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On the Positive Definiteness of Limiting Coderivative for Set-Valued Mappings

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Abstract
This paper concerns the interconnection between the positive definiteness of limiting coderivative and the local strong maximal monotonicity of set-valued mappings suspected in Mordukhovich and Nghia (SIAM J. Optim. **26**, 1032–1059, 2016, Conjecture 3.6). We disprove the conjecture by a counterexample and provide some special classes at which it is true. However, the positive definiteness of limiting coderivative characterizes a new property called nearly strong monotonicity. Consequently, we show that the strong metric regularity of set-valued mappings could be obtained under the positive definiteness of limiting coderivative.

Keywords Local maximal monotone · Positive definiteness · Limiting coderivative · Strong metric regularity · Set-valued mappings · Variational analysis

Mathematics Subject Classification (2010) Primary 49J53 · Secondary 49J52

1 Introduction

Maximal monotonicity is a fundamental notion in mathematics with vast applications in different areas such as partial differential equations [10], control theory [2], optimization and algorithms [1, 3, 25, 28]. In the recent years, a local version of maximal monotonicity

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[22] has been used to study solution stability of optimization, equilibrium, and variational problems [18, 19, 22, 23, 26, 27]. In these papers, the local maximal monotonicity of a special set-valued mapping that is the subdifferential of nonsmooth and nonconvex functions is mainly investigated.

Traditionally, maximal monotonicity of a mapping has strong connection with the positive semi-definitness of its "derivative". Roughly speaking, a differentiable single-valued mapping is (maximal) monotone if and only if its derivative is positive semi-definite at any point. When a single-valued mapping is not differentiable, extension of this classical result could be found in [5, 11, 13]. For set-valued mappings, e.g., the limiting subdifferential mapping to a nonsmooth function, the first necessary condition for maximal monotonicity was established by Poliquin and Rockafellar [23, Theorem 2.1] via the semi-definiteness of *limiting coderivatives* acting on these mappings at any point on their graphs. Recently, Chieu et al. [6] showed that this condition turns to be sufficient under the assumption of hypomonotonicity on the set-valued mappings. Both [6, 23] literally use the limiting coderivative, one of the most important notions in variational analysis introduced by Mordukhovich; see his monographs [15, 16] for its detail developments and vast applications to optimization and optimal control.

For local maximal monotonicity, similar connections could be established around the point in question [18, 22]. But when dealing with a local property, a pointwise characterization is usually more favorable. Even in the case of a continuously differentiable single-valued mapping, the semi-positive definiteness of its derivative at one point is not enough to guarantee the local maximal monotonicity of this mapping. However, the pointwise positive definiteness of its derivative is equivalent to a stronger concept, the local *strong* maximal monotonicity; see, e.g. [18, 23]. In the case of nondifferentiable Lipschitz continuous single-valued mappings, Mordukhovich and Nghia [18, Corollary 3.5] showed that such monotonicity could be characterized by the positive definiteness of limiting coderivative at the point in question. They also conjectured in [18, Conjecture 3.6] that the criteria remains valid for *hypomonotone* set-valued mappings. There are several evidences of this conjecture established in [18] for broad classes of set-valued mappings containing limiting subdifferential of nonsmooth nonconvex functions.

Our paper mainly surrounds the aforementioned conjecture with more emphasis on the positive definiteness of limiting coderivative for set-valued mappings. We indeed provide a counterexample, but show that the pointbased positive definiteness of limiting coderivative is equivalent to a property that is strongly related to the local maximal strong monotonicity. This allows us to establish some special classes different from those in [18] where the conjecture is true. Furthermore, we show that the positive definiteness of limiting coderivative is sufficient for *strong metric regularity*. This important property of set-valued mappings introduced by Robinson [24] is usually characterized via *strict graphical derivative* [28] and *degree theory* [9], both of which are usually difficult to check. We also connect the pointwise semi-definiteness of limiting coderivative with the so-called *strict 2-submonotone* property relating to the terminology introduced by Spingarn [29].

The rest of the paper is organized as follows. To be self-contained, Section 2 recalls some basic definitions in variational analysis and generalized differentiation used in our paper. In Section 3, we analyze the mentioned [18, Conjecture 3.6], introduce and characterize the so-called *nearly strong monotonicity* of set-valued mappings via positive definiteness of the limiting coderivative. Examples will be given to demonstrate our results. We conclude this section by showing that [18, Conjecture 3.6] is indeed true in one dimension or under some extra symmetric assumptions. Section 4 gives a few conclusions and leaves some open questions for future work.

2 Preliminaries

In this section we recall notations from variational analysis that will be used in sequel; see [15, 28] for further details. Let $\Omega \subset \mathbb{R}^n$ be locally closed around $\bar{x} \in \Omega$, the (Fréchet) *regular normal cone* to Ω at $\bar{x} \in \Omega$ is defined by:

$$\widehat{N}(\bar{x};\Omega) := \left\{ v \in \mathbb{R}^n \middle|\ \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le 0 \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means $x \to \bar{x}$ and $x \in \Omega$. Another important normal cone structure mainly used in our paper is the Mordukhovich *limiting normal cone* to Ω at $\bar{x} \in \Omega$:

$$N(\bar{x};\Omega) := \left\{ v \in \mathbb{R}^n \middle|\ \exists\, x_k \xrightarrow{\Omega} \bar{x},\, v_k \in \widehat{N}(x_k;\Omega),\, v_k \to v \right\}.$$

Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ then its domain and graph are given respectively by:

$$\operatorname{dom} F := \left\{ x \in \mathbb{R}^n \,|\, F(x) \neq \emptyset \right\} \quad \text{and} \quad \operatorname{gph} F := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \middle|\ y \in F(x) \right\}.$$

Suppose that gph F is locally closed around $(\bar{x},\bar{y}) \in \operatorname{gph} F$, we also recall here the *regular* and *limiting coderivatives* of F at $(\bar{x},\bar{y})$, respectively, by:

$$\begin{aligned}
\widehat{D}^* F(\bar{x}|\bar{y})(w) &:= \left\{ z \in \mathbb{R}^n \middle|\ (z,-w) \in \widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{y}) \right\}, \quad w \in \mathbb{R}^n,\\
D^* F(\bar{x}|\bar{y})(w) &:= \left\{ z \in \mathbb{R}^n \middle|\ (z,-w) \in N_{\operatorname{gph} F}(\bar{x},\bar{y}) \right\}, \quad w \in \mathbb{R}^n,
\end{aligned}$$

where $\bar{y}$ is skipped from the coderivative notation when F is single-valued around $\bar{x}$.

A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *Lipschitz-like* (Aubin property) around $(\bar{x},\bar{y}) \in \operatorname{gph} F$ with modulus $\ell > 0$ if there exists a neighborhood $U \times V \subset \mathbb{R}^n \times \mathbb{R}^n$ of $(\bar{x},\bar{y})$ such that

$$F(x) \cap V \subset F(u) + \ell\|x-u\|\mathbb{B} \ \text{ for all } \ x, u \in U.$$

The infimum of all such ℓ is known as the *exact Lipschitzian bound* around $(\bar{x},\bar{y})$ and denoted by $\operatorname{lip} F(\bar{x},\bar{y})$. When the graph of F is locally closed around $(\bar{x},\bar{y})$, this Lipschitz-like property is fully characterized by the Mordukhovich coderivative criterion [14, Corollary 5.4]

$$D^* F(\bar{x}|\bar{y})(0) = \{0\}. \tag{2.1}$$

Next, we present the standard version of single-valued localization of set-valued mapping used in this paper; see, e.g., [8, 17]. For a different purpose, our definition needs the single-valued localization $\widehat{F}$ to have full domain U, which is not required in [8].

Definition 2.1 (**single-valued localizations**) Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping and let $(\bar{x},\bar{y}) \in \operatorname{gph} F$. We say that F admits a SINGLE-VALUED LOCALIZATION around $(\bar{x},\bar{y})$ if there is a neighborhood $U \times V \subset \mathbb{R}^n \times \mathbb{R}^n$ of $(\bar{x},\bar{y})$ such that the mapping $\widehat{F} : U \to V$ defined by $\operatorname{gph} \widehat{F} := \operatorname{gph} F \cap (U \times V)$ is single-valued on U with $\operatorname{dom} \widehat{F} = U$. In this case we say that $\widehat{F}$ is a single-valued localization of F at $\bar{x}$ for $\bar{y}$. If in addition $\widehat{F}$ is Lipschitz continuous on U, then F admits a LIPSCHITZ CONTINUOUS SINGLE-VALUED LOCALIZATION at $\bar{x}$ for $\bar{y}$.

Finally, if the inverse F^{-1} of a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ admits a Lipschitz continuous single-valued localization with constant $\kappa > 0$ around $(\bar{v},\bar{x}) \in \operatorname{gph} F^{-1}$, we say F is *strongly metrically regular* around $(\bar{x},\bar{v})$ with modulus $\kappa > 0$; see [8, 24] for further discussions.

3 Positive Definiteness of Limiting Coderivative and Local Strong Maximal Monotonicity

Recall that a set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be (globally) *monotone* if

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \quad \text{whenever} \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T.$$

A monotone operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *maximally monotone* if $\operatorname{gph} T = \operatorname{gph} S$ for any monotone operator $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfying the inclusion $\operatorname{gph} T \subset \operatorname{gph} S$. We present next some local monotonicity notions (cf. [18, 21, 23]) considered in this section.

Definition 3.1 (**local monotonicity**) Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and let $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. We say that:

- T is LOCALLY MONOTONE around $(\bar{x}, \bar{v})$ if there is a neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \quad \text{for all} \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times V). \tag{3.1}$$

  T is LOCALLY MAXIMALLY MONOTONE around $(\bar{x}, \bar{v})$ if there is a neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that (3.1) holds and that $\operatorname{gph} T \cap (U \times V) = \operatorname{gph} S \cap (U \times V)$ for any monotone operator $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfying $\operatorname{gph} T \cap (U \times V) \subset \operatorname{gph} S$.
- T is LOCALLY HYPOMONOTONE around $(\bar{x}, \bar{v})$ if there exists a neighborhood $U \times V$ of this point together with a positive number $r > 0$ such that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2 \quad \text{for all} \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times V). \tag{3.2}$$

- T is LOCALLY STRONGLY MONOTONE around $(\bar{x}, \bar{v})$ with modulus $\kappa > 0$ if there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq \kappa\|u_1 - u_2\|^2 \quad \text{for all} \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times V).$$

  Finally, we say T is LOCALLY STRONGLY MAXIMALLY MONOTONE around $(\bar{x}, \bar{v})$ with modulus $\kappa > 0$ if it is locally strongly monotone and locally maximally monotone around $(\bar{x}, \bar{v})$.

We recall the following important and useful result taken from [18, Lemma 3.3 and Theorem 3.4] giving us necessary and sufficient conditions for the local strong maximal monotonicity of a set-valued mapping.

Theorem 3.2 (**necessary and sufficient conditions for local strongly maximal monotonicity,** [18]) *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. Suppose that the graph of T is locally closed around $(\bar{x}, \bar{v})$. The following statements are equivalent:*

(i) *T is locally strongly maximally monotone around $(\bar{x}, \bar{v})$ with modulus κ.*

(ii) *T admits a Lipschitz continuous single-valued localization $\vartheta : V \to U$ of T^{-1} relative to a neighborhood $V \times U$ of $(\bar{v}, \bar{x})$ that satisfies*

$$\langle v_1 - v_2, \vartheta(v_1) - \vartheta(v_2) \rangle \geq \kappa\|\vartheta(v_1) - \vartheta(v_2)\|^2 \quad \textit{for all} \quad v_1, v_2 \in V.$$

(iii) *T is locally hypomonotone around $(\bar{x}, \bar{v})$ and there exists $\eta > 0$ such that*

$$\langle z, w \rangle \geq \kappa\|w\|^2 \quad \textit{for all} \quad z \in \widehat{D}^*T(u|v)(w), (u, v) \in \operatorname{gph} T \cap \mathbb{B}_\eta(\bar{x}, \bar{v}). \tag{3.3}$$

Although (3.3) is a nice infinitesimal characterization for the local strong maximal monotonicity, it is natural to question whether a similar *pointwise* condition holds as in [18, Conjecture 3.6].

Conjecture 3.3 (**limiting coderivative characterization of local strong maximal monotonicity for set-valued mappings,** [18, Conjecture 3.6]) *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. The following assertions are equivalent:*

(i) *T is locally strongly maximally monotone around $(\bar{x}, \bar{v})$.*
(ii) *T is locally hypomonotone around $(\bar{x}, \bar{v})$ and $D^*T(\bar{x}|\bar{v})$ is positive definite in the sense that*

$$\langle z, w\rangle > 0 \quad \textit{whenever} \quad z \in D^*T(\bar{x}|\bar{v})(w),\ w \neq 0. \tag{3.4}$$

It is shown in [18] that the conjecture is valid in many classes, e.g., when T is either single-valued and Lipschitz continuous around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$ or $T = f + \partial g$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable mapping around $\bar{x}$ and $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a *continuously subdifferentiable* and *prox-regular* function at $\bar{x}$ for $\bar{v} - \nabla f(\bar{x})$ in the sense of [23]. The following result, a direct consequence of Theorem 3.2 also provides another evidence for this conjecture.

Corollary 3.4 (**validity of Conjecture 3.3 for Lipschitz-like mappings**) *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. Then the implication [**(i)**$\Rightarrow$**(ii)**] holds in Conjecture 3.3. If, in addition, T is Lipschitz-like around $(\bar{x}, \bar{v})$, the converse implication is also true.*

Proof To justify [**(i)**$\Rightarrow$**(ii)**], suppose that T is locally strongly maximally monotone around $(\bar{x}, \bar{v})$. It is obvious that T is locally hypomonotone around $(\bar{x}, \bar{v})$. Pick any $(w, z) \in \operatorname{gph} D^*T(\bar{x}|\bar{v})$ with $w \neq 0$, we find sequences $(x_k, v_k) \xrightarrow{\operatorname{gph} T} (\bar{x}, \bar{v})$ and $(w_k, z_k) \to (w, z)$ with $z_k \in \widehat{D}^*T(x_k|v_k)(w_k)$. By Theorem 3.2, there exists some $\kappa > 0$ such that $\langle z_k, w_k\rangle \geq \kappa\|w_k\|^2$ when k is large enough. Taking $k \to \infty$, we have $\langle z, w\rangle \geq \kappa\|w\|^2 > 0$. This clearly verifies **(ii)** of Conjecture 3.3.

To prove the converse implication, suppose that **(ii)** of Conjecture 3.3 is valid and that T is Lipschitz-like around $(\bar{x}, \bar{v})$. By Theorem 3.2, it suffices to show that condition (3.3) holds. By contradiction, we find sequences $(x_k, v_k) \xrightarrow{\operatorname{gph} T} (\bar{x}, \bar{v})$ and $(w_k, z_k) \to (w, z)$ with $z_k \in \widehat{D}^*T(x_k|v_k)(w_k)$ such that $\langle z_k, w_k\rangle < \frac{1}{k}\|w_k\|^2$. This implies that $w_k \neq 0$. Define $\bar{w}_k := w_k/\|w_k\|$ and $\bar{z}_k := z_k/\|w_k\|$, we have $\langle \bar{z}_k, \bar{w}_k\rangle < \frac{1}{k}$ and $\bar{z}_k \in \widehat{D}^*T(x_k|v_k)(\bar{w}_k)$. Since T is Lipschitz-like around $(\bar{x}, \bar{v})$, there exists some $\ell > 0$ such that

$$\|\bar{z}_k\| \leq \ell\|\bar{w}_k\| = \ell \quad \text{for sufficient large k}$$

by [15, Theorem 4.7]. Without loss of generality, we assume that $(\bar{z}_k, \bar{w}_k)$ converges to some $(\bar{z}, \bar{w})$. It follows that

$$\bar{z} \in D^*T(\bar{x}|\bar{v})(\bar{w}), \quad \langle \bar{z}, \bar{w}\rangle \leq 0, \quad \text{and} \quad \|\bar{w}\| = 1,$$

which contradicts (3.4). Hence condition (3.3) holds. By Theorem 3.2, **(i)** of Conjecture 3.3 is satisfied. □

Despite of many evidences of Conjecture 3.3 as discussed above, the next example shows that Conjecture 3.3 is indeed not true in general. More precisely, we construct a set-valued hypomonotone mapping T satisfying a stronger condition:

$$\langle z, w\rangle \geq \ell\|w\|^2 \quad \text{whenever} \quad z \in D^*T(\bar{x}|\bar{v})(w),\ w \in \mathbb{R}^n \tag{3.5}$$

with some constant $\ell > 0$, but T is even not locally monotone around $(\bar{x}, \bar{v})$.

Example 3.5 (**a counterexample of** [18, Conjecture 3.6]) Consider the following mapping $T:\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$

$$T(x)=\begin{cases}\left(x_1,\dfrac{x_2}{x_1^2}\right) & \text{if} \quad (x_1,x_2)\in\mathbb{R}^2, x_1\neq 0,\\ \{0\}\times\mathbb{R} & \text{if} \quad (x_1,x_2)=(0,0),\\ \emptyset & \text{if} \quad x_1=0, x_2\neq 0.\end{cases} \tag{3.6}$$

Define $\bar{x}:=(0,0)$ and $\bar{v}:=(0,1)$, note that the graph of T is closed around $(\bar{x},\bar{v})$. Let us check that T is locally hypomonotone around $(\bar{x},\bar{v})$ by showing that

$$\langle (y_1,y_2)-(v_1,v_2),(x_1,x_2)-(u_1,u_2)\rangle \geq -(2\varepsilon+\varepsilon^2)\|(x_1,x_2)-(u_1,u_2)\|^2 \tag{3.7}$$

for any $\varepsilon\in(0,1)$ and $((x_1,x_2),(y_1,y_2)),((u_1,u_2),(v_1,v_2))\in \operatorname{gph} T\cap(U\times V)$ with $U=(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)$ and $V=(-\varepsilon,\varepsilon)\times(1-\varepsilon,1+\varepsilon)$. Let us consider the following three cases:

Case 1. Both (x_1,x_2) and (u_1,u_2) are $(0,0)$. Then (3.7) is trivial.
Case 2. Only one of (x_1,x_2) and (u_1,u_2) is $(0,0)$. Without loss of generality, suppose that $(x_1,x_2)\neq(u_1,u_2)=(0,0)$. From the definition of T, $v_1=0$. Since $v_2\in V\subset(0,2)$, we have

$$\begin{aligned}\langle (x_1,x_2)-(u_1,u_2),(y_1,y_2)-(v_1,v_2)\rangle &= \langle (x_1,x_2),(x_1,\frac{x_2}{x_1^2}-v_2)\rangle = x_1^2+\frac{x_2^2}{x_1^2}-v_2x_2\\ &\geq x_1^2+\frac{x_2^2}{x_1^2}-2|x_2|\geq 0,\end{aligned}$$

which also verifies (3.7).
Case 3. Both (x_1,x_2) and (u_1,u_2) are different from $(0,0)$. Without loss of generality, suppose that $|x_1|\geq|u_1|$, it follows from (3.6) that

$$\begin{aligned}&\langle (y_1,y_2)-(v_1,v_2),(x_1,x_2)-(u_1,u_2)\rangle = \langle (x_1-u_1,y_2-v_2),(x_1-u_1,x_1^2y_2-u_1^2v_2)\rangle\\ &=(x_1-u_1)^2+(x_1^2y_2-u_1^2v_2)(y_2-v_2)=(x_1-u_1)^2+\left[x_1^2(y_2-v_2)+(x_1^2-u_1^2)v_2\right](y_2-v_2)\\ &\geq -(2\varepsilon+\varepsilon^2)(x_1-u_1)^2+\left[(1+\varepsilon)^2(x_1-u_1)^2+x_1^2(y_2-v_2)^2\right]-|x_1^2-u_1^2|\cdot|v_2|\cdot|y_2-v_2|\\ &\geq -(2\varepsilon+\varepsilon^2)(x_1-u_1)^2+2(1+\varepsilon)|x_1-u_1|\cdot|x_1|\cdot|y_2-v_2|-|x_1^2-u_1^2|\cdot|v_2|\cdot|y_2-v_2|\\ &\geq -(2\varepsilon+\varepsilon^2)(x_1-u_1)^2+[2(1+\varepsilon)|x_1|-|x_1+u_1|\cdot|v_2|]\,|x_1-u_1|\cdot|y_2-v_2|\\ &\geq -(2\varepsilon+\varepsilon^2)(x_1-u_1)^2,\end{aligned}$$

where the last inequality holds due to $|x_1+u_1|\leq|x_1|+|u_1|\leq 2|x_1|$ and $|v_2|\leq 1+\varepsilon$. This clearly verifies (3.7) and completes the proof for local hypomonotonicity of T around $(\bar{x},\bar{v})$.

We claim next that T is not locally monotone around $(\bar{x},\bar{v})$. Indeed, define the sequences

$$x^k=(x_1^k,x_2^k):=\left((1-k^{-1})k^{-1},(1-k^{-1})k^{-2}\right) \quad \text{and} \quad u^k=(u_1^k,u_2^k):=\left(k^{-1},k^{-2}\right) \quad \text{with} \quad k\in\mathbb{N}.$$

It follows that

$$y^k=(y_1^k,y_2^k):=((1-k^{-1})k^{-1},(1-k^{-1})^{-1})\in T(x^k) \quad \text{and} \quad v^k=(v_1^k,v_2^k):=(k^{-1},1)\in T(u^k).$$

Moreover, we have

$$\begin{aligned}\langle y^k - v^k, x^k - u^k\rangle &= (x_1^k - u_1^k)^2 + (y_2^k - v_2^k)(x_2^k - u_2^k)\\ &= k^{-4} - ((1-k^{-1})^{-1} - 1)k^{-3}\\ &= k^{-4} - (1-k^{-1})^{-1}k^{-4}\\ &= -(1-k^{-1})^{-1}k^{-5} < 0.\end{aligned}$$

Since $x^k, u^k \to \bar{x}$ and $y^k, v^k \to \bar{v}$ as $k \to \infty$, T is not locally monotone around $(\bar{x}, \bar{v})$.

Finally, we check that the positive definiteness of $D^*T(\bar{x}|\bar{v})$ in both (3.4) and (3.5) are satisfied. Observe that $T(x) = P^{-1}(x)$, where $P : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $P(x_1, x_2) = (x_1, x_1^2 x_2)$. Moreover, P is continuously differentiable with $\nabla P(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 2x_1x_2 & x_1^2 \end{pmatrix}$ for any $(x_1, x_2) \in \mathbb{R}^2$. Note further that $z \in D^*T(\bar{x}|\bar{v})(w)$ if any only if

$$-w \in D^*T^{-1}(\bar{v}|\bar{x})(-z) = D^*P(\bar{v}|\bar{x})(-z) = \nabla P(\bar{v})^*(-z) = (-z_1, 0).$$

It follows that

$$\langle z, w\rangle = \|z_1\|^2 = \|w\|^2,$$

which clearly verifies both (3.4) and (3.5).

From the above example, the local hypomonotonicity of T together with (3.5) and (3.4) is not enough to verify local strong monotonicity. However, in the following result, we show that condition (3.5) characterizes a close property, which is called *nearly strong monotonicity*.

Theorem 3.6 (**positive definiteness of limiting coderivative and nearly strong monotonicity**) *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in$ gph T. Suppose that T is locally hypomonotone around $(\bar{x}, \bar{v})$. The following assertions are equivalent:*

(i) *T is nearly strongly monotone around $(\bar{x}, \bar{v})$ with modulus $\kappa > 0$ in the sense that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathbb{B}_\delta(\bar{v}) \subset$ dom T^{-1} and that*

$$\langle v_1 - v_2, u_1 - u_2\rangle \ge \kappa\|u_1 - u_2\|^2 - \varepsilon\|v_1 - v_2\|^2 \ \textit{for all} \ (u_1, v_1), (u_2, v_2) \in \text{gph}\, T \cap \mathbb{B}_\delta(\bar{x}, \bar{v}). \tag{3.8}$$

(ii) *$D^*T(\bar{x}|\bar{v})$ is positive definite with modulus $\kappa > 0$ in the sense that*

$$\langle z, w\rangle \ge \kappa\|w\|^2 \quad \textit{whenever} \quad z \in D^*T(\bar{x}|\bar{v})(w),\ w \in \mathbb{R}^n. \tag{3.9}$$

Proof Suppose that **(i)** is valid, i.e., for any $\varepsilon > 0$ there exists some $\delta > 0$ such that (3.8) holds. Define $\vartheta : \mathbb{B}_\delta(\bar{v}) \rightrightarrows \mathbb{R}^n$ by gph ϑ = gph $T^{-1} \cap (\mathbb{B}_\delta(\bar{v}) \times \mathbb{R}^n)$. For any $(v_1, u_1), (v_2, u_2) \in$ gph ϑ, it follows from (3.8) that

$$0 \ge \kappa\|u_1 - u_2\|^2 - \|v_1 - v_2\| \cdot \|u_1 - u_2\| - \varepsilon\|v_1 - v_2\|^2,$$

which yields

$$\|u_1 - u_2\| \le \frac{1 + \sqrt{1 + 4\varepsilon\kappa}}{2\kappa}\|v_1 - v_2\|. \tag{3.10}$$

Since $\mathbb{B}_\delta(\bar{v}) \subset \operatorname{dom} T^{-1}$, we derive from (3.10) that ϑ is single-valued and Lipschitz-continuous on $\mathbb{B}_\delta(\bar{v})$ with modulus $L_\varepsilon := \dfrac{1+\sqrt{1+4\varepsilon\kappa}}{2\kappa}$.

Pick any $(u, v) \in \operatorname{gph} T \cap (\mathbb{B}_\delta(\bar{x}) \times \mathbb{B}_\delta(\bar{v}))$ and $z \in \widehat{D}^*T(u|v)(w)$, we claim that

$$\langle z, w\rangle \geq \kappa\|w\|^2 - \varepsilon\|z - 2\kappa w\|^2. \tag{3.11}$$

Indeed, since $z \in \widehat{D}^*T(u|v)(w)$, for any $\nu > 0$ there exist some η with $\mathbb{B}_\eta(u, v) \subset \mathbb{B}_\delta(\bar{x}, \bar{v})$ such that

$$\nu\,[\|x-u\| + \|y-v\|] \geq \langle z, x-u\rangle - \langle w, y-v\rangle \quad \text{for all} \quad (x, y) \in \mathbb{B}_\eta(u, v). \tag{3.12}$$

Define $v_t := v + t(z - 2\kappa w) \in \mathbb{B}_\eta(v) \subset \mathbb{B}_\delta(\bar{v})$ for $t > 0$ sufficiently small and $u_t := \vartheta(v_t)$. Since ϑ is Lipschitz continuous on $\mathbb{B}_\delta(\bar{v})$, we have $u_t := \vartheta(v_t) \in B_\eta(v)$ when $t > 0$ is small enough. It follows from (3.8) that

$$\begin{aligned}
\langle z, u_t - u\rangle - \langle w, v_t - v\rangle &= \langle t^{-1}(v_t - v) + 2\kappa w, u_t - u\rangle - t\langle w, z - 2\kappa w\rangle \\
&= \frac{\langle v_t - v, u_t - u\rangle}{t} + 2\kappa\langle w, u_t - u\rangle - t\langle w, z\rangle + 2t\kappa\|w\|^2 \\
&\geq \frac{\kappa}{t}\|u_t - u\|^2 - \frac{\varepsilon}{t}\|v_t - v\|^2 + 2\kappa\langle w, u_t - u\rangle - t\langle w, z\rangle + 2t\kappa\|w\|^2 \\
&= \left[\frac{\kappa}{t}\|u_t - u\|^2 + 2\kappa\langle w, u_t - u\rangle + t\kappa\|w\|^2\right] - t\langle w, z\rangle \\
&\quad + t\kappa\|w\|^2 - \frac{\varepsilon}{t}\|v_t - v\|^2 \\
&\geq -t\langle w, z\rangle + t\kappa\|w\|^2 - \frac{\varepsilon}{t}\|v_t - v\|^2 \\
&= -t\langle w, z\rangle + t\kappa\|w\|^2 - t\varepsilon\|z - 2\kappa w\|^2.
\end{aligned}$$

Observe further from the Lipschitz continuity of ϑ that

$$\begin{aligned}
\nu\,(\|u_t - u\| + \|v_t - v\|) &= \nu\,(\|\vartheta(v_t) - \vartheta(v)\| + \|v_t - v\|) \\
&\leq \nu\,(L_\varepsilon\|v_t - v\| + \|v_t - v\|) \\
&= \nu(L_\varepsilon + 1)t\|z - 2\kappa w\|.
\end{aligned}$$

This together with the above inequalities and (3.12) tells us that

$$\nu(L_\varepsilon + 1)\|z - 2\kappa w\| \geq -\langle z, w\rangle + \kappa\|w\|^2 - \varepsilon\|z - 2\kappa w\|^2$$

for any $\nu > 0$. Letting $\nu \to 0$, we derive (3.11) as claimed.

Now we prove the desired positive definiteness of $D^*T(\bar{x}|\bar{v})$ in (3.9). Indeed, pick any $z \in D^*T(\bar{x}|\bar{v})(w)$, $w \in \mathbb{R}^n$. Hence there is a sequence $(u_k, v_k, w_k, z_k) \in (\mathbb{R}^n)^4$ such that $(u_k, v_k) \overset{\operatorname{gph}T}{\to} (\bar{x}, \bar{v})$ and $(w_k, z_k) \to (w, z)$ with $z_k \in \widehat{D}^*T(u_k|v_k)(w_k)$. Thanks to (3.11), we have

$$\langle z_k, w_k\rangle \geq \kappa\|w_k\|^2 - \varepsilon\|z_k - 2\kappa w_k\|^2 \quad \text{for sufficiently large } k.$$

Passing $k \to \infty$, this inequality gives us

$$\langle z, w\rangle \geq \kappa\|w\|^2 - \varepsilon\|z - 2\kappa w\|^2.$$

Since (z, w) is independent from the arbitrary small $\varepsilon > 0$, we derive (3.9) from the latter and complete the proof for [**(i)**$\Rightarrow$**(ii)**].

To proceed the converse implication, suppose that (3.9) holds. Since T is hypomonotone around $(\bar{x}, \bar{v})$, by (3.2) we find some $r > 0$ and neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that

$$\langle v_1 - v_2, u_1 - u_2\rangle \geq -r\|u_1 - u_2\|^2 \quad \text{for all} \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times V). \tag{3.13}$$

Pick any $s > r + 1$ and define $J_s(v, u) := (u, v - s(u - \bar{x}))$ for any $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ and denote by $\mathbb{I}$ the identity mapping in $\mathbb{R}^n$. We show next that $F_s := (T + s\mathbb{I} - s\bar{x})^{-1}$ has a localization around $(\bar{v}, \bar{x})$ that is single-valued and Lipschitz continuous around $\bar{v}$ with a modulus $(\kappa + s - \alpha)^{-1}$ for some arbitrarily small $\alpha \in (0, \kappa + r) \subset (0, \kappa + s)$. Pick any $w \in D^*F_s(\bar{v}|\bar{x})(z)$, we have

$$-z \in D^*(T + s\mathbb{I} - s\bar{x})(\bar{x}|\bar{v})(-w) = D^*(T + s\mathbb{I})(\bar{x}|\bar{v} + s\bar{x})(-w) = D^*T(\bar{x}|\bar{v})(-w) - sw,$$

which implies that $-z + sw \in D^*T(\bar{x}|\bar{v})(-w)$. It follows from (3.9) that $\langle -z + sw, -w\rangle \geq \kappa\|w\|^2$ and thus

$$\|z\| \cdot \|w\| \geq \langle z, w\rangle \geq (\kappa + s)\|w\|^2.$$

Hence we have $\|z\| \geq (\kappa + s)\|w\|$ for any $w \in D^*F_s(\bar{v}|\bar{x})(z)$. Thanks to [15, Theorem 4.10], F_s is Lipschitz-like around $(\bar{v}, \bar{x})$ with the exact Lipschitzian bound smaller or equal to $(\kappa + s)^{-1}$. Hence for any $\alpha \in (0, \kappa + r)$ there exists some small number $\eta > 0$ depending on α and s such that

$$F_s(v_1) \cap \mathbb{B}_\eta(\bar{x}) \subset F_s(v_2) + (\kappa + s - \alpha)^{-1}\|v_1 - v_2\|\mathbb{B}_1 \quad \text{for all} \quad v_1, v_2 \in \mathbb{B}_\eta(\bar{v}). \tag{3.14}$$

This gives us that

$$\bar{x} \in F_s(\bar{v}) \cap \mathbb{B}_\eta(\bar{x}) \subset F_s(v) + (\kappa + s - \alpha)^{-1}\|v - \bar{v}\|\mathbb{B}_1 \ \text{ for } \ v \in \mathbb{B}_\eta(\bar{v}). \tag{3.15}$$

Hence $F_s(v) \neq \emptyset$ for all $v \in \mathbb{B}_\eta(\bar{v})$. Define $\varphi_s : \mathbb{B}_\eta(\bar{v}) \rightrightarrows \mathbb{R}^n$ with

$$\operatorname{gph} \varphi_s = \operatorname{gph} F_s \cap (\mathbb{B}_\eta(\bar{v}) \times \mathbb{B}_{(\kappa+s-\alpha)^{-1}\eta}(\bar{x}))$$

and thus $\operatorname{dom} \varphi_s = \mathbb{B}_\eta(\bar{v})$ due to (3.15).

By choosing η sufficiently small, we may suppose that $J_s(\mathbb{B}_\eta(\bar{v}, \bar{x})) \subset U \times V$. Pick $(v_i, u_i) \in \operatorname{gph} \varphi_s$, $i = 1, 2$, we have $u_i \in \mathbb{B}_{(\kappa+s-\alpha)^{-1}\eta}(\bar{x}) \subset \mathbb{B}_\eta(\bar{x})$, since $\kappa + s - \alpha \geq s - r > 1$. It follows that $(u_i, v_i - s(u_i - \bar{x})) = J_s(v_i, u_i) \in \operatorname{gph} T \cap (U \times V)$. By (3.13), we obtain

$$\|v_1 - v_2\| \cdot \|u_1 - u_2\| \geq \langle v_1 - v_2, u_1 - u_2\rangle \geq (s - r)\|u_1 - u_2\|^2,$$

which tells us that φ_s is single-valued and Lipschitz continuous with modulus $(s - r)^{-1}$.

By (3.14) there exists $\widehat{u}_2 \in F_s(v_2)$ such that

$$\|\varphi_s(v_1) - \widehat{u}_2\| \leq (\kappa + s - \alpha)^{-1}\|v_1 - v_2\|. \tag{3.16}$$

Hence we have

$$\begin{aligned}
\|\bar{x} - \widehat{u}_2\| &\leq \|\varphi_s(v_1) - \bar{x}\| + (\kappa + s - \alpha)^{-1}\|v_1 - v_2\| \\
&= \|\varphi_s(v_1) - \varphi_s(\bar{v})\| + (\kappa + s - \alpha)^{-1}\|v_1 - v_2\| \\
&\leq (s - r)^{-1}\|v_1 - \bar{v}\| + (\kappa + s - \alpha)^{-1}\|v_1 - v_2\|
\end{aligned}$$

Define $\nu := \min\left\{\dfrac{(\kappa + s - \alpha)^{-1}}{(s - r)^{-1} + 2(\kappa + s - \alpha)^{-1}}\eta, \eta\right\}$ and restrict $v_1, v_2 \in \mathbb{B}_\nu(\bar{v})$, we get from the above inequalities that $\widehat{u}_2 \in \mathbb{B}_{(\kappa+s-a)^{-1}\eta}(\bar{x})$, which means $\widehat{u}_2 = \varphi_s(v_2)$. This together with (3.16) gives us that

$$\|\varphi_s(v_1) - \varphi_s(v_2)\| \leq (\kappa + s - \alpha)^{-1}\|v_1 - v_2\| \quad \text{for all} \quad v_1, v_2 \in \mathbb{B}_\nu(\bar{v}). \tag{3.17}$$

Thus F_s admits a single-valued and Lipschitz continuous localization around $(\bar{v}, \bar{x})$ with modulus $(\kappa + s - \alpha)^{-1}$.

We are ready to prove (3.8). Pick any $\varepsilon > 0$ and choose $\delta > 0$ sufficiently small such that $J_s(\mathbb{B}_\delta(\bar{x}, \bar{v})) \subset \mathbb{B}_\nu(\bar{v}) \times \mathbb{B}_{(\kappa+s-a)^{-1}\eta}(\bar{x})$. For any $(u_i, v_i) \in \operatorname{gph} T \cap \mathbb{B}_\delta(\bar{x}, \bar{v})$, $i = 1, 2$, we have

$$(v_i + s(u_i - \bar{x}), u_i) \in \operatorname{gph} F_s \cap (\mathbb{B}_\nu(\bar{v}) \times \mathbb{B}_{(\kappa+s-a)^{-1}\eta}(\bar{x})).$$

This together with (3.17) tells us that

$$(\kappa+s-\alpha)^2\|u_1-u_2\|^2 \le \|v_1-v_2+s(u_1-u_2)\|^2 = \|v_1-v_2\|^2+2s\langle v_1-v_2, u_1-u_2\rangle+s^2\|u_1-u_2\|^2.$$

It follows that

$$\langle v_1 - v_2, u_1 - u_2\rangle \ge \left(\frac{(\kappa-\alpha)^2}{2s} + \kappa - \alpha\right)\|u_1 - u_2\|^2 - \frac{1}{2s}\|v_1 - v_2\|^2. \tag{3.18}$$

Since $\alpha > 0$ and $s > r + 1$ could be chosen arbitrarily small and large respectively, we could suppose innitially that

$$\frac{(\kappa-\alpha)^2}{2s} + \kappa - \alpha \ge \kappa \quad \text{and} \quad -\frac{1}{2s} \ge -\varepsilon.$$

This together with (3.18) justifies (3.8). To complete the proof of the theorem, we only need to show that $\bar{v} \in \operatorname{int}(\operatorname{dom} T^{-1})$. This is indeed trivial due to the Lipschitz-like property of T^{-1} around $(\bar{v}, \bar{x})$ obtained by (3.9) and the Mordukhovich coderivative criterion (2.1). □

Remark 3.7 When the mapping T admits a Lipschitz continuous localization at $\bar{x}$ for $\bar{v}$, it is obvious that the nearly strong monotonicity of T (3.8) around $(\bar{x}, \bar{v})$ is equivalent to the local strong maximal monotonicity of T around $(\bar{x}, \bar{v})$. Moreover, if T could be represented by $T = f + \partial g$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ being a continuously differentiable mapping around $\bar{x}$ and $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ being a continuously subdifferentiable and prox-regular function at $\bar{x}$ for $\bar{v} - \nabla f(\bar{x})$ in the sense of [23], [18, Corollary 3.5] together with the above theorem tells us that the nearly strong monotonicity and the local strong maximal monotonicity of T around $(\bar{x}, \bar{v})$ are also the same. The difference between these two types of monotonicity may appear for set-valued mappings that involves the subdifferentials in some indirect ways. As in Example 3.5 and Theorem 3.6, the mapping in (3.6) is nearly strongly monotone but it is *not* even locally monotone around the point in question.

Remark 3.8 Full calculus of the limiting coderivative D^*T for set-valued mappings T is well-known; see, e.g., [15, Chapter 3]. When T involves subdifferential/normal cone mappings, full calculation for D^*T is a research challenge especially when the corresponding system is *degenerate*. This topic belonging to the area of second-order variational analysis [28, Chapter 13] is out of scope of our paper; see also [16, Chapter 3] for a brief discussion about recent developments in this direction.

Given a set-valued mapping T that is locally hypomonotone around $(\bar{x}, \bar{v})$, [18, Theorem 3.4] shows that the positive definiteness of $\widehat{D}^*T$ *around* $(\bar{x}, \bar{v})$ with some modulus $\kappa > 0$ ensures the strong metric regularity [24] of T around $(\bar{x}, \bar{v})$. Our next result, a consequence of Theorem 3.6 provides a pointbased sufficient condition for such a property on T via the positive definiteness of $D^*T(\bar{x}|\bar{v})$.

Corollary 3.9 (**sufficient condition for strong metric regularity I**) *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. Suppose that T is hypomonotone around $(\bar{x}, \bar{v})$ and that $D^*T(\bar{x}|\bar{v})$ is positive definite with modulus $\kappa > 0$ in the sense of* (3.9)*. Then T is strongly metrically regular around $(\bar{x}, \bar{v})$ with $\operatorname{lip} T^{-1}(\bar{v}, \bar{x}) \le \kappa^{-1}$.*

Proof Under the above assumptions, T is nearly strongly monotone around $(\bar{x}, \bar{v})$ as in (3.8) due to Theorem 3.6. Observe from the proof of the implication [**(i)**$\Rightarrow$**(ii)**] in Theorem 3.6, e.g., (3.10) that T^{-1} admits a Lipschitz continuous and single-valued localization with modulus $L_\varepsilon = \dfrac{1+\sqrt{1+4\varepsilon\kappa}}{2\kappa}$, which is arbitrarily close to κ^{-1} for sufficiently small $\varepsilon > 0$. Then T is strongly metrically regular around $(\bar{x}, \bar{v})$ with $\operatorname{lip} T^{-1}(\bar{v}, \bar{x}) \le \kappa^{-1}$. $\square$

As a counterpart of Theorem 3.6, the positive semi-definiteness of hypomonotone mapping T ensures a close property to local monotonicity.

Corollary 3.10 (**positive semidefiniteness of limiting coderivative**) *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. Suppose that T is locally hypomonotone around $(\bar{x}, \bar{v})$. If $D^*T(\bar{x}|\bar{v})$ is positive semidefinite:*

$$\langle z, w\rangle \ge 0 \quad \textit{whenever} \quad z \in D^*T(\bar{x}|\bar{v})(w),\ w \in \mathbb{R}^n \tag{3.19}$$

then T is nearly monotone around $(\bar{x}, \bar{v})$ in the sense that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle v_1 - v_2, u_1 - u_2\rangle \ge -\varepsilon\|v_1 - v_2\|^2 \quad \textit{for all} \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap \mathbb{B}_\delta(\bar{x}, \bar{v}). \tag{3.20}$$

Proof To justify, pick any $\varepsilon > 0$ and fix any $s > 0$. Define $T_s = s\mathbb{I} + T - s\bar{x}$. It is clear that T_s is locally hypomonotone around $(\bar{x}, \bar{v})$ and $D^*T_s(\bar{x}|\bar{v})$ is positive definite with modulus s when (3.19) is valid. By applying Theorem 3.6 on T_s, there exists some $\eta > 0$ such that

$$\langle v_1 - v_2, u_1 - u_2\rangle \ge s\|u_1 - u_2\|^2 - \varepsilon\|v_1 - v_2\|^2 \quad \text{for all} \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T_s \cap \mathbb{B}_\eta(\bar{x}, \bar{v}).$$

This is equivalent to (3.20) after translation. The proof is complete. $\square$

The above result also tells us that the positive semi-definiteness of limiting coderivative $D^*T(\bar{x}|\bar{v})$ is sufficient for the *strict 2-submonotonicity* on T^{-1} around $(\bar{v}, \bar{x})$ in the sense that

$$\liminf_{\substack{(u_i, v_i) \xrightarrow{\operatorname{gph} T} (\bar{x}, \bar{v}),\, i=1,2 \\ v_1 \ne v_2}} \frac{\langle v_1 - v_2, u_1 - u_2\rangle}{\|v_1 - v_2\|^2} \ge 0.$$

provided that T is locally hypomonotone around $(\bar{x}, \bar{v})$. It is worth noting that the strict 2-submonotonicity on T^{-1} above implies that T^{-1} is also *strictly submonotone* around $(\bar{v}, \bar{x})$ as follows:

$$\liminf_{\substack{(u_i, v_i) \xrightarrow{\operatorname{gph} T} (\bar{x}, \bar{v}),\, i=1,2 \\ v_1 \ne v_2}} \frac{\langle v_1 - v_2, u_1 - u_2\rangle}{\|v_1 - v_2\|} \ge 0. \tag{3.21}$$

The terminology of strict submonotonicity was first introduced by Spingarn in [29] and usually used on subdifferential mappings to characterize approximate convexity on functions [7, 20]. Our definition (3.21) is slightly different when we restrict the local property on both $\bar{x}$ and $\bar{v}$.

Thanks to Corollary 3.10, we can replace the positive definiteness with modulus κ (3.9) by the stronger one (3.4) to obtain the strong metric regularity of a local hypomonotone mapping as in Corollary 3.9. This result is significant in our later analysis.

Corollary 3.11 (**sufficient condition for strong metric regularity II**) *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. Suppose that T is* *hypomonotone around* $(\bar{x}, \bar{v})$ *and that* $D^*T(\bar{x}|\bar{v})$ *is positive definite in the sense of* (3.4). *Then* T *is strongly metrically regular around* $(\bar{x}, \bar{v})$.

Proof Suppose that $D^*T(\bar{x}|\bar{v})$ is positive definite in the sense of (3.4). Then, $D^*T(\bar{x}|\bar{v})$ is also positive semidefinite. Hence, by Corollary 3.10, for any $\varepsilon > 0$ there exists $\delta > 0$ such that (3.20) is satisfied.

Note further that the positive definiteness (3.4) implies the Lipschitz-like property of T^{-1} around $(\bar{v}, \bar{x})$ due to Mordukhovich coderivative criterion (2.1). Fix $\varepsilon > 0$ and define the mapping $\bar{T}_s := T^{-1} + s\mathbb{I} - s\bar{v}$ where $s > \varepsilon$. It is easy to check that gph $\bar{T}_s$ is closed around $(\bar{v}, \bar{x})$. Moreover, by (3.20) $\bar{T}_s$ is strongly monotone around $(\bar{v}, \bar{x})$ with modulus $(s - \varepsilon)$. For any $w \in D^*\bar{T}_s(\bar{v}|\bar{x})(0)$, we obtain

$$w \in D^*T^{-1}(\bar{v}|\bar{x})(0),$$

which means $0 \in D^*T(\bar{x}|\bar{v})(-w)$. It follows from (3.4) that $w = 0$. By Mordukhovich coderivative criterion (2.1) again, $\bar{T}_s$ is Lipschitz-like around $(\bar{v}, \bar{x})$. Since $\bar{T}_s$ is strongly monotone around $(\bar{v}, \bar{x})$, it is also single-valued around $(\bar{v}, \bar{x})$. As a result, $\bar{T}_s$ admits a Lipschitz continuous and single-valued localization around $(\bar{v}, \bar{x})$, and so does T^{-1}. Therefore, T is strongly metrically regular around $(\bar{x}, \bar{v})$. The proof is complete. □

Corollaries 3.9 and 3.11 indeed tell us that studying positive definiteness of $D^*T(\bar{x}|\bar{v})$ in (3.4) or (3.9) only makes sense under the strong metric regularity. That is the reason why the following lemma established by Poliquin and Rockafellar [22, Lemma 5.6] is useful in our study.

Lemma 3.12 (**strong monotonicity of inverse mapping**) *Suppose that* P *is a Lipschitz continuous mapping from an open convex set* $\mathcal{O}$ *into* $\mathbb{R}^n$. *The following are equivalent for any* $\alpha > 0$:

(i) $T = P^{-1}$ *is strongly monotone with modulus* α, *which means*

$$\langle v_1 - v_2, P(v_1) - P(v_2)\rangle \geq \alpha\|P(v_1) - P(v_2)\|^2 \quad \textit{for all} \quad v_1, v_2 \in \mathcal{O}. \tag{3.22}$$

(ii) *For any* $v \in \mathcal{O}$ *where* P *is differentiable, the Jacobian matrix satisfies*

$$\langle z, \nabla P(v)z\rangle \geq \alpha\|\nabla P(v)z\|^2 \quad \textit{for all} \quad z \in \mathbb{R}^n.$$

It is worth recalling here from Example 3.5 that even T^{-1} admits a continuously differentiable localization and T satisfies (3.9), T may be not locally strongly maximally monotone. In the next theorem, we add some additional conditions to show that the positive definiteness of $D^*T(\bar{x}|\bar{v})$ in (3.9) could be sufficient for local strong maximal monotonicity.

Theorem 3.13 (**sufficient condition for local strong maximal monotone under symmetry**) *Let* $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a set-valued mapping with closed graph around* $(\bar{x}, \bar{v}) \in$ *gph* T. *Suppose that* T *is hypomonotone around* $(\bar{x}, \bar{v})$ *and that* $D^*T(\bar{x}|\bar{v})$ *is positive definite with modulus* $\kappa > 0$ *in the sense of* (3.9). *Let* P *be a single-valued and Lipschitz continuous localization of* T^{-1} *around* $(\bar{v}, \bar{x})$ *described in Corollary 3.9. Assume that* $\nabla P(v)$ *(if exist) are symmetric matrices for all* v *near* $\bar{v}$. *Then* T *is locally strongly maximally monotone around* $(\bar{x}, \bar{v})$ *with any modulus in* $(0, \kappa)$.

Proof Under the assumptions in this theorem, suppose without loss of generality that $\operatorname{gph} P_\eta := \operatorname{gph} T^{-1} \cap \mathbb{B}_\eta(\bar{v}, \bar{x}) \subset \operatorname{gph} P$ with some $\eta > 0$ such that $\nabla P_\eta(v)$ is symmetric for any $v \in \mathbb{B}_\eta(\bar{v})$ at which P_η is differentiable. For any $\varepsilon \in (0, \kappa)$, we assume from Corollary 3.9 that P_η is Lipschitz continuous with modulus $(\kappa - \varepsilon)^{-1}$.

By Theorem 3.2, to obtain the local strong maximal monotonicity of T with modulus $\kappa - \varepsilon$, we only need to prove (3.22) with $\alpha = \kappa - \varepsilon$ when η is sufficiently small. By contradiction and Lemma 3.12, suppose that there exists a sequence $v_k \to \bar{v}$ at which P is differentiable and $z_k \in \mathbb{R}^n$ such that

$$\langle z_k, \nabla P(v_k) z_k \rangle < (\kappa - \varepsilon)\|\nabla P(v_k) z_k\|^2 \tag{3.23}$$

and that $\nabla P(v_k)$ is symmetric. Due to the hypomonotonicity of T around $(\bar{x}, \bar{v})$ with modulus $r > 0$ we may suppose that

$$\langle v - v_k, P(v) - P(v_k) \rangle \geq -r\|P(v) - P(v_k)\|^2 \quad \text{for all} \quad v \in \mathbb{B}_\eta(\bar{v}).$$

For any $z \in \mathbb{R}^n$, substituting $v = v_k + tz \in \mathbb{B}_\eta(\bar{v})$ with $t > 0$ into the above inequality gives us that

$$\langle tz, t\nabla P(v_k)z + o(t) \rangle \geq -r\|t\nabla P(v_k)z + o(t)\|^2.$$

By dividing both sides by t^2 and taking $t \downarrow 0$, we obtain

$$\langle z, \nabla P(v_k) z \rangle \geq -r\|\nabla P(v_k) z\|^2 \quad \text{for all} \quad z \in \mathbb{R}^n. \tag{3.24}$$

Since $A_k := \nabla P(v_k)$ is symmetric, from the spectral decomposition and (3.24) all eigenvalues $\lambda_i(A_k)$, $i = 1, \ldots, n$ satisfy either $\lambda_i(A_k) \geq 0$ or $\lambda_i(A_k) \leq -\frac{1}{r}$. It follows that for any $s > r$, $\lambda_i(\mathbb{I} + sA_k) \geq 1$ or $\lambda_i(\mathbb{I} + sA_k) \leq 1 - \frac{s}{r} < 0$. Hence we have $\mathbb{I} + sA_k$ is invertible. Observe further from (3.24) that

$$\langle (\mathbb{I} + sA_k)z, A_k z \rangle \geq (s - r)\|A_k z\|^2 \quad \text{for all} \quad z \in \mathbb{R}^n.$$

Replacing z in this inequality by $(\mathbb{I} + sA_k)^{-1}z$, we have

$$\langle z, B_k z \rangle \geq (s - r)\|B_k z\|^2 \quad \text{for all} \quad z \in \mathbb{R}^n \quad \text{with} \quad B_k := A_k(\mathbb{I} + sA_k)^{-1}. \tag{3.25}$$

Note that

$$B_k = \frac{1}{s}(\mathbb{I} + sA_k - \mathbb{I})(\mathbb{I} + sA_k)^{-1} = \frac{1}{s}(\mathbb{I} - (\mathbb{I} + sA_k)^{-1}). \tag{3.26}$$

is also symmetric. It follows from (3.25) that B_k is positive semidefinite. Moreover, we get from (3.23) that

$$\langle (\mathbb{I} + sA_k)z_k, A_k z_k \rangle < (s + \kappa - \varepsilon)\|A_k z_k\|^2.$$

Define $\bar{z}_k := (\mathbb{I} + sA_k)z_k$, we derive from the latter and (3.25) that

$$\langle \bar{z}_k, B_k \bar{z}_k \rangle < (s + \kappa - \varepsilon)\|B_k \bar{z}_k\|^2.$$

Let $U_k(\operatorname{diag}(\lambda_i(B_k)))U_k^*$ be the spectral decomposition of B_k. Define $w_k := U_k^* \bar{z}_k$, we obtain from the above inequalities that

$$0 \leq \sum_{i=1}^n \lambda_i(B_k) w_{k_i}^2 < (s + \kappa - \varepsilon)\left[\sum_{i=1}^n \lambda_i^2(B_k) w_{k_i}^2\right] \leq (s + \kappa - \varepsilon)\lambda_{\max}(B_k)\left[\sum_{i=1}^n \lambda_i(B_k) w_{k_i}^2\right].$$

It follows that $\sum_{i=1}^n \lambda_i(B_k) w_{k_i}^2 > 0$ and

$$\|B_k\| = \lambda_{\max}(B_k) \geq \frac{1}{s + \kappa - \varepsilon}. \tag{3.27}$$

Furthermore, since P_η is Lipschitz continuous around $\bar{v}$ with modulus $(\kappa-\varepsilon)^{-1}$, we have $\|\nabla P(v_k)\| \le (\kappa-\varepsilon)^{-1}$ for all k. By passing to a subsequence, suppose that $A_k = \nabla P(v_k)$ converges to a symmetric matrix A as $k \to \infty$. Note further that $\nabla P_\eta(v_k)^* z = \widehat{D}^* P_\eta(v_k)(z)$ for any $z \in \mathbb{R}^n$. It follows that $A^* z \in D^* P_\eta(\bar{v})(z)$, i.e., $-z \in D^* T(\bar{x}|\bar{v})(-A^* z)$. This together with (3.9) tells us that

$$\langle z, Az\rangle = \langle z, A^* z\rangle \ge \kappa \|A^* z\|^2 = \kappa\|Az\|^2 \quad \text{for all} \quad z \in \mathbb{R}^n,$$

which implies $0 \le \lambda_i(A) \le \frac{1}{\kappa}$ due to the spectral decomposition and that $\mathbb{I}+sA$ is invertible. Define $B := A(\mathbb{I}+sA)^{-1}$, it is similar to (3.26) and (3.25) that B is symmetric and

$$\langle z, Bz\rangle \ge (s+\kappa)\|Bz\|^2 \quad \text{for all} \quad z \in \mathbb{R}^n.$$

By the spectral decomposition again for B, we obtain that

$$\min_{z\in\mathbb{R}^n} \frac{\langle z, Bz\rangle}{\|Bz\|^2} = \frac{1}{\lambda_{\max}(B)} = \frac{1}{\|B\|} \ge s+\kappa. \tag{3.28}$$

Moreover, since $A_k \to A$, we have $B_k \to B$. Note from (3.27) that $\|B\| \ge (s+\kappa-\varepsilon)^{-1}$, which contradicts the inequality $\|B\|^{-1} \ge (s+\kappa)$ in (3.28). Thus inequality (3.23) could not be satisfied. This completes the proof of the theorem. $\square$

In Theorem 3.6, the positive definiteness of coderivative (3.9) is characterized via nearly strong monotonicity around the point in question. Characterization for its variant (3.4) as desired in [18, Conjecture 3.6] is still missing. At this moment we do not know whether (3.4) and (3.9) are equivalent in general, but under some special circumstances , e.g., T is the subdifferential mapping to a lower semicontinuous extended real-valued function [17].

Theorem 3.14 (**equivalence between two kinds of positive definiteness of limiting coderivative**) *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be strongly metrically regular at $\bar{x} \in \operatorname{dom} T$ for $\bar{v} \in T(\bar{x})$ and let P be a Lipschitz continuous localization of T^{-1} around $(\bar{v}, \bar{x})$. Conditions (3.4) and (3.9) are equivalent when one of the following three statements holds:*

(i) *The Bouligand generalized Jacobian*

$$\nabla_B P(\bar{v}) := \{A \in \mathbb{R}^{n\times n} \mid \exists \{v_k\} \to \bar{v},\ P \textit{ is differentiable at } v_k,\ \nabla P(v_k) \to A\}$$

contains finitely many elements.

(ii) *All matrices in $\nabla_B P(\bar{v})$ have the same rank.*

(iii) *All matrices in $\nabla_B P(\bar{v})$ are symmetric.*

Proof It suffices to prove the implication [(3.4) $\Rightarrow$(3.9)].

(i) Suppose that $\nabla_B P(\bar{v})$ contains finitely many elements. Pick any $A \in \nabla_B P(\bar{v})$ and $z \in \mathbb{R}^n$. Since P is Lipschitz continuous around $\bar{v}$, we have

$$A^T z \in \partial\langle z, P\rangle(\bar{v}) = D^* T^{-1}(\bar{v}|\bar{x})(z),$$

which implies $-z \in D^* T(\bar{x}|\bar{v})(-A^T z)$. It follows from (3.4) that

$$\langle z, A^T z\rangle > 0 \quad \text{whenever} \quad A^T z \ne 0. \tag{3.29}$$

For any $u \in \operatorname{Ker} A^T$ and $v \in \operatorname{Im} A$, we have

$$0 \le \langle (u+v), A^T(u+v)\rangle = \langle u, A^T v\rangle + \langle v, A^T v\rangle.$$

Replacing u above by λu with any $\lambda \in \mathbb{R}$ gives us $\langle u, A^T v\rangle = 0$. Moreover, since $\operatorname{Ker} A^T \cap \operatorname{Im} A = \{0\}$, we get from (3.4) that $\langle v, A^T v\rangle > 0$ for any $v \in \operatorname{Im} A \setminus \{0\}$. Hence there exists a positive constant c such that $\langle v, A^T v\rangle \geq c\|v\|^2$ for all $v \in \operatorname{Im} A$.

Since $\operatorname{Ker} A^T \oplus \operatorname{Im} A = \mathbb{R}^n$, for any $z \in \mathbb{R}^n$, we find $u \in \operatorname{Ker} A^T$ and $v \in \operatorname{Im} A$ with $z = u + v$. It follows that

$$\langle z, A^T z\rangle = \langle u, A^T v\rangle + \langle v, A^T v\rangle = \langle v, A^T v\rangle \geq c\|v\|^2 \geq \frac{c}{\|A^T\|^2}\|A^T v\|^2 = \frac{c}{\|A^T\|^2}\|A^T z\|^2.$$

Since there are finitely many elements in $\nabla_B P(\bar{v})$, we could find a constant $\kappa > 0$ such that

$$\langle z, A^T z\rangle \geq \kappa\|A^T z\|^2 \quad \text{for all} \quad A \in \nabla_B P(\bar{v}), z \in \mathbb{R}^n. \tag{3.30}$$

Define $JP(\bar{v}) := \operatorname{co}\nabla_B P(\bar{v})$ as the Clarke *generalized Jacobian*; [4, Definition 2.6.1]. Due to the convexity of $\|\cdot\|^2$, it is easy to obtain from (3.30) that

$$\langle z, A^T z\rangle \geq \kappa\|A^T z\|^2 \quad \text{for all} \quad A \in JP(\bar{v}), z \in \mathbb{R}^n. \tag{3.31}$$

Pick any $(w, z) \in \mathbb{R}^n \times \mathbb{R}^n$ with $z \in D^*T(\bar{x}|\bar{v})(w)$, we have $-w \in D^*P(\bar{v}|\bar{x})(-z)$. It follows from the Lipschitz continuity of P at $\bar{v}$ that obtain that

$$-w \in \partial\langle -z, P\rangle(\bar{v}) \subset \partial_C\langle -z, P\rangle(\bar{v}) = JP(\bar{v})^T(-z),$$

where ∂_C denotes the Clarke's *generalized gradient*; see, e.g., [4, Theorem 2.6.6]. Thanks to the above inclusion and (3.31), there exists $A \in JP(\bar{v})$ such that $w = A^T z$ and

$$\langle w, z\rangle = \langle A^T z, z\rangle \geq \kappa\|A^T z\|^2 = \kappa\|w\|^2,$$

which clearly verifies (3.9).

(ii) All matrices in $\nabla_B P(\bar{v})$ have the same rank r. Thanks to (3.29), we have

$$\langle z, A^T z\rangle = 0 \quad \text{implies} \quad A^T z = 0 \quad \text{for all} \quad A \in \nabla_B P(\bar{v}).$$

A matrix $A \in \nabla_B P(\bar{v})$ satisfying the above property is called *positive semidefinite plus* [9]. By [12, Proposition 1], A could be decomposed by

$$A = U^T\begin{pmatrix}\Lambda_r + N & 0\\ 0 & 0\end{pmatrix}U,$$

where $U^T\begin{pmatrix}\Lambda_r & 0\\ 0 & 0\end{pmatrix}U$ is the spectral decomposition of $\frac{1}{2}(A + A^T)$ with $\Lambda_r = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$ due to (3.29) and N is an $r \times r$ skew-symmetric matrix with $\frac{1}{2}(A - A^T) = U^T\begin{pmatrix}N & 0\\ 0 & 0\end{pmatrix}U$. For any $z \in \mathbb{R}^n$, define $u = Uz$, we have

$$\begin{aligned}
\langle z, A^T z\rangle &= \langle z, \frac{1}{2}(A + A^T)z\rangle = \langle z, U^T\begin{pmatrix}\Lambda_r & 0\\ 0 & 0\end{pmatrix}Uz\rangle = \sum_{i=1}^r \lambda_i u_i^2\\
&\geq \min\left\{\frac{\lambda_1}{\lambda_1^2 + \|N\|^2}, \frac{\lambda_r}{\lambda_r^2 + \|N\|^2}\right\}\sum_{i=1}^r(\lambda_i^2 + \|N\|^2)u_i^2\\
&\geq \min\left\{\frac{0.5\lambda_1}{\lambda_1^2 + \|N\|^2}, \frac{0.5\lambda_r}{\lambda_r^2 + \|N\|^2}\right\}\|\begin{pmatrix}\Lambda_r - N & 0\\ 0 & 0\end{pmatrix}u\|^2\\
&= \min\left\{\frac{0.5\lambda_1}{\lambda_1^2 + 0.25\|A - A^T\|^2}, \frac{0.5\lambda_r}{\lambda_r^2 + 0.25\|A - A^T\|^2}\right\}\|A^T z\|^2
\end{aligned}$$

for all $A \in \nabla_B P(\bar{v})$. Since $\lambda_1 = \lambda_1(\frac{1}{2}(A + A^T))$ and $\lambda_r = \lambda_r(\frac{1}{2}(A + A^T))$ are continuous functions with respect to A, the quantity

$$\min\left\{\frac{0.5\lambda_1(0.5(A+A^T))}{\lambda_1^2(0.5(A+A^T)) + 0.25\|A - A^T\|^2}, \frac{0.5\lambda_r(0.5(A+A^T))}{\lambda_r^2(0.5(A+A^T)) + 0.25\|A - A^T\|^2}\right\} > 0$$

attains a positive minimum value on the compact set $\nabla_B P(\bar{v})$. This tells us that (3.30) is also true with some constant $\kappa > 0$. Imitating the proof of **(i)** after (3.30), we also derive (3.9).

(iii) All matrices in $\nabla_B P(\bar{v})$ are symmetric. Similarly to (3.29), all matrices in $\nabla_B P(\bar{v})$ are semidefinite. Following the proof in case **(i)**, we only need to prove the existence of $\kappa > 0$ in (3.30). Indeed, due to the spectral decomposition, we obtain that

$$\langle z, Az\rangle \geq \frac{1}{\lambda_{\max}(A)}\|Az\|^2 \quad \text{for all} \quad A \in \nabla_B P(\bar{v}). \tag{3.32}$$

Since $\nabla_B P(\bar{v})$ is a compact set and $\lambda_{\max}(\cdot)$ is a continuous function, $\lambda_{\max}(\cdot)$ is bounded above in $\nabla_B P(\bar{v})$. This together with (3.32) verifies (3.30) and thus (3.9). The proof is complete.

□

The assumption of strong metric regularity on T in the above theorem is not restrictive at all, since as either (3.4) or (3.9) is satisfied, such property is automatically valid due to Corollaries 3.9 and 3.11. Furthermore, if one could construct a Lipschitz continuous mapping $P : \mathbb{R}^2 \to \mathbb{R}^2$ around $\bar{v} \in \mathbb{R}^2$ satisfying

$$\nabla_B P(\bar{v}) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{n} \\ -\frac{1}{n} & \frac{1}{n^4} \end{pmatrix}, n = 1, 2 \ldots\right\}$$

then (3.4) is satisfied with $T = P^{-1}$ and $\bar{x} = P(\bar{v})$, but (3.9) is not necessarily valid, since there is no uniform $\kappa > 0$ such that (3.30) holds for all $A \in \nabla_B P(\bar{v})$. Unfortunately, at this moment we do not know either how to construct such mapping P or whether such a mapping P exists.

Let us finish this section by showing that Conjecture 3.3 is indeed true for the case of one dimension by applying Theorems 3.13 and 3.14.

Corollary 3.15 (**Validity of Conjecture 3.3 in one dimension**) *Let $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. Suppose that T is locally hypomonotone around $(\bar{x}, \bar{v})$. Then the assertions in Conjecture 3.3 are equivalent.*

Proof Since the implication [**(i)**⇒**(ii)**] is already proved in Corollary 3.4, we only need to show the converse implication. Suppose that T is hypomonotone around $(\bar{x}, \bar{v})$ and $D^*T(\bar{x}|\bar{v})$ is positive definite in the sense of (3.4). By Corollary 3.11, T is strongly metrically regular around $(\bar{x}, \bar{v})$. Let $P : \mathbb{R} \to \mathbb{R}$ be a single-valued and Lipschitz continuous localization of T^{-1} around $(\bar{v}, \bar{x})$. Since $\nabla_B P(\bar{v}) \subset \mathbb{R}$, the assertion **(iii)** in Theorem 3.14 holds and thus T satisfies (3.9). It follows from Theorem 3.13 that T is locally strongly maximally monotone around $(\bar{x}, \bar{v})$, which verifies **(i)**. □

4 Conclusion

In this paper, we disprove [18, Conjecture 3.6], which speculates the interconnection between the positive definiteness of limiting coderivative and the local strong maximal monotonicity. The conjecture is only true in one dimension or under some extra assumptions. However, we are able to characterize the positive definiteness of limiting coderivative by the nearly strong monotonicity, which is even not locally monotone. Consequently, the positive definiteness of limiting coderivative is sufficient for strong metric regularity. The pointwise infinitesimal characterization of local strong maximal monotonicity for set-valued mapping remains open. Whether two kinds of positive definiteness of limiting coderivative (3.4) and (3.9) are equivalent in general is also not known yet. In the future, we plan to use our results, e.g., Theorem 3.6 and Corollary 3.10 to study the nearly strong monotonicity and strong metric regularity for variational systems [15] and generalized equations [24] that may involve further advances on second-order variational analysis; especially when the corresponding system is degenerate.

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