a strong local minimizer of } f \text{ with modulus } \kappa\big\}.$$

In this section, we first introduce several new sufficient and necessary conditions for the QGC (1.1) by using the second-order construction defined in (2.11). The following fact, taken from [13, Corollary 3.3], which provides a sufficient condition for the QGC of f at $\bar{x}$ (1.1) via strong metric subregularity (SMS) of the subdifferential (1.2), is a significant tool in our analysis. We will show later that SMS of the subdifferential (1.2) at a local minimizer $\bar{x}$ is also a necessary condition for the QGC of f at $\bar{x}$ in several classes of optimization.

LEMMA 3.1 (SMS of the subdifferential and QGC [13, Corollary 3.3]). *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be an l.s.c. proper function, and let $\bar{x} \in \operatorname{dom} f$ with $0 \in \partial f(\bar{x})$. Suppose that the subgradient mapping ∂f is strongly metrically subregular at $\bar{x}$ for 0 with modulus $\kappa > 0$ and there are real numbers $r \in (0, \kappa^{-1})$ and $\delta > 0$ such that*

$$f(x) \geq f(\bar{x}) - \frac{r}{2}\|x - \bar{x}\|^2 \quad \text{for all} \quad x \in \mathbb{B}_\delta(\bar{x}). \tag{3.1}$$

Then, for any $\alpha \in (0, \kappa^{-1})$, there exists a real number $\eta > 0$ such that

$$f(x) \geq f(\bar{x}) + \frac{\alpha}{2}\|x - \bar{x}\|^2 \quad \text{for all} \quad x \in \mathbb{B}_\eta(\bar{x}). \tag{3.2}$$

When the function f is convex, the QGC could be fully characterized via the positive definiteness of the subgradient graphical derivative defined by (2.11) [2, Corollary 3.7]. Without convexity, we show in the next result that such a property is sufficient for the QGC.

THEOREM 3.2 (sufficient conditions for the QGC via the positive definiteness of the subgradient graphical derivative). *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper l.s.c. function with $\bar{x} \in \operatorname{dom} f$. Consider the following assertions:*

(i) *The quadratic growth condition* (1.1) *is satisfied.*

(ii) *$\bar{x}$ is a local minimizer and ∂f is strongly metrically subregular at $\bar{x}$ for 0.*

(iii) *$0 \in \partial_p f(\bar{x})$ and $D(\partial f)(\bar{x}|0)$ is positive definite in the following sense:*

$$\langle z, w\rangle > 0 \quad \text{for all} \quad z \in D(\partial f)(\bar{x}|0)(w) \quad \text{and} \quad w \in \operatorname{dom} D(\partial f)(\bar{x}|0) \setminus \{0\}. \tag{3.3}$$

(iv) *$0 \in \partial_p f(\bar{x})$ and there exists some real number $c > 0$ such that*

$$\langle z, w\rangle \geq c\|w\|^2 \quad \text{for all} \quad z \in D(\partial f)(\bar{x}|0)(w) \quad \text{and} \quad w \in \operatorname{dom} D(\partial f)(\bar{x}|0). \tag{3.4}$$

Then we have the implications [(iv)⇒(iii)⇒(ii)⇒(i)]. *Moreover, if* (iv) *is valid, we have*

$$\mathrm{QG}(f;\bar{x}) \geq \inf\left\{ \frac{\langle z, w\rangle}{\|w\|^2} \,\middle|\, z \in D(\partial f)(\bar{x}|0)(w), w \in \operatorname{dom} D(\partial f)(\bar{x}|0) \right\} \tag{3.5}$$

with the convention that $0/0 = \infty$.

Proof. Let us start the proof by assuming that (iv) is valid, i.e., $0 \in \partial_p f(\bar{x})$ and condition (3.4) is satisfied. We claim that (3.5) holds.

Since $0 \in \partial_p f(\bar{x})$, we have $0 \in \partial f(\bar{x})$ and there exist $r, \gamma > 0$ such that

$$f(x) - f(\bar{x}) \geq -\frac{r}{2}\|x - \bar{x}\|^2 \quad \text{for all} \quad x \in \mathbb{B}_\gamma(\bar{x}). \tag{3.6}$$

To proceed, pick any $s > r$ and define $g(x) := f(x) + \frac{s}{2}\|x - \bar{x}\|^2$, and then it is clear that

$$g(x) - g(\bar{x}) \geq \frac{s - r}{2}\|x - \bar{x}\|^2 \quad \text{for all} \quad x \in \mathbb{B}_\gamma(\bar{x}). \tag{3.7}$$

Note further that $\partial g(x) = \partial f(x) + s(x - \bar{x})$ and thus $0 \in \partial g(\bar{x})$. Thanks to the sum rule of the graphical derivative [10, Proposition 4A.2], we have

$$D(\partial g)(\bar{x}|0)(w) = D(\partial f)(\bar{x}|0)(w) + sw \quad \text{for all} \quad w \in \mathbb{R}^n. \tag{3.8}$$

Take any $(z,w)\in\mathbb{R}^n\times\mathbb{R}^n$ with $z\in D(\partial g)(\bar{x}|0)(w)$, i.e., $z-sw\in D(\partial f)(\bar{x}|0)(w)$. It follows from (3.4) that $\langle z-sw,w\rangle\ge c\|w\|^2$, which implies

$$\|z\|\cdot\|w\|\ge\langle z,w\rangle\ge(c+s)\|w\|^2. \tag{3.9}$$

We obtain that $D(\partial g)(\bar{x},0)^{-1}(0)=\{0\}$, i.e., ∂g is strongly metrically subregular at $\bar{x}$ for 0 by (2.5). Moreover, (2.6) tells us that

$$\operatorname{subreg}\partial g(\bar{x}|0)\le(c+s)^{-1}.$$

Since $\bar{x}$ is a local minimizer of g by (3.7), it follows from Lemma 3.1 that for any $\varepsilon>0$ there exists $\eta\in(0,\gamma)$ such that

$$\begin{aligned} g(x)&\ge g(\bar{x})+\tfrac{1}{2(\operatorname{subreg}\partial g(\bar{x}|0)+\varepsilon)}\|x-\bar{x}\|^2\\ &\ge g(\bar{x})+\tfrac{1}{2((c+s)^{-1}+\varepsilon)}\|x-\bar{x}\|^2\end{aligned}$$

for all $x\in\mathbb{B}_\eta(\bar{x})$. Since $f(x)=g(x)-\frac{s}{2}\|x-\bar{x}\|^2$, we obtain from the latter that

$$f(x)\ge f(\bar{x})+\frac{1}{2}\Big[\frac{1}{(c+s)^{-1}+\varepsilon}-s\Big]\|x-\bar{x}\|^2=f(\bar{x})+\frac{1}{2}\frac{\frac{c}{c+s}-s\varepsilon}{(c+s)^{-1}+\varepsilon}\|x-\bar{x}\|^2. \tag{3.10}$$

By choosing $\varepsilon>0$ sufficiently small, $\bar{x}$ is a strong local minimizer of f with a positive modulus being smaller than but arbitrarily close to c. Moreover, inequality (3.5) follows from (3.10) when taking $\varepsilon\downarrow 0$ and $c\to$ the right-hand side of (3.5).

Let us verify next [(iv)⇒(iii)⇒(ii)⇒(i)]. Obviously, [(iv)⇒(iii)] and [(ii)⇒(i)] follow directly from Lemma 3.1. It suffices to prove [(iii)⇒(ii)]. Suppose that (iii) is valid, i.e., $0\in\partial_p f(\bar{x})$ and condition (3.3) is satisfied. It follows that

$$D(\partial f)(\bar{x}|0)^{-1}(0)=\{0\}.$$

By (2.5), ∂f is strongly metrically subregular at $\bar{x}$ for 0 with any modulus $\kappa>\operatorname{subreg}\partial f(\bar{x}|0)\ge 0$. Since $0\in\partial_p f(\bar{x})$, we find some $r,\gamma>0$ satisfying (3.6) again. Pick any $s>r$, and define $g(x)=f(x)+\frac{s}{2}\|x-\bar{x}\|^2$ as above. For any $(z,w)\in\mathbb{R}^n\times\mathbb{R}^n$ with $z\in D(\partial g)(\bar{x}|0)(w)$, we derive from (3.8) that $z-sw\in D(\partial f)(\bar{x}|0)(w)$. It follows from (3.3) that $\langle z-sw,w\rangle\ge 0$, which means

$$\|z\|\cdot\|w\|\ge\langle z,w\rangle\ge s\|w\|^2.$$

This together with (2.5) and (2.6) tells us that ∂g is strongly metrically subregular at $\bar{x}$ for 0 with

$$\operatorname{subreg}\partial g(\bar{x}|0)\le s^{-1}.$$

Since $\bar{x}$ is a local minimizer of g by (3.7), it follows from Lemma 3.1 again that for any $\varepsilon>0$ with $\frac{s\varepsilon}{s^{-1}+\varepsilon}<\kappa^{-1}$, there exists $\delta\in(0,\gamma)$ such that

$$\begin{aligned} g(x)&\ge g(\bar{x})+\tfrac{1}{2(\operatorname{subreg}\partial g(\bar{x}|0)+\varepsilon)}\|x-\bar{x}\|^2\\ &\ge g(\bar{x})+\tfrac{1}{2(s^{-1}+\varepsilon)}\|x-\bar{x}\|^2\end{aligned}\qquad\text{for all}\quad x\in\mathbb{B}_\delta(\bar{x}).$$

Since $f(x)=g(x)-\frac{s}{2}\|x-\bar{x}\|^2$, we derive

$$f(x)\ge f(\bar{x})+\frac{1}{2}\Big[\frac{1}{s^{-1}+\varepsilon}-s\Big]\|x-\bar{x}\|^2=f(\bar{x})-\frac{1}{2}\frac{s\varepsilon}{s^{-1}+\varepsilon}\|x-\bar{x}\|^2\quad\text{for all}\quad x\in\mathbb{B}_\delta(\bar{x}).$$

Since $\frac{s\varepsilon}{s^{-1}+\varepsilon}<\kappa^{-1}$ and ∂f is strongly metrically subregular at $\bar{x}$ for 0 with modulus κ, $\bar{x}$ is a (strong) local minimizer of f by Lemma 3.1 again. The proof is complete. □

It is worth noting that the implication [(iii)⇒(ii)] is the main part in the above result. The validity of (iii) implies SMS of the subdifferential is straightforward from Levy–Rockafellar criterion (2.5), but the proof that turns the proximal stationary point $\bar{x}$ in (iii) to a local minimizer in (ii) is not trivial.

When the function f is twice differentiable, the above theorem recovers the classical second-order sufficient condition, which says if $\nabla f(\bar{x}) = 0$ and there exists some $c > 0$ such that

$$\langle \nabla^2 f(\bar{x})w, w\rangle \geq c\|w\|^2 \quad \text{for all} \quad w \in \mathbb{R}^n,$$

then $\bar{x}$ is a strong local minimizer of f. The above condition is usually written as an equivalent form by

$$\langle \nabla^2 f(\bar{x})w, w\rangle > 0 \quad \text{for all} \quad w \in \mathbb{R}^n, w \neq 0,$$

which is (3.3) in this case. In the nondifferential case as in Theorem 3.2, it is also natural to question whether condition (3.4) is equivalent to (3.3). Obviously, (3.3) is a consequence of (3.4). In the general case, we do not know yet whether the converse implication is also true. But the equivalence between them will be clarified later in Theorems 3.7, 3.10, and 4.7 for several broad classes of nondifferentiable functions.

As far as we know, the first idea of using the subgradient graphical derivative to study the QGC was initiated by Eberhard and Wenczel [14], in which they introduced the sufficient conditions of the second and third kinds (3.11).

DEFINITION 3.3 (sufficient conditions of the second and third kinds [14]). *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper l.s.c. function with $\bar{x} \in \operatorname{dom} f$ and $0 \in \partial_p f(\bar{x})$. We say the sufficient condition of the second kind holds at $\bar{x}$ if there exists some $c > 0$ such that for any $w \in \operatorname{dom} D(\partial_p f)(\bar{x}|0)$ with $\|w\| = 1$,*

$$\exists\, z \in D(\partial_p f)(\bar{x}|0)(w) \ \text{ satisfying } \ \langle z, w\rangle \geq c, \tag{3.11}$$

where $\partial_p f$ is the proximal subdifferential *of f.*

Moreover, the sufficient condition of the third kind is said to hold at $\bar{x}$ when there exists $\kappa > 0$ such that

$$\langle z, w\rangle \geq \kappa \quad \textit{for all} \quad z \in \widehat{D}^*(\partial_p f)(\bar{x}|0)(w) \quad \textit{and} \quad w \in \operatorname{dom} \widehat{D}^*(\partial f)(\bar{x}|0), \|w\| = 1. \tag{3.12}$$

Precisely, Theorem 71(2) in [14] claims that when the function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is l.s.c., *prox-bounded*, and *proximally stable*, the sufficient condition of the second kind at $\bar{x}$ with $0 \in \partial_p f(\bar{x})$ ensures the QGC of f at $\bar{x}$. Moreover, the sufficient condition of the third kind at $\bar{x}$ is also sufficient for the QGC due to [14, Theorem 69]. However, it seems to us that these two results are incorrect due to the following example.[1]

EXAMPLE 3.4 (inaccuracy of [14, Theorems 69 and 71(2)]). *Define the function $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} x & \textit{if} \quad x > 1, \\ \alpha_{n+1}x + \beta_{n+1} & \textit{if} \quad \alpha_{n+1} < x \leq \alpha_n,\ n = 0, 1, 2, \ldots, \\ \beta & \textit{if} \quad x = 0, \\ +\infty & \textit{if} \quad x < 0, \end{cases} \tag{3.13}$$

[1]This example modified from our previous one in the original submission by cutting one branch on its graph was suggested by one of the referees.

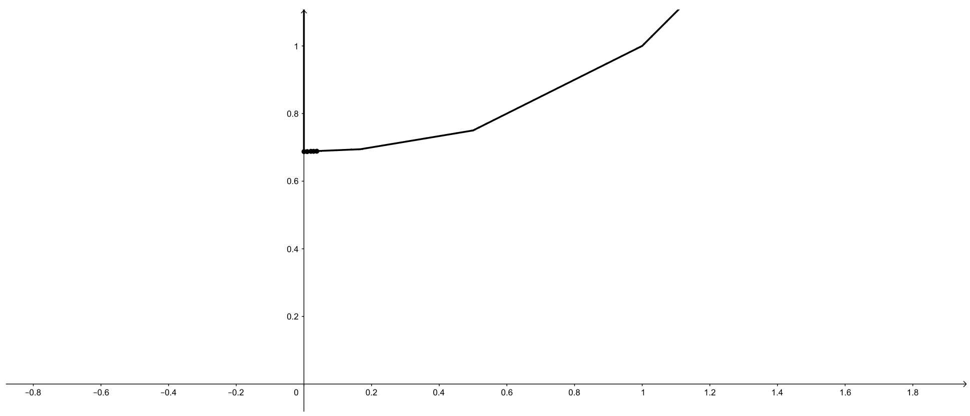

FIG. 1. *Graph of the l.s.c. convex function* f.

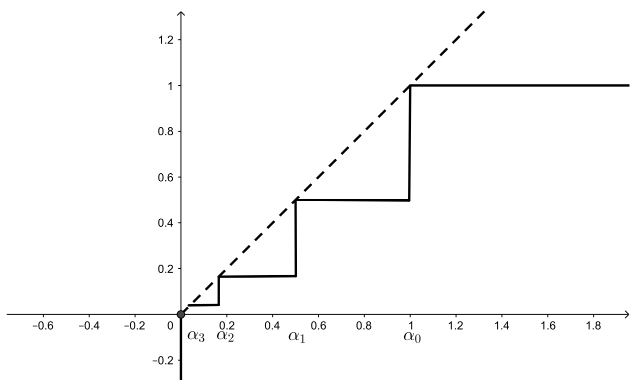

FIG. 2. *Graph of the subgradient mapping* $\partial_p f$.

where $\alpha_n = 1/(n+1)!$, $\beta_{n+1} := \sum_{k=0}^{n} \frac{1}{k!(k+2)!}$ *with* $n = 0, 1, 2, \ldots$, $\beta_0 = 0$, *and* $\beta = \lim_{n\to\infty} \beta_n$ *(see Figure* 1 *for the graph of* f*). It is easy to see that* f *is an l.s.c. convex function with global optimal solution* $\bar{x} := 0$*, which clearly implies that* f *is* prox-bounded *and* proximally stable *at* $\bar{x}$ *in the sense of* [14].

Moreover, direct computation on $\partial_p f$ *gives us that*

$$\text{(3.14)}\qquad \partial_p f(x) = \partial f(x) = \begin{cases} \{1\} & \text{if } \ x > 1, \\ [\alpha_{n+1}, \alpha_n] & \text{if } \ x = \alpha_n,\ n = 0, 1, 2, \ldots, \\ \{\alpha_{n+1}\} & \text{if } \ \alpha_{n+1} < x < \alpha_n,\ n = 0, 1, 2, \ldots, \\ \mathbb{R}_- & \text{if } \ x = 0, \\ \emptyset & \text{if } \ x < 0. \end{cases}$$

See Figure 2 *for the graph of* $\partial_p f$*. Define further* $K := \{(w, z) |\ 0 \le z \le w\} \cup \{0\} \times \mathbb{R}_-$*; then we have* $\operatorname{gph} \partial_p f \subset K$ *and*

$$\text{(3.15)}\qquad T := T_{\operatorname{gph} \partial_p f}(\bar{x}, 0) \subset T_K(\bar{x}, 0) = K.$$

Next we verify the "$\supset$*" inclusion in* (3.15)*. Take any* (w, z) *with* $0 \le z \le w$*, and consider the following three cases:*

- *Case* 1: $(w, z) = (0, 0)$ *clearly belongs to* T.
- *Case* 2: $z = 0 < w$. *Choose* $t_n = \alpha_n/w \downarrow 0$ *as* $n \to \infty$*; then we have* $t_n(w, \frac{w}{n+2}) = (\alpha_n, \alpha_{n+1}) \in \operatorname{gph} \partial_p f$, *and thus* $(w, \frac{w}{n+2}) \to (w, 0) \in T$.
- *Case* 3: $0 < z \le w$. *Fix* $k \in \mathbb{N}$ *satisfying* $1/(k+2) \le z/w$, *and define* $t_n := \alpha_n/w$ *for* $n \ge k$*; then we have* $t_n(w, z) = (\alpha_n, \alpha_n z/w) \in \{\alpha_n\} \times [\alpha_{n+1}, \alpha_n] \subset \operatorname{gph} \partial_p f$.

It follows that $\{(w, z)|\ 0 \le z \le w\} \subset T$. *Moreover, for any* $z \in \mathbb{R}_-$, $n \in \mathbb{N}$, *we have* $\frac{1}{n} z \in \partial_p f(0)$ *for any* n, *which means* $(0, z) \in T$, *i.e.,* $\{0\} \times \mathbb{R}_- \subset T$. *Combining this inclusion with* (3.15) *ensures the equality in* (3.15). *Note further that* $1 \in D(\partial_p f)(\bar{x}|0)(1)$, *which verifies the sufficient condition of the second kind* (3.11) *at* $\bar{x}$ *with* $\kappa = 1$. *However, both* (3.3) *and* (3.4) *are not satisfied and the quadratic growth condition* (1.1) *is not valid at* $\bar{x}$. *This tells us that the statement of* [14, *Theorem* 71(2)] *is not accurate even in the convex case.*

Observe further that $z \in \widehat{D}^*(\partial_p f)(\bar{x}|0)(w)$ *if and only if*

$$(z, -w) \in \widehat{N}_{\operatorname{gph} \partial_p f}(\bar{x}, 0) = T^\circ = \{(x, y)|\ -x \ge y \ge 0\}, \tag{3.16}$$

which means $z \le w \le 0$. *It follows that*

$$\langle z, w \rangle \ge \|w\|^2$$

for any $z \in \widehat{D}^*(\partial_p f)(\bar{x}|0)(w)$. *Hence, the sufficient condition of the third kind is also satisfied, but as mentioned above,* $\bar{x}$ *is not a strong local minimizer. It means* [14, *Theorem* 69] *is incorrect too.*

Analyzing the proofs of [14, *Theorems* 69 *and* 71(2)] *reveals that the condition*

$$\operatorname{ri}\left(\operatorname{dom} \widehat{D}^*(\partial_p f)(\bar{x}|0)\right) \ne \{0\} \tag{3.17}$$

observed from [14, *Proposition* 64] *is missed in their statements, where* $\operatorname{ri}(\Omega)$ *represents the relative interior of a set* $\Omega \subset \mathbb{R}^m$.[2] *However, note that* (3.17) *is also satisfied for our function* f, *since*

$$\operatorname{dom} \widehat{D}^*(\partial_p f)(\bar{x}|0) = \mathbb{R}_-$$

thanks to (3.16).

As discussed in the introduction, the QGC and strong local minimizer could be fully characterized via several different types of second-order directional derivatives [3, 7, 24, 25, 27]. For instance, it follows from [24, Theorem 13.24] that $\bar{x}$ is a strong local minimizer to a proper function $f : \mathbb{R}^n \to \mathbb{R}$ if and only if $0 \in \partial f(\bar{x})$ (or $0 \in \partial_p f(\bar{x})$) and the second subderivative (2.10) of f at $\bar{x}$ for 0 is *positive definite* in the sense that

$$d^2 f(\bar{x}|0)(w) > 0 \quad \text{for all} \quad w \in \mathbb{R}^n,\ w \ne 0. \tag{3.18}$$

It is clear that the second subderivative (2.10) is a construction on primal space, while the subgradient graphical derivative (2.11) includes both primal and dual spaces. Connection between these two constructions could be found in (2.12) for a special class of twice epidifferentiable functions. Despite the simplicity of the second subderivative and the full characterization of QGC (3.18), computing $d^2 f(\bar{x}|0)$ could be challenging under some strong regularity conditions. On the other hand, the subgradient graphical

[2] This nice observation was pointed out by one of the referees.

derivative is fully computed in many broad classes of optimization problems [8, 9, 16] under milder assumptions.

Unlike (3.18), both of our conditions (3.3) and (3.4) are not generally necessary conditions for strong local minimizers, as shown in the following example.

EXAMPLE 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows:*

$$f(x) = \begin{cases} x & \text{if } x \in \{0\} \cup [1, +\infty), \\ \frac{1}{2^n} & \text{if } x \in [\frac{3}{2^{n+2}}, \frac{1}{2^n}),\ n = 0, 1, 2, \dots, \\ 2x - \frac{1}{2^{n+1}} & \text{if } x \in [\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}}),\ n = 0, 1, 2, \dots, \\ f(-x) & \text{if } x < 0. \end{cases}$$

We see that $f(x) \geq f(0) + |x|^2$ for all $x \in [-1, 1]$, which means that $\bar{x} = 0$ is a strong local minimizer of f. On the other hand, since

$$\left(\bigcup_{n=0}^{\infty} \left(-\frac{1}{2^n}, -\frac{3}{2^{n+2}} \right) \cup \{0\} \cup \bigcup_{n=0}^{\infty} \left(\frac{3}{2^{n+2}}, \frac{1}{2^n} \right) \right) \times \{0\} \subset \operatorname{gph} \partial f,$$

it follows that $\mathbb{R} \times \{0\} \subset T_{\operatorname{gph} \partial f}(\bar{x}, 0)$. Therefore, for $w \in \mathbb{R} \setminus \{0\}$ and $z = 0$, we have

$$z \in D(\partial f)(\bar{x}|0)(w) \quad \text{and} \quad \langle z, w \rangle = 0.$$

This shows that (3.3) and (3.4) are not necessary conditions for strong local minimizers.

Our next aim is to present several classes of functions at which both (3.3) and (3.4) are also necessary conditions for strong local minimizers. To this end, we first need the following lemma.

LEMMA 3.6. *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function. Suppose that h is* positively homogeneous of degree 2 *in the sense that $h(\lambda w) = \lambda^2 h(w)$ for all $\lambda > 0$ and $w \in \operatorname{dom} h$. Then, for any $w \in \operatorname{dom} h$ and $z \in \partial h(w)$, we have $\langle z, w \rangle = 2h(w)$.*

Proof. For any $z \in \partial h(w)$ with $w \in \operatorname{dom} h$, by (2.7) and (2.8) we find sequences $\{w_k\} \subset \operatorname{dom} h$, $z_k \in \partial_p h(w_k)$, and $\varepsilon_k, r_k > 0$ such that $w_k \to w$, $h(w_k) \to h(w)$, and $z_k \to z$ and that

$$h(u) - h(w_k) \geq \langle z_k, u - w_k \rangle - \frac{r_k}{2} \|u - w_k\|^2 \quad \text{for all} \quad u \in \mathbb{B}_{\varepsilon_k}(w_k).$$

By choosing $u = \lambda w_k \in \mathbb{B}_{\varepsilon_k}(w_k)$ with $0 < \lambda$ and $\|w_k\| \cdot |\lambda - 1| < \varepsilon_k$ in the above inequality, the positive homogeneity of degree 2 of h tells us that

$$(\lambda^2 - 1) h(w_k) = h(\lambda w_k) - h(w_k) \geq \langle z_k, (\lambda - 1) w_k \rangle - \frac{r_k}{2} (\lambda - 1)^2 \|w_k\|^2. \tag{3.19}$$

When $\lambda > 1$ satisfying $\|w_k\| \cdot (\lambda - 1) < \varepsilon_k$, we get from inequality (3.19) that

$$(\lambda + 1) h(w_k) \geq \langle z_k, w_k \rangle - \frac{r_k}{2} (\lambda - 1) \|w_k\|^2.$$

Taking $\lambda \downarrow 1$ gives us that $2h(w_k) \geq \langle z_k, w_k \rangle$. Similarly, when $0 < \lambda < 1$ with $\|w_k\| \cdot (1 - \lambda) < \varepsilon_k$, we derive from (3.19) that

$$(\lambda + 1) h(w_k) \leq \langle z_k, w_k \rangle - \frac{r_k}{2} (\lambda - 1) \|w_k\|^2.$$

By letting $\lambda \uparrow 1$, the latter implies $2h(w_k) \leq \langle z_k, w_k \rangle$. Thus, we have $\langle z_k, w_k \rangle = 2h(w_k)$, which clearly yields $\langle z, w \rangle = 2h(w)$ when $k \to \infty$ due to the choice of z_k, w_k at the beginning. ∎

THEOREM 3.7 (equivalence between the QGC and SMS of the subdifferential at a local minimizer for prox-regular and twice epidifferentiable functions). *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be an l.s.c. proper function with* $\bar{x} \in \operatorname{dom} f$. *Suppose that* $0 \in \partial f(\bar{x})$ *and* f *is subdifferentially continuous, prox-regular, and twice epidifferentiable at* $\bar{x}$ *for* 0. *Then the following assertions are equivalent:*

(i) *The QGC* (1.1) *is satisfied at* $\bar{x}$.

(ii) $\bar{x}$ *is a local minimizer and* ∂f *is strongly metrically subregular at* $\bar{x}$ *for* 0.

(iii) $D(\partial f)(\bar{x}|0)$ *is positive definite in the sense of* (3.3).

(iv) $D(\partial f)(\bar{x}|0)$ *is positive definite in the sense of* (3.4).

Moreover, if one of the assertions (i)–(iv) *holds, then*

$$\text{QG}(f;\bar{x}) = \inf\left\{\frac{\langle z,w\rangle}{\|w\|^2}\,\middle|\, z \in D(\partial f)(\bar{x}|0)(w), w \in \operatorname{dom} D(\partial f)(\bar{x}|0)\right\}. \tag{3.20}$$

Proof. Since f is subdifferentially continuous and prox-regular at $\bar{x}$ for $0 \in \partial f(\bar{x})$, we have $0 \in \partial_p f(\bar{x})$. Thus, implications [(iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)] follow from Theorem 3.2. It remains to verify that [(i) $\Rightarrow$ (iv)] and (3.20) is valid. To this end, suppose that $\bar{x}$ is a strong local minimizer with modulus κ as in (1.1). We derive from (1.1) and (2.10) that

$$d^2 f(\bar{x}|0)(w) \geq \kappa\|w\|^2 \quad \text{for all} \quad w \in \mathbb{R}^n. \tag{3.21}$$

Since f is subdifferentially continuous, prox-regular, and twice epidifferentiable at $\bar{x}$ for $0 \in \partial f(\bar{x})$, it follows from (2.12) that

$$D(\partial f)(\bar{x}|0) = \partial h \quad \text{with} \quad h(\cdot) := \frac{1}{2} d^2 f(\bar{x}|0)(\cdot). \tag{3.22}$$

Note from (2.10) and (3.21) that h is proper and positively homogeneous of degree 2. By Lemma 3.6, for any $z \in D(\partial f)(\bar{x}|0)(w) = \partial h(w)$, we obtain from (3.21) and (3.22) that

$$\langle z, w\rangle = 2h(w) = d^2 f(\bar{x}|0)(w) \geq \kappa\|w\|^2, \tag{3.23}$$

which clearly verifies (iv) and

$$\kappa \leq \inf\left\{\frac{\langle z,w\rangle}{\|w\|^2}\,\middle|\, z \in D(\partial f)(\bar{x}|0)(w),\ w \in \operatorname{dom} D(\partial f)(\bar{x}|0)\right\}.$$

Since κ is an arbitrary modulus of the strong local minimizer $\bar{x}$, the latter implies that

$$\text{QG}(f;\bar{x}) \leq \inf\left\{\frac{\langle z,w\rangle}{\|w\|^2}\,\middle|\, z \in D(\partial f)(\bar{x}|0)(w)\right\}.$$

This along with (3.5) justifies (3.20) and finishes the proof. ∎

Besides the full characterization of strong local minimizers in terms of (3.4) and (3.3) for a class of prox-regular and twice epidifferentiable functions, the above theorem also tells us the equivalence between the QGC and SMS of the subdifferential at a local minimizer for 0. This correlation has been also established for different classes of functions in [1, 11, 13].

The above theorem allows us to recover [14, Corollary 73].

COROLLARY 3.8. *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be an l.s.c. proper function with* $\bar{x} \in \operatorname{dom} f$. *Suppose that* $0 \in \partial f(\bar{x})$ *and that* f *is subdifferentially continuous, prox-regular, and twice epidifferentiable at* $\bar{x}$ *for* 0. *Then the following assertions are equivalent:*

(i) *$\bar{x}$ is a strong local minimizer.*
(ii) *The sufficient condition of the second kind in Definition* 3.3 *holds at $\bar{x}$.*

Proof. Since f is subdifferentially continuous, prox-regular, and twice epidifferentiable at $\bar{x}$ for $0 \in \partial f(\bar{x})$, the proof of Theorem 3.7, e.g., (3.23) tells us that

$$\langle z, w\rangle = d^2 f(\bar{x}|0)(w) \quad \text{for all} \quad z \in D(\partial f)(\bar{x}|0)(w) = D(\partial_p f)(\bar{x}|0)(w).$$

Hence, (ii) in this corollary is equivalent to Theorem 3.7(iv). The proof is complete via Theorem 3.7. ∎

The following concept of variational convexity was introduced recently by Rockafellar [23, Definition 2].

DEFINITION 3.9 (variational convexity). *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be an l.s.c. proper function, and let $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. One says that f is variationally convex at $\bar{x}$ for $\bar{v}$ if there exist an open neighborhood $X \times V$ of $(\bar{x}, \bar{v})$ and a convex l.s.c. function $\widehat{f} \le f$ on X and $\varepsilon > 0$ such that*

$$[X_\varepsilon \times V] \cap \operatorname{gph} \partial f = [X \times V] \cap \operatorname{gph} \partial \widehat{f},$$

and $f(x) = \widehat{f}(x)$ for every $x \in \Pi_X\big([X_\varepsilon \times V] \cap \operatorname{gph} \partial f\big)$, where $X_\varepsilon := \{x \in X | f(x) < f(\bar{x}) + \varepsilon\}$ and $\Pi_X : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the mapping given by $\Pi_X(x, v) = x$ for $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

The class of variationally convex functions includes convex functions. However, it may contain nonconvex functions [22]. Note further that variational convexity implies prox-regularity and subdifferential continuity [22]. The following result resembles Theorem 3.7 for the class of variationally convex functions.

THEOREM 3.10 (equivalence between the QGC and SMS of the subdifferential at a local minimizer for variationally convex functions). *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be an l.s.c. proper function with $\bar{x} \in \operatorname{dom} f$. Suppose that $0 \in \partial f(\bar{x})$ and that f is variationally convex at $\bar{x}$ for* 0. *Then the following assertions are equivalent:*
(i) *The QGC* (1.1) *is satisfied at $\bar{x}$.*
(ii) *$\bar{x}$ is a local minimizer and ∂f is strongly metrically subregular at $\bar{x}$ for* 0.
(iii) *$D(\partial f)(\bar{x}|0)$ is positive definite in the sense of* (3.3).
(iv) *$D(\partial f)(\bar{x}|0)$ is positive definite in the sense of* (3.4).
Moreover, if one of the assertions (i)–(iv) *holds, then*

$$\text{(3.24)} \quad \operatorname{QG}(f; \bar{x}) \ge \inf\left\{ \frac{\langle z, w\rangle}{\|w\|^2} \,\Big|\, z \in D(\partial f)(\bar{x}|0)(w), w \in \operatorname{dom} D(\partial f)(\bar{x}|0) \right\} \ge \frac{1}{2}\operatorname{QG}(f; \bar{x}).$$

Proof. Note from the variational convexity of f at $\bar{x}$ for $0 \in \partial f(\bar{x})$ and (2.9) that $0 \in \partial \widehat{f}(\bar{x})$ and $0 \in \partial_p f(\bar{x})$. Similarly to the proof of Theorem 3.7, we only need to verify [(i) ⇒ (iv)] and the right inequality in (3.24) due to (3.5). Suppose that $\bar{x}$ is a strong local minimizer with modulus κ, that is, there is a number $\gamma > 0$ such that

$$\text{(3.25)} \qquad f(x) - f(\bar{x}) \ge \frac{\kappa}{2}\|x - \bar{x}\|^2 \quad \text{for all} \quad x \in \mathbb{B}_\gamma(\bar{x}).$$

The variational convexity of f at $\bar{x}$ for 0 allows us to find an open neighborhood $X \times V$ of $(\bar{x}, \bar{v})$ and a convex l.s.c. function $\widehat{f} \le f$ on X with

$$[X \times V] \cap \operatorname{gph} \partial f = [X \times V] \cap \operatorname{gph} \partial \widehat{f},$$

and $f(x) = \widehat{f}(x)$ for every $x \in \Pi_X\big([X \times V] \cap \operatorname{gph} \partial f\big)$. Pick any $z \in D\partial f(\bar{x}|0)(w)$; then, by (2.11), there exist $t_k \downarrow 0$ and $(z_k, w_k) \to (z, w)$ such that

$$(\bar{x}, 0) + t_k(w_k, z_k) \in [X \times V] \cap \operatorname{gph} \partial f = [X \times V] \cap \operatorname{gph} \partial \widehat{f}.$$

Note that $\bar{x}, \bar{x} + t_k w_k \in \Pi_X\big([X \times V] \cap \operatorname{gph} \partial f\big)$. It follows from (3.25) that

$$\widehat{f}(\bar{x} + t_k w_k) = f(\bar{x} + t_k w_k) \geq f(\bar{x}) + \frac{\kappa}{2} t_k^2 \|w_k\|^2 = \widehat{f}(\bar{x}) + \frac{\kappa}{2} t_k^2 \|w_k\|^2 \quad \text{for all } k \text{ sufficiently large.}$$

Furthermore, since $\widehat{f}$ is convex and $(\bar{x}, 0) + t_k(w_k, z_k) \in \operatorname{gph} \partial \widehat{f}$, we have

$$\widehat{f}(\bar{x}) - \widehat{f}(\bar{x} + t_k w_k) \geq -\langle t_k z_k, t_k w_k \rangle \quad \text{for all} \quad k.$$

Combining the above two inequalities gives us that $\langle z_k, w_k \rangle \geq \frac{\kappa}{2}\|w_k\|^2$ for sufficiently large k . Letting $k \to \infty$, we get $\langle z, w \rangle \geq \frac{\kappa}{2}\|w\|^2$, which clearly clarifies [(i) $\Rightarrow$ (iv)] and the right inequality in (3.24). □

4. Equivalence between the quadratic growth and SMS of the subdifferentials for conic programs. In this section, let us consider the following constrained optimization problem:

$$(P) \qquad \min_{x \in \mathbb{R}^n} \quad g(x) \quad \text{s.t.} \quad q(x) \in \Theta, \tag{4.1}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^n \to \mathbb{R}^m$ with $q(x) = \big(q_1(x), ..., q_m(x)\big)$ are twice continuously differentiable, and Θ is a nonempty closed convex subset of $\mathbb{R}^m$.

Define $\Gamma := \{x \in \mathbb{R}^n \mid q(x) \in \Theta\}$ as the feasible solution set to problem (4.1), and fix $\bar{x} \in \Gamma$ with $\bar{y} := q(\bar{x})$. Put

$$f(x) := g(x) + \delta_\Gamma(x) \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{4.2}$$

where $\delta_\Gamma(x)$ is the indicator function to Γ, which equals 0 when $x \in \Gamma$ and ∞ otherwise. Problem (4.1) can be rewritten as an unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x).$$

The given point $\bar{x} \in \Gamma$ is said to be a *strong local minimizer* to problem (4.1) if there exist numbers $\kappa > 0, \gamma > 0$ such that

$$g(x) \geq g(\bar{x}) + \frac{\kappa}{2}\|x - \bar{x}\|^2 \quad \text{for all} \quad x \in \Gamma \cap \mathbb{B}_\gamma(\bar{x}), \tag{4.3}$$

that is, $\bar{x}$ is a strong local minimizer of the function f defined above. In the case of (4.3), we say the QGC holds at $\bar{x}$ to problem (4.1). Moreover, the exact modulus for problem (4.1) at $\bar{x}$ denoted by $QG((P); \bar{x})$ is the supremum of all κ such that (4.3) is satisfied, i.e., $QG((P); \bar{x}) = QG(f; \bar{x})$.

We call $\bar{x} \in \Gamma$ a *local minimizer* to (4.1) if condition (4.3) holds with some $\gamma > 0$ and $\kappa = 0$. Furthermore, $\bar{x}$ is a *stationary point* when there exists a Lagrange multiplier $\lambda \in N_\Theta(q(\bar{x}))$ such that

$$0 = \nabla g(\bar{x}) + \nabla q(\bar{x})^T \lambda. \tag{4.4}$$

The set of Lagrange multipliers satisfying (4.4) is denoted by $\Lambda(\bar{x})$.

In what follows, we always assume that the closed convex set Θ is C^2-*cone reducible* at $\bar{y}$ to a *pointed* closed convex cone $C \subset \mathbb{R}^l$ in the sense that there exists a neighborhood $V \subset \mathbb{R}^m$ of $\bar{y}$ and a twice continuously differentiable mapping $h : V \to \mathbb{R}^l$ such that

$$h(\bar{y}) = 0,\ \nabla h(\bar{y}) \text{ is surjective, and } \Theta \cap V = \big\{y \in V \mid h(y) \in C\big\}. \tag{4.5}$$

For more information on the concept of C^2-cone reducibility, we refer the reader to the paragraph right after Definition 3.135 in [7]. Note that the assumption of reducible sets allows us to cover a wide range of optimization problems, including nonlinear programming, semidefinite programming, and second-order cone programming; see, e.g., [7, section 3.4.4].

Furthermore, we assume that the *metric subregularity constraint qualification* (MSCQ) [15, 16] holds at $\bar{x}$, which means the set-valued mapping $F(x) := q(x) - \Theta$, $x \in \mathbb{R}^n$, is metrically subregular at $\bar{x}$ for 0. This condition is well known to be stable around $\bar{x}$ and strictly weaker than the notable Robinson constraint qualification (RCQ) at $\bar{x}$:

$$0 \in \text{int}\,\{q(\bar{x}) + \nabla q(\bar{x})\mathbb{R}^n - \Theta\}. \tag{4.6}$$

Furthermore, if the MSCQ is satisfied at $\bar{x}$, there exists some $\eta > 0$ such that the normal cone $N_\Gamma(x)$ is identical with $\partial_p \delta_\Gamma(x)$ for all $x \in \mathbb{B}_\eta(\bar{x})$ and is computed by

$$N_\Gamma(x) = \partial_p \delta_\Gamma(x) = \{\nabla q(x)^T\lambda |\ \lambda \in N_\Theta(q(x))\} \quad \text{for} \quad x \in \mathbb{B}_\eta(\bar{x}). \tag{4.7}$$

The main purpose of this section is to establish that for the function f given by (4.2), the quadratic growth, SMS of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are three equivalent properties.

To this end, we need to calculate the subgradient graphical derivative on the function f defined in (4.2) at the given point $\bar{x}$ for 0 due to Theorem 3.2 under the assumption that $0 \in \partial_p f(\bar{x})$. Under the MSCQ at $\bar{x}$, it follows from (4.7) that

$$\partial_p f(x) = \nabla g(x) + \partial_p \delta_\Gamma(x) = \partial f(x) \quad \text{for} \quad x \in \mathbb{B}_\eta(\bar{x}) \tag{4.8}$$

with the same η in (4.7). By (4.8), observe that $0 \in \partial_p f(\bar{x})$ if and only if $\bar{x}$ is a stationary point to (4.4). Furthermore, we have

$$D(\partial f)(\bar{x}|0)(w) = D(\nabla g + N_\Gamma)(\bar{x}|0)(w) = \nabla^2 g(\bar{x})w + DN_\Gamma(\bar{x}| - \nabla g(\bar{x}))(w). \tag{4.9}$$

The following result taken from [16, Corollary 5.4] is helpful in our study.

LEMMA 4.1 (see [16, Corollary 5.4 and Theorem 3.3]). *Let $\bar{x}$ be a stationary point to problem* (4.1). *Suppose that the MSCQ is satisfied at $\bar{x}$ and that the set Θ is $\mathcal{C}^2$-reducible at $\bar{y} = q(\bar{x})$. Then we have*

$$DN_\Gamma(\bar{x}, -\nabla g(\bar{x}))(w) = \Big\{\big(\nabla^2(\lambda^T q)(\bar{x}) + \widetilde{\mathcal{H}}_\lambda\big)w \mid \lambda \in \overline{\Lambda}(\bar{x}; w)\Big\} + N_{\overline{K}}(w) \ \ \textit{for} \ \ w \in \mathbb{R}^n, \tag{4.10}$$

where $\overline{K}$ is the critical cone defined by

$$\overline{K} := K\big(\bar{x}, -\nabla g(\bar{x})\big) := T_\Gamma(\bar{x}) \cap \{-\nabla g(\bar{x})\}^\perp \tag{4.11}$$

and the set $\overline{\Lambda}(\bar{x}; w)$ is written by

$$\overline{\Lambda}(\bar{x}; w) := \text{argmax}\big\{w^T\nabla^2(\lambda^T q)(\bar{x})w + w^T\widetilde{\mathcal{H}}_\lambda w |\ \lambda \in \Lambda\big(\bar{x}\big)\big\} \neq \emptyset \tag{4.12}$$

with

$$\widetilde{\mathcal{H}}_\lambda w := \nabla q(\bar{x})^T \nabla^2 \left\langle \left(\nabla h\big(q(\bar{x})\big)^*\right)^{-1}(\lambda), h(\cdot)\right\rangle \big(q(\bar{x})\big)\nabla q(\bar{x})w. \tag{4.13}$$

Pick any $z \in D(\partial f)(\bar{x}|0)(w)$; then it follows from (4.9) and (4.10) that $w \in \operatorname{dom} D(\partial f)(\bar{x}|0) = \overline{K}$ and there exist $\bar{\lambda} \in \overline{\Lambda}(\bar{x}; w)$ and $u \in N_{\overline{K}}(w)$ such that

$$z = \nabla^2 g(\bar{x})w + \big(\nabla^2(\bar{\lambda}^T q)(\bar{x}) + \widetilde{\mathcal{H}}_{\bar{\lambda}}\big)w + u.$$

Hence, we have

$$\langle z, w\rangle = \langle \nabla^2 g(\bar{x})w, w\rangle + w^T\nabla^2(\bar{\lambda}^T q)w + w^T\widetilde{\mathcal{H}}_{\bar{\lambda}}w + \langle u, w\rangle.$$

Since $\overline{K}$ is a cone, $\langle u, w\rangle = 0$. We derive from the latter and (4.12) that

$$\langle z, w\rangle = \max\big\{w^T\nabla^2 L(\bar{x},\lambda)w + w^T\widetilde{\mathcal{H}}_\lambda w \mid \lambda \in \Lambda(\bar{x})\big\} \text{ for } z \in D(\partial f)(\bar{x}|0)(w),\, w \in \overline{K}, \tag{4.14}$$

with the Lagrange function $L(x, \lambda) := g(x) + \langle \lambda, q(x)\rangle$ for $x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m$. This together with Theorem 3.2 tells us that the second-order condition

$$\max\big\{w^T\nabla^2 L(\bar{x},\lambda)w + w^T\widetilde{\mathcal{H}}_\lambda w \mid \lambda \in \Lambda(\bar{x})\big\} > 0 \quad \text{for all} \quad w \in \overline{K} \tag{4.15}$$

is sufficient for strong local minimizer $\bar{x}$ when the MSCQ holds at the stationary point $\bar{x}$. This is a classical fact [7, Theorem 3.137] established under the strictly stronger RCQ. Moreover, (4.15) is necessary for the strong local minimizer $\bar{x}$ under the RCQ. In order to show this fact under the MSCQ, we need a few lemmas in preparation.

Let $Q : q^{-1}(V) \to \mathbb{R}^\ell$ be given by $Q(x) = h \circ q(x)$ with h and V taken from (4.5).

Lemma 4.2 (see [16, Proposition 3.1]). *Suppose that the MSCQ holds for the system $q(x) \in \Theta$ at $\bar{x} \in \Gamma$. Then with any $w \in \mathbb{R}^n$ satisfying $\nabla Q(\bar{x})w \in C$ one can find a positive number κ such that for any $(z, y) \in \mathbb{R}^n \times \mathbb{R}^\ell$ with*

$$\nabla Q(\bar{x})z + \langle w, \nabla^2 Q(\bar{x})w\rangle + y \in T_C(\nabla Q(\bar{x})w),$$

there exists $\widetilde{z} \in \mathbb{R}^n$ satisfying the conditions

$$\nabla Q(\bar{x})\widetilde{z} + \langle w, \nabla^2 Q(\bar{x})w\rangle \in T_C(\nabla Q(\bar{x})w) \quad \textit{and} \quad \|\widetilde{z} - z\| \le \kappa\|y\|.$$

The latter condition can be reformulated as the upper Lipschitzian property

$$\Psi(y) \subset \Psi(0) + \kappa\|y\|\mathbb{B}_{\mathbb{R}^n} \quad \textit{for all} \quad y \in \mathbb{R}^\ell,$$

where

$$\Psi(y) := \big\{z \in \mathbb{R}^n \mid \nabla Q(\bar{x})z + \langle w, \nabla^2 Q(\bar{x})w\rangle + y \in T_C(\nabla Q(\bar{x})w)\big\}.$$

Lemma 4.3 (see [7, p. 242]). *If $w \in \mathbb{R}^n$ with $\nabla Q(\bar{x})w \in C$, then*

$$T_C^2\big(Q(\bar{x}), \nabla Q(\bar{x})w\big) = T_C\big(\nabla Q(\bar{x})w\big),$$

where

$$T_C^2\big(Q(\bar{x}), \nabla Q(\bar{x})w\big) := \left\{v \mid \exists t_k \searrow 0 \text{ s.t. } d\left(Q(\bar{x}) + t_k w + \frac{1}{2}t_k^2 v; C\right) = o(t_k^2)\right\}$$

is the outer second-order tangent cone to the set C at $Q(\bar{x})$ and in the direction $w \in T_C(Q(\bar{x}))$.

LEMMA 4.4. *Let $\bar{x}$ be a local minimizer of* (4.1) *satisfying the MSCQ. Then, for each $w \in K\big(\bar{x}, -\nabla g(\bar{x})\big)$ and $z \in \mathbb{R}^n$ with*

$$\nabla Q(\bar{x})z + \langle w, \nabla^2 Q(\bar{x})w\rangle \in T_C^2\big(Q(\bar{x}), \nabla Q(\bar{x})w\big), \tag{4.16}$$

we have

$$\nabla g(\bar{x})z + \langle w, \nabla^2 g(\bar{x})w\rangle \geq 0.$$

Proof. Take any $w \in K\big(\bar{x}, -\nabla g(\bar{x})\big)$ and $z \in \mathbb{R}^n$ satisfying (4.16). Then there exists $t_k \searrow 0$ such that

$$d\bigg(Q(\bar{x}) + t_k\nabla Q(\bar{x})w + \frac{1}{2}t_k^2\big(\nabla Q(\bar{x})z + \langle w, \nabla^2 Q(\bar{x})w\rangle\big); C\bigg) = o(t_k^2). \tag{4.17}$$

Defining $x(t_k) := \bar{x} + t_k w + \frac{1}{2}t_k^2 z$, we have

$$d\Big(Q\big(x(t_k)\big); C\Big) = d\bigg(Q(\bar{x}) + t_k\nabla Q(\bar{x})w + \frac{1}{2}t_k^2\big(\nabla Q(\bar{x})z + \langle w, \nabla^2 Q(\bar{x})w\rangle\big) + o(t_k^2), C\bigg) = o(t_k^2).$$

Since the MSCQ holds at $\bar{x}$, the set-valued mapping $M_Q(x) = Q(x) - C$ is metrically subregular at $\bar{x}$ for $0 = Q(\bar{x}) \in \mathbb{R}^\ell$ by [16, Lemma 5.2]. Then there exist a neighborhood U of $\bar{x}$ and a real number $\kappa > 0$ such that

$$d\big(x; \Gamma \cap q^{-1}(V)\big) = d\big(x; M_Q^{-1}(0)\big) \leq \kappa d\big(0; M_Q(x)\big) = \kappa d\big(Q(x); C\big) \quad \text{for all} \quad x \in U.$$

We assume without loss of generality that $x(t_k) \in U$ for all k. Consequently,

$$d\big(x(t_k); \Gamma \cap q^{-1}(V)\big) \leq \kappa d\Big(Q\big(x(t_k)\big); C\Big) = o(t_k^2).$$

So for each k one can find $\widetilde{x}(t_k) \in \Gamma \cap q^{-1}(V)$ with $\|x(t_k) - \widetilde{x}(t_k)\| = o(t_k^2)$. Noting that $x(t_k) \to \bar{x}$ as $k \to \infty$ and $\bar{x}$ is a local optimal solution to (4.1), we may assume that $g\big(\widetilde{x}(t_k)\big) \geq g(\bar{x})$ for all k. On the other hand, by the Taylor expansion,

$$g\big(x(t_k)\big) = g(\bar{x}) + t_k\nabla g(\bar{x})w + \frac{1}{2}t_k^2\big(\nabla g(\bar{x})z + \langle w, \nabla^2 g(\bar{x})w\rangle\big) + o(t_k^2),$$

and $g\big(x(t_k)\big) = g\big(\widetilde{x}(t_k)\big) + o(t_k^2)$. We have

$$g(\bar{x}) + t_k\nabla g(\bar{x})w + \frac{1}{2}t_k^2\big(\nabla g(\bar{x})z + \langle w, \nabla^2 g(\bar{x})w\rangle\big) + o(t_k^2) \geq g(\bar{x}) \quad \text{for all} \quad k.$$

Combining this with $\nabla g(\bar{x})w = 0$ yields

$$\frac{1}{2}t_k^2\big(\nabla g(\bar{x})z + \langle w, \nabla^2 g(\bar{x})w\rangle\big) + o(t_k^2) \geq 0.$$

This implies that $\nabla g(\bar{x})z + \langle w, \nabla^2 g(\bar{x})w\rangle \geq 0$. ∎

LEMMA 4.5 (second-order necessary optimality conditions for local minimizers of conic program under the MSCQ and $\mathcal{C}^2$-cone reducibility constraint). *Suppose that $\bar{x}$ is a stationary point of* (4.1) *at which the MSCQ is satisfied and that Θ is $\mathcal{C}^2$-cone reducible at $\bar{y} = q(\bar{x})$ to a pointed closed convex cone C. Consider the following assertions:*

(i) *$\bar{x}$ is a local minimizer for* (4.1).

(ii) $D(\partial f)(\bar{x}|0)$ *is positive semidefinite in the sense that*

$$\langle z, w\rangle \geq 0 \quad \text{for all} \quad z \in D(\partial f)(\bar{x}|0)(w),\ w \in \operatorname{dom} D(\partial f)(\bar{x}|0). \tag{4.18}$$

(iii) *For each* $w \in \operatorname{dom} D(\partial f)(\bar{x}|0)$, *there exists* $z \in D(\partial f)(\bar{x}|0)(w)$ *such that*

$$\langle z, w\rangle \geq 0. \tag{4.19}$$

(iv) *For each* $w \in \overline{K}$, *one has*

$$\max\big\{w^T\nabla^2 L(\bar{x},\lambda)w + w^T\widetilde{\mathcal{H}}_\lambda w \mid \lambda \in \Lambda\big(\bar{x}\big)\big\} \geq 0$$

with $\widetilde{H}_\lambda$ *defined in* (4.13).

Then we have [(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)].

Proof. [(i) $\Rightarrow$ (iv)]. Suppose $\bar{x}$ is a local minimizer for (4.1). For each $w \in \overline{K}$, let us consider the linear conic problem $(\widetilde{P})$ defined as

$$(\widetilde{P}) \qquad \begin{array}{ll} \inf\limits_{z\in\mathbb{R}^n} & \nabla g(\bar{x})z + \langle w, \nabla^2 g(\bar{x})w\rangle \\ \text{s.t.} & \nabla Q(\bar{x})z + \langle w, \nabla^2 Q(\bar{x})w\rangle \in T_C\big(\nabla Q(\bar{x})w\big), \end{array}$$

and its parametric dual $(\widetilde{D})$ given by

$$(\widetilde{D}) \qquad \begin{array}{ll} \sup\limits_{\mu\in\mathbb{R}^\ell} & w^T\nabla^2_x\mathcal{L}(\bar{x},\mu)w \\ \text{s.t.} & \mu \in N_C\big(\nabla Q(\bar{x})w\big), \nabla Q(\bar{x})^T\mu = -\nabla g(\bar{x}), \end{array}$$

where $\mathcal{L}(x,\mu) = g(x) + \langle \mu, Q(x)\rangle$, $(x,\mu) \in \mathbb{R}^n \times \mathbb{R}^\ell$. By Lemma 4.2, the feasible set of $(\widetilde{P})$ is nonempty. Moreover, by Lemmas 4.3 and 4.4, $\operatorname{val}(\widetilde{P})$ is finite and $\operatorname{val}(\widetilde{P}) \geq 0$, where $\operatorname{val}(\widetilde{P})$ is the optimal value of $(\widetilde{P})$. Due to the upper Lipschitz continuity of the mapping Ψ in Lemma 4.2, we derive from [7, Propositions 2.147 and 2.186] that $\operatorname{val}(\widetilde{P}) = \operatorname{val}(\widetilde{D})$ and that the optimal solution set to $(\widetilde{D})$ is nonempty. With $\lambda = \nabla h\big(q(\bar{x})\big)^T\mu$, note that

$$\begin{aligned} & w^T\nabla^2_x\mathcal{L}(\bar{x},\mu)w \\ &= w^T\nabla^2_x L(\bar{x},\lambda)w + \Big\langle \mu, (\nabla q(\bar{x})w)^T\nabla^2 h\big(q(\bar{x})\big)(\nabla q(\bar{x})w)\Big\rangle \\ &= w^T\nabla^2_x L(\bar{x},\lambda)w + \Big\langle \Big(\nabla h\big(q(\bar{x})\big)^*\Big)^{-1}(\lambda), (\nabla q(\bar{x})w)^T\nabla^2 h\big(q(\bar{x})\big)(\nabla q(\bar{x})w)\Big\rangle \\ &= w^T\nabla^2_x L(\bar{x},\lambda)w + w^T\widetilde{\mathcal{H}}_\lambda w. \end{aligned}$$

We claim further that

$$\{\lambda = \nabla h\big(q(\bar{x})\big)^T\mu |\ \mu \in N_C\big(\nabla Q(\bar{x})w\big), \nabla Q(\bar{x})^T\mu = -\nabla g(\bar{x})\} = \Lambda(\bar{x}). \tag{4.20}$$

To justify the "$\subset$" inclusion, pick any $\mu \in N_C\big(\nabla Q(\bar{x})w\big) \subset C^\circ$ with $\nabla Q(\bar{x})^T\mu = -\nabla g(\bar{x})$ and define $\lambda := \nabla h\big(q(\bar{x})\big)^T\mu$. Since C is a convex cone and $Q(\bar{x}) = h\big(q(\bar{x})\big) = h(\bar{y}) = 0$, we have

$$\lambda = \nabla h\big(q(\bar{x})\big)^T\mu \in \nabla h\big(q(\bar{x})\big)^T C^\circ = \nabla h\big(q(\bar{x})\big)^T N_C(Q(\bar{x})) = N_\Theta(q(\bar{x})).$$

Moreover, it is clear that

$$\nabla q(\bar{x})^T\lambda = \nabla q(\bar{x})^T\nabla h\big(q(\bar{x})\big)^T\mu = \nabla Q(\bar{x})^T\mu = -\nabla g(\bar{x}). \tag{4.21}$$

This shows that $\lambda \in \Lambda(\bar{x})$ and thus verifies the "$\subset$" inclusion in (4.20).

To ensure the opposite inclusion in (4.20), take any $\lambda \in \Lambda(\bar{x})$. Since $\lambda \in N_\Theta(q(\bar{x})) = \nabla h(q(\bar{x}))^T N_C\big(Q(\bar{x})\big)$, we find $\mu \in N_C(Q(\bar{x})) = C^\circ$ with $\lambda = \nabla h(q(\bar{x}))^T \mu$. It is similar to (4.21) that $\nabla Q(\bar{x})^T \mu = -\nabla g(\bar{x})$. Moreover, note from the fact $w \in \overline{K} := T_\Gamma(\bar{x}) \cap \{-\nabla g(\bar{x})\}^\perp$ that

$$\langle \mu, \nabla Q(\bar{x}) w\rangle = \langle \nabla Q(\bar{x})^T \mu, w\rangle = \langle -\nabla g(\bar{x}), w\rangle = 0,$$

showing that $\mu \in \{\nabla Q(\bar{x})w\}^\perp$. Since C is a convex cone, $N_C(\nabla Q(\bar{x})w) = C^\circ \cap \{\nabla Q(\bar{x})w\}^\perp$. Thus, $\mu \in N_C(\nabla Q(\bar{x})w)$. This clearly verifies the "$\supset$" inclusion in (4.20).

Note further from (4.5) that $\nabla h(q(\bar{x}))^T$ is injective; then we derive that

$$\text{(4.22)} \qquad \operatorname{val}(\widetilde{D}) = \max\big\{w^T \nabla^2 L(\bar{x}, \lambda) w + w^T \widetilde{\mathcal{H}}_\lambda w \mid \lambda \in \Lambda(\bar{x})\big\}.$$

Since $\operatorname{val}(\widetilde{D}) = \operatorname{val}(\widetilde{P}) \geq 0$, the latter implies assertion (iv).

[(iv) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (ii)]. Take any $z \in D(\partial f)(\bar{x}|0)(w)$ with $w \in \operatorname{dom} D(\partial f)(\bar{x}|0)$. We obtain from (4.9) and Lemma 4.1 that $\operatorname{dom} D\partial f(\bar{x}|0) = \overline{K}$. The equivalence [(iv) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (ii)] simply follows from (4.14). The proof is complete. ☐

REMARK 4.6. *For the nonlinear programming, the mapping h in the definition of C^2-conic reducible sets can be chosen to be an affine mapping, and thus $\widetilde{\mathcal{H}}_\lambda u = 0$ for all $u \in \mathbb{R}^n$. In this case, the implication* [(i) $\Rightarrow$ (iv)] *was established by Guo, Lin, and Ye* [18, *Theorem* 2.1] *under the calmness condition, which is weaker than the MSCQ.*

The main result of this section reads as follows.

THEOREM 4.7 (equivalence between the QGC and SMS of the subdifferential at a local minimizer for $\mathcal{C}^2$-reducible conic programs under the MSCQ). *Suppose that $\bar{x}$ is a stationary point of* (4.1) *satisfying MSCQ and that Θ is $\mathcal{C}^2$-cone reducible at $\bar{y} = q(\bar{x})$ to a pointed closed convex cone C. The following assertions are equivalent:*

(i) *f satisfies QGC* (1.1) *at $\bar{x}$.*

(ii) *$\bar{x}$ is a local minimizer to* (4.1) *and ∂f is strongly metrically subregular at $\bar{x}$ for* 0.

(iii) *$D(\partial f)(\bar{x}|0)$ is positive definite in the sense of* (3.3).

(iv) *There exists $\kappa > 0$ such that*

$$\text{(4.23)} \qquad \langle z, w\rangle \geq \kappa\|w\|^2 \quad \textit{for all} \quad z \in D(\partial f)(\bar{x}|0)(w), w \in \operatorname{dom} D(\partial f)(\bar{x}|0).$$

Moreover, if one of the assertions (i)–(iv) *holds, then*

$$\text{(4.24)} \qquad \operatorname{QG}(f;\bar{x}) = \inf\left\{\frac{\langle z, w\rangle}{\|w\|^2}\,\middle|\, z \in D(\partial f)(\bar{x}|0)(w),\ w \in \operatorname{dom} D(\partial f)(\bar{x}|0)\right\}.$$

Proof. We see that $\bar{x}$ is a strong local minimizer for (4.1) with modulus κ if and only if $\bar{x}$ is a local minimizer of the function

$$g_\kappa(x) := g(x) - \frac{\kappa}{2}\|x - \bar{x}\|^2 \quad \text{over } \Gamma.$$

Applying Lemma 4.5 by replacing the function g there by $g_\kappa(x)$ along with using the sum rules of the graphical derivative, the latter gives us that

$$\langle z, w\rangle \geq \kappa\|w\|^2 \quad \text{for all} \quad z \in D(\partial f)(\bar{x}|0)(w),\ w \in \operatorname{dom} D(\partial f)(\bar{x}|0),$$

which verifies [(i) $\Rightarrow$ (iv)] and

$$\mathrm{QG}(f;\bar{x}) \le \inf\left\{ \frac{\langle z,w\rangle}{\|w\|^2} \,\middle|\, z \in D(\partial f)(\bar{x}|0)(w), w \in \mathrm{dom}\, D(\partial f)(\bar{x}|0) \right\}.$$

Note further from (4.7) and (4.8) that $\partial_p f(x) = \partial f(x)$ for x around $\bar{x}$. Thus, the implication [(iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)] and the "$\ge$" inequality in (4.24) follows from Theorem 3.2. This verifies the equivalence of (i)–(iv) and the validity of (4.24). The proof is complete. ∎

REMARK 4.8 (comparison of the positive definiteness of the subgradient graphical derivative with other second-order optimality conditions). *Under the assumption of Theorem* 4.7, *consider the following assertions:*

(v) *The sufficient condition of the second kind in* Definition 3.3 *holds at $\bar{x}$; i.e., there exists $\kappa > 0$ such that for each $w \in \mathrm{dom}\, D(\partial_p f)(\bar{x}|0)$ with $\|w\| = 1$ condition* (3.11) *holds.*

(vi) *The* second-order sufficient condition *holds at $\bar{x}$ in the sense that for each $w \in \overline{K} \setminus \{0\}$ one has*

$$\max\left\{ w^T \nabla^2 L(\bar{x},\lambda) w + w^T \widetilde{\mathcal{H}}_\lambda w \mid \lambda \in \Lambda(\bar{x}) \right\} > 0.$$

(vii) *There exists $\kappa > 0$ such that*

$$\max\left\{ w^T \nabla^2 L(\bar{x},\lambda) w + w^T \widetilde{\mathcal{H}}_\lambda w \mid \lambda \in \Lambda(\bar{x}) \right\} \ge \kappa \|w\|^2 \quad \textit{for all } w \in \overline{K}. \tag{4.25}$$

Using (4.14) *and the equality* $\mathrm{dom}\, D(\partial f)(\bar{x}|0) = \overline{K}$, *we see that*

$$[(\mathrm{v}) \Leftrightarrow (\mathrm{iv}) \Leftrightarrow (\mathrm{vii}) \Rightarrow (\mathrm{vi}) \Rightarrow (\mathrm{iii})].$$

So, by Theorem 4.7, *assertions* (i)–(vii) *are indeed equivalent. Furthermore, it is easy to check that*

$$\mathrm{QG}((P);\bar{x}) := \mathrm{QG}(f;\bar{x}) = \inf_{w\in\overline{K},\|w\|=1} \max\left\{ w^T \nabla^2 L(\bar{x},\lambda) w + w^T \widetilde{\mathcal{H}}_\lambda w \mid \lambda \in \Lambda(\bar{x}) \right\}.$$

The equivalence between (i) *and* (vi) *above was established under the RCQ in Bonnans and Shapiro* [7, *Theorem* 1.137]. *An earlier version of this no gap second-order optimality condition for nonlinear programming was proved in Ioffe* [19] *and Ben-Tal* [5] *under the Mangasarian–Fromovitz constraint qualification (MFCQ), a particular of RCQ* (4.6).

Next let us provide an example where the MSCQ holds while RCQ (4.6) does not, but assertions (i)–(iv) in Theorem 4.7 are satisfied.

EXAMPLE 4.9. *Consider the problem* $(EP1)$ *as follows:*

$$(EP1) \qquad \min_{x\in\mathbb{R}^3} \; g(x) \quad s.t. \quad q(x) \in \mathcal{Q}_3, \tag{4.26}$$

where $\mathcal{Q}_3 = \left\{(s_0,s_1,s_2) \in \mathbb{R}^3 \mid s_0 \ge \sqrt{s_1^2+s_2^2}\right\}$ *is the second-order cone in* $\mathbb{R}^3$, *and the objective function and the constraint mapping are given, respectively, by*

$$g(x) := \frac{1}{2}x_1^2 + x_2^2 \quad \textit{and} \quad q(x) := \left(2x_2^2, x_2^2 - x_3, x_2^2 + x_3\right)$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. It is well known that the second-order cone $\mathcal{Q}_3$ is $\mathcal{C}^2$-cone reducible. Moreover, define

$$\Gamma := \{x |\ q(x) \in \mathcal{Q}_3\} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_2^2 \geq |x_3|\}.$$

With $\bar{x} := (0, 0, 0)$, we see that

$$\begin{aligned} g(x) - g(\bar{x}) &= \tfrac{1}{2}x_1^2 + x_2^2 \\ &= \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_2^2 \\ &\geq \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}|x_3| \\ &\geq \tfrac{1}{2}(x_1^2 + x_2^2 + x_3^2) \\ &= \tfrac{1}{2}\|x\|^2 \end{aligned}$$

for all $x = (x_1, x_2, x_3) \in \Gamma$ with $|x_3| \leq 1$. So $\bar{x}$ is a strong local minimizer to problem (4.26).

It is easy to check that

$$\nabla q(\bar{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_{\mathcal{Q}_3}\big(q(\bar{x})\big) = -\mathcal{Q}_3.$$

Hence we have

$$N_{\mathcal{Q}_3}\big(q(\bar{x})\big) \cap \ker \nabla q(\bar{x})^T = \bigcup_{t \in \mathbb{R}} (-\infty, -\sqrt{2}|t|] \times \{t\} \times \{t\},$$

which shows that RCQ (4.6) *is not satisfied at $\bar{x}$.*

Next let us verify the validity of the MSCQ at $\bar{x}$. Since $q(x) \in -\mathcal{Q}_3$ if and only if $x_2 = x_3 = 0$, we have $d\big(q(x); \mathcal{Q}_3\big) = 0$ when $q(x) \in \mathcal{Q}_3 \cup (-\mathcal{Q}_3)$. When $q(x) \notin \mathcal{Q}_3 \cup (-\mathcal{Q}_3)$, note that

$$\sqrt{(x_2^2 - x_3)^2 + (x_2^2 + x_3)^2} - 2x_2^2 = \sqrt{2x_2^4 + 2x_3^2} - 2x_2^2 \geq (|x_3| + x_2^2) - 2x_2^2 = |x_3| - x_2^2 > 0.$$

It follows that

$$d\big(q(x); \mathcal{Q}_3\big) = \begin{cases} 0 & \text{if } q(x) \in \mathcal{Q}_3 \cup (-\mathcal{Q}_3), \\ \frac{1}{\sqrt{2}}\big(\sqrt{2x_2^4 + 2x_3^2} - 2x_2^2\big) & \text{if } q(x) \notin \mathcal{Q}_3 \cup (-\mathcal{Q}_3). \end{cases}$$

When $q(x) \in \mathcal{Q}_3 \cup (-\mathcal{Q}_3)$, we have $d(x; \Gamma) = d\big(q(x); \mathcal{Q}_3\big) = 0$. When $q(x) \notin \mathcal{Q}_3 \cup (-\mathcal{Q}_3)$, define $u := \big(x_1, x_2, \frac{x_3}{|x_3|}x_2^2\big) \in \Gamma$ and observe that

$$d(x; \Gamma) \leq \|x - u\| = \left| x_3 - \frac{x_3}{|x_3|}x_2^2 \right| = \big||x_3| - x_2^2\big| = |x_3| - x_2^2 \leq \sqrt{2}d\big(q(x), \mathcal{Q}_3\big).$$

This shows that the MSCQ holds at $\bar{x}$. By Theorem 4.7, *assertions* (i)–(iv) *hold.*

Under the setting of Theorem 4.7, condition (vi) is equivalent to the sufficient condition given in [7, Theorem 3.86], ensuring that not only is the point in question a strong local minimizer, but the subdifferential is all strongly metrically subregular. As shown by the following example [9], in the absence of the MSCQ, the condition given in [7, Theorem 3.86] is sufficient for the point to be a strong local minimizer alone, without SMS of the subdifferential.

EXAMPLE 4.10. *Consider the problem* $(EP2)$ *as follows:*

$$(EP2) \qquad \min_{x\in\mathbb{R}} \quad g(x) := \tfrac{1}{2}x^2 \quad s.t. \quad q(x) \in \Theta, \tag{4.27}$$

where

$$q(x) = \begin{cases} x^6 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and $\Theta = \{0\}$. *Put*

$$\Gamma := q^{-1}(\Theta) = \{0\} \cup \left\{ \frac{1}{n\pi} : n = \pm 1, \pm 2, \pm 3, \dots \right\}.$$

For $x_n = \frac{1}{2}\big(\frac{1}{n\pi} + \frac{1}{(n+1)\pi}\big) = \frac{2n+1}{2n(n+1)\pi}$, $n = 1, 2, \dots$, *we see that*

$$d(x_n, \Gamma) = \frac{1}{2n(n+1)\pi} \quad \text{and} \quad d\big(q(x_n), \Theta\big) = |x_n^6 \sin \frac{1}{x_n}| \leq \left(\frac{2n+1}{2n(n+1)\pi}\right)^6.$$

Hence,

$$\lim_{n\to\infty} \frac{d\big(q(x_n), \Theta\big)}{d(x_n, \Gamma)} = 0,$$

proving that the MSCQ is not satisfied at $\bar{x} := 0 \in \Gamma$. *Obviously,* $\overline{K} = \mathbb{R}$, $\Lambda(\bar{x}) = \mathbb{R}$, *and*

$$\begin{aligned} \Lambda^G(\bar{x}) &:= \big\{(\alpha, \lambda) \in \mathbb{R}_+ \times \mathbb{R} : \nabla_x L^G(\bar{x}, \alpha, \lambda) = 0, (\alpha, \lambda) \neq (0, 0)\big\} \\ &= \big\{(\alpha, \lambda) \in \mathbb{R}_+ \times \mathbb{R} : (\alpha, \lambda) \neq (0, 0)\big\}, \end{aligned}$$

where $L^G(x, \alpha, \lambda) := \alpha g(x) + \langle \lambda, q(x) \rangle$. *For* $\mathcal{T}(h) := T^2_\Theta\big(q(\bar{x}), \nabla q(\bar{x})h\big) = \{0\}$, *we have*

$$\sigma\big(\lambda, \mathcal{T}(h)\big) = 0 \quad \text{for all} \quad h \in \mathbb{R},$$

and thus

$$h^T \nabla_x^2 L^G(\bar{x}, \alpha, \lambda)h - \sigma\big(\lambda, \mathcal{T}(h)\big) = |h|^2 > 0 \quad \text{for all} \quad h \in \overline{K}\backslash\{0\}, (\alpha, \lambda) \in \Lambda^G(\bar{x}),$$

where $\sigma\big(\lambda, \mathcal{T}(h)\big) := \sup_{v\in\mathcal{T}(h)} \langle \lambda, v \rangle$. *Moreover, for each* $h \in \overline{K}$, $\Theta := \{0\}$ *is outer second-order regular at the point* $q(\bar{x})$ *in the direction* $\nabla q(\bar{x})h$ *and with respect to* $\nabla q(\bar{x})$ *in the sense of* [7, *Theorem* 3.86]. *Therefore,* [7, *Theorem* 3.86] *can be used to justify that* $\bar{x}$ *is a strong local minimizer. Note that, for* $f(x) := g(x) + \delta_\Gamma(x)$, $x \in \mathbb{R}$, *we see that*

$$\partial f(x) = \begin{cases} \mathbb{R} & \text{if } x \in \Gamma, \\ \emptyset & \text{if } x \in \mathbb{R}\backslash\Gamma, \end{cases}$$

and thus $\partial f(\cdot)$ *is not strongly metrically subregular at* $\bar{x}$ *for* $0 \in \partial f(\bar{x})$. *This shows, in the absence of the MSCQ, that the sufficient condition formulated in* [7, *Theorem* 3.86] *guarantees that* $\bar{x}$ *is a strong local minimizer alone, without SMS of the subdifferential.*

5. Conclusion. For an extended-real-valued lower semicontinuous proper function, we have shown that the positive definiteness of the subgradient graphical derivative at a proximal stationary point is sufficient for the point to be a local minimizer at which the subdifferential is strongly subregular for 0. The latter was known to imply the quadratic growth. When the function is either a subdifferentially continuous, prox-regular, twice epidifferentiable function, a variationally convex function, or the sum of the objective function and the indicator function of the feasible set of a $\mathcal{C}^2$-cone reducible constrained program satisfying the MSCQ, we have proved that the quadratic growth, SMS of the subdifferential at a local minimizer, and the positive definiteness of the subgradient graphical derivative at a stationary point are equivalent. In the future, we intend to extend our applications to different optimization problems, including composite functions [6, 7, 24] and mathematical programs with equilibrium constraints [17, 18]. This extension will require further computation on the second-order structures of the subgradient graphical derivative presented in our paper. Another direction that caught our attention is to study the epidifferentiability of composite functions in order to use the advantage of Theorem 3.7.

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