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Nonlinear stability of diffusive contact wave for a chemotaxis model

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Abstract

We consider a 2×2 system of hyperbolic-parabolic balance laws. Our system is the converted form under inverse Hopf-Cole transformation of a Keller-Segel type chemotaxis model with logistic growth, logarithmic sensitivity, non-diffusive chemical signal and density-dependent production/consumption rate. We study Cauchy problem when the Cauchy data are near a diffusive contact wave. The contact wave connects two different end-states as $x \to \pm\infty$, reflecting the situation when the logarithmic singularity plays an intrinsic role in the original chemotaxis model. We establish global existence of solution and study time asymptotic behavior of the solution. Consequently, we obtain nonlinear stability of the diffusive contact wave. Our result shows a significant difference when comparing our model to Euler equations with damping. In our case, there exists a secondary wave in the asymptotic ansatz. Therefore, the solution to Cauchy problem converges to the diffusive contact wave slower than in the case of Euler equations with damping. Besides its own physical relevance, our model is a prototype of a general system of hyperbolic-parabolic balance laws. Our results shed light on the future study of nonlinear stability of elementary waves for a general system.
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1. Introduction

We consider the following initial value problem of a system of partial differential equations in the unknowns v and u:

$$\begin{cases} v_t + u_x = 0, \\ u_t + (uv)_x = u_{xx} + ru(1-u), \end{cases} \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$(v, u)(x, 0) = (v_0, u_0)(x), \tag{1.2}$$

where the parameter $r > 0$ is a constant, and the initial data satisfy

$$\lim_{x\to\pm\infty} v_0(x) = v_\pm, \qquad \lim_{x\to\pm\infty} u_0(x) = 1, \tag{1.3}$$

for two constants $v_- \neq v_+$. The goal is to establish the existence of solution global in time and to study the time asymptotic behavior of the solution under appropriate assumptions.

1.1. Background

System (1.1) is derived from a Keller-Segel type chemotaxis model with logistic growth, logarithmic sensitivity, non-diffusive chemical signal and density-dependent production/consumption rate,

$$\begin{cases} s_t = -\mu u s - \sigma s, \\ u_t = D u_{xx} - \chi[u(\ln s)_x]_x + au(1 - \frac{u}{K}), \end{cases} \quad x \in \mathbb{R},\ t > 0. \tag{1.4}$$

Here the unknown functions are $s = s(x,t)$ and $u = u(x,t)$ for the concentration of a chemical signal and the density of a cellular population, respectively. The system parameters have the following meaning: $\mu \neq 0$ is the coefficient of density-dependent production/consumption rate of chemical signal; $\sigma \geq 0$ the natural degradation rate of the signal; $D > 0$ the diffusion coefficient of cellular population; $\chi \neq 0$ the coefficient of chemotactic sensitivity; $a > 0$ the natural growth rate of the cellular population; and $K > 0$ the typical carrying capacity for the population. All these parameters are constants. Interested readers are referred to [35,36] and references therein for a more detailed discussion on the model (1.4).

A commonly adopted approach to remove the logarithmic function in (1.4) is by the inverse Hopf-Cole transformation [9]:

$$v = (\ln s)_x = \frac{s_x}{s}. \tag{1.5}$$

Under the new variables v and u, the reaction-diffusion system (1.4) becomes a system of hyperbolic-parabolic balance laws,

$$\begin{cases} v_t + \mu u_x = 0, \\ u_t + \chi(uv)_x = D u_{xx} + au(1 - \frac{u}{K}). \end{cases} \tag{1.6}$$

We assume $\chi\mu > 0$, which implies $\chi, \mu > 0$, or $\chi, \mu < 0$. The former is interpreted as cells are attracted to and consume the chemical, while the latter indicates that cells deposit the signal to modify the local environment for succeeding passages [22]. Mathematically, the non-diffusive, non-reactive part of (1.6) is hyperbolic in biologically relevant regimes when $\chi\mu > 0$, while it may change type when $\chi\mu < 0$ [35].

Under the assumption $\chi\mu > 0$, we introduce rescaled and dimensionless variables,

$$\tilde{t} = \frac{\chi\mu K}{D} t, \qquad \tilde{x} = \frac{\sqrt{\chi\mu K}}{D} x, \qquad \tilde{v} = \operatorname{sign}(\chi)\sqrt{\frac{\chi}{\mu K}}\, v, \qquad \tilde{u} = \frac{u}{K}. \tag{1.7}$$

This simplifies (1.6) to

$$\begin{cases} \tilde{v}_{\tilde{t}} + \tilde{u}_{\tilde{x}} = 0, \\ \tilde{u}_{\tilde{t}} + (\tilde{u}\tilde{v})_{\tilde{x}} = \tilde{u}_{\tilde{x}\tilde{x}} + r\tilde{u}(1-\tilde{u}), \end{cases}$$

where $r = aD/(\chi\mu K) > 0$. Dropping the tilde accent we obtain (1.1).

Corresponding to (1.2), we impose initial condition

$$(s, u)(x, 0) = (s_0, u_0)(x), \quad x \in \mathbb{R} \tag{1.8}$$

to (1.4). Here s_0 is related to v_0 by the transformation (1.5) and the rescaling (1.7). For $\chi > 0$ it is easy to see that without loss of generality and for simplicity we may bypass (1.7) and have

$$v_0(x) = v(x, 0) = (\ln s_0)'(x) = \frac{s_0'(x)}{s_0(x)}.$$

This implies

$$s_0(x) = s_0(0) e^{\int_0^x v_0(y)dy}, \quad s_0(0) > 0. \tag{1.9}$$

We are interested in the Cauchy problem (1.1), (1.2), with Cauchy data satisfying (1.3). This includes (but is not limited to) the following special cases.

(i) $0 < v_- < \infty$ and $\int_0^\infty |v_0(y)|\,dy < \infty$. In this case, (1.3) and (1.9) imply

$$\lim_{x\to-\infty} s_0(x) = 0, \qquad \lim_{x\to\infty} s_0(x) = s_+ < \infty.$$

(ii) $-\infty < v_+ < 0$ and $\int_{-\infty}^0 |v_0(y)|\,dy < \infty$. Similarly, (1.3) and (1.9) imply

$$\lim_{x\to-\infty} s_0(x) = s_- < \infty, \qquad \lim_{x\to\infty} s_0(x) = 0.$$

(iii) $0 < v_- < \infty$ and $-\infty < v_+ < 0$. In this case,

$$\lim_{x\to\pm\infty} s_0(x) = 0.$$

We observe that in those special cases, s_0 is not bounded away from zero while $v_- > v_+$. If $\chi < 0$ in (1.7), there are also special cases where s_0 is not bounded away from zero while $v_- < v_+$. In other words, the singularity of the logarithmic function in (1.4) is intrinsic, which is further reflected into technical difficulties associated with $v_- \neq v_+$ in (1.1)-(1.3).

1.2. Connection with existing literature

When $a = 0$ in (1.4), the nongrowth model

$$\begin{cases} s_t = -\mu u s - \sigma s, \\ u_t = D u_{xx} - \chi [u(\ln s)_x]_x \end{cases} \tag{1.10}$$

was proposed in [22] for describing the movement of chemotactic populations that deposit non-diffusive chemical signals that modify the local environment for succeeding passages, and later found applications in cancer research [10]. Under the transformation (1.5) and rescaling (1.7), one gets the following hyperbolic-parabolic system of conservation laws:

$$\begin{cases} v_t + u_x = 0, \\ u_t + (uv)_x = u_{xx}. \end{cases} \tag{1.11}$$

Since the model was proposed in the late 1990s, the qualitative behavior of (1.10) has been analyzed to a large extent. In the pioneering works of [9,22], explicit and numerical solutions to (1.10) were constructed to exhibit chemotactic aggregation or collapsing. A series of papers followed, in which a number of topics were studied for (1.11). These include global well-posedness and long time behavior of large data classical solutions [3,11], stability of traveling waves [13,14], boundary layer formation [12] and others. Also see references therein.

Growth and death are important factors in population dynamics. Therefore, sophisticated chemotaxis models include reaction terms to describe those mechanisms. Among them logistic growth is a popular choice. Many research results for chemotaxis models with logistic growth are for constant rate production and degradation of the chemical signal. Among them many are with regular sensitivity as well. For instance, global well-posedness of large data classical solutions has been studied for those models on bounded domains in all space dimensions under suitable conditions, see [27,29,4] and references therein. Also see [1] for systems with singular sensitivity functions. Cauchy problem is considered in $\mathbb{R}^2$ in [21]. In addition, we refer readers to [7,28,30] and references therein for the existence of weak solutions.

Systems with density-dependent production/consumption rate of the chemical signal, singular chemotactic sensitivity and logistic growth of cells have been studied as well in recent years. For example, global existence and asymptotic behavior have been studied in multi-dimensional bounded domains, see [39,8] and references therein. The systems are similar to (1.4) but for a diffusive chemical signal. The existing works implicate an interest in understanding the role of logistic growth in different chemotaxis models. Besides its biological significance, the logistic growth resembles common phenomena such as damping and relaxation in kinetic theory. This is to be seen in Section 1.3 below.

Cauchy problem (1.1), (1.2) has been considered in [35–38,24,25] under appropriate assumptions, which all imply

$$\lim_{x\to\pm\infty} v_0(x) = 0, \qquad \lim_{x\to\pm\infty} u_0(x) = 1. \tag{1.12}$$

Here in [35], global well-posedness, long-time behavior and vanishing coefficient limits have been studied. The relevant results are obtained for large data solutions, i.e., under the assumption that $\|(v_0, u_0 - 1)\|_{H^2}$ is finite and $u_0 > 0$. In view of (1.12), say, assuming $v_0 \in L^1(\mathbb{R})$, (1.9) implies

$$\lim_{x\to\pm\infty} s_0(x) = s_\pm > 0. \tag{1.13}$$

Therefore, in [35] Cauchy problem (1.1), (1.2) is studied away from the singularity of the logarithmic function in (1.4).

Under an additional assumption that $v_0(x)$ (hence $v(x,t)$) is of zero mass, which is translated into

$$\lim_{x\to\pm\infty} s_0(x) = \bar{s} > 0,$$

explicit decay rates of the solution and its derivatives are obtained in [35]. The non-optimal rates serve as a starting point in an iteration scheme developed in [36], where optimal rates of solutions to the original system (1.4) and to the converted system (1.1) have been obtained. The results on (1.4) are applicable to the border case when $s(x,t)$ neither exponentially decays nor exponentially grows. Also see [37,38].

Under the assumption (1.12), if we consider small solutions we may obtain a very detailed picture of solution behavior for large time. In that scenario we do not need zero mass assumption (in comparison to [36]). Specifically, let $|(v_0, u_0 - 1)|(x) = O(1)(x^2+1)^{-\alpha}$ with $\alpha > 1/2$, and $\sup_{x\in\mathbb{R}}[(x^2+1)^\alpha|(v_0, u_0-1)|(x)] + \|(v_0, u_0-1)\|_{H^2}$ be sufficiently small. We identify the time asymptotic solution of $v(x,t)$ as a heat kernel $\theta(x,t)$ determined by the parameter $r > 0$ in (1.1) and by the mass of v_0. The corresponding time asymptotic solution of $u(x,t)$ is $1 - \frac{1}{r}\theta_x(x,t)$. The error between (v,u) and $(\theta, 1 - \frac{1}{r}\theta_x)$ is given pointwisely in x and t, which leads to optimal time decay rates in L^p, $1 \le p \le \infty$. See [25,24] for details.

1.3. A prototype of systems of hyperbolic-parabolic balance laws

A general system of hyperbolic-parabolic balance laws takes the form

$$w_t + f(w)_x = [B(w)w_x]_x + g(w), \tag{1.14}$$

where $w, f, g \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times n}$. Here w is the unknown density function, f the flux function, g the reaction term, and B the viscosity matrix. The reaction term g is for external force, relaxation, chemical reaction, etc, while the viscosity matrix B is for viscosity, heat conduction, species diffusion and so forth. We assume that f, g and B are smooth functions of w. In physical applications, the Jacobian matrix f' has real, distinct eigenvalues or can be symmetrized with an entropy function, and the viscosity matrix B is rank deficient. Thus we say that (1.14) is hyperbolic-parabolic. In many important applications, the Jacobian matrix g' is also rank deficient. The system describes the balance of physical quantities dictated by laws in physics. A prominent example is the system for physical gas dynamics, which includes translational and thermal/chemical non-equilibrium.

A special case of (1.14) is when $B = 0$, which gives a system of hyperbolic balance laws

$$w_t + f(w)_x = g(w). \tag{1.15}$$

An important example is Euler equations with damping.

Another special case of (1.14) is when $g = 0$, which gives a system of hyperbolic-parabolic conservation laws,

$$w_t + f(w)_x = [B(w)w_x]_x. \tag{1.16}$$

In this case, physical quantities are conserved. A well-known example is Navier-Stokes equations for a compressible flow.

We observe that (1.1) is a prototype of (1.14), with nontrivial B and g. Here

$$w = \begin{pmatrix} v \\ u \end{pmatrix}, \quad f(w) = \begin{pmatrix} u \\ uv \end{pmatrix}, \quad B(w) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad g(w) = \begin{pmatrix} 0 \\ ru(1-u) \end{pmatrix}. \tag{1.17}$$

It is clear that both B and g' are rank deficient. It is also clear that

$$f'(w) = \begin{pmatrix} 0 & 1 \\ u & v \end{pmatrix}$$

has two real, distinct eigenvalues $\lambda_\pm = \frac{1}{2}(v \pm \sqrt{v^2 + 4u})$ in its biologically relevant regime $u > 0$.

Cauchy problem of the general system (1.14) has been studied for small solutions around a constant equilibrium state $\bar{w}$, $g(\bar{w}) = 0$, in [32,34]. Here is [32] a set of structural conditions have been proposed, under which existence of global in time solutions has been established for (1.14) and its multi-dimensional counterpart. L^p $(p \geq 2)$ decay rates have been obtained in [34], also see [33] for similar results in multi-space dimensions. The results apply to (1.1) and the much more complicated system of physical gas dynamics. Here in view of (1.17), a stable, constant equilibrium state of (1.1) is $(\bar{v}, 1)$ for a constant $\bar{v}$. For physically interesting scenario $\lim_{x\to\pm\infty} s_0(x) = s_\pm < \infty$, (1.9) implies $\bar{v} = 0$, see [35] for details. Therefore, the above mentioned results apply to (1.1), (1.2) with regard to small solutions around the state $(0, 1)$.

To the best of the author's knowledge, Cauchy problem of (1.14), on the other hand, has not been studied when w connects two different end-states as $x \to \pm\infty$. Since (1.1) is a prototype, results from this paper on (1.1)-(1.3) may shed light on the study of (1.14). We are particularly interested in the similar situation for physical gas dynamics.

In contrast to (1.14), there is a rich literature on Cauchy problems of the special cases (1.15) and (1.16) where Cauchy data connect two different end-states as $x \to \pm\infty$. Most results are on the existence and/or stability of an elementary wave.

For instance, let $w_\pm$ be two constant states that form a shock wave with speed σ of the hyperbolic system

$$w_t + f(w)_x = 0. \tag{1.18}$$

Under appropriate assumptions, we may show that there exists a traveling wave solution of (1.16) connecting w_- to w_+ and with speed σ [2]. The traveling wave is called a viscous shock wave of (1.16). We may then consider the stability of the viscous shock wave. For example, it is show in [18] that a weak shock of (1.16) is stable under a generic perturbation and physical assumptions.

Similarly, let (w_-, w_+, σ) be a contact discontinuity of (1.18). Under appropriate assumptions, there is a viscous contact wave that is a smooth solution of (1.16), connecting w_- to w_+ and having a center moving at the speed σ. It takes states on the contact-wave curve of (1.18) from w_- to w_+, parameterized by the solution of a heat equation with Riemann data at $t = -1$. Interested readers are referred to [15,31] and references therein.

The scenario of viscous rarefaction waves is much more intricate. As indicated and explained in [16], there is no exact, explicit representation of a viscous rarefaction wave, though there is an accurate approximation of the wave for (1.16) using the scalar Burgers equation.

In the context of elementary waves and their stability, the special case (1.15) turns out to be quite different from (1.16). This is because now those waves are not associated with (1.18) (the frozen system) but with the reduced system (the equilibrium system). To be relevant to this paper, below we focus on the specific model of Euler equations with damping.

1.4. Comparison to Euler equations with damping

We consider Euler equations with damping for isentropic flows. Under Lagrangian coordinates they read

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -ru, \end{cases} \qquad x \in \mathbb{R},\ t > 0, \tag{1.19}$$

where $r > 0$ is a constant. The unknown functions v and u are for the specific volume and velocity, respectively. On the other hand, p is the pressure, a given smooth function of v, satisfying $p'(v) < 0$. The model describes a compressible flow through a porous medium. It is shown in [5] that solutions of (1.19) time asymptotically behave as those of the porous medium equation and the Darcy's law,

$$\begin{cases} v_t = -\frac{1}{r} p(v)_{xx}, \\ p(v)_x = -ru. \end{cases} \tag{1.20}$$

Also see [19,20,23] and references therein.

The equilibrium manifold of (1.19) is $u = 0$. Substituting it into the first equation we obtain the equilibrium equation of (1.19),

$$v_t = 0, \tag{1.21}$$

which is also known as the reduced equation. A better (the next order) approximation is obtained by first dropping u_t, the higher order term in time decay rate, in the second equation of (1.19). This gives the second equation of (1.20). Then we substitute it into the first equation of (1.19). We thus arrive at the first equation of (1.20). The idea employed here is Chapman-Enskog expansion.

In [5] Cauchy problem is considered with Cauchy data $(v, u)(x, 0) \to (v_\pm, u_\pm)$ as $x \to \pm\infty$, with $v_- \neq v_+$. To focus on the key difference between (1.1) and (1.19) in their solution behavior, we simplify the setting to $u_- = u_+ = 0$. Thus we consider (1.19) under initial condition (1.2), where

$$(v_0, u_0)(x) \to (v_\pm, 0) \quad \text{as} \quad x \to \pm\infty, \tag{1.22}$$

with $v_- \neq v_+$. That is, the end-states are equilibrium states.

Following [5], the primary wave in the solution of (1.19), (1.2) is

$$(\bar{v}, \bar{u})(x + x_0, t), \quad \bar{u} = -\frac{1}{r} p(\bar{v})_x. \tag{1.23}$$

Here $\bar{v}$ is the unique self-similar solution, $\bar{v}(x,t) = \phi(x/\sqrt{t+1})$, of the porous medium equation

$$\bar{v}_t = -\frac{1}{r} p(\bar{v})_{xx}, \tag{1.24}$$

satisfying the boundary condition

$$\lim_{x \to \pm\infty} \bar{v}(x,t) = v_\pm. \tag{1.25}$$

The translation x_0 is uniquely determined by

$$\int_{-\infty}^{\infty} [v(x,t) - \bar{v}(x + x_0, t)]\, dx = \int_{-\infty}^{\infty} [v_0(x) - \bar{v}(x + x_0, 0)]\, dx = 0, \tag{1.26}$$

noting the equations for v and $\bar{v}$ are conservation laws.

The end-states $v_\pm$ from a contact discontinuity with speed zero of the reduced equation (1.21):

$$\hat{v}(x,t) = \begin{cases} v_- & \text{if } x < 0, \\ v_+ & \text{if } x > 0, \end{cases}$$

see [26]. The primary wave $\bar{v}$ defined by (1.24), (1.25) can be regarded as a diffusive version of $\hat{v}$. Thus we call $\bar{v}$ a diffusive contact wave.

Based on (1.26) we introduce new variables,

$$V(x,t) = \int_{-\infty}^{x} [v(y,t) - \bar{v}(y + x_0, t)]\, dy,$$
$$U(x,t) = u(x,t) - \bar{u}(x + x_0, t).$$

Thus,

$$V_x(x,t) = v(x,t) - \bar{v}(x + x_0, t).$$

Nonlinear stability of a weak diffusive contact wave is studied in [5]. That is, if $|v_+ - v_-|$ is small and $V(x,0)$ and $U(x,0) = V_t(x,0)$ are small in $H^3(\mathbb{R})$ and $H^2(\mathbb{R})$, respectively, there exists a global in time solution of (1.19), (1.2), (1.22). The solution converges in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to $(\bar{v}, \bar{u})(x + x_0, t)$ time asymptotically, with $\|(V_x, U)(t)\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})}$ decaying at the rate $(t+1)^{-\frac{1}{2}}$. The decay rate is improved to optimal ones, $\|V_x(t)\|_{L^2(\mathbb{R})} \sim (t+1)^{-\frac{3}{4}}$, $\|U(t)\|_{L^2(\mathbb{R})} \sim (t+1)^{-\frac{5}{4}}$, $\|V_x(t)\|_{L^\infty(\mathbb{R})} \sim (t+1)^{-1}$, and $\|U(t)\|_{L^\infty(\mathbb{R})} \sim (t+1)^{-\frac{3}{2}}$, under a variety of assumptions on the initial data [19,20].

For our model (1.1), similarly, we can derive an approximate system that is the counterpart of Darcy's law (1.20). As a consequence, the primary wave in the time asymptotic ansatz of the solution to (1.1)-(1.3) is a diffusive contact wave. The purpose of this paper is to study its stability assuming that the wave is weak.

Our result, however, reveals a significant difference between the solutions to (1.1) and (1.19) in their long-time behavior. We show that there exists a secondary wave in the asymptotic ansatz of the solution to (1.1)-(1.3). The v-component of the wave has zero mass and decays like a heat kernel. The u-component, on the other hand, decays like the first derivative of a heat kernel. We are able to show that the remainder of the solution after taking out the primary and secondary waves is higher order in $L^2(\mathbb{R})$. This implies that the secondary wave is the leading term in the time-asymptotic error when approximating the solution by the diffusive contact wave. Therefore, the contact wave is stable, and the L^2-convergence rate to it is $(t+1)^{-\frac{1}{4}}$ for the v-component and $(t+1)^{-\frac{3}{4}}$ for the u-component. This is to compare with $(t+1)^{-\frac{3}{4}}$ and $(t+1)^{-\frac{5}{4}}$, respectively, for (1.19).

The existence of the secondary wave hence the slower convergence rates to the diffusive contact wave come from the fact that the nonlinear flux in the equation for u in (1.1) contains both v and u. By contrast, in (1.19) it is $p(v)$, a function in v only. On the other hand, the extra diffusion term u_{xx} in (1.1) does not affect the structure of the primary and the secondary waves. It only contributes to the error of the approximation by those waves, see (2.3) and (2.10) below. Thus, we believe that in the general case, (1.15) or (1.14), there are similar secondary waves in the time-asymptotic ansatz when studying the stability of contact waves.

1.5. The goal of the paper

In this paper we establish the global existence of solution to (1.1)-(1.3) when Cauchy data are small perturbations of a diffusive contact wave. We identify and justify the leading term, a secondary wave, in the time-asymptotic error. This leads to nonlinear stability of contact wave and large time behavior of solution to (1.1)-(1.3). Our main focus is an innovative insight of the asymptotic solution for general systems like (1.14) or (1.15), beyond what has been understood through the Euler equations with damping. Therefore, we are content to achieve our goal in L^2 space via energy and weighted energy methods. Using more sophisticated methods it is possible to obtain results in L^p spaces, $1 \le p \le \infty$. This is left to a future work.

The plan of the paper is as follows. Section 2 is for preliminaries and the statement of main results. In Section 3 we discuss local existence of solution. In Section 4 we prove Theorem 2.3 to establish global existence of solution. This is done by energy estimate. In Section 5 we prove Theorem 2.4, which gives convergence rates of the solution to the asymptotic solution and thus justifies the asymptotic solution. It is done by weighted energy estimate. Finally, in the Appendix we prove Proposition 2.2, which describes the behavior of the secondary wave in the asymptotic solution.

2. Preliminaries and main results

We first consider the primary wave, a diffusive contact wave, for (1.1)-(1.3). Introduce the perturbation $\tilde{u}$ of u,

$$\tilde{u} = u - 1 \quad \text{or} \quad u = 1 + \tilde{u}. \tag{2.1}$$

The second equation in (1.1) becomes

$$\tilde{u}_t + v_x + (\tilde{u}v)_x = \tilde{u}_{xx} - r\tilde{u} - r\tilde{u}^2. \tag{2.2}$$

By identifying the leading terms with respect to time asymptotic decay rates in the first equation of (1.1) and (2.2), we define the leading term $(\bar{v}, \bar{u})$ of $(v, \tilde{u})$ as a solution to

$$\begin{cases} \bar{v}_t + \bar{u}_x = 0, \\ \bar{v}_x = -r\bar{u}. \end{cases} \tag{2.3}$$

Substituting the second equation in (2.3) into the first one, we have

$$\begin{cases} \bar{v}_t = \frac{1}{r}\bar{v}_{xx}, \\ \bar{u} = -\frac{1}{r}\bar{v}_x. \end{cases} \tag{2.4}$$

The equations in (2.4) are the counterparts of the porous medium equation and Darcy's law for Euler equations with damping, see (1.20). We define $\bar{v}$ as the self-similar solution with

$$\lim_{x\to\pm\infty} \bar{v}(x,t) = v_{\pm}. \tag{2.5}$$

Then $\bar{u}$ is determined by the second equation of (2.4). Explicitly,

$$\begin{aligned} \bar{v}(x,t) &= \frac{v_-}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4(t+1)/r}}}^{\infty} e^{-y^2}\,dy + \frac{v_+}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4(t+1)/r}}} e^{-y^2}\,dy \\ &= \frac{v_- + v_+}{2} - \frac{v_- - v_+}{2}\operatorname{erf}\Big(\frac{x}{\sqrt{4(t+1)/r}}\Big), \\ \bar{u}(x,t) &= \frac{v_- - v_+}{\sqrt{4\pi r(t+1)}} e^{-\frac{rx^2}{4(t+1)}}. \end{aligned} \tag{2.6}$$

We note that $\bar{v}(x,t)$ given in (2.6) is a diffusive contact wave of the heat equation in (2.4), with Riemann data at $t = -1$.

We observe that the self-similar solution (a function in $x/\sqrt{t+1}$) of (2.4) satisfying the boundary condition (2.5) is unique up to a translation x_0. We determine the constant x_0 by the excess mass of v. That is, we choose x_0 such that

$$\int_{\mathbb{R}} [v_0(x) - \bar{v}(x + x_0, 0)]\, dx = 0, \tag{2.7}$$

i.e.

$$x_0 = \frac{1}{v_+ - v_-} \int_{\mathbb{R}} [v_0(x) - \bar{v}(x, 0)]\, dx. \tag{2.8}$$

From (1.1) and (2.3) we have

$$\frac{d}{dt} \int_{\mathbb{R}} [v(x,t) - \bar{v}(x + x_0, t)]\, dx = 0.$$

Combining with (2.7) we further have

$$\int_{\mathbb{R}} [v(x,t) - \bar{v}(x + x_0, t)]\, dx = 0. \tag{2.9}$$

Now $(\bar{v}, 1 + \bar{u})(x + x_0, t)$, with $\bar{v}$ and $\bar{u}$ defined in (2.6), is the primary wave in the solution of (1.1)-(1.3). However, it is not sufficiently accurate. We thus construct a secondary wave. For this we substitute (v, u) in (1.1) by $(\bar{v} + v^*, 1 + \bar{u} + u^*)$, apply (2.3) and keep the leading terms only. We arrive at

$$\begin{cases} v_t^* + u_x^* = 0, \\ v_x^* + r u^* = -R(x,t), \end{cases} \tag{2.10}$$

where

$$R(x,t) = (\bar{u}_x \bar{v})(x + x_0, t). \tag{2.11}$$

Substituting the second equation in (2.10) into the first one gives us

$$\begin{cases} v_t^* = \frac{1}{r} v_{xx}^* + \frac{1}{r} R_x(x,t), \\ u^* = -\frac{1}{r} v_x^* - \frac{1}{r} R(x,t). \end{cases} \tag{2.12}$$

Noting (2.7) we set

$$v^*(x, 0) = 0. \tag{2.13}$$

The secondary wave in the asymptotic ansatz of the solution to (1.1)-(1.3) is set as (v^*, u^*), the solution to (2.12), (2.13).

We solve (2.12), (2.13) explicitly by Duhamel's principle:

$$v^*(x,t) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi r(t-\tau)}} e^{-\frac{r(x-y)^2}{4(t-\tau)}} R_y(y,\tau)\, dy d\tau, \tag{2.14}$$

and $u^*(x,t)$ is given by the second equation of (2.12). The exact formulation (2.14), however, does not provide a clear, convenient picture of the behavior of v^*. Next, we give such a picture by optimal, pointwise estimation. For comparison we give similar estimates on $\bar{v}$ and $\bar{u}$ first.

Lemma 2.1. *Let $0 < r' < r/4$ be an arbitrarily fixed constant. For $x \in \mathbb{R}$, $t \ge 0$ we have*

$$\begin{aligned} &|\bar{v} - v_-| < |v_- - v_+|, \\ &\frac{\partial^l \bar{v}}{\partial x^l}(x,t) = O(1)|v_- - v_+|(t+1)^{-\frac{l}{2}} e^{-\frac{r'x^2}{t+1}}, \quad l \ge 1, \\ &\frac{\partial^l \bar{u}}{\partial x^l}(x,t) = O(1)|v_- - v_+|(t+1)^{-\frac{l+1}{2}} e^{-\frac{r'x^2}{t+1}}, \quad l \ge 0. \end{aligned} \tag{2.15}$$

Proof. From (2.3) we have $\bar{v}_x = -r\bar{u}$ where $\bar{u}$ is given in (2.6). Thus $\bar{v}_x \lesseqgtr 0$ if $v_- \gtreqless v_+$. That is, for a fixed $t \ge 0$, $\bar{v}(x,t)$ monotonically decreases or increases from v_- to v_+. The first estimate in (2.15) follows. Other estimates in (2.15) are direct consequence of $\bar{u}(x,t)$ in (2.6), together with $\bar{v}_x = -r\bar{u}$. □

The following proposition gives similar estimates on v^* and u^*. We postpone its proof to the Appendix.

Proposition 2.2. *Let $0 < r' < r/4$ be an arbitrarily fixed constant and $l \ge 0$ be an integer. For $x \in \mathbb{R}$, $t \ge 0$ we have*

$$\begin{aligned} &\frac{\partial^l v^*}{\partial x^l}(x,t) = O(1)|v_- - v_+|(t+1)^{-\frac{l+1}{2}} e^{-\frac{r'(x+x_0)^2}{t+1}}, \\ &\frac{\partial^l u^*}{\partial x^l}(x,t) = O(1)|v_- - v_+|(t+1)^{-\frac{l+2}{2}} e^{-\frac{r'(x+x_0)^2}{t+1}}. \end{aligned} \tag{2.16}$$

With x_0 given in (2.8), we have (2.7) hence define a function

$$V_0(x) = \int_{-\infty}^{x} [v_0(y) - \bar{v}(y + x_0, 0)]\, dy. \tag{2.17}$$

From (2.10) and (2.13) we have

$$\int_{\mathbb{R}} v^*(x,t)\, dx = \int_{\mathbb{R}} v^*(x,0)\, dx = 0.$$

Combining with (2.9) we arrive at

$$\int_{\mathbb{R}} [v(x,t) - \bar{v}(x + x_0, t) - v^*(x,t)]\, dx = 0.$$

Thus we define a new variable

$$V(x,t)=\int_{-\infty}^{x}[v(y,t)-\bar{v}(y+x_0,t)-v^*(y,t)]\,dy. \tag{2.18}$$

It is clear that

$$V_x(x,t)=v(x,t)-\bar{v}(x+x_0,t)-v^*(x,t). \tag{2.19}$$

Correspondingly, we define

$$U(x,t)=\tilde{u}(x,t)-\bar{u}(x+x_0,t)-u^*(x,t)=u(x,t)-1-\bar{u}(x+x_0,t)-u^*(x,t). \tag{2.20}$$

From (2.13) it is clear that

$$V(x,0)=V_0(x). \tag{2.21}$$

We introduce the following notations to abbreviate the norms of Sobolev spaces with respect to x:

$$\|\cdot\|_m=\|\cdot\|_{H^m(\mathbb{R})},\quad \|\cdot\|=\|\cdot\|_{L^2(\mathbb{R})}. \tag{2.22}$$

Our first result is on global existence when Cauchy data are small perturbations of a weak diffusive contact wave.

Theorem 2.3. *Let $m\geq 3$ be an integer, $V_0\in H^{m+1}(\mathbb{R})$ and $u_0-1\in H^m(\mathbb{R})$. Then there exists a constant $\varepsilon_0>0$, such that if*

$$|v_- - v_+|+\|V_0\|_{m+1}+\|u_0-1\|_m\leq\varepsilon_0,$$

the Cauchy problem (1.1)-(1.3) has a unique global solution (v,u). The solution satisfies $V\in C(0,\infty;H^{m+1}(\mathbb{R}))\cap C^1(0,\infty;H^m(\mathbb{R}))$, $U\in C(0,\infty;H^m(\mathbb{R}))\cap C^1(0,\infty;H^{m-2}(\mathbb{R}))\cap L^2(0,\infty;H^{m+1}(\mathbb{R}))$, and the following energy estimate,

$$\begin{aligned}&\sup_{t\geq 0}\{\|V(t)\|_{m+1}^2+\|U(t)\|_m^2\}+\int_0^\infty[\|V_x(t)\|_m^2+\|U(t)\|_{m+1}^2]\,dt\\ \leq&\, C(\|V_0\|_{m+1}^2+\|u_0-1\|_m^2+|v_- - v_+|^2),\end{aligned} \tag{2.23}$$

where $C>0$ is a constant.

Our second result gives time decay rates of the solution towards the asymptotic solution.

Theorem 2.4. *Under the same assumptions as in Theorem 2.3, and with sufficiently small ε_0, the global solution (v,u) of (1.1)-(1.3) has the following estimates for $t\geq 0$,*

$$\begin{aligned}&(t+1)[\|V_x(t)\|_m^2+\|U(t)\|_m^2]+(t+1)^2[\|V_{xx}(t)\|_{m-1}^2+\|U_x(t)\|_{m-1}^2]\\&+\int_0^t(\tau+1)[\|V_{xx}(\tau)\|_{m-1}^2+\|U(\tau)\|_{m+1}^2]\,d\tau+\int_0^t(\tau+1)^2\|U_x(\tau)\|_m^2\,d\tau\\&\le C(\|V_0\|_{m+1}^2+\|u_0-1\|_m^2+|v_--v_+|^2),\end{aligned}\tag{2.24}$$

$$\|U(t)\|\le C(\|V_0\|_{m+1}+\|u_0-1\|_m+|v_--v_+|)(t+1)^{-1},\tag{2.25}$$

where $C>0$ is a constant.

Remark 2.5. We are able to obtain $L^\infty(\mathbb{R})$ decay rates via Sobolev inequality, see (3.9). From (2.23)-(2.25) we have

$$\begin{aligned}\|V(t)\|_{L^\infty(\mathbb{R})}&\le C(\|V_0\|_{m+1}+\|u_0-1\|_m+|v_--v_+|)(t+1)^{-1/4},\\\|V_x(t)\|_{L^\infty(\mathbb{R})}&\le C(\|V_0\|_{m+1}+\|u_0-1\|_m+|v_--v_+|)(t+1)^{-3/4},\\\|U(t)\|_{L^\infty(\mathbb{R})}&\le C(\|V_0\|_{m+1}+\|u_0-1\|_m+|v_--v_+|)(t+1)^{-1},\end{aligned}$$

where $C>0$ is a constant. Here the rate for $\|U(t)\|_{L^\infty(\mathbb{R})}$ can be improved to $(t+1)^{-5/4}$ if we carry out an estimate similar to (2.25) for U_x.

Recall (2.19) and (2.20), which give us

$$\begin{cases}v(x,t)=\bar v(x+x_0,t)+v^*(x,t)+V_x(x,t),\\u(x,t)=1+\bar u(x+x_0,t)+u^*(x,t)+U(x,t).\end{cases}\tag{2.26}$$

From (2.6), $\bar v$ is a diffusive contact wave while $\bar u$ is a heat kernel. From (2.16) we have L^2 decay rates of $(t+1)^{-\frac14}$ and $(t+1)^{-\frac34}$ for v^* and u^*, respectively. Here (2.24) and (2.25) give us L^2 decay rates $(t+1)^{-\frac12}$ for V_x and $(t+1)^{-1}$ for U. This gives the nonlinear stability of the diffusive contact wave. It also justifies that (v^*,u^*) indeed is a secondary wave in the time asymptotic ansatz of the solution (v,u) to (1.1)-(1.3). The rates for V_x and U can be improved to the optimal ones by a different set of analytic tools. It is left to a future work since the main purpose of this paper is the global existence of solution, the stability of the diffusive contact wave, and the identification of the asymptotic ansatz.

3. Local existence of solution

We rewrite (1.1)-(1.3) in terms of the new variables V and U as defined in (2.18) and (2.20). Substituting (2.26) into (1.1) and applying (2.3) and (2.10), we have

$$\begin{cases}V_{xt}+U_x=0,\\U_t+(UV_x)_x+V_{xx}-U_{xx}+rU^2+rU=F,\end{cases}\tag{3.1}$$

where

$$\begin{aligned}&F=F_1+F_2+F_3,\\&F_1=-[(\bar u+u^*)V_x]_x,\\&F_2=-[(\bar v+v^*)U]_x-2r(\bar u+u^*)U,\\&F_3=-\bar u_t-u^*_t-[(\bar u+u^*)(\bar v+v^*)]_x-(\bar v+v^*)_x\\&\qquad+(\bar u+u^*)_{xx}-r(\bar u+u^*)^2-r(\bar u+u^*).\end{aligned}\tag{3.2}$$

Applying (2.3), (2.10) and (2.11) we further simplify F_3 as

$$\begin{aligned}F_3=&\frac{1}{r}\bar v_{xt}+\frac{1}{r}(v^*_x+\bar u_x\bar v)_t-(\bar u\bar v)_x-[\bar uv^*+u^*(\bar v+v^*)]_x\\&+\bar u_x\bar v+(\bar u+u^*)_{xx}-r(\bar u^2+2\bar uu^*+u^{*2})\\=&-\frac{1}{r}\bar u_{xx}-\frac{1}{r}u^*_{xx}+\frac{1}{r}(\bar u_x\bar v)_t-[\bar uv^*+u^*(\bar v+v^*)]_x+(\bar u+u^*)_{xx}-r(2\bar uu^*+u^{*2})\\=&\frac{1}{r}(\bar u_x\bar v)_t-[\bar uv^*+u^*(\bar v+v^*)]_x+(1-\frac{1}{r})(\bar u+u^*)_{xx}-r(2\bar u+u^*)u^*.\end{aligned}\tag{3.3}$$

From (3.2) we see that $F=F(V_x,U,V_{xx},U_x,x,t)$, where the explicit dependence on x and t is via $(\bar v,\bar u)$ and (v^*,u^*), defined in (2.6) and (2.11)-(2.13). We solve (3.1) for (V_x,U) with initial data

$$\begin{cases}V_x(x,0)=v_0(x)-\bar v(x+x_0,0)=V_0'(x),\\U(x,0)=u_0(x)-1-\bar u(x+x_0,0)-\frac{1}{r}(\bar u_x\bar v)(x+x_0,0)\equiv U_0(x),\end{cases}\tag{3.4}$$

see (2.21), (2.17), (2.20) and (2.11)-(2.13).

Local existence of solutions for Cauchy problems has been established by Kawashima for a quite general class of hyperbolic-parabolic systems, with applications in continuum mechanics [6]. Kawashima's theory, however, does not apply directly to (3.1), (3.4), due to the explicit dependence of F on x and t. In this section we modify Kawashima's argument, specifically for (3.1), to establish our local existence theory. We focus on the part that is related to the (x,t)-dependence of F, and briefly outline the rest. More details can be found in [6].

Since a local solution of (3.1), (3.4) is constructed as the limit of a successive approximation sequence, we consider Cauchy problem of the linear system that produces the sequence. This is

$$Z_{1t}=f_1(x,t),\quad Z_1(x,0)=Z_{10}(x),\tag{3.5}$$

$$Z_{2t}-Z_{2xx}=f_2(x,t),\quad Z_2(x,0)=Z_{20}(x).\tag{3.6}$$

Note that Z_1 and Z_2 are decoupled in this system. We cite Lemma 2.6 and Proposition 2.7 in [6], simplified for the special case (3.5), (3.6).

Lemma 3.1 *([6] energy estimate for linearized equation). Let $m \geq 2$ be an integer and $T > 0$ be a constant.*

(i) Let $0 \leq l \leq m$ be an integer and $f_1 \in L^\infty(0, T; H^{l-1}(\mathbb{R})) \cap L^2(0, T; H^l(\mathbb{R}))$. Assume that Z_1 is a solution of (3.5) satisfying $Z_1 \in L^\infty(0, T; H^l(\mathbb{R}))$ and $Z_{1t} \in L^\infty(0, T; H^{l-1}(\mathbb{R}))$. Then we have $Z_1 \in C(0, T; H^l(\mathbb{R}))$, satisfying the energy inequality

$$\|Z_1(t)\|_l^2 \leq 2[\|Z_{10}\|_l^2 + t\int_0^t \|f_1(\tau)\|_l^2 d\tau], \quad 0 \leq t \leq T. \tag{3.7}$$

(ii) Let $1 \leq l \leq m$ be an integer and $f_2 \in L^\infty(0, T; H^{l-1}(\mathbb{R}))$. Assume that Z_2 is a solution of (3.6) satisfying $Z_2 \in L^\infty(0, T; H^l(\mathbb{R}))$ and $Z_{2t} \in L^\infty(0, T; H^{l-2}(\mathbb{R}))$. Then $Z_2 \in C(0, T; H^l(\mathbb{R})) \cap L^2(0, T; H^{l+1}(\mathbb{R}))$, satisfying the energy inequality

$$\|Z_2(t)\|_l^2 + \int_0^t \|Z_2(\tau)\|_{l+1}^2 d\tau \leq e^{2t}(\|Z_{20}\|_l^2 + 2t\int_0^t \|f_2(\tau)\|_{l-1}^2 d\tau), \quad 0 \leq t \leq T. \tag{3.8}$$

Proposition 3.2 *([6] existence of solutions for linearized equations). Let $m \geq 2$ be an integer and $T > 0$ be a constant.*

(i) Let $1 \leq l \leq m$ be an integer and $f_1 \in C(0, T; H^{l-1}(\mathbb{R})) \cap L^2(0, T; H^l(\mathbb{R}))$. If $Z_{10} \in H^l(\mathbb{R})$, then (3.5) has a unique solution $Z_1 \in C(0, T; H^l(\mathbb{R})) \cap C^1(0, T; H^{l-1}(\mathbb{R}))$, satisfying the energy estimate (3.7).

(ii) Let $2 \leq l \leq m$ be an integer and $f_2 \in C(0, T; H^{l-1}(\mathbb{R}))$. If $Z_{20} \in H^l(\mathbb{R})$, then (3.6) has a unique solution $Z_2 \in C(0, T; H^l(\mathbb{R})) \cap C^1(0, T; H^{l-2}(\mathbb{R})) \cap L^2(0, T; H^{l+1}(\mathbb{R}))$, satisfying the energy estimate (3.8).

We need some analytic tools such as Sobolev inequality.

Lemma 3.3.

(i) Let $u \in H^1(\mathbb{R})$. Then $u \in L^\infty(\mathbb{R})$ with

$$\|u\|_{L^\infty} \leq \sqrt{2}\|u\|^{\frac{1}{2}}\|u'\|^{\frac{1}{2}} \leq \sqrt{2}\|u\|_1. \tag{3.9}$$

(ii) Let $m \geq 1$ and $0 \leq l \leq m$ be integers. If $u \in H^m(\mathbb{R})$ and $v \in H^l(\mathbb{R})$, the $uv \in H^l(\mathbb{R})$ with

$$\|uv\|_l \leq C\|u\|_m\|v\|_l, \tag{3.10}$$

where $C > 0$ is a constant.

(iii) Let $m \geq 2$ and $1 \leq l \leq m$ be integers. Suppose $u \in H^m(\mathbb{R})$ and $v \in H^{l-1}(\mathbb{R})$. Then for $0 \leq k \leq l$, we have the commutator $[\partial_x^k, u]v \equiv \partial_x^k(uv) - u\partial_x^k v \in L^2(\mathbb{R})$ and

$$\sum_{k=0}^{l} \|[\partial_x^k, u]v\| \leq C\|u_x\|_{m-1}\|v\|_{l-1}, \tag{3.11}$$

where $C > 0$ is a constant.

Following the formulation in [6] we write (3.1), (3.4) as

$$\begin{cases} Z_{1t} = f_1(Z_{2x}) \\ Z_{2t} - Z_{2xx} = f_2(Z_1, Z_2, Z_{1x}, Z_{2x}, x, t), \\ (Z_1, Z_2)(x, 0) = (V_0', U_0)(x). \end{cases} \tag{3.12}$$

Here (Z_1, Z_2) stands for (V_x, U), and

$$f_1(Z_{2x}) = -Z_{2x},$$
$$f_2(Z_1, Z_2, Z_{1x}, Z_{2x}, x, t) = -(Z_1Z_2)_x - Z_{1x} - rZ_2^2 - rZ_2 + F(Z_1, Z_2, Z_{1x}, Z_{2x}, x, t). \tag{3.13}$$

Interested readers can compare (3.12) with (2.1), (2.2) in [6].

To set up the successive approximation we further study the linear system

$$\begin{cases} \hat{Z}_{1t} = f_1(Z_{2x}) \\ \hat{Z}_{2t} - \hat{Z}_{2xx} = f_2(Z_1, Z_2, Z_{1x}, Z_{2x}, x, t), \\ (\hat{Z}_1, \hat{Z}_2)(x, 0) = (Z_1, Z_2)(x, 0) = (V_0', U_0)(x), \end{cases} \tag{3.14}$$

where f_1 and f_2 are defined in (3.13). An invariant set is to be built for the mapping $(Z_1, Z_2) \to (\hat{Z}_1, \hat{Z}_2)$.

For a constant $T > 0$ we denote $Q_T = \mathbb{R} \times [0, T]$. For $(Z_1, Z_2)(x, t)$ defined on Q_T, with an integer $m \geq 2$ we assume the following,

$$Z_1 \in C(0, T; H^m(\mathbb{R})), \quad Z_{1t} \in C(0, T; H^{m-1}(\mathbb{R})), \tag{3.15}$$

$$\begin{cases} Z_2 \in C(0, T; H^m(\mathbb{R})) \cap L^2(0, T; H^{m+1}(\mathbb{R})), \\ Z_{2t} \in C(0, T; H^{m-2}(\mathbb{R})) \cap L^2(0, T; H^{m-1}(\mathbb{R})), \end{cases} \tag{3.16}$$

$$\sup_{0\leq t\leq T} \|(Z_1, Z_2)(t)\|_m^2 + \int_0^T \|Z_2(t)\|_{m+1}^2\, dt \leq M^2, \tag{3.17}$$

$$\int_0^T \|(Z_{1t}, Z_{2t})(t)\|_{m-1}^2\, dt \leq M_1^2. \tag{3.18}$$

Here M and M_1 are positive constants. We denote by $X_T^m(M, M_1)$ the set of functions $(Z_1, Z_2)(x,t)$ satisfying (3.15)-(3.18).

We determine the constants M, M_1 and T such that for $(Z_1, Z_2) \in X_T^m(M, M_1)$, the solution $(\hat{Z}_1, \hat{Z}_2)$ of (3.14) is in the same $X_T^m(M, M_1)$.

Lemma 3.4. *Let $m \geq 2$ be an integer and $(Z_1, Z_2) \in X_T^m(M, M_1)$.*

(i) For f_1 and f_2 defined by (3.13) and (3.2) we have

$$\begin{aligned}&\|f_1(t)\|_{m-1} = \|Z_{2x}(t)\|_{m-1} \leq M,\\ &\|f_2(t)\|_{m-1} \leq B(M, A),\end{aligned} \tag{3.19}$$

where B is a positive constant depending on M and an upper bound A of $|v_-| + |v_+|$.

(ii) Let $(\hat{Z}_1, \hat{Z}_2)$ be a solution of (3.14) and satisfy (3.15)-(3.17), with M in (3.17) replaced by $\hat{M}$. Then

$$\int_0^T \|(\hat{Z}_{1t}, \hat{Z}_{2t})(t)\|_{m-1}^2\, dt \leq 2\hat{M}^2 + T(M^2 + 2B^2). \tag{3.20}$$

Proof. Under the assumption $(Z_1, Z_2) \in X_T^m(M, M_1)$, from (3.13) we have

$$\|f_1(t)\|_{m-1} = \|-Z_{2x}(t)\|_{m-1} \leq \|Z_2(t)\|_m \leq M, \quad t \in [0, T]. \tag{3.21}$$

This gives us the first estimate in (3.19). Also with (3.2) and (3.10) we have

$$\begin{aligned}\|f_2(t)\|_{m-1} \leq & \|Z_1Z_2\|_m + \|Z_1\|_m + r\|Z_2^2\|_{m-1} + r\|Z_2\|_{m-1} + \|(\bar{u}+u^*)Z_1\|_m\\ &+ \|(\bar{v}+v^*)Z_2\|_m + 2r\|(\bar{u}+u^*)Z_2\|_{m-1} + \|F_3(t)\|_{m-1}\\ \leq & C\|Z_1\|_m\|Z_2\|_m + \|Z_1\|_m + rC\|Z_2\|_{m-1}^2 + r\|Z_2\|_{m-1} + C\|\bar{u}+u^*\|_m\|Z_1\|_m\\ &+ C\|\bar{v}+v^*\|_{W^{m,\infty}}\|Z_2\|_m + 2rC\|\bar{u}+u^*\|_{m-1}\|Z_2\|_{m-1} + \|F_3(t)||_{m-1}\\ \leq & C(M + 1 + r + \|\bar{u}+u^*\|_m + \|\bar{v}+v^*\|_{W^{m,\infty}})M + \|F_3(t)||_{m-1},\end{aligned} \tag{3.22}$$

where $C > 1$ is a universal constant. In (3.22) we note that $\bar{v} \notin L^2(\mathbb{R})$ thus we use L^∞ norm instead.

From (3.3), (2.6), (2.15) and (2.16) we have

$$\begin{aligned}\|\partial_x^l F_3(t)\| \leq & \frac{1}{r}\|\partial_x^l \partial_t(\bar{u}_x\bar{v})\| + \|\partial_x^{l+1}(\bar{u}v^*)\| + \|\partial_x^{l+1}[u^*(\bar{v}+v^*)]\|\\ &+ |1 - \frac{1}{r}|\|\partial_x^{l+2}(\bar{u}+u^*)\| + r\|\partial_x^l[(2\bar{u}+u^*)u^*]\|\\ \leq & C|v_- - v_+|(t+1)^{-\frac{l}{2}-\frac{5}{4}}\end{aligned} \tag{3.23}$$

for an integer $l \geq 0$. Here $C > 0$ in (3.23) is a constant depending only on an upper bound A of $|v_-| + |v_+|$. This implies

$$\|F_3(t)\|_{m-1} \leq C|v_- - v_+|(t+1)^{-\frac{5}{4}}. \tag{3.24}$$

Substituting (3.24) into (3.22) and noting $\|\bar{u} + u^*\|_m + \|\bar{v} + v^*\|_{W^{m,\infty}} \leq C(|v_-| + |v_+|)$ give us the second estimate in (3.19).

Suppose $(\hat{Z}_1, \hat{Z}_2)$ is a solution of (3.14) satisfying (3.15)-(3.17) with M replaced by $\hat{M}$. Then with (3.19) we have

$$\begin{aligned}
&\int_0^T \|\hat{Z}_{1t}(t)\|_{m-1}^2\, dt + \int_0^T \|\hat{Z}_{2t}(t)\|_{m-1}^2\, dt \\
&= \int_0^T \|f_1(t)\|_{m-1}^2\, dt + \int_0^T \|(\hat{Z}_{2xx} + f_2)(t)\|_{m-1}^2\, dt \\
&\leq \int_0^T [\|f_1(t)\|_{m-1}^2 + 2\|f_2(t)\|_{m-1}^2]\, dt + \int_0^T 2\|\hat{Z}_{2xx}(t)\|_{m-1}^2\, dt \\
&\leq \int_0^T [M^2 + 2B^2]\, dt + 2\hat{M}^2 = (M^2 + 2B^2)T + 2\hat{M}. \quad \square
\end{aligned}$$

The following proposition gives us an invariant set under iterations.

Proposition 3.5. *Let $m \geq 2$ be an integer, and A and A_0 be positive constants. Suppose $|v_-| + |v_+| \leq A$ and $(V_0', U_0) \in H^m(\mathbb{R})$ with $\|(V_0', U_0)\|_m \leq A_0$. Then there exists a constant $T_0 > 0$, depending only on A and A_0, such that if $(Z_1, Z_2) \in X_{T_0}^m(\sqrt{8}A_0, \sqrt{17}A_0)$, the initial value problem (3.14) has a unique solution $(\hat{Z}_1, \hat{Z}_2)$ in the same $X_{T_0}^m(\sqrt{8}A_0, \sqrt{17}A_0)$.*

Proof. We set $M = \sqrt{8}A_0$ and $M_1 = \sqrt{17}A_0$. If $(Z_1, Z_2) \in X_{T_0}^m(M, M_1)$, then by (3.13) we have $f_1 = -Z_{2x} \in C(0, T_0; H^{m-1}(\mathbb{R})) \cap L^2(0, T_0; H^m(\mathbb{R}))$. Similarly, we also have $f_2 \in C(0, T_0; H^{m-1}(\mathbb{R}))$. Applying Proposition 3.2, there is a unique solution $(\hat{Z}_1, \hat{Z}_2)$ of (3.14),

$$\hat{Z}_1 \in C(0, T_0; H^m(\mathbb{R})) \cap C^1(0, T_0; H^{m-1}(\mathbb{R})),$$

$$\hat{Z}_2 \in C(0, T_0; H^m(\mathbb{R})) \cap C^1(0, T_0; H^{m-2}(\mathbb{R})) \cap L^2(0, T_0; H^{m+1}(\mathbb{R})).$$

The solution satisfies the energy estimate

$$\begin{aligned}&\|\hat{Z}_1(t)\|_m^2+\|\hat{Z}_2(t)\|_m^2+\int_0^t\|\hat{Z}_2(\tau)\|_{m+1}^2\,d\tau\\&\le 2[\|V_0'\|_m^2+t\int_0^t\|f_1(\tau)\|_m^2\,d\tau]+e^{2t}[\|U_0\|_m^2+2\int_0^t\|f_2(\tau)\|_{m-1}^2\,d\tau]\end{aligned}\tag{3.25}$$

for $0\le t\le T_0$.

To show $(\hat{Z}_1,\hat{Z}_2)\in X^m_{T_0}(M,M_1)$ we only need to obtain (3.17) and (3.18) for $(\hat{Z}_1,\hat{Z}_2)$. For this we apply (3.19) and (3.17) to bound the right-hand side of (3.25) by

$$\begin{aligned}&2\|V_0'\|_m^2+2T_0\int_0^{T_0}\|Z_{2x}(t)\|_m^2\,dt+e^{2T_0}[\|U_0\|_m^2+2T_0B^2]\\&\le(2\|V_0'\|_m^2+e^{2T_0}\|U_0\|_m^2)+2T_0(M^2+e^{2T_0}B^2),\end{aligned}$$

where B is a positive constant depending on M (hence on A_0) and A. Now we choose $T_0>0$ such that

$$e^{2T_0}\le 2\quad\text{and}\quad 2T_0(M^2+2B^2)\le 2A_0^2.\tag{3.26}$$

This gives us

$$\sup_{0\le t\le T_0}\|(\hat{Z}_1,\hat{Z}_2)(t)\|_m^2+\int_0^{T_0}\|\hat{Z}_2(t)\|_{m+1}^2\,dt\le 8A_0^2=M^2,$$

which is (3.17) for $(\hat{Z}_1,\hat{Z}_2)$. Here from (3.26) we note that T_0 depends only on A and A_0.

Next we apply (3.20) with $\hat{M}$ taken as M to have

$$\int_0^{T_0}\|(\hat{Z}_{1t},\hat{Z}_{2t})(t)\|_{m-1}^2\,dt\le 2M^2+T_0(M^2+2B^2)=17A_0^2=M_1^2.$$

This is (3.18) for $(\hat{Z}_1,\hat{Z}_2)$. □

The following theorem establishes the existence of local solution to (3.12) hence to (3.1), (3.4).

Theorem 3.6. *Let $m\ge 2$ be an integer, and A and A_0 be positive constants. Suppose $|v_-|+|v_+|\le A$ and $(V_0',U_0)\in H^m(\mathbb{R})$ with $\|(V_0',U_0)\|_m\le A_0$. Then there exists a positive constant T_1 ($\le T_0$), depending only on A and A_0, such that the initial value problem (3.12) has a unique solution (Z_1,Z_2), satisfying*

$$\begin{aligned}&Z_1\in C(0,T_1;H^m(\mathbb{R}))\cap C^1(0,T_1;H^{m-1}(\mathbb{R})),\\&Z_2\in C(0,T_1;H^m(\mathbb{R}))\cap C^1(0,T_1;H^{m-2}(\mathbb{R}))\cap L^2(0,T_1;H^{m+1}(\mathbb{R})).\end{aligned}$$

Proof. We define a successive approximation sequence $\{(Z_1^k, Z_2^k)(x,t)\}$ via the following iteration scheme,

$$(Z_1^0, Z_2^0) = (0,0),$$
$$\begin{cases} Z_{1t}^{k+1} = f_1(Z_{2x}^k), \\ Z_{2t}^{k+1} - Z_{2xx}^{k+1} = f_2(Z_1^k, Z_2^k, Z_{1x}^k, Z_{2x}^k, x, t), \quad k \geq 0, \\ (Z_1^{k+1}, Z_2^{k+1})(x,0) = (V_0', U_0)(x). \end{cases} \tag{3.27}$$

By Proposition 3.5, for all $k \geq 0$, $(Z_1^k, Z_2^k)(x,t)$ is well-defined on Q_{T_0} and $(Z_1^k, Z_2^k)(x,t) \in X_{T_0}^m(M, M_1)$, where $M = \sqrt{8}A_0$ and $M_1 = \sqrt{17}A_0$.

Next we prove that $\{(Z_1^k, Z_2^k)(x,t)\}$ is a Cauchy sequence. Let

$$\bar{Z}_1^k = Z_1^{k+1} - Z_1^k, \quad \bar{Z}_2^k = Z_2^{k+1} - Z_2^k. \tag{3.28}$$

From (3.27), (3.13) and (3.2), for $k \geq 1$ we have

$$\begin{cases} \bar{Z}_{1t}^k = \bar{f}_1^k, \\ \bar{Z}_{2t}^k - \bar{Z}_{2xx}^k = \bar{f}_2^k, \\ (\bar{Z}_1^k, \bar{Z}_2^k)(x,0) = (0,0), \end{cases} \tag{3.29}$$

where

$$\begin{aligned} \bar{f}_1^k =& f_1(Z_{2x}^k) - f_1(Z_{2x}^{k-1}) = -\bar{Z}_{2x}^{k-1}, \\ \bar{f}_2^k =& f_2(Z_1^k, Z_2^k, Z_{1x}^k, Z_{2x}^k, x, t) - f_2(Z_1^{k-1}, Z_2^{k-1}, Z_{1x}^{k-1}, Z_{2x}^{k-1}, x, t) \\ =& -\bar{Z}_{1x}^{k-1} - r\bar{Z}_2^{k-1} - (Z_{1x}^k Z_2^k - Z_{1x}^{k-1} Z_2^{k-1}) - (Z_1^k Z_{2x}^k - Z_1^{k-1} Z_{2x}^{k-1}) \\ & - r[(Z_2^k)^2 - (Z_2^{k-1})^2] - (\bar{u}_x + u_x^*)\bar{Z}_1^{k-1} - (\bar{u} + u^*)\bar{Z}_{1x}^{k-1} \\ & - (\bar{v}_x + v_x^*)\bar{Z}_2^{k-1} - (\bar{v} + v^*)\bar{Z}_{2x}^{k-1} - 2r(\bar{u} + u^*)\bar{Z}_2^{k-1}. \end{aligned}$$

Since $(Z_1^k, Z_2^k) \in X_{T_0}^m(M, M_1)$, together with (3.9) and (3.10) we have

$$\begin{aligned} \|\bar{f}_1^k\|_{m-1} =& \|\bar{Z}_{2x}^{k-1}\|_{m-1} \leq \|\bar{Z}_{2x}^{k-1}\|_m, \\ \|\bar{f}_2^k\|_{m-2} \leq& \|\bar{Z}_{1x}^{k-1}\|_{m-2} + r\|\bar{Z}_2^{k-1}\|_{m-2} + \|\bar{Z}_{1x}^{k-1} Z_2^k\|_{m-2} + \|Z_{1x}^{k-1}\bar{Z}_2^k\|_{m-2} \\ & + \|\bar{Z}_1^{k-1} Z_{2x}^k\|_{m-2} + \|Z_1^{k-1}\bar{Z}_{2x}^{k-1}\|_{m-2} + r\|(Z_2^k + Z_2^{k-1})\bar{Z}_2^{k-1}\|_{m-2} \\ & + \|(\bar{u}_x + u_x^*)\bar{Z}_1^{k-1}\|_{m-2} + \|(\bar{u} + u^*)\bar{Z}_{1x}^{k-1}\|_{m-2} + \|(\bar{v}_x + v_x^*)\bar{Z}_2^{k-1}\|_{m-2} \\ & + \|(\bar{v} + v^*)\bar{Z}_{2x}^{k-1}\|_{m-2} + 2r\|(\bar{u} + u^*)\bar{Z}_2^{k-1}\|_{m-2} \\ \leq& C\|(\bar{Z}_1^{k-1}, \bar{Z}_2^{k-1})\|_{m-1}, \end{aligned} \tag{3.30}$$

where $C > 0$ is a constant depending only on M and A (hence only on A_0 and A).

From (3.28) it is clear that $(Z_1^k, Z_2^k) \in X_{T_0}^m(M, M_1)$ implies $(\bar{Z}_1^k, \bar{Z}_2^k) \in X_{T_0}^m(2M, 2M_1)$. Then (3.29) implies $\bar{f}_1^k \in C(0, T_0; H^{m-2}(\mathbb{R})) \cap L^2(0, T_0; H^{m-1}(\mathbb{R}))$ and $\bar{f}_2^k \in C(0, T_0; H^{m-2}(\mathbb{R}))$. Applying Lemma 3.1 to (3.29) with $l = m-1$, from (3.7), (3.8) and (3.30) we have

$$
\begin{aligned}
&\|\bar{Z}_1^k(t)\|_{m-1}^2 + \|\bar{Z}_2^k(t)\|_{m-1}^2 + \int_0^t \|\bar{Z}_2^k(\tau)\|_m^2 \, d\tau \\
\le & 2t \int_0^t \|\bar{f}_1^k(\tau)\|_{m-1}^2 \, d\tau + 2e^{2t} \int_0^t \|\bar{f}_2^k(\tau)\|_{m-2}^2 \, d\tau \\
\le & 2t \int_0^t \|\bar{Z}_2^{k-1}(\tau)\|_m^2 \, d\tau + Ce^{2t} \int_0^t \|(\bar{Z}_1^{k-1}, \bar{Z}_2^{k-1})(\tau)\|_{m-1}^2 \, d\tau
\end{aligned}
$$

for $0 \le t \le T_0$. Here $C > 1$ is a constant depending only on A and A_0. Thus for $0 \le t \le T_0$,

$$
\begin{aligned}
&\sup_{0\le\tau\le t} \|(\bar{Z}_1^k, \bar{Z}_2^k)(\tau)\|_{m-1}^2 + \int_0^t \|\bar{Z}_2^k(\tau)\|_m^2 \, d\tau \\
\le & 2Ce^{2T_0} t \sup_{0\le\tau\le t} \|(\bar{Z}_1^{k-1}, \bar{Z}_2^{k-1})(\tau)\|_{m-1}^2 + 4t \int_0^t \|\bar{Z}_2^{k-1}(\tau)\|_m^2 \, d\tau.
\end{aligned}
\tag{3.31}
$$

Now we take a positive constant T_1 such that

$$
T_1 \le T_0, \quad \rho \equiv 4CT_1 < 1. \tag{3.32}
$$

Clearly $T_1 > 0$ is a constant depending only on A and A_0. Noting (3.26) we simplify (3.31) to

$$
\begin{aligned}
&\sup_{0\le\tau\le t} \|(\bar{Z}_1^k, \bar{Z}_2^k)(\tau)\|_{m-1}^2 + \int_0^t \|\bar{Z}_2^k(\tau)\|_m^2 \, d\tau \\
\le & \rho\Big[\sup_{0\le\tau\le t} \|(\bar{Z}_1^{k-1}, \bar{Z}_2^{k-1})(\tau)\|_{m-1}^2 + \int_0^t \|\bar{Z}_2^{k-1}(\tau)\|_m^2 \, d\tau\Big], \quad 0 \le t \le T_1.
\end{aligned}
\tag{3.33}
$$

The rest is completely parallel to the proof of Theorem 2.9 in [6]. It follows from (3.33) that $\{(Z_1^k, Z_2^k)\}$ is a Cauchy sequence in $C(0, T_1; H^{m-1}(\mathbb{R}))$. Therefore, there is $(Z_1, Z_2) \in C(0, T_1; H^{m-1}(\mathbb{R}))$ such that $(Z_1^k - Z_1, Z_2^k - Z_2) \to 0$ strongly in $C(0, T_1; H^{m-1}(\mathbb{R}))$ as $k \to \infty$. Since $(Z_1^k, Z_2^k) \in X_{T_0}^m(M, M_1) \subset X_{T_1}^m(M, M_1)$, we conclude that there is a subsequence $\{k'\}$ of $\{k\}$ such that $Z_2^{k'} - Z_2 \rightharpoonup 0$ weakly in $L^2(0, T_1; H^{m-1}(\mathbb{R}))$.

Similarly, for each $t \in [0, T_1]$, there is a subsequence $\{k''\} = \{k''\}(t)$ of $\{k'\}$ such that $(Z_1^{k''} - Z_1, Z_2^{k''} - Z_2) \rightharpoonup 0$ weakly in $H^m(\mathbb{R})$. Thus we have a solution (Z_1, Z_2) of (3.12), satisfying

$$
\begin{aligned}
&Z_1 \in L^\infty(0, T_1; H^m(\mathbb{R})),\\
&Z_2 \in L^\infty(0, T_1; H^m(\mathbb{R})) \cap L^2(0, T_1; H^{m+1}(\mathbb{R})).
\end{aligned}
$$

Moreover, it follows that

$$
\begin{aligned}
&Z_{1t} \in L^\infty(0, T_1; H^{m-1}(\mathbb{R})),\\
&Z_{2t} \in L^\infty(0, T_1; H^{m-2}(\mathbb{R})) \cap L^2(0, T_1; H^{m-1}(\mathbb{R})).
\end{aligned}
$$

By Lemma 3.1, we improve the regularity to

$$
(Z_1, Z_2) \in C(0, T_1; H^m(\mathbb{R})),
$$

hence

$$
Z_{1t} \in C(0, T_1; H^{m-1}(\mathbb{R})), \quad Z_{2t} \in C(0, T_1; H^{m-2}(\mathbb{R})).
$$

The uniqueness of (Z_1, Z_2) follows from the regularity and Proposition 3.2. □

4. Global existence of solution

In this section we prove Theorem 2.3, the existence of a solution global in time for (1.1)-(1.3). Based on Theorem 3.6, we only need to prove the following proposition. From there a standard continuity argument gives us Theorem 2.3.

Proposition 4.1. *Let $m \geq 3$ be an integer, $V_0 \in H^{m+1}(\mathbb{R})$ and $u_0 - 1 \in H^m(\mathbb{R})$. Suppose (V_x, U) is a solution of (3.1), (3.4), satisfying*

$$
\begin{aligned}
&V \in C(0, T; H^{m+1}(\mathbb{R})) \cap C^1(0, T; H^m(\mathbb{R})),\\
&U \in C(0, T; H^m(\mathbb{R})) \cap C^1(0, T; H^{m-2}(\mathbb{R})) \cap L^2(0, T; H^{m+1}(\mathbb{R})).
\end{aligned}
$$

Let

$$
N_m^2(t) = \sup_{0\leq\tau\leq t} \{\|V(\tau)\|_{m+1}^2 + \|U(\tau)\|_m^2\} + \int_0^t [\|V_x(\tau)\|_m^2 + \|U(\tau)\|_{m+1}^2]\, d\tau, \quad t \in [0, T]. \tag{4.1}
$$

Then there exist constants $\delta_0, \delta_1 > 0$, such that if $|v_- - v_+| \leq \delta_0$ and $N_m(T) \leq \delta_1$, the following a priori estimate holds:

$$
N_m^2(T) \leq C(\|V_0\|_{m+1}^2 + \|U_0\|_m^2 + |v_- - v_+|^2), \tag{4.2}
$$

where $C > 0$ is a constant.

Proof. In the following C denotes a universal positive constant. In particular, it is independent of T. For $0 \leq l \leq m$, we apply ∂_x^l to the second equation in (3.1) to have

$$
\partial_x^l U_t + \partial_x^{l+2} V - \partial_x^{l+2} U + r\partial_x^l U = -\partial_x^{l+1}(U V_x) - r\partial_x^l U^2 + \partial_x^l F. \tag{4.3}
$$

Multiply (4.3) by $\partial_x^l U$ and integrate with respect to x. After integration by parts we have

$$\frac{d}{dt}[\frac{1}{2}\|\partial_x^l U\|^2 + \frac{1}{2}\|\partial_x^{l+1} V\|^2] + \|\partial_x^{l+1} U\|^2 + r\|\partial_x^l U\|^2 = I_1 + I_2 + I_3, \tag{4.4}$$

$$I_1 = -\int_{\mathbb{R}} \partial_x^l U \partial_x^{l+1}(UV_x)\,dx, \quad I_2 = -r\int_{\mathbb{R}} \partial_x^l U \partial_x^l U^2\,dx, \quad I_3 = \int_{\mathbb{R}} \partial_x^l U \partial_x^l F\,dx. \tag{4.5}$$

Here we have applied the first equation in (3.1) to obtain the second term on the left-hand side of (4.4).

For I_1 using the commutator notation in Lemma 3.3 and by integration by parts, we have

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}} (\partial_x^l U)U(\partial_x^{l+2} V)\,dx - \int_{\mathbb{R}} (\partial_x^l U)[\partial_x^{l+1}, U]V_x\,dx \\ &= \int_{\mathbb{R}} U(\partial_x^{l+1} U)(\partial_x^{l+1} V)\,dx + \int_{\mathbb{R}} U_x(\partial_x^l U)(\partial_x^{l+1} V)\,dx - \int_{\mathbb{R}} (\partial_x^l U)[\partial_x^{l+1}, U]V_x\,dx \\ &\equiv I_{11} + I_{12} + I_{13}. \end{aligned} \tag{4.6}$$

Here for I_{11} again we apply (3.1) to have

$$\begin{aligned} I_{11} &= -\int_{\mathbb{R}} U(\partial_x^{l+1} V_t)(\partial_x^{l+1} V)\,dx = -\frac{1}{2}\int_{\mathbb{R}} U\frac{\partial}{\partial t}(\partial_x^{l+1} V)^2\,dx \\ &= \frac{d}{dt}[-\frac{1}{2}\int_{\mathbb{R}} U(\partial_x^{l+1} V)^2\,dx] \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} [-(UV_x)_x - V_{xx} + U_{xx} - rU^2 - rU + F](\partial_x^{l+1} V)^2\,dx. \end{aligned}$$

Applying (3.9) we bound the right-hand side to have

$$I_{11} \le \frac{d}{dt}[-\frac{1}{2}\int_{\mathbb{R}} U(\partial_x^{l+1} V)^2\,dx] + C(\|V_{xx}\|_1 + \|U\|_3 + \|F\|_{L^\infty})\|\partial_x^{l+1} V\|^2.$$

From (3.2), (3.3), (2.15), (2.16) and (2.6) we have

$$\begin{aligned} \|F\|_{L^\infty} &\le \|F_1\|_{L^\infty} + \|F_2\|_{L^\infty} + |F_3\|_{L^\infty} \\ &\le C(\|V_x\|_1 + \|V_{xx}\|_1 + \|U\|_1 + \|U_x\|_1 + |v_- - v_+|). \end{aligned}$$

This gives us

$$I_{11} \le \frac{d}{dt}[-\frac{1}{2}\int_{\mathbb{R}} U(\partial_x^{l+1} V)^2\,dx] + C(\|V_x\|_2 + \|U\|_3 + |v_- - v_+|)\|\partial_x^{l+1} V\|^2. \tag{4.7}$$

Similarly, for I_{12} and I_{13} in (4.6) we have

$$I_{12} \le C\|U_x\|_1 \|\partial_x^l U\| \|\partial_x^{l+1} V\|, \quad I_{13} \le \|\partial_x^l U\| \|[\partial_x^{l+1}, U]V_x\|. \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.6) and noting (4.1), we arrive at

$$\begin{aligned} I_1 \le & \frac{d}{dt}[-\frac{1}{2}\int_{\mathbb{R}} U(\partial_x^{l+1} V)^2\, dx] + C[N_m(t) + |v_- - v_+|]\|\partial_x^{l+1} V\|^2 \\ & + \frac{1}{2} N_m(t)\|\partial_x^l U\|^2 + \|U\|_m \|[\partial_x^{l+1}, U]V_x\|. \end{aligned} \tag{4.9}$$

For I_2 in (4.5) we apply (3.10) to have

$$I_2 \le r\|\partial_x^l U\| \|\partial_x^l U^2\| \le r\|\partial_x^l U\| \|U^2\|_l \le C\|\partial_x^l U\| \|U\|_m \|U\|_l. \tag{4.10}$$

For I_3, with (3.2) we write

$$I_3 = \int_{\mathbb{R}} \partial_x^l U \partial_x^l F_1\, dx + \int_{\mathbb{R}} \partial_x^l U \partial_x^l F_2\, dx + \int_{\mathbb{R}} \partial_x^l U \partial_x^l F_3\, dx \equiv I_{31} + I_{32} + I_{33}. \tag{4.11}$$

Here applying (3.2) and using the commutator we have

$$\begin{aligned} I_{31} &= -\int_{\mathbb{R}} \partial_x^l U \partial_x^{l+1}[(\bar{u} + u^*)V_x]\, dx \\ &= -\int_{\mathbb{R}} (\partial_x^l U)(\bar{u} + u^*)\partial_x^{l+1} V_x\, dx - \int_{\mathbb{R}} (\partial_x^l U)[\partial_x^{l+1}, \bar{u} + u^*]V_x\, dx \\ &\equiv I_{311} + I_{312}. \end{aligned}$$

By integration by parts and applying (3.1), (2.6), (2.11), (2.12), (2.15) and (2.16) we have

$$\begin{aligned} I_{311} = & -\int_{\mathbb{R}} (\partial_x^{l+1} V_t)(\bar{u} + u^*)\partial_x^{l+1} V\, dx + \int_{\mathbb{R}} (\partial_x^l U)(\bar{u}_x + u_x^*)\partial_x^{l+1} V\, dx \\ = & \frac{d}{dt}[-\frac{1}{2}\int_{\mathbb{R}} (\bar{u} + u^*)(\partial_x^{l+1} V)^2\, dx] + \frac{1}{2}\int_{\mathbb{R}} (\bar{u}_t + u_t^*)(\partial_x^{l+1} V)^2\, dx \\ & + \int_{\mathbb{R}} (\partial_x^l U)(\bar{u}_x + u_x^*)\partial_x^{l+1} V\, dx \\ \le & \frac{d}{dt}[-\frac{1}{2}\int_{\mathbb{R}} (\bar{u} + u^*)(\partial_x^{l+1} V)^2\, dx] + C|v_- - v_+|(\|\partial_x^{l+1} V\|^2 + \|\partial_x^l U\|^2). \end{aligned}$$

It is clear that

$$I_{312} \le \|\partial_x^l U\| \|[\partial_x^{l+1}, \bar{u} + u^*]V_x\|.$$

Therefore,

$$\begin{aligned} I_{31} \le & \frac{d}{dt}[-\frac{1}{2}\int_{\mathbb{R}} (\bar{u} + u^*)(\partial_x^{l+1} V)^2 \, dx] + C|v_- - v_+|(\|\partial_x^{l+1} V\|^2 + \|\partial_x^l U\|^2) \\ & + \|\partial_x^l U\| \|[\partial_x^{l+1}, \bar{u} + u^*]V_x\|. \end{aligned} \tag{4.12}$$

Similarly, the other terms in (4.11) have the following estimates,

$$\begin{aligned} I_{32} = & \int_{\mathbb{R}} \partial_x^l U \partial_x^l F_2 \, dx \\ = & -\int_{\mathbb{R}} \partial_x^l U \partial_x^{l+1}[(\bar{v} + v^*)U] \, dx - 2r \int_{\mathbb{R}} \partial_x^l U \partial_x^l [(\bar{u} + u^*)U] \, dx \\ = & -\int_{\mathbb{R}} (\bar{v} + v^*)[\frac{1}{2}(\partial_x^l U)^2]_x \, dx - \int_{\mathbb{R}} (\partial_x^l U)[\partial_x^{l+1}, \bar{v} + v^*]U \, dx \\ & - 2r \int_{\mathbb{R}} \partial_x^l U \partial_x^l [(\bar{u} + u^*)U] \, dx \\ \le & C|v_- - v_+| \|\partial_x^l U\| \|U\|_l + \|\partial_x^l U\| \|[\partial_x^{l+1}, \bar{v} + v^*]U\|, \end{aligned} \tag{4.13}$$

$$I_{33} \le \|\partial_x^l U\| \|\partial_x^l F_3\| \le C|v_- - v_+|(t+1)^{-\frac{l}{2}-\frac{5}{4}} \|\partial_x^l U\|. \tag{4.14}$$

Here is (4.14) we have used (3.3), (2.4), (2.15) and (2.16) to conclude that

$$|\partial_x^l F_3| \le C|v_- - v_+|(t+1)^{-\frac{l+3}{2}} e^{-\frac{r'(x+x_0)^2}{t+1}} \tag{4.15}$$

for some $r' > 0$. Combining (4.11)-(4.14) we arrive at

$$\begin{aligned} I_3 \le & \frac{d}{dt}[-\frac{1}{2}\int_{\mathbb{R}} (\bar{u} + u^*)(\partial_x^{l+1} V)^2 \, dx] + C|v_- - v_+|(\|\partial_x^{l+1} V\|^2 + \|\partial_x^l U\| \|U\|_l) \\ & + \|\partial_x^l U\|(\|[\partial_x^{l+1}, \bar{u} + u^*]V_x\| + \|[\partial_x^{l+1}, \bar{v} + v^*]U\|) + C|v_- - v_+|(t+1)^{-\frac{l}{2}-\frac{5}{4}} \|\partial_x^l U\|. \end{aligned} \tag{4.16}$$

Now we substitute (4.9), (4.10) and (4.16) into (4.4). This gives us

$$\begin{aligned}&\frac{d}{dt}[\frac{1}{2}\|\partial_x^l U\|^2+\frac{1}{2}\|\partial_x^{l+1} V\|^2+\frac{1}{2}\int_{\mathbb{R}} U(\partial_x^{l+1}V)^2\,dx+\frac{1}{2}\int_{\mathbb{R}}(\bar{u}+u^*)(\partial_x^{l+1}V)^2\,dx]\\&+\|\partial_x^{l+1}U\|^2+r\|\partial_x^l U\|^2\\ \le& C[N_m(t)+|v_--v_+|](\|\partial_x^{l+1}V\|^2+\|\partial_x^l U\|\|U\|_l)\\&+\|U\|_m(\|[\partial_x^{l+1},U]V_x\|+\|[\partial_x^{l+1},\bar{u}+u^*]V_x\|+\|[\partial_x^{l+1},\bar{v}+v^*]U\|)\\&+C|v_--v_+|(t+1)^{-\frac{l}{2}-\frac{5}{4}}\|\partial_x^l U\|.\end{aligned}\tag{4.17}$$

We sum up (4.17) for $0\le l\le m$ and apply (3.11) to have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[\|U\|_m^2+\|V_x\|_m^2+\int_{\mathbb{R}}(U+\bar{u}+u^*)\sum_{l=0}^{m}(\partial_x^{l+1}V)^2\,dx]+\|U_x\|_m^2+r\|U\|_m^2\\ \le& C[N_m(t)+|v_--v_+|](\|V_x\|_m^2+\|U\|_m^2)\\&+C\|U\|_m(\|U_x\|_m\|V_x\|_m+\|\bar{u}_x+u_x^*\|_m\|V_x\|_m+\|\bar{v}_x+v_x^*\|_m\|U\|_m)\\&+C|v_--v_+|(t+1)^{-\frac{5}{4}}\|U\|_m\\ \le& C[N_m(t)+|v_--v_+|](\|V_x\|_m^2+\|U\|_m^2)+C\|U\|_m\|U_x\|_m\|V_x\|_m\\&+C|v_--v_+|(t+1)^{-\frac{5}{4}}\|U\|_m.\end{aligned}\tag{4.18}$$

Integrating (4.18) with respect to time on $[0,t]$ for $0\le t\le T$, we further have

$$\begin{aligned}&[\|U(t)\|_m^2+\|V_x(t)\|_m^2+\int_0^t\|U(\tau)\|_{m+1}^2\,d\tau\\ \le& C\{\|U_0\|_m^2+\|V_0'\|_m^2+\|(U+\bar{u}+u^*)(0)\|_{L^\infty}\|V_0'\|_m^2+\|(U+\bar{u}+u^*)(t)\|_{L^\infty}\|V_x(t)\|_m^2\\&+[N_m(t)+|v_--v_+|]N_m^2(t)+|v_--v_+|\sup_{0\le\tau\le t}\|U(\tau)\|_m\}\\ \le& C\{\|U_0\|_m^2+\|V_0'\|_m^2+[N_m(t)+|v_--v_+|]N_m^2(t)+|v_--v_+|\sup_{0\le\tau\le t}\|U(\tau)\|_m\}.\end{aligned}\tag{4.19}$$

Next, for $0\le l\le m-1$, we multiply (4.3) by $\partial_x^{l+2}V$ and integrate with respect to x. This gives us

$$\|\partial_x^{l+2}V\|^2=I_4+I_5+I_6+I_7+I_8,\tag{4.20}$$

where

$$I_4 = -\int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l U_t \, dx, \quad I_5 = \int_{\mathbb{R}} \partial_x^{l+2} V (\partial_x^{l+2} U - r \partial_x^l U) \, dx,$$

$$I_6 = -\int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^{l+1} (U V_x) \, dx, \quad I_7 = -r \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l U^2 \, dx, \quad I_8 = \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l F \, dx.$$

For I_4 by integration by parts and (3.1),

$$\begin{aligned} I_4 &= \int_{\mathbb{R}} \partial_x^{l+1} V \partial_x^{l+1} U_t \, dx = \frac{d}{dt} \int_{\mathbb{R}} \partial_x^{l+1} V \partial_x^{l+1} U \, dx - \int_{\mathbb{R}} \partial_x^{l+1} V_t \partial_x^{l+1} U \, dx \\ &= \frac{d}{dt} \int_{\mathbb{R}} \partial_x^{l+1} V \partial_x^{l+1} U \, dx + \|\partial_x^{l+1} U\|^2. \end{aligned} \tag{4.21}$$

It is clear that

$$I_5 \leq \|\partial_x^{l+2} V\| (\|\partial_x^{l+2} U\| + r \|\partial_x^l U\|) \leq \frac{c}{2} \|\partial_x^{l+2} V\|^2 + \frac{1}{2c} (\|\partial_x^{l+2} U\| + r \|\partial_x^l U\|)^2, \tag{4.22}$$

where $c > 0$ is a constant to be determined. Similarly, with (3.9) and (3.10) we have

$$\begin{aligned} I_6 &= -\int_{\mathbb{R}} (\partial_x^{l+2} V)^2 U \, dx - \int_{\mathbb{R}} (\partial_x^{l+2} V) [\partial_x^{l+1}, U] V_x \, dx \\ &\leq \sqrt{2} \|U\|_1 \|\partial_x^{l+2} V\|^2 + \|\partial_x^{l+2} V\| \|[\partial_x^{l+1}, U] V_x\|, \end{aligned} \tag{4.23}$$

$$I_7 \leq r \|\partial_x^{l+2} V\| \|\partial_x^l U^2\| \leq C \|\partial_x^{l+2} V\| \|U\|_{m-1} \|U\|_l. \tag{4.24}$$

For I_8 from (3.2) we have

$$I_8 = \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l F_1 \, dx + \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l F_2 \, dx + \int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^l F_3 \, dx \equiv I_{81} + I_{82} + I_{83}. \tag{4.25}$$

Applying (2.15) and (2.16), it is straightforward to have

$$\begin{aligned} I_{81} &= -\int_{\mathbb{R}} \partial_x^{l+2} V \partial_x^{l+1} [(\bar{u} + u^*) V_x] \, dx \leq \|\partial_x^{l+2} V\| \|\partial_x^{l+1} [(\bar{u} + u^*) V_x]\| \\ &\leq C |v_- - v_+| (t+1)^{-\frac{1}{2}} \|\partial_x^{l+2} V\| \|V_x\|_{l+1}, \\ I_{82} &\leq \|\partial_x^{l+2} V\| \{ \|\partial_x^{l+1} [(\bar{v} + v^*) U]\| + 2r \|\partial_x^l [(\bar{u} + u^*) U]\| \} \\ &\leq C \|\partial_x^{l+2} V\| \|\partial_x^{l+1} U\| + C |v_- - v_+| (t+1)^{-\frac{1}{2}} \|\partial_x^{l+2} V\| \|U\|_l \\ &\leq \frac{c}{2} \|\partial_x^{l+2} V\|^2 + \frac{C^2}{2c} \|\partial_x^{l+1} U\|^2 + C |v_- - v_+| (t+1)^{-\frac{1}{2}} \|\partial_x^{l+2} V\| \|U\|_l. \end{aligned}$$

With (4.15) we also have

$$I_{83} \le \|\partial_x^{l+2}V\|\|\partial_x^l F_3\| \le C|v_- - v_+|(t+1)^{-\frac{l}{2}-\frac{5}{4}}\|\partial_x^{l+2}V\|.$$

Substituting these estimates into (4.25) gives us

$$\begin{aligned} I_8 \le& \frac{c}{2}\|\partial_x^{l+2}V\|^2 + C\|\partial_x^{l+1}U\|^2 + C|v_- - v_+|(t+1)^{-\frac{1}{2}}\|\partial_x^{l+2}V\|(\|V_x\|_{l+1} + \|U\|_l) \\ &+ C|v_- - v_+|(t+1)^{-\frac{l}{2}-\frac{5}{4}}\|\partial_x^{l+2}V\|. \end{aligned} \tag{4.26}$$

Now we combine (4.20)-(4.24) and (4.26) to have

$$\begin{aligned} \|\partial_x^{l+2}V\|^2 \le& \frac{d}{dt}\int_{\mathbb{R}} \partial_x^{l+1}V\partial_x^{l+1}U\,dx + C\|U\|_{l+2}^2 + c\|\partial_x^{l+2}V\|^2 + C\|U\|_1\|\partial_x^{l+2}V\|^2 \\ &+ C\|\partial_x^{l+2}V\|(\|[\partial_x^{l+1}, U]V_x\| + \|U\|_{m-1}\|U\|_l) \\ &+ C|v_- - v_+|(t+1)^{-\frac{1}{2}}\|\partial_x^{l+2}V\|(\|V_x\|_{l+1} + \|U\|_l) \\ &+ C|v_- - v_+|(t+1)^{-\frac{l}{2}-\frac{5}{4}}\|\partial_x^{l+2}V\|. \end{aligned} \tag{4.27}$$

We take $c = \frac{1}{2}$ in (4.27), and sum up the inequality for $0 \le l \le m-1$. Applying (3.11) we have

$$\begin{aligned} \frac{1}{2}\|V_{xx}\|_{m-1}^2 \le& \frac{d}{dt}\sum_{l=0}^{m-1}\int_{\mathbb{R}} \partial_x^{l+1}V\partial_x^{l+1}U\,dx + C\|U\|_{m+1}^2 + C\|U\|_1\|V_{xx}\|_{m-1}^2 \\ &+ C\|V_{xx}\|_{m-1}(\|U_x\|_{m-1}\|V_x\|_{m-1} + \|U\|_{m-1}^2) \\ &+ C|v_- - v_+|(t+1)^{-\frac{1}{2}}\|V_{xx}\|_{m-1}(\|V_x\|_m + \|U\|_{m-1}) \\ &+ C|v_- - v_+|(t+1)^{-\frac{5}{4}}\|V_{xx}\|_{m-1}. \end{aligned} \tag{4.28}$$

Integrating with respect to time on $[0, t]$ for $0 \le t \le T$ gives us

$$\begin{aligned} \int_0^t \|V_{xx}(\tau)\|_{m-1}^2\,d\tau \le& 2\sum_{l=0}^{m-1}[\|\partial_x^{l+1}V(t)\|\|\partial_x^{l+1}U(t)\| + \|\partial_x^{l+1}V(0)\|\|\partial_x^{l+1}U(0)\|] \\ &+ C\int_0^t \|U(\tau)\|_{m+1}^2\,d\tau + CN_m^3(t) + C|v_- - v_+|[N_m^2(t) + N_m(t)] \\ \le& \|V_x(t)\|_{m-1}^2 + \|U_x(t)\|_{m-1}^2 + \|V_0'\|_{m-1}^2 + \|U_0'\|_{m-1}^2 \\ &+ C\int_0^t \|U(\tau)\|_{m+1}^2\,d\tau + CN_m^3(t) + C|v_- - v_+|[N_m^2(t) + N_m(t)]. \end{aligned}$$

Substituting (4.19) into the right-hand side we arrive at

$$\int_0^t \|V_{xx}(\tau)\|_{m-1}^2\,d\tau \le C\{\|U_0\|_m^2 + ||V_0'\|_m^2 + [N_m(t) + |v_- - v_+|]N_m^2(t) + |v_- - v_+|N_m(t)\}. \tag{4.29}$$

We still need to find energy estimate for V. From (3.1) we have

$$V_t + U = 0, \tag{4.30}$$

which implies

$$\frac{d}{dt}(\frac{1}{2}\|V\|^2) + \int_{\mathbb{R}} VU\,dx = 0. \tag{4.31}$$

From the second equation in (3.1) we have

$$U = \frac{1}{r}[-U_t - (UV_x)_x - V_{xx} + U_{xx} - rU^2 + F].$$

Thus integration by parts and (4.30) give us

$$\begin{aligned}\int_{\mathbb{R}} VU\,dx =&\frac{1}{r}\int_{\mathbb{R}}[-(VU)_t + V_tU]\,dx + \frac{1}{r}\int_{\mathbb{R}} V_x[UV_x + V_x - U_x]\,dx\\ &-\int_{\mathbb{R}} VU^2\,dx + \frac{1}{r}\int_{\mathbb{R}} VF\,dx\\ =&\frac{d}{dt}[-\frac{1}{r}\int_{\mathbb{R}} VU\,dx] - \frac{1}{r}\|U\|^2 + \frac{1}{r}\|V_x\|^2 + \frac{1}{r}\int_{\mathbb{R}}(UV_x^2 - V_xU_x)\,dx\\ &-\int_{\mathbb{R}} VU^2\,dx + \frac{1}{r}\int_{\mathbb{R}} VF\,dx.\end{aligned} \tag{4.32}$$

Now we substitute (4.32) into (4.31) to have

$$\begin{aligned}&\frac{d}{dt}(\frac{1}{2}\|V\|^2 - \frac{1}{r}\int_{\mathbb{R}} VU\,dx) + \frac{1}{r}\|V_x\|^2\\ =&\frac{1}{r}\|U\|^2 - \frac{1}{r}\int_{\mathbb{R}}(UV_x^2 - V_xU_x)\,dx + \int_{\mathbb{R}} VU^2\,dx - \frac{1}{r}\int_{\mathbb{R}} VF\,dx.\end{aligned} \tag{4.33}$$

For the last term on the right-hand side, we have the following from (3.2), (2.15), (2.16), (4.30), (2.6) and (4.15), and by integration by parts,

$$
\begin{aligned}
-\frac{1}{r}\int_{\mathbb{R}} VF\,dx =& -\frac{1}{r}[\int_{\mathbb{R}} (\bar{u}+u^*)(V_x)^2\,dx + \int_{\mathbb{R}} V_x(\bar{v}+v^*)U\,dx \\
& - 2r\int_{\mathbb{R}} V\bar{u}U\,dx - 2r\int_{\mathbb{R}} Vu^*U\,dx + \int_{\mathbb{R}} VF_3\,dx] \\
\le& -2\int_{\mathbb{R}} \bar{u}VV_t\,dx + C\|V_x\|\|U\| + C|v_- - v_+|\|V_x\|^2 \\
& + C|v_- - v_+|(t+1)^{-1}\|V\|\|U\| + \|V\|\|F_3\| \\
\le& \frac{d}{dt}[-\int_{\mathbb{R}} \bar{u}V^2\,dx] + \frac{1}{4r}\|V_x\|^2 + C\|U\|^2 + C|v_- - v_+|[(t+1)^{-\frac{3}{2}}\|V\|^2 \\
& + \|V_x\|^2 + (t+1)^{-1}\|V\|\|U\| + (t+1)^{-\frac{5}{4}}\|V\|].
\end{aligned} \tag{4.34}
$$

Substituting (4.34) into (4.33), we arrive at

$$
\begin{aligned}
&\frac{d}{dt}(\frac{1}{2}\|V\|^2 - \frac{1}{r}\int_{\mathbb{R}} VU\,dx + \int_{\mathbb{R}} \bar{u}V^2\,dx) + \frac{1}{r}\|V_x\|^2 \\
\le& \frac{1}{r}\|U\|^2 + C(\|U\|_1\|V_x\|^2 + \|U_x\|^2 + \|V\|_1\|U\|^2) + \frac{1}{2r}\|V_x\|^2 + C\|U\|^2 \\
& + C|v_- - v_+|[(t+1)^{-\frac{3}{2}}\|V\|^2 + \|V_x\|^2 + (t+1)^{-1}\|V\|\|U\| + (t+1)^{-\frac{5}{4}}\|V\|].
\end{aligned}
$$

After simplifying, we integrate the above estimate with respect to time on $[0,t]$ for $0 \le t \le T$. Then we have

$$
\begin{aligned}
&\frac{1}{2}\|V(t)\|^2 + \frac{1}{2r}\int_0^t \|V_x(\tau)\|^2\,d\tau \\
\le& C\|V_0\|^2 + \frac{1}{4}\|V(t)\|^2 + \frac{1}{r}\|V_0\|\|U_0\| + C[\|U(t)\|^2 + \int_0^t \|U(\tau)\|_1^2\,d\tau] \\
& + C[|v_- - v_+|N_0^2(t) + N_0^3(t) + |v_- - v_+|N_0(t)].
\end{aligned} \tag{4.35}
$$

Substituting (4.19) into (4.35) and simplifying, we have

$$
\begin{aligned}
&\|V(t)\|^2 + \int_0^t \|V_x(\tau)\|^2\,d\tau \\
\le& C\{\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + [N_m(t) + |v_- - v_+|]N_m^2(t) + |v_- - v_+|N_m(t)\}.
\end{aligned} \tag{4.36}
$$

Finally, combining (4.19), (4.29) and (4.36) gives us

$$\|U(t)\|_m^2 + \|V(t)\|_{m+1}^2 + \int_0^t [\|U(\tau)\|_{m+1}^2 + \|V_x(\tau)\|_m^2]\, d\tau$$
$$\le C\{\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + [N_m(t) + |v_- - v_+|]N_m^2(t) + |v_- - v_+|N_m(t)\}.$$

This implies

$$N_m^2(T) \le C\{\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + [N_m(T) + |v_- - v_+|]N_m^2(T)\} + C|v_- - v_+|N_m(T).$$

Since the last term on the right-hand side is bounded by $\frac{1}{2}N_m^2(T) + C|v_- - v_+|^2$, we further have

$$N_m^2(T) \le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2) + C[N_m(T) + |v_- - v_+|]N_m^2(T).$$

That is,

$$\{1 - C[N_m(T) + |v_- - v_+|]\}N_m^2(T) \le C(\|U_0\|_m^2 + \|V_0\|_{m+1}^2 + |v_- - v_+|^2). \tag{4.37}$$

Now we take positive δ_0 and δ_1 sufficiently small such that

$$C(\delta_0 + \delta_1) \le \frac{1}{2}.$$

Then (4.37) implies (4.2). □

5. Asymptotic behavior of solution

In this section we prove Theorem 2.4, which justifies $(\bar{v}(x + x_0, t) + v^*(x, t), 1 + \bar{u}(x + x_0, t) + u^*(x, t))$ as an asymptotic solution to (1.1)-(1.3). This is done by weighted energy estimate. We continue to use C as a generic positive constant.

For $0 \le l \le m$, we multiply (4.4) in the energy estimate by a weight $(t + 1)$. This gives us

$$\begin{aligned} &\frac{d}{dt}[\frac{1}{2}(t+1)\|\partial_x^l U\|^2 + \frac{1}{2}(t+1)\|\partial_x^{l+1} V\|^2] - \frac{1}{2}\|\partial_x^l U\|^2 - \frac{1}{2}\|\partial_x^{l+1} V\|^2 \\ &+ (t+1)\|\partial_x^{l+1} U\|^2 + r(t+1)\|\partial_x^l U\|^2 = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \tilde{I}_1 &= (t+1)I_1 = -(t+1)\int_{\mathbb{R}} \partial_x^l U \partial_x^{l+1}(UV_x)\, dx, \\ \tilde{I}_2 &= (t+1)I_2 = -r(t+1)\int_{\mathbb{R}} \partial_x^l U \partial_x^l U^2\, dx, \\ \tilde{I}_3 &= (t+1)I_3 = (t+1)\int_{\mathbb{R}} \partial_x^l U \partial_x^l F\, dx. \end{aligned} \tag{5.2}$$

Integrating (5.1) on $[0, t]$ with respect to time and applying (2.23) give us

$$\begin{aligned} &\frac{1}{2}(t+1)[\|\partial_x^l U(t)\|^2 + \|\partial_x^{l+1} V(t)\|^2] + \int_0^t (\tau+1)[\|\partial_x^{l+1} U(\tau)\|^2 + r\|\partial_x^l U(\tau)\|^2]\,d\tau \\ =&\frac{1}{2}[\|\partial_x^l U(0)\|^2 + \|\partial_x^{l+1} V(0)\|^2] + \frac{1}{2}\int_0^t [\|\partial_x^l U(\tau)\|^2 + \|\partial_x^{l+1} V(\tau)\|^2]\,d\tau \\ &+ \int_0^t (\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3)(\tau)\,d\tau \\ \le& C(\|V_0\|_{m+1}^2 + \|u_0 - 1\|_m^2 + |v_- - v_+|^2) + \int_0^t (\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3)(\tau)\,d\tau. \end{aligned} \tag{5.3}$$

From (4.6), (3.9) and (3.11) we have

$$\begin{aligned} \int_0^t \tilde{I}_1(\tau)\,d\tau \le& C\int_0^t (\tau+1)(\|U\|_1\|\partial_x^{l+1} U\| + \|U_x\|_1\|\partial_x^l U\|)(\tau)\|\partial_x^{l+1} V(\tau)\|\,d\tau \\ &+ C\int_0^t (\tau+1)\|\partial_x^l U(\tau)\|\|U_x(\tau)\|_m\|V_x(\tau)\|_l\,d\tau \\ \le& C\sup_{0\le\tau\le t}\|V_x(\tau)\|_l \int_0^t (\tau+1)\|U(\tau)\|_m\|U_x(\tau)\|_m\,d\tau. \end{aligned} \tag{5.4}$$

Similarly, with (3.10), (3.2), (2.15), (2.16) and (4.15) we have

$$\int_0^t \tilde{I}_2(\tau)\,d\tau \le C\sup_{0\le\tau\le t}\|U(\tau)\|_m \int_0^t (\tau+1)\|\partial_x^l U(\tau)\|\|U(\tau)\|_l\,d\tau, \tag{5.5}$$

$$\begin{aligned} \int_0^t \tilde{I}_3(\tau)\,d\tau =& \int_0^t (\tau+1)\int_{\mathbb{R}} \partial_x^{l+1} U \partial_x^l[(\bar{u}+u^*)V_x]\,dxd\tau \\ &- \int_0^t (\tau+1)\int_{\mathbb{R}} (\bar{v}+v^*)\partial_x^l U \partial_x^{l+1} U\,dxd\tau \\ &- \int_0^t (\tau+1)\int_{\mathbb{R}} \partial_x^l U[\partial_x^{l+1}, \bar{v}+v^*]U\,dxd\tau \end{aligned}$$

$$
\begin{aligned}
&- 2r\int_0^t (\tau+1)\int_{\mathbb{R}} \partial_x^l U \partial_x^l[(\bar{u}+u^*)U]\,dxd\tau \\
&+ \int_0^t (\tau+1)\int_{\mathbb{R}} \partial_x^l U \partial_x^l F_3\,dxd\tau \\
\le & \int_0^t (\tau+1)\{\|\partial_x^{l+1}U\|\|\partial_x^l[(\bar{u}+u^*)V_x]\| + \frac{1}{2}\|\bar{v}_x+v_x^*\|_{L^\infty}\|\partial_x^l U\|^2 \\
&+ \|\partial_x^l U\|\|[\partial_x^{l+1},\bar{v}+v^*]U\| + 2r\|\partial_x^l U\|\|\partial_x^l[(\bar{u}+u^*)U]\| + \|\partial_x^l U\|\|\partial_x^l F_3\|\}\,d\tau \\
\le & C\int_0^t (\tau+1)\{\|\partial_x^{l+1}U\|\|\bar{u}+u^*\|_{W^{l,\infty}}\|V_x\|_l + \|\bar{v}_x+v_x^*\|_{L^\infty}\|\partial_x^l U\|^2 \\
&+ \|\partial_x^l U\|\|\bar{v}_x+v_x^*\|_{W^{l,\infty}}\|U\|_l + \|\partial_x^l U\|\|\bar{u}+u^*\|_{W^{l,\infty}}\|U\|_l \\
&+ \|\partial_x^l U\|\|\partial_x^l F_3\|\}(\tau)\,d\tau \\
\le & C|v_- - v_+|\int_0^t (\tau+1)^{\frac{1}{2}}[\|\partial_x^{l+1}U\|\|V_x\|_l + \|\partial_x^l U\|\|U\|_l](\tau)\,d\tau \\
&+ C|v_- - v_+|\int_0^t (\tau+1)^{-\frac{l}{2}-\frac{1}{4}}\|\partial_x^l U(\tau)\|\,d\tau. \qquad (5.6)
\end{aligned}
$$

We substitute (5.4)-(5.6) into (5.3), sum up for $0 \le l \le m$, and apply (2.23). These give us

$$
\begin{aligned}
&(t+1)[\|U(t)\|_m^2 + \|V_x(t)\|_m^2] + \int_0^t (\tau+1)\|U(\tau)\|_{m+1}^2\,d\tau \\
\le & C(\|V_0\|_{m+1}^2 + \|u_0-1\|_m^2 + |v_- - v_+|^2) \\
&+ C[\sup_{0\le\tau\le t}\|V_x(\tau)\|_m + \sup_{0\le\tau\le t}\|U(\tau)\|_m]\int_0^t (\tau+1)\|U(\tau)\|_{m+1}^2\,d\tau \\
&+ C|v_- - v_+|\int_0^t (\tau+1)^{\frac{1}{2}}\|U(\tau)\|_{m+1}[\|V_x(\tau)\|_m + \|U(\tau)\|_m + (\tau+1)^{-\frac{3}{4}}]\,d\tau \\
\le & C(\|V_0\|_{m+1}^2 + \|u_0-1\|_m^2 + |v_- - v_+|^2) \\
&+ C(\|V_0\|_{m+1} + \|u_0-1\|_m + |v_- - v_+|)\int_0^t (\tau+1)\|U(\tau)\|_{m+1}^2\,d\tau
\end{aligned}
$$

$$+\frac{1}{4}\int_0^t (\tau+1)\|U(\tau)\|_{m+1}^2\,d\tau + C|v_- - v_+|^2\int_0^t [\|V_x(\tau)\|_m^2 + \|U(\tau)\|_m^2 + (\tau+1)^{-\frac{3}{2}}]\,d\tau. \tag{5.7}$$

Letting $\varepsilon_0 \le 1/(4C)$ in Theorem 2.3, we simplify (5.7) and apply (2.23) one more time to have

$$\begin{aligned}&(t+1)[\|U(t)\|_m^2 + \|V_x(t)\|_m^2] + \int_0^t (\tau+1)\|U(\tau)\|_{m+1}^2\,d\tau\\ \le& C(\|V_0\|_{m+1}^2 + \|u_0 - 1\|_m^2 + |v_- - v_+|^2).\end{aligned} \tag{5.8}$$

Next we multiply (4.28) by the weight $(t+1)$ and integrate with respect to time on $[0,t]$. Similar to the derivation of (5.8) we have

$$\begin{aligned}&\frac{1}{2}\int_0^t (\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau\\
\le&(t+1)\sum_{l=0}^{m-1}\|\partial_x^{l+1}V(t)\|\|\partial_x^{l+1}U(t)\| + \sum_{l=0}^{m-1}\|\partial_x^{l+1}V(0)\|\|\partial_x^{l+1}U(0)\|\\
&+C\int_0^t \|V_x(\tau)\|_{m-1}\|U_x(\tau)\|_{m-1}\,d\tau\\
&+C\int_0^t (\tau+1)[\|U\|_{m+1}^2 + \|U\|_1\|V_{xx}\|_{m-1}^2 + \|V_{xx}\|_{m-1}(\|U_x\|_{m-1}\|V_x\|_{m-1} + \|U\|_{m-1}^2)\\
&+|v_- - v_+|(\tau+1)^{-\frac{1}{2}}\|V_{xx}\|_{m-1}(\|V_x\|_m + \|U\|_{m-1})]\,d\tau\\
&+C|v_- - v_+|\int_0^t (\tau+1)^{-\frac{1}{4}}\|V_{xx}(\tau)\|_{m-1}\,d\tau\\
\le&\frac{1}{2}(t+1)[\|V_x(t)\|_{m-1}^2 + \|U_x(t)\|_{m-1}^2] + C(\|V_0\|_m^2 + \|U_0\|_m^2)\\
&+C\int_0^t [\|V_x(\tau)\|_{m-1}^2 + \|U_x(\tau)\|_{m-1}^2]\,d\tau + C\int_0^t (\tau+1)\|U(\tau)\|_{m+1}^2\,d\tau\\
&+C[\sup_{0\le\tau\le t}\|U(\tau)\|_1 + \sup_{0\le\tau\le t}\|V_x(\tau)\|_{m-1} + |v_- - v_+|]\int_0^t (\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau\\
&+C\sup_{0\le\tau\le t}\|V_x(\tau)\|_m\int_0^t (\tau+1)\|U(\tau)\|_m^2\,d\tau + C|v_- - v_+|\int_0^t [\|V_x(\tau)\|_m^2 + \|U(\tau)\|_{m-1}^2]\,d\tau\end{aligned}$$

$$+\frac{1}{4}\int_0^t(\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau+C|v_--v_+|^2\int_0^t(\tau+1)^{-\frac{3}{2}}\,d\tau.$$

Applying both (2.23) and (5.8) we have

$$\begin{aligned}\frac{1}{2}\int_0^t(\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau &\le C(\|V_0\|_{m+1}^2+\|u_0-1\|_m^2+|v_--v_+|^2)\\ &+C(\|V_0\|_{m+1}+\|u_0-1\|_m+|v_--v_+|)\int_0^t(\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau\\ &+\frac{1}{4}\int_0^t(\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau.\end{aligned}$$

Simplifying and assuming $\varepsilon_0\le 1/(8C)$, we arrive at

$$\int_0^t(\tau+1)\|V_{xx}(\tau)\|_{m-1}^2\,d\tau\le C(\|V_0\|_{m+1}^2+\|u_0-1\|_m^2+|v_--v_+|^2). \tag{5.9}$$

Estimates of the other terms on the left-hand side of (2.24) are obtained similarly. For $1\le l\le m$ we multiply (4.4) by the weight $(t+1)^2$. Then we integrate the result on $[0,t]$ with respect to time, and apply (5.8) and (5.9). We have

$$\begin{aligned}&\frac{1}{2}(t+1)^2[\|\partial_x^lU(t)\|^2+\|\partial_x^{l+1}V(t)\|^2+\int_0^t(\tau+1)^2(\|\partial_x^{l+1}U\|^2+r\|\partial_x^lU\|^2)(\tau)\,d\tau\\ \le &C(\|V_0\|_{m+1}^2+\|u_0-1\|_m^2+|v_--v_+|^2)+\int_0^t(\tau+1)^2(I_1+I_2+I_3)(\tau)\,d\tau,\end{aligned}\tag{5.10}$$

where I_1, I_2 and I_3 are defined in (4.5).

Similar to (5.4) and with the updated estimate (5.8) we have

$$\begin{aligned}\int_0^t(\tau+1)^2I_1(\tau)\,d\tau &\le C\sup_{0\le\tau\le t}[(\tau+1)^{\frac{1}{2}}\|V_x(\tau)\|_l]\int_0^t(\tau+1)^{\frac{3}{2}}\|U(\tau)\|_m\|U_x(\tau)\|_m\,d\tau\\ &\le C\sup_{0\le\tau\le t}[(\tau+1)^{\frac{1}{2}}\|V_x(\tau)\|_l][\int_0^t(\tau+1)\|U(\tau)\|_m^2\,d\tau+\int_0^t(\tau+1)^2\|U_x(\tau)\|_m^2\,d\tau]\\ &\le C(\|V_0\|_{m+1}^2+\|u_0-1\|_m^2+|v_--v_+|^2)\end{aligned}$$

$$+ C(\|V_0\|_{m+1} + \|u_0 - 1\|_m + |v_- - v_+|) \int_0^t (\tau+1)^2 \|U_x(\tau)\|_m^2 \, d\tau. \tag{5.11}$$

Similar to (5.5) and (5.6) but noting $l \geq 1$ (thus replacing $\|V_x\|_l$ on the right-hand side of (5.6) by $\|Vxx\|_{l-1} + (\tau+1)^{-\frac{1}{2}}\|V_x\|$), with (2.23) and the updated estimates (5.8) and (5.9) we also have

$$\begin{aligned} \int_0^t (\tau+1)^2 I_2(\tau)\, d\tau \leq & C(\|V_0\|_{m+1}^2 + \|u_0 - 1\|_m^2 + |v_- - v_+|^2) \\ & + C(\|V_0\|_{m+1} + \|u_0 - 1\|_m + |v_- - v_+|) \int_0^t (\tau+1)^2 \|U_x(\tau)\|_{m-1}^2 \, d\tau, \end{aligned} \tag{5.12}$$

$$\begin{aligned} \int_0^t (\tau+1)^2 I_3(\tau)\, d\tau \leq & C(\|V_0\|_{m+1}^2 + \|u_0 - 1\|_m^2 + |v_- - v_+|^2) \\ & + C|v_- - v_+| \int_0^t (\tau+1)^2 [\|\partial_x^{l+1} U(\tau)\|^2 + \|\partial_x^l U(\tau)\|^2]\, d\tau \\ & + \frac{r}{2} \int_0^t (\tau+1)^2 \|\partial_x^l U(\tau)\|^2 \, d\tau. \end{aligned} \tag{5.13}$$

We substitute (5.11)-(5.13) into (5.10), sum up for $1 \leq l \leq m$ and simplify. We arrive at

$$\begin{aligned} & (t+1)^2[\|U_x(t)\|_{m-1}^2 + \|V_{xx}(t)\|_{m-1}^2] + \int_0^t (\tau+1)^2 \|U_x(\tau)\|_m^2 \, d\tau \\ \leq & C(\|V_0\|_{m+1}^2 + \|u_0 - 1\|_m^2 + |v_- - v_+|^2) \\ & + C(\|V_0\|_{m+1} + \|u_0 - 1\|_m + |v_- - v_+|) \int_0^t (\tau+1)^2 \|U_x(\tau)\|_m^2 \, d\tau. \end{aligned}$$

Letting $\varepsilon_0 \leq 1/(2C)$ gives us

$$\begin{aligned} & (t+1)^2[\|U_x(t)\|_{m-1}^2 + \|V_{xx}(t)\|_{m-1}^2] + \int_0^t (\tau+1)^2 \|U_x(\tau)\|_m^2 \, d\tau \\ \leq & C(\|V_0\|_{m+1}^2 + \|u_0 - 1\|_m^2 + |v_- - v_+|^2). \end{aligned} \tag{5.14}$$

Now we combine (5.8), (5.9) and (5.14) to have (2.24).

To derive (2.25), which improves the rate of $\|U(t)\|$, we multiply the second equation of (3.1) by U and integrate with respect to x to have

$$\frac{d}{dt}[\frac{1}{2}\|U(t)\|^2]+r\|U(t)\|^2+\|U_x(t)\|^2=-\int_{\mathbb{R}} U[(UV_x)_x+V_{xx}+rU^2-F]dx$$

$$\leq\|U(t)\|\|(UV_x)_x+V_{xx}+rU^2-F\|(t)\leq\frac{r}{2}\|U(t)\|^2+\frac{1}{2r}\|(UV_x)_x+V_{xx}+rU^2-F\|^2(t).$$

This implies

$$\frac{d}{dt}[\frac{1}{2}\|U(t)\|^2]+\frac{r}{2}\|U(t)\|^2\leq\frac{1}{2r}\|(UV_x)_x+V_{xx}+rU^2-F\|^2(t).$$

Applying (5.8), (5.14), (3.9), (3.2), (2.15), (2.16) and (4.15), we bound the right-hand side by

$$C(\|V_0\|^2_{m+1}+\|u_0-1\|^2_m+|v_--v_+|^2)(t+1)^{-2}.$$

By Gronwall inequality,

$$\frac{1}{2}\|U(t)\|^2\leq\frac{1}{2}e^{-rt}\|U_0\|^2+C(\|V_0\|^2_{m+1}+\|u_0-1\|^2_m+|v_--v_+|^2)\int_0^t e^{-r(t-\tau)}(\tau+1)^{-2}d\tau$$

$$\leq C(\|V_0\|^2_{m+1}+\|u_0-1\|^2_m+|v_--v_+|^2)(t+1)^{-2}.$$

We thus obtain (2.25).

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Appendix A. Proof of Proposition 2.2

To prove Proposition 2.2 we cite a result from [17]. The following lemma is a simplified version of Lemma 3.2 therein.

Lemma A.1 *([17]). Let $r'>0$ be a constant. For $x\in\mathbb{R}$, $t\geq 0$, we have*

$$\int_0^t\int_{\mathbb{R}}(t-\tau)^{-1}e^{-\frac{r'(x-y)^2}{t-\tau}}(\tau+1)^{-1}e^{-\frac{r'y^2}{\tau+1}}dyd\tau=O(1)(t+1)^{-\frac{1}{2}}e^{-\frac{r'x^2}{t+1}}. \tag{A.1}$$

Proof. The key of the proof is completing the square for the sum of the two exponents,

$$-\frac{r'(x-y)^2}{t-\tau}-\frac{r'y^2}{\tau+1}=-\frac{r'(t+1)}{(t-\tau)(\tau+1)}[y-\frac{(\tau+1)x}{t+1}]^2-\frac{r'x^2}{t+1}. \tag{A.2}$$

Applying (A.2) to the left-hand side of (A.1) and integrating with respect to y give us

$$\int_0^t \int_{\mathbb{R}} (t-\tau)^{-1} e^{-\frac{r'(x-y)^2}{t-\tau}} (\tau+1)^{-1} e^{-\frac{r'y^2}{\tau+1}}\, dy d\tau = \sqrt{\frac{\pi}{r'}} (t+1)^{-\frac{1}{2}} e^{-\frac{r'x^2}{t+1}} \int_0^t (t-\tau)^{-\frac{1}{2}} (\tau+1)^{-\frac{1}{2}}\, d\tau.$$

We evaluate the time integral on the right-hand side to obtain (A.1). □

We now prove Proposition 2.2. Applying (2.11) and (2.15), for $l \geq 0$ we have

$$\frac{\partial^l}{\partial x^l} R(x,t) = \frac{\partial^l}{\partial x^l} [(\bar{u}_x \bar{v})(x+x_0,t)] = O(1)|v_- - v_+|(t+1)^{-\frac{l+2}{2}} e^{-\frac{r'(x+x_0)^2}{t+1}}. \tag{A.3}$$

Consider $t \geq 1$ first. From (2.14) and by integration by parts we have

$$\frac{\partial^l}{\partial x^l} v^*(x,t) = \int_0^{\frac{t}{2}} \int_{\mathbb{R}} \frac{\partial^{l+1}}{\partial x^{l+1}} [\frac{1}{\sqrt{4\pi r(t-\tau)}} e^{-\frac{r(x-y)^2}{4(t-\tau)}}] R(y,\tau)\, dy d\tau$$
$$+ \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \frac{\partial}{\partial x} [\frac{1}{\sqrt{4\pi r(t-\tau)}} e^{-\frac{r(x-y)^2}{4(t-\tau)}}] \frac{\partial^l}{\partial y^l} R(y,\tau)\, dy d\tau.$$

Applying (A.3) gives us

$$\frac{\partial^l}{\partial x^l} v^*(x,t) = \int_0^{\frac{t}{2}} \int_{\mathbb{R}} O(1)(t-\tau)^{-\frac{l+2}{2}} e^{-\frac{r'(x-y)^2}{t-\tau}} |v_- - v_+|(\tau+1)^{-1} e^{-\frac{r'(y+x_0)^2}{\tau+1}}\, dy d\tau$$
$$+ \int_{\frac{t}{2}}^t \int_{\mathbb{R}} O(1)(t-\tau)^{-1} e^{-\frac{r'(x-y)^2}{t-\tau}} |v_- - v_+|(\tau+1)^{-\frac{l+2}{2}} e^{-\frac{r'(y+x_0)^2}{\tau+1}}\, dy d\tau$$
$$= O(1)|v_- - v_+|(t+1)^{-\frac{l}{2}} \int_0^t \int_{\mathbb{R}} (t-\tau)^{-1} e^{-\frac{r'(x-y)^2}{t-\tau}} (\tau+1)^{-1} e^{-\frac{r'(y+x_0)^2}{\tau+1}}\, dy d\tau.$$

After change of the variable of integration, $y + x_0 \to y$, and applying (A.1), we obtain the first equation in (2.16).

For $0 \leq t \leq 1$, similarly, from (2.14), (A.1) and (A.3), and by integration by parts, we have

$$\frac{\partial^l}{\partial x^l} v^*(x,t) = \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x} [\frac{1}{\sqrt{4\pi r(t-\tau)}} e^{-\frac{r(x-y)^2}{4(t-\tau)}}] \frac{\partial^l}{\partial y^l} R(y,\tau)\, dy d\tau$$
$$= O(1)|v_- - v_+| \int_0^t \int_{\mathbb{R}} (t-\tau)^{-1} e^{-\frac{r'(x-y)^2}{t-\tau}} (\tau+1)^{-1} e^{-\frac{r'(y+x_0)^2}{\tau+1}}\, dy d\tau$$
$$= O(1)|v_- - v_+|(t+1)^{-\frac{1}{2}} e^{-\frac{r'(x+x_0)^2}{t+1}} = O(1)|v_- - v_+|(t+1)^{-\frac{l+1}{2}} e^{-\frac{r'(x+x_0)^2}{t+1}},$$

noting $(t+1)^{\frac{l}{2}} = O(1)$. This also gives us the first equation in (2.16).

The second equation in (2.16) is obtained by observing $u^* = -\frac{1}{r}v^* - \frac{1}{r}R(x,t)$ in (2.12).

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