Correction to "Provable Low Rank Phase Retrieval"

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***Abstract*—This note corrects a few errors in the proof of the main result of the paper "Provable Low Rank Phase Retrieval". The result itself has no change. This paper introduced an alternating minimization solution, called AltMinLowRaP, for solving the Low Rank Phase Retrieval (LRPR) problem: recover an $n \times q$ matrix $\boldsymbol{X}^*$ of rank r from $\boldsymbol{y}_k := |\boldsymbol{A}_k{}' \boldsymbol{x}_k^*|, k = 1, 2, \ldots, q$ when the measurement matrices $\boldsymbol{A}_k$ are mutually independent. Here $\boldsymbol{y}_k$ is an m length vector, $\boldsymbol{A}_k$ is an $n \times m$ matrix, and $'$ denotes transpose.**

This note corrects a few errors in the proof of the main result of the paper "Provable Low Rank Phase Retrieval" [1]. The result itself has no change. We repeat below the problem studied in [1], followed by summarizing the AltMin algorithm to solve it and the guarantee for it from [1]. In Sec II, we correct the proof errors.

I. Summary of the original paper [1]

1) Low Rank PR (LRPR) problem: The goal is to recover an $n \times q$ rank-r matrix $\boldsymbol{X}^* := [\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_k^*, \ldots, \boldsymbol{x}_q^*]$ from

$$\boldsymbol{y}_{ik} := |\langle \boldsymbol{a}_{ik}, \boldsymbol{x}_k^* \rangle|, \ i \in [m], k \in [q], \tag{1}$$

when all the $\boldsymbol{a}_{ik}$'s are iid standard (real-valued) Gaussian vectors. By defining the m-length vector $\boldsymbol{y}_k := [\boldsymbol{y}_{1,k}, \boldsymbol{y}_{2,k}, \ldots, \boldsymbol{y}_{m,k}]'$ and the $n \times m$ matrix $\boldsymbol{A}_k := [\boldsymbol{a}_{1,k}, \boldsymbol{a}_{2,k}, \ldots, \boldsymbol{a}_{m,k}]$, the above measurement model can also be rewritten as $\boldsymbol{y}_k := |\boldsymbol{A}_k{}' \boldsymbol{x}_k^*|, \ k = 1, 2, \ldots, q$. Here $'$ denotes transpose, $[m] := \{1, 2, \ldots, m\}$, and $|\boldsymbol{z}|$ denotes element-wise magnitude of a vector.

Let $\boldsymbol{X}^* \overset{\text{SVD}}{=} \boldsymbol{U}^* \boldsymbol{\Sigma}^* \boldsymbol{B}^*$ denote its reduced singular value decomposition (SVD) so that $\boldsymbol{U}^* \in \mathbb{R}^{n \times r}$, $\boldsymbol{B}^* \in \mathbb{R}^{r \times q}$, and $\boldsymbol{\Sigma}^* \in \mathbb{R}^{r \times r}$ is a diagonal matrix. We use $\sigma_{\max}^*, \sigma_{\min}^*$ to denote the maximum, minimum singular values of $\boldsymbol{X}^*$ and $\kappa = \sigma_{\max}^* / \sigma_{\min}^*$ to denote the condition number of $\boldsymbol{\Sigma}^*$. Finally, we let

$$\tilde{\boldsymbol{B}}^* := \boldsymbol{\Sigma}^* \boldsymbol{B}^*.$$

Right singular vectors' Incoherence. Since we have do not have global measurements of the entire matrix $\boldsymbol{X}^*$, in order to correctly recover $\boldsymbol{X}^*$ while needing $m < n$, we need an assumption that allows for correct interpolation across the rows. As explained in [1], incoherence of the right singular vectors (henceforth referred to as "right incoherence") suffices for this purpose. In our notation, this means that we assume

$$\max_k \|\boldsymbol{b}_k^*\|^2 \leq \mu^2 \frac{r}{q}, \tag{2}$$

with $\mu \geq 1$ being a constant. Clearly, this implies that

$$\|\boldsymbol{x}_k^*\|^2 = \|\tilde{\boldsymbol{b}}_k^*\|^2 \leq {\sigma_{\max}^*}^2 \mu^2 \frac{r}{q} = \kappa^2 {\sigma_{\min}^*}^2 \mu^2 \frac{r}{q} \leq \kappa^2 \mu^2 \frac{\|\boldsymbol{X}^*\|_F^2}{q} \tag{3}$$

Algorithm 1 AltMin-LowRaP: Alt-Min for Phaseless Low Rank Recovery

Parameters: T, $T_{RWF,t}$, ω.
Partition the m_{tot} measurements and design vectors for each $\boldsymbol{x}_k^*$ into one set for initialization and $2T$ disjoint sets for the main loop.
Set $\hat{r}$ as the largest index j for which $\lambda_j(\boldsymbol{Y}_U) - \lambda_n(\boldsymbol{Y}_U) \geq \omega$ where

$$\boldsymbol{Y}_U = \frac{1}{mq} \sum_{k=1}^{q} \sum_{i=1}^{m} \boldsymbol{y}_{ik}^2 \boldsymbol{a}_{ik} \boldsymbol{a}_{ik}' \mathbb{1}_{\left\{ \boldsymbol{y}_{ik}^2 \leq 9\kappa^2\mu^2 \frac{1}{mq} \sum_{ik} \boldsymbol{y}_{ik}^2 \right\}}$$

$\boldsymbol{U}^0 \leftarrow \hat{\boldsymbol{U}}^0 \leftarrow$ top $\hat{r}$ singular vectors of $\boldsymbol{Y}_U$ defined above.
for $t = 0 : T$ **do**
  $\hat{\boldsymbol{b}}_k^t \leftarrow \text{RWF}(\{\boldsymbol{y}_k^{(t)}, \boldsymbol{U}^{t\prime} \boldsymbol{A}_k^{(t)}\}, T_{RWF,t})$ for each $k = 1, 2, \cdots, q$ (RWF*: Reshaped Wirtinger Flow or any algorithm to solve standard PR).*
  $\hat{\boldsymbol{x}}_k^t \leftarrow \boldsymbol{U}^t \hat{\boldsymbol{b}}_k^t$ for each $k = 1, 2, \cdots, q$.
  $\hat{\boldsymbol{C}}_k \leftarrow \text{Phase}\left(\boldsymbol{A}_k^{(T+t)\prime} \hat{\boldsymbol{x}}_k^t \right)$ for each $k = 1, 2, \cdots, q$.
  Get $\boldsymbol{B}^t$ by QR decomp: $\hat{\boldsymbol{B}}^t \overset{\text{QR}}{=} \boldsymbol{R}_B^t \boldsymbol{B}^t$.
  $\hat{\boldsymbol{U}}^{t+1} \leftarrow \arg\min_{\tilde{\boldsymbol{U}}} \sum_{k=1}^{q} \|\hat{\boldsymbol{C}}_k \boldsymbol{y}_k^{(T+t)} - \boldsymbol{A}_k^{(T+t)\prime} \tilde{\boldsymbol{U}} \boldsymbol{b}_k^t\|^2$.
  Get $\boldsymbol{U}^{t+1}$ by QR decomp: $\hat{\boldsymbol{U}}^{t+1} \overset{\text{QR}}{=} \boldsymbol{U}^{t+1} \boldsymbol{R}_U^{t+1}$.
end for

2) Notation: We use $\|.\|$ to denotes the (induced) l_2-norm and $\|.\|_F$ to denote the Frobenius norm. We use $\mathbb{1}_{\text{statement}}$ to denote the indicator function; it takes the value one if statement is true and is zero otherwise. A tall matrix with orthonormal columns is referred to as a *"basis matrix"*. For two basis matrices $\boldsymbol{U}_1, \boldsymbol{U}_2$, we define the subspace error (distance) as $\sin\Theta(\boldsymbol{U}_1, \boldsymbol{U}_2) := \|(\boldsymbol{I} - \boldsymbol{U}_1 \boldsymbol{U}_1') \boldsymbol{U}_2\|$. This measures the sine of the largest principal angle between the two subspaces. For real-valued vectors, the phase-invariant distance is computed as $\text{dist}(\boldsymbol{x}^*, \hat{\boldsymbol{x}}) = \min(\|\boldsymbol{x}^* - \hat{\boldsymbol{x}}\|, \|\boldsymbol{x}^* + \hat{\boldsymbol{x}}\|)$. We *reuse* the letters c, C to denote different numerical constants in each use, with the convention $C \geq 1$ and $c < 1$.

3) Algorithm and Guarantee: Algorithm 1 was introduced and studied in [1] where we proved the following guarantee for it. The guarantee itself is correct although, with our corrected proof, it can be improved slightly.

Theorem 1.1 (Guarantee for AltMinLowRaP from [1])**.** *Consider Algorithm 1. Assume right singular vectors' incoherence holds. Set $T := C \log(1/\epsilon)$, $T_{RWF,t} = C(\log r + \log \kappa + t(\log(0.7)/\log(1-c)))$, $\omega = 1.3{\sigma_{\min}^*}^2/q$. Assume that, for the initialization step and for each new update, we use a new* *set of m measurements with m satisfying $mq \geq C\kappa^{12}\mu^4\ nr^4$ and $m \geq C\max(r, \log q, \log n)$. Then, with probability (w.p.) at least $1 - Cn^{-10}$, the algorithm converges geometrically, i.e., after $T = C\log(1/\epsilon)$ iterations,*

$$\sin\Theta(\boldsymbol{U}^*, \boldsymbol{U}^T) \leq \epsilon,$$

and $\mathrm{dist}(\hat{\boldsymbol{x}}_k^T, \boldsymbol{x}_k^*) \leq \epsilon\|\boldsymbol{x}_k^*\|$ for each k. The time complexity is $mqnr\log^2(1/\epsilon)$.

With the corrected proof given below, we can actually prove a marginally stronger result, we need only nr^3 total samples for the AltMin iterations, but still need nr^4 for initialization. Thus the total number of measurements per column, m_{tot}, needed to obtain an ϵ-accurate estimate needs to satisfy $m_{tot}q \geq C_{\kappa,\mu}nr^3(r + \log(1/\epsilon))$ and $m_{tot} \geq C\max(r, \log q, \log n)\log(1/\epsilon)$.

II. Proving the Theorem: corrections

A. Two main claims that prove Theorem 1.1

The proof for the subspace error bound is an immediate consequence of the next two claims. The bound on $\mathrm{dist}(\hat{\boldsymbol{x}}_k, \boldsymbol{x}_k^*)$ follows by Lemma 2.8 given below. The claim statements were essentially correct in [1]. The error was in the proof of the Claim 2.2.

Claim 2.1 (Rank estimation and Initialization of $\boldsymbol{U}^*$). *Let $\boldsymbol{U}_{\mathrm{init}} = \hat{\boldsymbol{U}}^0$. Pick a $\delta_{\mathrm{init}} < 0.25$. Assume $mq \geq \kappa^8\mu^4 nr^2/\delta_{\mathrm{init}}^2$. Then, w.p. at least $1 - 6n^{-10}$, the rank is correctly estimated and $\sin\Theta(\boldsymbol{U}_{\mathrm{init}}, \boldsymbol{U}^*) \leq \delta_{\mathrm{init}}$.*

Claim 2.2 (Descent). *At iteration t, assume that $\sin\Theta(\boldsymbol{U}^*, \boldsymbol{U}^t) \leq \delta_t$ and $\delta_t \leq \delta_{\mathrm{init}} = c/r\kappa^2$. If $mq \geq C\kappa^6\mu^2 nr^3$ and $m \geq C\max(r, \log n, \log q)$ then w.p. at least $1 - C\exp(-nr) - n^{-10}$, $\sin\Theta(\boldsymbol{U}^{t+1}, \boldsymbol{U}^*) \leq 0.7\delta_t := \delta_{t+1}$.*

Proof of Theorem. Claim 2.2 requires $mq \geq C\kappa^6\mu^2 nr^3$, $m \geq C\max(r, \log n, \log q)$, and $\delta_{\mathrm{init}} = c/r\kappa^2$. By Claim 2.1, if $mq \geq C\kappa^{12}\mu^4 nr^4$, then $\sin\Theta(\boldsymbol{U}_{\mathrm{init}}, \boldsymbol{U}^*) \leq \delta_{\mathrm{init}} = c/r\kappa^2$.

Thus, combining these two claims, if, in each iteration, we have $mq \geq C\kappa^{12}\mu^4 nr^4$, and $m \geq C\max(r, \log n, \log q)$, then, w.p. $\geq 1 - 10n^{-10}$, the estimates converge geometrically, i.e., $\sin\Theta(\boldsymbol{U}^T, \boldsymbol{U}^*) \leq 0.7^T\delta_{\mathrm{init}}$. Therefore, we need $T = C\log(1/\epsilon)$ iterations to achieve ϵ accuracy. By Lemma 2.8, the other conclusions follow. □

B. Corrected Proof of Claim 2.2

The proof is an easy consequence of the next four lemmas. In this section, we remove the superscripts ^{t} or $^{T+t}$ except where essential, e.g., we let $\boldsymbol{a}_{ik} := \boldsymbol{a}_{ik}^{(T+t)}$ and $\boldsymbol{y}_{ik} := \boldsymbol{y}_{ik}^{(T+t)}$ when updating $\boldsymbol{U}$, and we let $\boldsymbol{a}_{ik} := \boldsymbol{a}_{ik}^{(t)}$ and $\boldsymbol{y}_{ik} := \boldsymbol{y}_{ik}^{(t)}$ when updating $\boldsymbol{B}$. We should also remind the reader of the following two points.

1) Because of this sample-splitting, (in each new iteration for updating either $\boldsymbol{U}$ or $\boldsymbol{B}$, we use a new *independent* set of measurement matrices $\boldsymbol{A}_k$ and measurements $\boldsymbol{y}_k$), we can assume that the $\boldsymbol{a}_{ik}$s used in the curent update are independent of the previous estimates of $\boldsymbol{U}$, or $\boldsymbol{B}$ that appear in the update equation.
2) Without loss of generality, as explained in detail in [1], and as also done in previous works on PR, in all the proofs below, when considering $\mathrm{dist}(\hat{\boldsymbol{x}}, \boldsymbol{x}^*)$ we assume that $\boldsymbol{x}^*$ is replaced by $\boldsymbol{x}^* sign(\langle\hat{\boldsymbol{x}}, \boldsymbol{x}^*\rangle)$. With this, $\mathrm{dist}(\hat{\boldsymbol{x}}, \boldsymbol{x}^*) = \|\hat{\boldsymbol{x}} - \boldsymbol{x}^*\|$.

Lemma 2.3. *[same as that in [1]] We have*

$$\sin\Theta(\boldsymbol{U}^{t+1}, \boldsymbol{U}^*) \leq \frac{\mathrm{MainTerm}}{\sigma_{\min}(\boldsymbol{U}^*\boldsymbol{\Sigma}^*\boldsymbol{B}^*\boldsymbol{B}') - \mathrm{MainTerm}} \tag{4}$$

where $\mathrm{MainTerm} :=$

$$\frac{\max_{\boldsymbol{W}\in\mathcal{S}_W}|\mathrm{Term1}(\boldsymbol{W})| + \max_{\boldsymbol{W}\in\mathcal{S}_W}|\mathrm{Term2}(\boldsymbol{W})|}{\min_{\boldsymbol{W}\in\mathcal{S}_W}\mathrm{Term3}(\boldsymbol{W})},$$

$$\mathrm{Term1}(\boldsymbol{W}) := \sum_{ik}\boldsymbol{b}_k{}'\boldsymbol{W}'\boldsymbol{a}_{ik}\boldsymbol{a}_{ik}{}'\boldsymbol{U}^*(\tilde{\boldsymbol{B}}^*\boldsymbol{B}'\boldsymbol{b}_k - \tilde{\boldsymbol{b}}_k^*),$$

$$\mathrm{Term2}(\boldsymbol{W}) := \sum_{ik}(\boldsymbol{c}_{ik}\hat{\boldsymbol{c}}_{ik} - 1)(\boldsymbol{a}_{ik}{}'\boldsymbol{W}\boldsymbol{b}_k)(\boldsymbol{a}_{ik}{}'\boldsymbol{x}_k^*),$$

$$\mathrm{Term3}(\boldsymbol{W}) := \sum_{ik}(\boldsymbol{a}_{ik}{}'\boldsymbol{W}\boldsymbol{b}_k)^2,$$

$$\mathcal{S}_W := \{\boldsymbol{W} \in \mathbb{R}^{n\times r} : \ \|\boldsymbol{W}\|_F = 1\}$$

and $\boldsymbol{c}_{ik}, \hat{\boldsymbol{c}}_{ik}$ are the phases (signs) of $\boldsymbol{a}_{ik}{}'\boldsymbol{x}_k^*$ and $\boldsymbol{a}_{ik}{}'\hat{\boldsymbol{x}}_k$.

Lemma 2.4 (minor correction from [1]). *Under the conditions of Theorem 1.1, and assuming that $\sin\Theta(\boldsymbol{U}^*, \boldsymbol{U}) \leq \delta_t$, with $\delta_t < 0.1$, w.p. at least $1 - 2\exp\left(nr(\log 17) - c\frac{\epsilon_3^2 mq}{\hat{\mu}^2 r}\right) - n^{-10}$,*

$$\min_{\boldsymbol{W}\in\mathcal{S}_{\boldsymbol{W}}}\mathrm{Term3}(\boldsymbol{W}) \geq 0.5(1 - \epsilon_3)m$$

and

$$\max_{\boldsymbol{W}\in\mathcal{S}_{\boldsymbol{W}}}\mathrm{Term3}(\boldsymbol{W}) \leq 1.5(1 + \epsilon_3)m.$$

The above lemma is the same as that in [1] except for the use of ϵ_3 where δ_t was used earlier. The proof is also essentially the same except for a minor change when applying the concentration bound (Lemma 2.7 below): one needs to set $t = m\epsilon_3\delta_t$ instead of $t = m\delta_t^2$.

Lemma 2.5 (correction from [1]). *Under the conditions of Theorem 1.1, and assuming that $\sin\Theta(\boldsymbol{U}^*, \boldsymbol{U}) \leq \delta_t$, with $\delta_t < 0.1$, w.p. at least $1 - 2\exp\left(nr(\log 17) - c\frac{\epsilon_1^2 mq}{\kappa^3\mu^2 r}\right) - n^{-10}$.*

$$\max_{\boldsymbol{W}\in\mathcal{S}_{\boldsymbol{W}}}\mathrm{Term1}(\boldsymbol{W}) \leq m\epsilon_1\delta_t\|\boldsymbol{X}^*\|_F.$$

This lemma statement has the following change from [1]: we now have $\epsilon_1\delta_t$ in the bound instead of δ_t^2 and ϵ_1^2 in the probability expression instead of δ_t^2. There was an error in its proof; we provide a corrected proof below.

Lemma 2.6 (correction from [1]). *Under the conditions of Theorem 1.1, and assuming $\sin\Theta(\boldsymbol{U}^*, \boldsymbol{U}) \leq \delta_t$ with $\delta_t < 0.1$, w.p. at least $1 - 2\exp\left(nr(\log 17) - c\frac{\epsilon_3^2 mq}{\hat{\mu}^2 r}\right) - 2\exp\left(-c\epsilon_2^2 mq\right) - n^{-10}$,*

$$\max_{\boldsymbol{W}\in\mathcal{S}_W}\mathrm{Term2}(\boldsymbol{W}) \leq m\ \sqrt{1 + \epsilon_3}\sqrt{\delta_t + \epsilon_2}\delta_t\|\boldsymbol{X}^*\|_F.$$

This has the following change from [1]: we now have $\sqrt{\delta_t + \epsilon_2}$ instead of $\sqrt{\delta_t}$ in the bound and ϵ_2^2 instead of δ_t^2

in the probability expression. The proof is almost the same, the main change is in applying the concentration bound and in a few steps after it. We provide a brief corrected proof below.

Proof of Claim 2.2. By Lemma 2.3 and using the simple fact that $\sigma_{\min}(\boldsymbol{U}^*\boldsymbol{\Sigma}^*\boldsymbol{B}^*\boldsymbol{B}') \geq \sigma_{\min}(\boldsymbol{U}^*)\sigma_{\min}(\boldsymbol{\Sigma}^*)\sigma_{\min}(\boldsymbol{B}^*)\sigma_{\min}(\boldsymbol{B}') \geq \sigma_{\min}(\boldsymbol{\Sigma}^*) = \sigma^*_{\min}$,

$$\sin\Theta(\boldsymbol{U}^{t+1}, \boldsymbol{U}^*) \leq \frac{\text{MainTerm}}{\sigma^*_{\min} - \text{MainTerm}}. \tag{5}$$

where $\text{MainTerm} := \frac{\max_{\boldsymbol{W}\in\mathcal{S}_W}|\text{Term1}(\boldsymbol{W})| + \max_{\boldsymbol{W}\in\mathcal{S}_W}|\text{Term2}(\boldsymbol{W})|}{\min_{\boldsymbol{W}\in\mathcal{S}_W}\text{Term3}(\boldsymbol{W})}$.

Combining Lemmas 2.5, 2.6, 2.4, and using $\|\boldsymbol{X}^*\|_F \leq \sqrt{r}\sigma^*_{\max}$, we conclude that, w.p. at least $1 - 2\exp\left(nr(\log 17) - c\frac{\epsilon_1^2 mq}{\kappa^3\mu^2 r}\right) - 2\exp\left(nr(\log 17) - c\frac{\epsilon_3^2 mq}{\kappa^2\mu^2 r}\right) - 2\exp\left(-c\epsilon_2^2 mq\right) - n^{-10}$,

$$\text{MainTerm} \leq C\frac{(\epsilon_1 + \sqrt{1+\epsilon_3}\sqrt{\delta_t + \epsilon_2})}{0.5(1-\epsilon_3)}\delta_t\sqrt{r}\sigma^*_{\max}.$$

In order to show that the RHS of (5) is less than $\delta_{t+1} := 0.7\delta_t$, it suffices to bound MainTerm by $c\delta_t\sigma^*_{\min}$ with $c = 0.4$.

To obtain $\text{MainTerm} \leq 0.4\delta_t\sigma^*_{\min}$, we can pick $\epsilon_3 = 0.1$, $\epsilon_1 = c/\sqrt{r}\kappa$, $\epsilon_2 = c/r\kappa^2$, and $\delta_t \leq c/r\kappa^2$. Since we assumed $\delta_t \leq \delta_{\text{init}}$, the bound on δ_t is ensured if we set $\delta_{\text{init}} = c/r\kappa^2$. With these choices of ϵ_j's and δ_{init}, if $mq \geq Cnr^2/\epsilon_1^2 = Cnr^3$, $mq \geq Cnr^2/\epsilon_3^2 = Cnr^2$, $mq \geq Cnr/\epsilon_2^2 = Cnr^3$, and $\delta_{\text{init}} = c/r\kappa^2$, then w.p. $1 - n^{-10} - 4\exp(-cnr)$, $\sin\Theta(\boldsymbol{U}^{t+1}, \boldsymbol{U}^*) \leq 0.7\delta_t$. ☐

C. Corrected proofs of Lemma 2.5 and Lemma 2.6

The proof uses the following concentration bound for sums of products of sub-Gaussian random variables [2].

Lemma 2.7 ([2]). *Let $\{X_i, Y_i\}$ be sub-Gaussian random variables with sub-Gaussian norm K_{X_i} and K_{Y_i} respectively and with $\mathbb{E}[X_iY_i] = 0$. Assume that for different i, $\{X_i, Y_i\}$ are mutually independent. Then* $\Pr\{|\sum_i X_iY_i| \geq t\}$ $\leq 2\exp\left(-c\min\left(\frac{t^2}{\sum_i K^2_{X_i}K^2_{Y_i}}, \frac{t}{\max_i|K_{X_i}K_{Y_i}|}\right)\right)$.

We also need the following two lemmas which were correct in the original paper [1].

Lemma 2.8 (Recovery of $\tilde{\boldsymbol{b}}^*_k$'s). *Let $\boldsymbol{g}_k := \boldsymbol{U}'\boldsymbol{x}^*_k$. At iteration t, assume that $\sin\Theta(\boldsymbol{U}^*, \boldsymbol{U}) \leq \delta_t$. Pick a $\delta_b < 1$. If $m \geq Cr$, and if we set $T_{RWF,t} = C\log\delta_t/\log(1-c)$, then, w.p. at least $1 - 2q\exp\left(-c\delta_b^2 m\right)$, the following bounds hold:*

$$\|\boldsymbol{g}_k - \hat{\boldsymbol{b}}_k\| \leq C\delta_t\|\tilde{\boldsymbol{b}}^*_k\| = C\delta_t\|\boldsymbol{x}^*_k\|,$$
$$\|\boldsymbol{G} - \hat{\boldsymbol{B}}\|_F \leq C\delta_t\|\tilde{\boldsymbol{B}}^*\|_F = C\delta_t\|\boldsymbol{X}^*\|_F,$$
$$\|\hat{\boldsymbol{x}}_k - \boldsymbol{x}^*_k\| \leq (C+1)\delta_t\|\boldsymbol{x}^*_k\| \tag{6}$$

for each $k = 1, 2, \cdots, q$. Here $C = \sqrt{1+\delta_b} + 1$. Thus, if $m \geq C\max(r, \log n, \log q)/\delta_b^2$, then the above bounds hold w.p. at least $1 - n^{-10}$.

Lemma 2.9 (Incoherence of $\boldsymbol{B}$). *Pick a $\delta_b < 0.1$ and assume that $m \geq C\max(r, \log n, \log q)/\delta_b^2$. At iteration t, assume that $\sin\Theta(\boldsymbol{U}^*, \boldsymbol{U}) \leq \delta_t$ with $\delta_t \leq \frac{0.25}{C\sqrt{r\kappa}}$. If $\boldsymbol{B}^*$ is μ-incoherent, then, w.p. at least $1 - n^{-10}$, $\boldsymbol{B}$ is $\hat{\mu}$-incoherent with $\hat{\mu} = C\kappa\mu$, i.e., $\max_k\|\boldsymbol{b}_k\| \leq \hat{\mu}\sqrt{r/q}$.*

Proof of Lemma 2.5. Recall that

$$\text{Term1}(\boldsymbol{W}) = \sum_{ik}\boldsymbol{b}_k{}'\boldsymbol{W}'\boldsymbol{a}_{ik}\boldsymbol{a}_{ik}{}'\boldsymbol{p}_k.$$

where

$$\boldsymbol{p}_k := \boldsymbol{U}^*\tilde{\boldsymbol{B}}^*\boldsymbol{B}'\boldsymbol{b}_k - \boldsymbol{U}^*\tilde{\boldsymbol{b}}^*_k = \boldsymbol{X}^*\boldsymbol{B}'\boldsymbol{b}_k - \boldsymbol{x}^*_k.$$

As shown while bounding Term1 in [1][1],

$$\mathbb{E}[\text{Term1}(\boldsymbol{W})] = 0.$$

The main error in [1] was in bounding $\|\boldsymbol{p}_k\|$ and $\|\boldsymbol{P}\|_F$ where $\boldsymbol{P} := [\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_q]$. We provide the corrected bounds next. Using $\hat{\boldsymbol{X}} = \boldsymbol{U}\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{B}} \overset{\text{QR}}{=} \boldsymbol{R}_B\boldsymbol{B}$, $\boldsymbol{B}\boldsymbol{B}' = I$,

$$\hat{\boldsymbol{X}}\boldsymbol{B}'\boldsymbol{b}_k = \boldsymbol{U}\boldsymbol{R}_B\boldsymbol{B}\boldsymbol{B}'\boldsymbol{b}_k = \boldsymbol{U}\boldsymbol{R}_B\boldsymbol{b}_k = \boldsymbol{U}\hat{\boldsymbol{b}}_k = \hat{\boldsymbol{x}}_k.$$

Thus,

$$\begin{aligned}\boldsymbol{p}_k &= (\boldsymbol{X}^* - \hat{\boldsymbol{X}} + \hat{\boldsymbol{X}})\boldsymbol{B}'\boldsymbol{b}_k - \boldsymbol{x}^*_k \\ &= (\boldsymbol{X}^* - \hat{\boldsymbol{X}})\boldsymbol{B}'\boldsymbol{b}_k + (\hat{\boldsymbol{x}}_k - \boldsymbol{x}^*_k)\end{aligned}$$

Using Lemma 2.8,

$$\begin{aligned}\|\boldsymbol{p}_k\| &\leq \|\boldsymbol{x}^*_k - \hat{\boldsymbol{x}}_k\| + \|\boldsymbol{X}^* - \hat{\boldsymbol{X}}\|\ \|\boldsymbol{B}\|\ \|\boldsymbol{b}_k\| \\ &\leq C\delta_t\|\boldsymbol{x}^*_k\| + \|\boldsymbol{X}^* - \hat{\boldsymbol{X}}\|\|\boldsymbol{b}_k\| \\ &\leq C\delta_t\sigma^*_{\max}\|\boldsymbol{b}^*_k\| + C\|\boldsymbol{X}^* - \hat{\boldsymbol{X}}\|\|\boldsymbol{b}_k\|.\end{aligned}$$

Writing $\boldsymbol{X}^* - \hat{\boldsymbol{X}} = (\boldsymbol{U}\boldsymbol{U}' + (\boldsymbol{I} - \boldsymbol{U}U'))(\boldsymbol{X}^* - \hat{\boldsymbol{X}})$, using Lemma 2.8, $\hat{\boldsymbol{X}} = \boldsymbol{U}\hat{\boldsymbol{B}}$, and $\boldsymbol{G} = \boldsymbol{U}'\boldsymbol{X}^*$,

$$\begin{aligned}\|\boldsymbol{X}^* - \hat{\boldsymbol{X}}\| &= \|\boldsymbol{U}(\boldsymbol{G} - \hat{\boldsymbol{B}}) + (\boldsymbol{I} - \boldsymbol{U}\boldsymbol{U}')\boldsymbol{X}^*\| \\ &\leq \|\boldsymbol{G} - \hat{\boldsymbol{B}}\| + \delta_t\|\boldsymbol{B}^*\| \\ &\leq \|\boldsymbol{G} - \hat{\boldsymbol{B}}\|_F + \delta_t\sigma^*_{\max} \\ &\leq C\delta_t\|\boldsymbol{X}^*\|_F + \delta_t\sigma^*_{\max} \leq C\delta_t\|\boldsymbol{X}^*\|_F\end{aligned}$$

The last inequality used $\sigma^*_{\max} \leq \|\boldsymbol{X}^*\|_F$. Thus,

$$\begin{aligned}\|\boldsymbol{p}_k\| &\leq C\delta_t(\sigma^*_{\max} + \|\boldsymbol{X}^*\|_F)\max(\|\boldsymbol{b}^*_k\|, \|\boldsymbol{b}_k\|) \\ &\leq C\delta_t\|\boldsymbol{X}^*\|_F\max(\|\boldsymbol{b}^*_k\|, \|\boldsymbol{b}_k\|)\end{aligned} \tag{7}$$

Next we bound $\|\boldsymbol{P}\|_F$. Observe that

$$\boldsymbol{P} := [\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_q] = \boldsymbol{X}^*(\boldsymbol{B}'\boldsymbol{B} - \boldsymbol{I}).$$

To bound this, we add and subtract $\hat{\boldsymbol{X}} = \boldsymbol{U}\hat{\boldsymbol{B}} = \boldsymbol{U}\boldsymbol{R}_B\boldsymbol{B}$ from $\boldsymbol{X}^*$ and use the facts that $\boldsymbol{B}(\boldsymbol{B}'\boldsymbol{B} - \boldsymbol{I}) = 0$ and $\|\boldsymbol{B}\boldsymbol{B}' - \boldsymbol{I}\| \leq 2$ (by triangle inequality and $\|\boldsymbol{B}\| = 1$). This gives

$$\begin{aligned}\|\boldsymbol{P}\|_F &= \|(\boldsymbol{X}^* - \hat{\boldsymbol{X}} + \hat{\boldsymbol{X}})(\boldsymbol{B}'\boldsymbol{B} - \boldsymbol{I})\|_F \\ &= \|(\boldsymbol{X}^* - \hat{\boldsymbol{X}})(\boldsymbol{B}'\boldsymbol{B} - \boldsymbol{I})\|_F \\ &\leq 2\|\boldsymbol{X}^* - \hat{\boldsymbol{X}}\|_F \leq C\delta_t\|\boldsymbol{X}^*\|_F\end{aligned} \tag{8}$$

We now use above bounds to apply the concentration bound Lemma 2.7. Let $X_{ik} = \boldsymbol{a}'_{ik}\boldsymbol{W}\boldsymbol{b}_k$ and $Y_{ik} = \boldsymbol{a}'_{ik}\boldsymbol{p}_k$. Both are sub-Gaussian with $K_{X_{ik}} = \|\boldsymbol{W}\boldsymbol{b}_k\| \leq \|\boldsymbol{W}\|_F\|\boldsymbol{b}_k\| \leq \|\boldsymbol{b}_k\|$,

[1] $\mathbb{E}[\text{Term1}(\boldsymbol{W})] = \mathbb{E}[\text{trace}(\text{Term1}(\boldsymbol{W})] = \mathbb{E}[\text{trace}(\boldsymbol{W}'\sum_{ik}\boldsymbol{a}_{ik}\boldsymbol{a}_{ik}{}'\boldsymbol{p}_k\boldsymbol{b}_k{}')] = m\text{trace}(\boldsymbol{W}'\sum_k\boldsymbol{p}_k\boldsymbol{b}_k{}')]$. Since $\boldsymbol{B}\boldsymbol{B}' = I$, $\sum_k\boldsymbol{p}_k\boldsymbol{b}_k{}' = \boldsymbol{X}^*\boldsymbol{B}'\boldsymbol{B}\boldsymbol{B}' - \boldsymbol{X}^*\boldsymbol{B}' = \boldsymbol{0}$.

and $K_{Y_{ik}} \leq \|\boldsymbol{p}_k\|$. Also, using Lemma 2.9, $\boldsymbol{b}_k$'s are incoherent w.p. at least $1 - 2q\exp(r\log(17) - \delta_b^2 m) \geq 1 - n^{-10}$ if $m \geq C\max(r, \log n, \log q)/\delta_b^2$, i.e.,

$$\|\boldsymbol{b}_k\|^2 \leq \hat{\mu}^2 r/q = C\kappa^2\mu^2 r/q.$$

Thus, using this and right singular vectors' incoherence, with above probability,

$$\max(\|\boldsymbol{b}_k^*\|^2, \|\boldsymbol{b}_k\|^2) \leq C\kappa^2\mu^2 r/q$$

Applying Lemma 2.7 with $t = m\epsilon_1\delta_t\|\boldsymbol{X}^*\|_F$, and using the above bounds on $\|\boldsymbol{p}_k\|$, $\|\boldsymbol{P}\|_F$ and $\max(\|\boldsymbol{b}_k^*\|^2, \|\boldsymbol{b}_k\|^2)$,

$$\begin{aligned}
\frac{t^2}{\sum_{ik} K_{X_{ik}}^2 K_{Y_{ik}}^2} &\geq \frac{m^2\epsilon_1^2\delta_t^2\|\boldsymbol{X}^*\|_F^2}{m\max_k\|\boldsymbol{b}_k\|^2\sum_k\|\boldsymbol{p}_k\|^2} \\
&= \frac{m\epsilon_1^2\delta_t^2\|\boldsymbol{X}^*\|_F^2}{\max_k\|\boldsymbol{b}_k\|^2\|\boldsymbol{P}\|_F^2} \geq c\frac{mq\epsilon_1^2}{\kappa^2\mu^2\ r}, \\
\frac{t}{\max_{ik} K_{X_{ik}} K_{Y_{ik}}} &= \frac{m\epsilon_1\delta_t\|\boldsymbol{X}^*\|_F}{\max_k\|\boldsymbol{b}_k\|\|\boldsymbol{p}_k\|} \\
&\geq c\frac{m\epsilon_1\delta_t\|\boldsymbol{X}^*\|_F}{\delta_t\|\boldsymbol{X}^*\|_F\max_k\max(\|\boldsymbol{b}_k^*\|, \|\boldsymbol{b}_k\|)} \\
&\geq c\frac{mq\epsilon_1}{\kappa^2\mu^2 r}
\end{aligned}$$

Thus,

$$\Pr\{|\text{Term1}(\boldsymbol{W})| \leq m\epsilon_1\delta_t\|\boldsymbol{X}^*\|_F\} \geq 1 - \exp\left(-c\frac{mq\epsilon_1^2}{\kappa^2\mu^2 r}\right)$$

Now we just need to extend our bound for all $\boldsymbol{W} \in \mathcal{S}_W$. This part is standard and exactly the same as the argument given in the original paper [1] □

Proof of Lemma 2.6. By Cauchy-Schwarz,

$$\begin{aligned}
\text{Term2}(\boldsymbol{W}) &:= \sum_{ik}(\boldsymbol{c}_{ik}\hat{\boldsymbol{c}}_{ik} - 1)(\boldsymbol{a}_{ik}{}'\boldsymbol{W}\boldsymbol{b}_k)(\boldsymbol{a}_{ik}{}'\boldsymbol{x}_k^*) \\
&\leq \sqrt{\sum_{ik}|\boldsymbol{a}_{ik}{}'\boldsymbol{W}\boldsymbol{b}_k|^2}\sqrt{\sum_{ik}|\boldsymbol{c}_{ik}\hat{\boldsymbol{c}}_{ik} - 1|^2|\boldsymbol{a}_{ik}{}'\boldsymbol{x}_k^*|^2}
\end{aligned} \tag{9}$$

We can bound the first term using Lemma 2.4. Consider the second term. Since $\boldsymbol{c}_{ik} = sign(\boldsymbol{a}_{ik}{}'\boldsymbol{x}_k^*)$ and $\hat{\boldsymbol{c}}_{ik} = sign(\boldsymbol{a}_{ik}{}'\hat{\boldsymbol{x}}_k)$, clearly $(\boldsymbol{c}_{ik}\hat{\boldsymbol{c}}_{ik} - 1)^2 = (4)\mathbb{1}_{\{\boldsymbol{c}_{ik} \neq \hat{\boldsymbol{c}}_{ik}\}}$. Define

$$Q_{ik} := \mathbb{1}_{\{\boldsymbol{c}_{ik} \neq \hat{\boldsymbol{c}}_{ik}\}} \cdot (\boldsymbol{a}_{ik}{}'\boldsymbol{x}_k^*)^2$$

Then we need to bound $\sum_{ik} 4Q_{ik}^2$. In the Term2 bound proof given in [1], we showed that, as long as the bound of Lemma 2.8 holds,

$$\sum_{ik}\mathbb{E}[Q_{ik}] \leq Cm\delta_t^3\|\boldsymbol{X}^*\|_F^2. \tag{10}$$

To bound $\sum_{ik} 4Q_{ik}^2$, the next step is to apply the concentration bound. This part is almost the same as that in [1] with the difference being the choice of t: we need to pick $t = m\epsilon_2\delta_t^2\|\boldsymbol{X}^*\|_F^2$.

As shown in [1] (taken from proof of Theorem 1 of [3]), $\boldsymbol{c}_{ik} \neq \hat{\boldsymbol{c}}_{ik}$ implies that $(\boldsymbol{a}_{ik}{}'\boldsymbol{x}_k^*)^2 \leq (\boldsymbol{a}_{ik}{}'(\boldsymbol{x}_k^* - \hat{\boldsymbol{x}}_k))^2$. Thus,

$$Q_{ik} \leq \mathbb{1}_{\{\boldsymbol{c}_{ik} \neq \hat{\boldsymbol{c}}_{ik}\}}\left(\boldsymbol{a}_{ik}{}'(\boldsymbol{x}_k^* - \hat{\boldsymbol{x}}_k)\right)^2 \leq \left(\boldsymbol{a}_{ik}{}'(\boldsymbol{x}_k^* - \hat{\boldsymbol{x}}_k)\right)^2.$$

Hence it is a sub-exponential r.v., or equivalently it is a product of sub-Gaussian r.v.'s $\sqrt{Q_{ik}}$ and we can apply Lemma 2.7 with $K_{X_{ik}} = K_{Y_{ik}} = \|(\boldsymbol{x}_k^* - \hat{\boldsymbol{x}}_k)\|$. Applying Lemma 2.8, and simplifying the exponent terms, we can conclude that, conditioned on the event that the bounds of Lemma 2.8 hold,

$$\begin{aligned}
&\Pr\left\{|\textstyle\sum_{ik} Q_{ik} - \sum_{ik}\mathbb{E}[Q_{ik}]| \geq m\epsilon_2\delta_t^2\|\boldsymbol{X}^*\|_F^2\right\} \\
&\qquad \leq 2\exp\left(-c\epsilon_2^2 mq/\kappa^2\mu^2\right).
\end{aligned} \tag{11}$$

The event of Lemma 2.8 holds w.p. $1 - n^{-10}$ as long as the lower bound on m holds. Thus, combining (9), Lemma 2.4, (11) and (10), and union bound if $m \geq C\max(r, \log n, \log q)/\delta_b^2$, w.p. at least $1 - n^{-10} - 2\exp\left(nr - c\epsilon_3^2\frac{mq}{\hat{\mu}^2 r}\right) - 2\exp\left(-c\epsilon_2^2\frac{mq}{\kappa^2\mu^2}\right)$,

$$\max_{\boldsymbol{W}\in\mathcal{S}_{\boldsymbol{W}}} \text{Term2}(\boldsymbol{W}) \leq Cm\sqrt{1+\epsilon_3}\sqrt{\delta_t+\epsilon_2}\,\delta_t\|\boldsymbol{X}^*\|_F.$$

□

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Biography

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