



Vertex Operators, Solvable Lattice Models and Metaplectic Whittaker Functions

Ben Brubaker¹ , Valentin Buciumas², Daniel Bump³,
Henrik P. A. Gustafsson^{3,4,5,6}

¹ School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA.
E-mail: brubaker@math.umn.edu

² School of Mathematics and Physics, The University of Queensland, St. Lucia, QLD 4072, Australia.
E-mail: valentin.buciumas@gmail.com

³ Department of Mathematics, Stanford University, Stanford, CA 94305-2125, USA.
E-mail: bump@math.stanford.edu

⁴ Present address: School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA.

⁵ Present address: Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA.

⁶ Present address: Department of Mathematical Sciences, University of Gothenburg and Chalmers University of Technology, 412 96 Gothenburg, Sweden. E-mail: gustafsson@ias.edu

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Abstract: We show that spherical Whittaker functions on an n -fold cover of the general linear group arise naturally from the quantum Fock space representation of $U_q(\widehat{\mathfrak{sl}}(n))$ introduced by Kashiwara, Miwa and Stern (KMS). We arrive at this connection by reconsidering the solvable lattice models known as “metaplectic ice” whose partition functions are metaplectic Whittaker functions. First, we show that a certain Hecke action on metaplectic Whittaker coinvariants agrees (up to twisting) with a Hecke action of Ginzburg, Reshetikhin, and Vasserot arising in quantum affine Schur-Weyl duality. This allows us to expand the framework of KMS by Drinfeld twisting to introduce Gauss sums into the quantum wedge, which are necessary for connections to metaplectic forms. Our main theorem interprets the row transfer matrices of this ice model as “half” vertex operators on quantum Fock space that intertwine with the action of $U_q(\widehat{\mathfrak{sl}}(n))$. In the process, we introduce new symmetric functions termed *metaplectic symmetric functions* and explain how they are related to Whittaker functions on an n -fold metaplectic cover of GL_r . These resemble *LLT polynomials* or *ribbon symmetric functions* introduced by Lascoux, Leclerc and Thibon, and in fact the metaplectic symmetric functions are (up to twisting) specializations of *supersymmetric LLT polynomials* defined by Lam. Indeed Lam constructed families of symmetric functions from Heisenberg algebra actions on the Fock space commuting with the $U_q(\widehat{\mathfrak{sl}}(n))$ -action. The Heisenberg algebra is independent of Drinfeld twisting of the quantum group. We explain that half vertex operators agree with Lam’s construction and this interpretation allows for many new identities for metaplectic symmetric and Whittaker functions, including Cauchy identities. While both metaplectic symmetric functions and LLT polynomials can be related to vertex operators on the quantum Fock space, only metaplectic symmetric functions are connected to solvable lattice models.

1. Introduction

This paper concerns two mechanisms by which the quantum groups $U_q(\hat{\mathfrak{g}})$, for $\mathfrak{g}$ a simple Lie algebra or superalgebra, produce families of special functions with a suite of interesting properties including functional equations, branching rules and unexpected algebraic relations. The first mechanism uses solvable lattice models associated to finite-dimensional modules of $U_q(\hat{\mathfrak{g}})$. The second mechanism uses actions of Heisenberg and Clifford algebras on a fermionic Fock space, as in the boson-fermion correspondence [27, 31, 45] with connections to soliton theory. We will use these two points of view to provide new insight into the theory of metaplectic Whittaker functions for the general linear group and relate them to LLT polynomials. To begin, we explain these two approaches to special functions from quantum affine groups in more detail.

If V is a finite-dimensional module of $\mathfrak{g}$, then since $U_q(\hat{\mathfrak{g}})$ is the quantization of a central extension of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, we obtain a family of *evaluation modules* V_z ($z \in \mathbb{C}^\times$) in which t is specialized to the value z . Using quasitriangularity, we have $U_q(\hat{\mathfrak{g}})$ -homomorphisms (almost always isomorphisms) $V_{z_1} \otimes V_{z_2} \longrightarrow V_{z_2} \otimes V_{z_1}$ dictated by an R -matrix $R(z_1, z_2)$ satisfying

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2), \quad (1.1)$$

in $\text{End}(V_{z_1} \otimes V_{z_2} \otimes V_{z_3})$. This identity is called the *parametrized (quantum) Yang-Baxter equation* with parameter group $\mathbb{C}^\times$. These are endomorphisms of $V_{z_1} \otimes V_{z_2} \otimes V_{z_3}$ and the subscripts R_{ij} mean that the matrix R is applied to the i -th and j -th component of the threefold tensor product.

Given any such matrix R , we may ask for a matrix $T(z)$ satisfying the “RTT” relation:

$$R(z_1, z_2)T(z_1)T(z_2) = T(z_2)T(z_1)R(z_1, z_2). \quad (1.2)$$

Typically, the matrix $T(z)$ arises as an endomorphism of $V_z \otimes W$ where W is a fixed object in the category of $U_q(\hat{\mathfrak{g}})$ -modules. If $W = V_{z_3}$ and $T(z) = R(z, z_3)$ then (1.2) is equivalent to (1.1). For arbitrary W , the existence of a $T(z) \in \text{End}(V_z, W)$ making (1.2) true follows from quasitriangularity.

Solutions $T(z)$ to (1.2) may also arise as a “row transfer matrix” in a solvable lattice model. For example in the case of the field-free six-vertex model, Baxter [2] demonstrates that the resulting partition function is a symmetric function in the z_i when its Boltzmann weights satisfy the Yang-Baxter equation (1.1). The underlying algebra was explained by Kulish and Reshetikhin [36], Sklyanin [47], Drinfeld [13] and Jimbo [25] and the relevant quantum group associated to the R -matrix is $U_q(\widehat{\mathfrak{sl}}_2)$. To connect to the presentation of the Yang-Baxter equation in the previous paragraph, each edge in the planar lattice model is associated to a two-dimensional evaluation module V_z and the local Boltzmann weights encode endomorphisms among them.

In [5], the first three authors considered examples of solvable square lattice models connected to R -matrices of evaluation modules for $U_q(\widehat{\mathfrak{gl}}(n|1))$. In these examples (Theorem 1 in [5]), the matrices $T(z)$ in (1.2) do not quite fit the standard paradigm. Each vertex in the square lattice receives a Boltzmann weight reflecting the action of $T(z)$ on basis elements determined by adjacent edges; while the horizontal edges may be identified with evaluation modules for $U_q(\widehat{\mathfrak{gl}}(n|1))$, the vertical edges represent a two-dimensional vector space with no known algebraic connection to this quantum group. The problem is that we are not aware of any candidate for a two dimensional module M of $U_q(\widehat{\mathfrak{gl}}(n|1))$ that would explain the matrix $T(z)$. In other words, we would like there to exist an M such that the R -matrix for $V_z \otimes M$ is the matrix for a set of Boltzmann

weights used in this paper. See Table 1 in Section 3. If no such two-dimensional module exists, then we have an example of a parametrized Yang-Baxter equation that is not explained by quasitriangularity. This is an important unresolved question.

Nevertheless in [5] the partition function of the model is shown to be solvable and equal the spherical Whittaker function on an n -fold metaplectic cover of the general linear group; this will be our primary example of the sort of special functions mentioned at the outset.

As we will explain in the present paper, an alternate algebraic interpretation is possible if we take $T(z)$ to be the row transfer matrix of an infinite grid; then a module explaining $T(z)$ does appear, and it is the quantum fermionic Fock space defined by Kashiwara, Miwa and Stern [32]. Thus instead of trying to interpret the vertically oriented edges (which can have only two states $\pm$) as 2-dimensional modules in the category, there is an alternative approach – one that takes us from the solvable lattice model point of view to the Heisenberg algebra point of view, our second mechanism for producing special functions. In this approach, an infinite sequence of vertical edges in a fixed row of our square lattice model parametrizes a vector in the *fermionic Fock space* $\mathfrak{F}$. The row transfer matrix for the model then becomes an operator $T(z) : \mathfrak{F} \rightarrow \mathfrak{F}$ with $z \in \mathbb{C}^\times$ a fixed parameter. The Yang-Baxter equation implies that the operators $T(z_i)$ and $T(z_j)$ commute for any i and j .

In these examples, the space $\mathfrak{F}$ is not the usual fermionic Fock space described (for example) in [31]. Instead it is the quantum Fock space $\mathfrak{F} = \mathfrak{F}_q^{(n)}$ of [32], which is a module for $U_q(\widehat{\mathfrak{sl}}_n)$. It will be a consequence of our main theorem that the operators $T(z)$ are $U_q(\widehat{\mathfrak{sl}}_n)$ -module homomorphisms. It also gives a proof, independent of the Yang-Baxter equation, that the operators $T(z)$ commute. Thus our method here succeeds in providing a quantum group interpretations to these problematic vertical edges in the metaplectic ice model.

We may picture the Fock space $\mathfrak{F}$ as follows. Similar to the way Dirac described the electron sea, consider a quantum particle with an infinite number of states, one for each energy level, and a system of such particles obeying the Pauli exclusion principle where the lowest energy levels are all occupied and the highest levels are unoccupied. Thus if u_i represents the particle in a state with energy i , then a basis of $\mathfrak{F}$ consists of vectors

$$u_{\mathbf{i}} := u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \quad (1.3)$$

where $\mathbf{i} = (i_m, i_{m-1}, \dots)$ is a strictly decreasing sequence such that $i_k = k$ for $k \ll 0$. Here $i_m, i_{m-1}, \dots$ are the energy levels of occupied states; we may arrange that $i_m > i_{m-1} > \dots$. The condition that $i_k = k$ for $k \ll 0$ ensures that all sufficiently low energy levels are occupied. The totality of such states for fixed m is the level m space $\mathfrak{F}_m$ and $\mathfrak{F} = \bigoplus_m \mathfrak{F}_m$.

If m is given, we may parametrize the semi-infinite monomials (1.3) by partitions: if $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition, then we may take $i_m = m + \lambda_1$, $i_{m-1} = m - 1 + \lambda_2$, etc. This gives a bijection between partitions and basis vectors of $\mathfrak{F}_m$. Thus we write

$$|\lambda\rangle = |\lambda; m\rangle := u_{m+\lambda_1} \wedge u_{m-1+\lambda_2} \wedge \cdots \quad (1.4)$$

In Section 2 we review and generalize the construction of the quantum Fock space of [32]. In Theorem 2.5 we relate the Hecke action that underlies this construction (due to Ginzburg, Reshetikhin and Vasserot [20]) to another Hecke action, which was motivated by the action in [6] on Whittaker coinvariants. Because of this we are able to easily build an action of the Hecke algebra modified by a Drinfeld twist. This generalization allows

us to introduce Gauss sums to the anticommutation rule for vectors in the Fock space. This twisting is needed for the application to metaplectic Whittaker functions, but is more general than what is needed for this application and so may be of importance for other purposes.

To connect the quantum Fock space to solvable lattice models, we introduce a grid, infinite in width, whose boundary edges encode vectors in the Fock space. The column edges of the solvable lattice model in [5] are likewise indexed by partitions, so may be viewed as semi-infinite wedge products according to the above correspondence. This point of view will be detailed further in Section 3.

Let us now explain our main theorem which considers two solvable lattice models connected to $U_q(\widehat{\mathfrak{gl}}(n|1))$ R -matrices called *Gamma ice* and *Delta ice* detailed in Section 4, and their row transfer matrices $T_\Gamma(z)$ and $T_\Delta(z)$.

In addition to being a $U_q(\widehat{\mathfrak{sl}}_n)$ -module, $\mathfrak{F}$ is a module for a Heisenberg Lie algebra, spanned by “current” operators J_k , and by a central vector 1. The operator J_k (denoted B_k in [32], and defined in (2.25)) shifts one fermion to a different level by changing its energy from i to $i - kn$. The operators J_k with $k > 0$ are thus right-moving operators, and those with $k < 0$ are left-moving. They satisfy $[J_k, J_l] = 0$ unless $k = -l$.

Introduce the operators $H_+(z)$ and $H_-(z)$ defined by

$$H_\pm(z) := \sum_{k=1}^{\infty} \frac{1}{k} (1 - v^k) z^{\pm nk} J_{\pm k}. \quad (1.5)$$

Our main theorem, which will be proved in Section 4, is:

Theorem A. *The operators $e^{H_+(z)}$ and $e^{H_-(z)}$ equal the row transfer matrices of Gamma and Delta ice:*

$$e^{H_+(z)} = T_\Delta(z), \quad e^{H_-(z)} = T_\Gamma(z). \quad (1.6)$$

Operators such as these occur in conformal field theory, and also other areas of mathematics such as soliton theory, “monstrous moonshine” and the abstract boson-fermion correspondence. Generally, we will call an operator of the form

$$\exp(H_+[a](z)), \quad H_+[a](z) = \sum_{k=1}^{\infty} a_k J_k z^k \quad (1.7)$$

or

$$\exp(H_-[b](z)), \quad H_-[b](z) = \sum_{k=1}^{\infty} b_{-k} J_{-k} z^{-k} \quad (1.8)$$

a *half-vertex operator*. We must be careful with $H_-[b]$, since $H_-[b](z)|\lambda\rangle$ is an infinite sum and not in $\mathfrak{F}_m$. Nevertheless the sum $\langle \mu | H_-[b](z) | \lambda \rangle$ is finite and therefore such expressions make sense; in fact just $\langle \mu | H_-[b](z) \rangle$ is a finite sum. (Here we use the usual Dirac notation for operators on $\mathfrak{F}_m$. If $H : \mathfrak{F}_m \longrightarrow \mathfrak{F}_m$ is an operator, we will denote by $\langle \mu | H | \lambda \rangle$ the inner product of $H|\lambda\rangle$ with $|\mu\rangle$.)

Operators of the form $\exp(H_-[b](z)) \cdot \exp(H_+[a](z))$ appear in mathematical physics. See for example [18, 33] Part II in Volume I or [27] (1.15). Subject to a locality assumption ([16, 30]), they are called *vertex operators*. In this paper we will deal mainly with half-vertex operators. Yet there are representation theory contexts in which Gamma ice and Delta ice occur together ([7, 8, 10, 21, 24]) leading to vertex operators as above. In Section 7 we show that the locality properties of such operators fit into the algebraic framework of Frenkel and Reshetikhin [17].

As mentioned above, the method of Baxter [2] based on the Yang-Baxter equation produces families of commuting row-transfer matrices. That is, $T_\Delta(z_1)T_\Delta(z_2) = T_\Delta(z_2)T_\Delta(z_1)$, and similarly for $T_\Gamma(z)$. On the other hand, the commutativity also follows from the identity (1.6), because J_k and J_l commute if k and l have the same sign. Note however that T_Δ does not commute with T_Γ . In Theorem 7.3 we compute precisely a scalar $C(z, w)$ such that $T_\Delta(z)T_\Gamma(w) = C(z, w)T_\Gamma(w)T_\Delta(z)$, and this calculation is essential to our discussion of locality.

In the paragraphs above we have described relationships between quantum groups, solvable lattice models, and Heisenberg algebras acting on a Fock space $\mathfrak{F}$. Using these we will make two connections to existing literature. First, it is shown in [5, 7] that the Boltzmann weights that we use in this paper can be used in finite systems whose partition functions are Whittaker functions on the n -fold metaplectic covers of GL_r over a local field. It is striking that for these, the relevant quantum group is $U_q(\widehat{\mathfrak{gl}}_n)$ or its relatives $U_q(\widehat{\mathfrak{gl}}(n|1))$ or $U_q(\widehat{\mathfrak{sl}}_n)$. The relationship between the degree n of the cover and the rank of the quantum group was very unexpected. For the application to metaplectic Whittaker functions, the quantum group must be modified by Drinfeld twisting in order to introduce Gauss sums into the comultiplication of $U_q(\widehat{\mathfrak{sl}}_n)$, and consequently into the R -matrix and quantum wedge relations in $\mathfrak{F}$.

Although the metaplectic Whittaker functions are not symmetric in the Langlands parameters $\mathbf{z} = (z_1, \dots, z_r)$, when we switch to the infinite grids and the Fock space $\mathfrak{F}$, we find expressions such as

$$\mathcal{M}_{\lambda/\mu}^n(\mathbf{z}) = \langle \mu | T_\Delta(z_1) \cdots T_\Delta(z_r) | \lambda \rangle = \left\langle \mu \left| \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} (1 - v^k) p_{nk}(\mathbf{z}) J_k \right) \right| \lambda \right\rangle, \quad (1.9)$$

where $p_{nk}(\mathbf{z}) = \sum_i z_i^{nk}$ is the power-sum symmetric function. (We use the notation $\mathcal{M}_\lambda^n$ if μ is the empty partition.) By Theorem A, $\mathcal{M}_{\lambda/\mu}^n$ can be interpreted as a partition function very similar to the metaplectic Whittaker functions. But unlike Whittaker functions, these polynomials are symmetric. We will call them *metaplectic symmetric functions*. In Theorem 6.3 we will show how metaplectic Whittaker functions (which are not symmetric) can be expressed in terms of the new metaplectic symmetric functions.

Thus we will show that the solvable models of [5, 7] admit an interpretation in terms of a Heisenberg algebra commuting with a $U_q(\widehat{\mathfrak{sl}}_n)$ action on Fock space. The case when $n = 1$, which reduces to the Shintani-Casselmann-Shalika formula for the general linear group (or Tokuyama's formula), was treated in Brubaker and Schultz [11]. In that case, values of Whittaker functions are Schur polynomials, and so recovers a result expressing Schur polynomials as partition functions of free-fermionic six-vertex models [9, 23, 52, 53].

This brings us to the second connection to existing literature. The quantum Fock space has in prior results [37, 38, 42] been applied in the theory of LLT polynomials, also known as *ribbon symmetric functions*. These are q -deformations of products of n Schur functions. If n is large, they become Hall-Littlewood polynomials. They are a reflection of the plethysm with power-sum symmetric functions (Adams operations) and are connected with algorithms in the (modular) representation theory of symmetric groups. They have reappeared in other contexts such as Schur positivity and affine Schubert calculus.

Lam [38] formalized a generalized boson-fermion correspondence that includes these examples and others such as the LLT polynomials. The bosonic Fock space $\mathfrak{B}$ may be identified with the ring Λ of symmetric polynomials and (over $\mathbb{Q}$) the power-sum

symmetric functions p_k generate. They give rise to a representation of the Heisenberg Lie algebra on $\mathfrak{B}$ in which multiplication by, or differentiation with respect to, the p_k correspond to the operators J_k on the fermionic Fock space. (See also [27, 31, 45].) Lam explained how to construct symmetric functions from any such Heisenberg algebra action and reinterpreted results of [42] to put LLT polynomials into this framework.

As we demonstrate in Section 5, Lam's symmetric function construction is equivalent to action by half-vertex operators. Thus LLT polynomials may be expressed in the form

$$\mathcal{G}_{\lambda/\mu}^n(\mathbf{z}^n) = \left\langle \mu \left| \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} p_{nk}(\mathbf{z}) J_k \right) \right| \lambda \right\rangle. \quad (1.10)$$

This is very similar to the metaplectic symmetric functions, and indeed we will show that the metaplectic symmetric functions are specializations of super LLT polynomials, presented in Definition 29 of [37]. One might suspect from this that LLT polynomials might likewise be expressible as partition functions of the solvable lattice models from [5] with boundary conditions determined by the pair of partitions λ and μ ; in fact this is not possible. It is only this very particular specialization of the super LLT polynomial that results in an appropriate cancellation of terms and permits the resulting function to be expressed using our solvable models.

Moreover in [38], Lam shows that these families of symmetric functions constructed from Heisenberg algebras satisfy a large collection of interesting identities, including Cauchy and Pieri identities. Thus, as a consequence of the main theorem, we are now able to use these same tools to prove analogous identities for metaplectic symmetric functions. As proof of concept, we prove a Cauchy identity for the new metaplectic symmetric functions. (See Theorem 5.10.) In the non-metaplectic setting, such Cauchy identities for Schur functions found application in the Rankin-Selberg method.

It seems an important question to find other theories that connect the two mechanisms of solvable lattice models and vertex operators. The well-known relationship between the Heisenberg spin-chain Hamiltonians and the field-free six and eight vertex models may be one example. (See Baxter [1].) Another place to look for an analog of our Main Theorem is in the theory of Hall-Littlewood polynomials. Thus in Jing [28, 29] a quantum boson-fermion correspondence is described, where the commuting actions of a Heisenberg Lie algebra with a quantum group is used to study Hall-Littlewood polynomials in the context of vertex operators. But on the other hand Korff [35], taking a point of view surprisingly close to ours, develops a theory of Hall-Littlewood polynomials using lattice models based on Boltzmann weights that connect with a q -deformed *bosonic* Fock space. Borodin and Wheeler [4] and Wheeler and Zinn-Justin [51] contain further developments of this viewpoint.

As noted above, the results described above concern mainly half-vertex operators, which have expansions in terms of the positive or negative Heisenberg generators J_k . However, it is also interesting to consider operators that involve both the positive and negative generators. Because Gamma ice and Delta ice occur together in several different contexts, it is natural to consider “fields” such as $V(z) = T_{\Gamma}(z)T_{\Delta}(z)$. We will look at these in Section 7, in particular investigating locality properties of the field $V(z)$. It is outside the scope of this paper to fully realize our operators in the language of vertex algebras, but it seems likely that this can be done using the framework of quantum vertex algebras [3, 15, 17] and we intend to revisit this in a subsequent paper. Additional future directions may include generalizations of our construction of solvable lattice models to other Cartan types, perhaps using the abstract Fock space built in the work of Lanini, Ram and Sobaje [40, 41].

2. The Fermionic Fock Space

This section reviews the definition of the fermionic Fock space following Kashiwara, Miwa and Stern [32]. As they showed, this is a module for the affine quantum group $U_q(\widehat{\mathfrak{sl}}_n)$. However we will require greater generality by giving the Fock space the structure of a module over a Drinfeld twist of this quantum group. Thus while we follow [32] very closely, sometimes we add some details to make clear the differences between working with $U_q(\widehat{\mathfrak{sl}}_n)$ or its Drinfeld twist. Theorem 2.5 appears to be new and it is a key ingredient that allows us to deduce the action of the affine Hecke algebra on the Drinfeld twist of the Fock space.

2.1. The quantum group. Let n be a positive integer, and let q be either a formal parameter or a *generic* complex number (i.e., not a root of unity). All the indices in the relations in this paper involving elements of the quantum group should be read modulo n .

We introduce the quantum group $U_q(\widehat{\mathfrak{sl}}_n)$ which acts on the fermionic Fock space, focusing on the quasitriangular bialgebra structure (it is also a Hopf algebra, but we will not be using the antipode anywhere). Let $[m]_q$ be the quantum integer associated to the integer m defined by

$$[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}.$$

Let $A = (a_{ij})_{0 \leq i, j \leq n-1}$ be the Cartan matrix of affine type $\widehat{A}_{n-1}$. Its non-zero entries are $a_{ii} = 2$ and $a_{ij} = -1$ when $i = j \pm 1$ for $n \geq 3$ (where we recall that the indices should be read modulo n). For $n = 2$ the second equality in the definition of the Cartan matrix is replaced by $a_{ij} = -2$.

The quantum group $U_q(\widehat{\mathfrak{sl}}_n)$ is the unital algebra generated by elements $E_i, F_i, K_i^{\pm}$ for $0 \leq i \leq n-1$, subject to the following relations (when $n \geq 3$ or $n = 1$):

$$\begin{aligned} K_i K_j &= K_j K_i, \quad K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i, \\ E_i E_j &= E_j E_i, \quad F_i F_j = F_j F_i \quad \text{if } i \neq j \pm 1, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 &= 0, \\ F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 &= 0. \end{aligned} \tag{2.1}$$

In the case $n = 2$, the last two relations are replaced by the following relations:

$$\begin{aligned} E_i^3 E_{i\pm 1} - [3]_q E_i^2 E_{i\pm 1} E_i + [3]_q E_i E_{i\pm 1} E_i^2 - E_{i\pm 1} E_i^3 &= 0, \\ F_i^3 F_{i\pm 1} - [3]_q F_i^2 F_{i\pm 1} F_i + [3]_q F_i F_{i\pm 1} F_i^2 - F_{i\pm 1} F_i^3 &= 0. \end{aligned} \tag{2.2}$$

The subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by $E_i, F_i, K_i^{\pm}$ for $1 \leq i \leq n-1$ is the finite quantum group $U_q(\mathfrak{sl}_n)$.

Remark 2.1. In the case $n = 1$, $U_q(\widehat{\mathfrak{sl}}_1)$ is the algebra generated by $K_0^{\pm 1}$. We will show that our method produces interesting six-vertex models even starting from this “trivial” quantum group.

Remark 2.2. The quantum group we denote by $U_q(\widehat{\mathfrak{sl}}_n)$ is denoted by $U'_q(\widehat{\mathfrak{sl}}_n)$ in [32]. Our quantum group does not contain a derivation d .

The comultiplication Δ on $U_q(\widehat{\mathfrak{sl}}_n)$ is defined as follows:

$$\begin{aligned}\Delta(K_i) &= K_i \otimes K_i, \\ \Delta(E_i) &= 1 \otimes E_i + E_i \otimes K_i, \\ \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i.\end{aligned}\tag{2.3}$$

Let V_n be an n -dimensional vector space with basis $\{v_1, \dots, v_n\}$. The natural module of $U_q(\widehat{\mathfrak{sl}}_n)$, which we denote by $V_n(z)$, is the vector space $V_n \otimes \mathbb{C}[z, z^{-1}]$ with basis $\{z^k v_i\}$ for $1 \leq i \leq n, k \in \mathbb{Z}$. Another useful basis is $\{u_j\}$ with $j \in \mathbb{Z}$ satisfying the relations

$$u_{j-kn} = z^k v_j, \tag{2.4}$$

for $1 \leq j \leq n, k \in \mathbb{Z}$. The action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $V_n(z)$ is as follows:

$$\begin{aligned}K_i z^k v_j &= q^{\delta_{i,j} - \delta_{i+1,j}} z^k v_j, \\ E_i z^k v_j &= \delta_{i,j-1} z^{k-\delta_{i,0}} v_{j-1}, \\ F_i z^k v_j &= \delta_{i,j} z^{k-\delta_{i,0}} v_{j+1}.\end{aligned}\tag{2.5}$$

There is a natural ordering on the basis $\{z^k v_j\}$:

$$\dots > z^{k-1} v_2 > z^{k-1} v_1 > z^k v_n > z^k v_{n-1} > \dots \tag{2.6}$$

Note that in the $\{u_j\}$ -basis, the ordering is just $u_{j+1} > u_j$.

There is an action of $U_q(\widehat{\mathfrak{sl}}_n)$ on tensor powers of the natural module $V_n(z)^{\otimes N}$ given by iterations of the comultiplication. An affine version of Schur-Weyl duality was studied in [20, 50], where it is shown that the centralizer of the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $V_n(z)^{\otimes N}$ is the Hecke algebra $\widehat{H}_N(v)$, for $v = q^2$. In [22], the affine quantum Schur algebra is introduced and a double centralizer property is proved (though note that the definition of the affine quantum group in [22] is slightly different from our definition).

2.2. The affine Hecke algebra. The (type A) affine Hecke algebra $\widehat{H}_N(q^2) =: \widehat{H}_N$ is the associative algebra with generators T_i for $1 \leq i \leq N-1$ and $y_j^\pm$ for $1 \leq j \leq N$ subject to the following relations:

$$\begin{aligned}T_i^2 &= (q^2 - 1)T_i + q^2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \\ T_i T_j &= T_j T_i \quad \text{if } |i - j| > 1, \\ y_i y_j &= y_j y_i, \\ y_j T_i &= T_i y_j \quad \text{if } i \neq j, j+1, \\ T_i y_i T_i &= q^2 y_{i+1}.\end{aligned}\tag{2.7}$$

The first relation in the definition of the Hecke algebra can be rewritten as $(T_i + 1)(T_i - q^2) = 0$, which allows one to decompose any space on which T_i acts into eigenspaces corresponding to its two eigenvalues: q^2 and -1 .

We denote by S_N the symmetric group on N strands. For $\sigma \in S_N$, let $\sigma = s_{i_1} \cdots s_{i_l}$ be a minimal length expression, where $s_i \in S_N$ are the simple permutations. It is then well known that the definition

$$T_\sigma := T_{s_{i_1}} \cdots T_{s_{i_l}} \quad (2.8)$$

is independent of which minimal length expression of σ we choose and that the set $\{T_\sigma, \sigma \in S_N\}$ is a basis for the *finite Hecke algebra* $H_N \subset \widehat{H}_N$, which by definition is the algebra generated by $T_1, \dots, T_{N-1}$.

We denote $(V_n(z))^{\otimes N} = V_n(z_1) \otimes V_n(z_2) \otimes \cdots \otimes V_n(z_N)$ to distinguish between the indeterminates corresponding to different copies of $V_n(z)$. The space $V_n(z)^{\otimes N}$ has a basis

$$v_{\mathbf{j}} \otimes \mathbf{z} := v_{j_1} \otimes \cdots \otimes v_{j_N} \otimes z_1^{k_1} \cdots z_N^{k_N} \quad (2.9)$$

where $\mathbf{j} = (j_1, \dots, j_N)$ and $\mathbf{z}$ is shorthand for $z_1^{k_1} \cdots z_N^{k_N}$, $k_i \in \mathbb{Z}$. The symmetric group S_N acts on all elements of the form $v_{\mathbf{j}}$ by permutation; it also acts on all elements of the form $\mathbf{z}$ as follows: $s_i : \cdots z_i^{k_i} z_{i+1}^{k_{i+1}} \cdots \mapsto \cdots z_{i+1}^{k_i} z_i^{k_{i+1}} \cdots$.

Remark 2.3. The notation $\mathbf{z}$ has a different meaning in this section than in the introduction. In this section (following [32]) $\mathbf{z}$ is defined by (2.9).

There is a right action of the Hecke algebra $\widehat{H}_N$ on the tensor product $V_n(z)^{\otimes N}$ which was first written down in [20]:

$$(v_{\mathbf{j}} \otimes \mathbf{z}) \cdot T_i = \begin{cases} (1 - q^2) v_{\mathbf{j}} \otimes \frac{z_{i+1} \mathbf{z}^{s_i} - z_i \mathbf{z}}{z_i - z_{i+1}} - q v_{s_i(\mathbf{j})} \otimes \mathbf{z}^{s_i} & \text{if } j_i < j_{i+1}, \\ (1 - q^2) v_{\mathbf{j}} \otimes \frac{z_i (\mathbf{z}^i - \mathbf{z})}{z_i - z_{i+1}} - v_{s_i(\mathbf{j})} \otimes \mathbf{z}^{s_i} & \text{if } j_i = j_{i+1}, \\ (1 - q^2) v_{\mathbf{j}} \otimes \frac{z_i (\mathbf{z}^i - \mathbf{z})}{z_i - z_{i+1}} - q v_{s_i(\mathbf{j})} \otimes \mathbf{z}^{s_i} & \text{if } j_i > j_{i+1}, \end{cases} \quad (2.10)$$

$$(v_{\mathbf{j}} \otimes \mathbf{z}) \cdot y_i = (v_{\mathbf{j}} \otimes \mathbf{z} \cdot z_i^{-1}).$$

A crucial fact in defining the quantum Fock space is the property that the right action of $\widehat{H}_N$ and the left action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $V_n(z)^{\otimes N}$ commute.

Let V_x , $x \in \mathbb{C}^\times$ be the *evaluation module* of $U_q(\widehat{\mathfrak{sl}}_n)$. It is the quotient of the natural module by the submodule spanned by elements $v_i z^{k+1} - x v_i z^k$. It is called the evaluation module because we “evaluate” the indeterminate z at $x \in \mathbb{C}^\times$. In [6, Section 3], the first three authors and Friedberg give examples of representations of the affine Hecke algebra on evaluation modules of quantum groups with applications to the study of metaplectic Whittaker functions. There is a “natural” lifting of the action in [6] to an action of $\widehat{H}_N$ on $V_n(z)^{\otimes N}$ which involves the affine R -matrix.

The quantum group $U_q(\widehat{\mathfrak{sl}}_n)$ is quasitriangular; this means there is an element living in (a completion of) $U_q(\widehat{\mathfrak{sl}}_n) \otimes U_q(\widehat{\mathfrak{sl}}_n)$ called the universal R -matrix, which we denote by $\mathcal{R}$, satisfying certain well-known properties. See Proposition 4.1 in [19] for a formula of the universal R -matrix of $U_q(\widehat{\mathfrak{sl}}_n)$. The action of $\mathcal{R}$ on $V_n(z)^{\otimes 2} = V_n(z_i) \otimes V_n(z_{i+1})$ is given by the affine R -matrix $R(z_i, z_{i+1}) \in \text{End}(V_n \otimes V_n) \otimes \mathbb{C}[z_i, z_{i+1}]$ defined as:

$$\begin{aligned} \tau R(z_i, z_{i+1}) = & \sum_i (q z_i - q^{-1} z_{i+1}) e_{ii} \otimes e_{ii} + \sum_{i \neq j} (z_i - z_{i+1}) e_{ji} \otimes e_{ij} \\ & + \sum_{i > j} (q - q^{-1}) z_i e_{jj} \otimes e_{ii} + (q - q^{-1}) z_{i+1} e_{ii} \otimes e_{jj}, \end{aligned} \quad (2.11)$$

where $e_{ij} \in \text{End}(V_n \otimes V_n)$ are the maps $e_{ij} : v_j \mapsto \delta_{ij} v_i$ and $\tau \in \text{End}(V_n \otimes V_n)$ is the flip map $\tau : v_i \otimes v_j \mapsto v_j \otimes v_i$. See the unnumbered equation between equations 30

and 31 in [19] and the preceding discussion for an explanation of the fact that the action of $\mathcal{R}$ on $V_n(z_1) \otimes V_n(z_2)$ is $R(z_1, z_2)$. Denote by $R(z) := R(1, z)$.

Remark 2.4. If we replace the indeterminates z_i and z_{i+1} in $R(z_i, z_{i+1})$ by complex numbers x_i and x_{i+1} , we obtain the affine (type A) R -matrix for evaluation modules discovered by Jimbo [26] before the work of Frenkel and Reshetikhin [19].

The natural version of the evaluation action of the affine Hecke algebra given in [6], Theorem 3.3 reads as follows:

$$\begin{aligned} (v_{\mathbf{j}} \otimes \mathbf{z}) \cdot T_i &= (q^2 - 1) \frac{z_i}{z_i - z_{i+1}} v_{\mathbf{j}} \otimes \mathbf{z} - \frac{q}{z_i - z_{i+1}} (\tau R_q)_{i,i+1}(z_i, z_{i+1}) v_{\mathbf{j}} \otimes \mathbf{z}^{s_i}, \\ (v_{\mathbf{j}} \otimes \mathbf{z}) \cdot y_i &= (v_{\mathbf{j}} \otimes \mathbf{z} \cdot z_i^{-1}). \end{aligned} \quad (2.12)$$

Theorem 2.5. *The actions of the affine Hecke algebra in equations (2.10) and (2.12) agree.*

Proof. This follows by the following computation:

$$\begin{aligned} ((v_{\mathbf{j}} \otimes \mathbf{z}) \cdot T_i)_{\text{eq. (2.10)}} &= \begin{cases} (1 - q^2) v_{\mathbf{j}} \otimes \frac{z_{i+1} \mathbf{z}^{s_i} - z_i \mathbf{z}}{z_i - z_{i+1}} - q v_{s_i(\mathbf{j})} \otimes \mathbf{z}^{s_i} & \text{if } j_i < j_{i+1} \\ (1 - q^2) v_{\mathbf{j}} \otimes \frac{z_i (\mathbf{z}^{s_i} - \mathbf{z})}{z_i - z_{i+1}} - v_{s_i(\mathbf{j})} \otimes \mathbf{z}^{s_i} & \text{if } j_i = j_{i+1} \\ (1 - q^2) v_{\mathbf{j}} \otimes \frac{z_i (\mathbf{z}^{s_i} - \mathbf{z})}{z_i - z_{i+1}} - q v_{s_i(\mathbf{j})} \otimes \mathbf{z}^{s_i} & \text{if } j_i > j_{i+1} \end{cases} \\ &= (q^2 - 1) \frac{z_i}{z_i - z_{i+1}} v_{\mathbf{j}} \\ &\quad \otimes \mathbf{z} - \frac{1}{z_i - z_{i+1}} \begin{cases} ((q^2 - 1) v_{\mathbf{j}} z_{i+1} + q(z_i - z_{i+1}) v_{s_i(\mathbf{j})}) \otimes \mathbf{z}^{s_i} & \text{if } j_i < j_{i+1} \\ ((q^2 - 1) v_{\mathbf{j}} z_i + (z_i - z_{i+1}) v_{s_i(\mathbf{j})}) \otimes \mathbf{z}^{s_i} & \text{if } j_i = j_{i+1} \\ ((q^2 - 1) v_{\mathbf{j}} z_i + q(z_i - z_{i+1}) v_{s_i(\mathbf{j})}) \otimes \mathbf{z}^{s_i} & \text{if } j_i > j_{i+1} \end{cases} \\ &= (q^2 - 1) \frac{z_i}{z_i - z_{i+1}} v_{\mathbf{j}} \\ &\quad \otimes \mathbf{z} - \frac{1}{z_i - z_{i+1}} \begin{cases} ((q^2 - 1) v_{\mathbf{j}} z_{i+1} + q(z_i - z_{i+1}) v_{s_i(\mathbf{j})}) \otimes \mathbf{z}^{s_i} & \text{if } j_i < j_{i+1} \\ ((q^2 - 1) v_{\mathbf{j}} z_i + (z_i - z_{i+1}) v_{s_i(\mathbf{j})}) \otimes \mathbf{z}^{s_i} & \text{if } j_i = j_{i+1} \\ ((q^2 - 1) v_{\mathbf{j}} z_i + q(z_i - z_{i+1}) v_{s_i(\mathbf{j})}) \otimes \mathbf{z}^{s_i} & \text{if } j_i > j_{i+1} \end{cases} \\ &= (q^2 - 1) \frac{z_i}{z_i - z_{i+1}} v_{\mathbf{j}} \\ &\quad \otimes \mathbf{z} - \frac{q}{z_i - z_{i+1}} (\tau R_q)_{i,i+1}(z_i, z_{i+1}) v_{\mathbf{j}} \otimes \mathbf{z}^{s_i} = ((v_{\mathbf{j}} \otimes \mathbf{z}) \cdot T_i)_{\text{eq. (2.12)}}. \end{aligned}$$

□

The importance of Theorem 2.5 is twofold. First it clarifies the relation between the two actions of the affine Hecke algebra which were discovered in different contexts. Secondly, it gives us a way to rewrite the action in [32], which is instrumental in the construction of the Fock space representation, in terms of the affine R -matrix. In the next sections we use a Drinfeld twist of the R -matrix to write down a different action of the affine Hecke algebra on $V_{\mathcal{R}}(z)^{\otimes N}$ (which commutes with the action of a Drinfeld twist of the quantum group $U_q(\mathfrak{sl}_n)$). This allows us to define the Drinfeld twist of the quantum Fock space.

2.3. Drinfeld twisting. The Drinfeld twist [14] is a deformation of the Hopf algebra structure of a quantum group that changes the comultiplication, antipode and the universal R -matrix, but leaves the multiplication, unit and counit intact. Drinfeld twisting produces new solutions of the Yang-Baxter equation.

Reshetikhin [46] proved that given a quantum group H and an element $F \in H \otimes H$ of the form $F = \sum_i f^i \otimes f_i$ satisfying certain properties ([46, Section 1]), one can define a Drinfeld twist of H , denoted H^F , with a new comultiplication and universal R -matrix given by

$$\begin{aligned}\Delta^F(a) &= F\Delta(a)F^{-1} \\ \mathcal{R}^F &= F_{21}\mathcal{R}F^{-1}\end{aligned}\tag{2.13}$$

where $F_{21} = \sum_i f_i \otimes f^i$. He then shows, in [46] Section 2, that

$$F = \exp\left(\sum_{1 \leq i < j \leq n} a_{ij}(H_i \otimes H_j - H_j \otimes H_i)\right)\tag{2.14}$$

satisfies the relations needed to produce a Drinfeld twist of $U_q(\mathfrak{sl}_n)$, where $K_i = q^{H_i}$ and $a_{ij} \in \mathbb{C}$.

Let $U_q^F(\widehat{\mathfrak{sl}}_n)$ be the quantum group obtained by applying a Drinfeld twist on $U_q(\widehat{\mathfrak{sl}}_n)$ using the element $F \in U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{sl}_n) \subset U_q(\widehat{\mathfrak{sl}}_n) \otimes U_q(\widehat{\mathfrak{sl}}_n)$ defined in equation (2.14). Its comultiplication and universal R -matrix will be given by equation (2.13). The twisted quantum group $U_q^F(\widehat{\mathfrak{sl}}_n)$ is the same as $U_q(\widehat{\mathfrak{sl}}_n)$ as algebras, however the coproduct, universal R -matrix and antipode are different. The twisted quantum group $U_q^F(\widehat{\mathfrak{sl}}_n)$ also has a natural module $V_n(z)$. (We will abuse notation, but it should be clear throughout the paper when $V_n(z)$ is the natural module of the twisted or untwisted quantum group.) Since the twisted and untwisted quantum groups are the same as algebras, the action $U_q^F(\widehat{\mathfrak{sl}}_n)$ on $V_n(z)$ is the same as the action given in (2.5). However, since the comultiplication is different, the action of the twisted and untwisted quantum groups on $V_n(z)^{\otimes N}$ for $N > 1$ will be different.

Remark 2.6. The quantum group in [46] is defined over $\mathbb{C}[[h]]$ as opposed to being defined over $\mathbb{C}(q)$ as in our case. It follows that F defined in (2.14) does not live in $U_q(\widehat{\mathfrak{sl}}_n) \otimes U_q(\widehat{\mathfrak{sl}}_n)$, but in a certain completion of the tensor product. Similarly, the universal R -matrix $\mathcal{R}$ also lives in a completion of $U_q(\widehat{\mathfrak{sl}}_n) \otimes U_q(\widehat{\mathfrak{sl}}_n)$. These facts will not be problematic for our purposes.

Recall the definition of u_i from equation (2.4) and denote the tensor product $u_i \otimes u_j$ by u_{ij} .

Lemma 2.7. *The elements F (defined in equation (2.14)) and F_{21} act on $V_n(z)^{\otimes 2}$ as follows:*

$$\begin{aligned}F : u_i \otimes u_j &\mapsto \sqrt{\alpha_{ij}}u_i \otimes u_j, \\ F_{21} : u_i \otimes u_j &\mapsto \sqrt{\alpha_{ji}}u_i \otimes u_j,\end{aligned}\tag{2.15}$$

where $\alpha_{ii} = 1$ and α_{ij} when $i \neq j$ is given by

$$\alpha_{ij} = \exp(2a_{i,j} - 2a_{i-1,j} - 2a_{i,j-1} + 2a_{i-1,j-1})\tag{2.16}$$

Proof. This follows from noting that $H_i : u_j \mapsto (\delta_{i,j} - \delta_{i,j-1})u_j$. $\square$

Proposition 2.8. *The affine R -matrix $R^F(z_i, z_{i+1})$ corresponding to $U_q^F(\widehat{\mathfrak{sl}}_n)$ is*

$$\begin{aligned} \tau R^F(z_i, z_{i+1}) = & \sum_i (qz_i - q^{-1}z_{i+1})e_{ii} \otimes e_{ii} + \sum_{i \neq j} \alpha_{ij}(z_i - z_{i+1})e_{ji} \otimes e_{ij} \\ & + \sum_{i > j} (q - q^{-1})z_i e_{jj} \otimes e_{ii} + (q - q^{-1})z_{i+1} e_{ii} \otimes e_{jj}. \end{aligned} \quad (2.17)$$

Proof. Note that $R^F(z_i, z_{i+1})$ is the action of the universal R -matrix $\mathcal{R}^F$ on the representation $V_n(z)^{\otimes 2}$. The result follows immediately after using the action of F^{-1} and F_{21} on $V_n(z)^{\otimes 2}$ from Lemma 2.7. $\square$

Given $U_q^F(\widehat{\mathfrak{sl}}_n)$ with R -matrix $R^F(z_i, z_{i+1})$ as in Proposition 2.8 which depends on complex numbers a_{ij} , denote by α the set of numbers α_{ij} , $1 \leq i, j \leq n$ obtained from a_{ij} using equation (2.16). From now on we will write the dependence of the Drinfeld twisting in terms of α instead of F (so we write $U_q^\alpha(\widehat{\mathfrak{sl}}_n)$ instead of $U_q^F(\widehat{\mathfrak{sl}}_n)$ and $R^\alpha(z_i, z_{i+1})$ instead of $R^F(z_i, z_{i+1})$). Even though there are different choices of F that produce the same set α , we will not distinguish between such quantum groups. For our purposes, Drinfeld twists by different F 's with the same α 's will correspond to the same six-vertex models in future sections.

One should keep in mind that for $\alpha_{ij} = 1$, $U_q^\alpha(\widehat{\mathfrak{sl}}_n)$ is the non-twisted quantum groups $U_q(\widehat{\mathfrak{sl}}_n)$ and that $\alpha_{ij}\alpha_{ji} = 1 = \alpha_{ii}$ for any α . A standard, though tedious, computation shows:

Proposition 2.9. *There is an action of the Hecke algebra $\widehat{H}_N$ on $V_n(z)^{\otimes N}$ where y_i acts by multiplication with z_i^{-1} and*

$$(v_j \otimes \mathbf{z}) \cdot T_i = (q^2 - 1) \frac{z_i}{z_i - z_{i+1}} v_j \otimes \mathbf{z} - \frac{q}{z_i - z_{i+1}} (\tau R_q^\alpha)_{i,i+1}(z_i, z_{i+1}) v_j \otimes \mathbf{z}^{s_i}. \quad (2.18)$$

Equation (2.18) can be rewritten, via the same process as in the proof of Theorem 2.5, as

$$(v_j \otimes \mathbf{z}) \cdot T_i = \begin{cases} (1 - q^2) v_j \otimes \frac{z_{i+1} \mathbf{z}^{s_i} - z_i \mathbf{z}}{z_i - z_{i+1}} - q \alpha_{j_i j_{i+1}} v_{s_i(j)} \otimes \mathbf{z}^{s_i} & \text{if } j_i < j_{i+1} \\ (1 - q^2) v_j \otimes \frac{z_i (\mathbf{z}^{s_i} - \mathbf{z})}{z_i - z_{i+1}} - v_{s_i(j)} \otimes \mathbf{z}^{s_i} & \text{if } j_i = j_{i+1} \\ (1 - q^2) v_j \otimes \frac{z_i (\mathbf{z}^{s_i} - \mathbf{z})}{z_i - z_{i+1}} - q \alpha_{j_i j_{i+1}} v_{s_i(j)} \otimes \mathbf{z}^{s_i} & \text{if } j_i > j_{i+1} \end{cases}. \quad (2.19)$$

Proposition 2.10. *The action of T_i in equation (2.19) is an $U_q^\alpha(\widehat{\mathfrak{sl}}_n)$ -module homomorphism.*

Proof. Note that $a \in U_q^\alpha(\widehat{\mathfrak{sl}}_n)$ acts on $V_n(z)^{\otimes 2}$ via $\Delta^F(a) = F\Delta(z)F^{-1}$ and using the action of F and F^{-1} on $V_n(z)^{\otimes 2}$ from equation (2.15), the proof becomes a routine calculation.

A non-computational proof goes as follows: by equation (2.18), the action of T_i on $V_n(z)^{\otimes N}$ is a linear combination of the identity map and $(\tau R)_{i,i+1}(z_i, z_{i+1})$, both of which are $U_q^\alpha(\widehat{\mathfrak{sl}}_n)$ -module homomorphisms. $\square$

It follows that the right action of $\widehat{H}_N$ from equation (2.19) (which depends on α) and the left action of $U_q^\alpha(\widehat{\mathfrak{sl}}_n)$ on $V_n(z)^{\otimes N}$ commute.

2.4. The quantum wedge. We now define the exterior product of $V_n(z)$ following [32].

Define the q -antisymmetrizing operator $A^{(N)}$ acting on $V_n(z)^{\otimes N}$ to be

$$A^{(N)} = \sum_{\sigma \in S_N} T_\sigma$$

where T_σ was defined in (2.8).

Proposition 2.11. *The $U_q(\widehat{\mathfrak{sl}}_n)$ -module $V_n(z)^{\otimes N}$ decomposes as*

$$V_n(z)^{\otimes N} = \text{Im } A^{(N)} \oplus \text{Ker } A^{(N)}$$

and the space $\text{Ker } A^{(N)}$ is the sum of the kernels of the operators $T_i + 1$ for $1 \leq i \leq N - 1$.

Proof. See Propositions 1.1 and 1.2 in [32]. Their proof goes through unchanged even for the new action of $\widehat{H}_N$ on $V_n(z)^{\otimes N}$ from equation (2.19). $\square$

In order to understand the spaces $\text{Ker}(T_i + 1)$ which determine $\text{Ker}(A^{(N)})$, take $N = 2$ and $T := T_1$.

Given integers m and l , let $k_1, k_2 \in \mathbb{Z}$ and $1 \leq j_1, j_2 \leq n$ be such that $l = j_1 - k_1 n$ and $m = j_2 - k_2 n$ so that $u_l = v_{j_1} z^{k_1}$ and $u_m = v_{j_2} z^{k_2}$. For such integers m and l , define $\alpha_{lm} := \alpha_{ij}$. Then the following elements in $V_n(z) \otimes V_n(z)$ are in $\text{Ker}(T + 1)$:

$$\begin{aligned} & u_l \otimes u_m + u_m \otimes u_l && \text{if } l \equiv m \pmod{n} \\ & u_l \otimes u_m + q\alpha_{l,m}u_m \otimes u_l + \\ & \quad + u_{m-i} \otimes u_{l+i} + q\alpha_{m-i,l+i}u_{l+i} \otimes u_{m-i} && \text{if } m - l \equiv i \pmod{n} \text{ and } 0 < i < n. \end{aligned} \quad (2.20)$$

Note that $\alpha_{l,m} = \alpha_{m-i,l+i}$ when $m - l \equiv i \pmod{n}$ and $0 < i < n$.

Define the quantum wedge $\Lambda^2 V_n(z)$ to be the quotient $V_n(z)^{\otimes 2} / \text{Ker } A^{(2)}$ and denote by $u_l \wedge u_m$ the image of $u_l \otimes u_m$ in $\Lambda^2 V_n(z)$. It is easy to see from equation (2.20) that the following relation holds in $\Lambda^2 V_n(z)$ when $m = l \pmod{n}$:

$$u_l \wedge u_m = -u_m \wedge u_l. \quad (2.21)$$

If m, l are integers such that $m > l$ and $m - l \equiv i \pmod{n}$, then consider the following sequence of ordered elements taken out of equation (2.6):

$$\cdots > u_m > u_{m-i} > u_{m-n} > u_{m-n-i} > \cdots > u_{l+n+i} > u_{l+n} > u_{l+i} > u_l \cdots \quad (2.22)$$

We say a wedge $u_l \wedge u_m$ is *normal-ordered* if $l > m$, so that $u_l > u_m$ in the order given by equation (2.6). For $u_m > u_l$, the following relation holds in $\Lambda^2 V_n(z)$:

$$\begin{aligned} u_l \wedge u_m = & -q\alpha_{l,m}u_m \wedge u_l + (q^2 - 1)(u_{m-i} \wedge u_{l+i} - q\alpha_{l,m}u_{m-n} \wedge u_{l+n} + \\ & q^2 u_{m-n-i} \wedge u_{l+n+i} - q^3 \alpha_{m,l}u_{m-2n} \wedge u_{l+2n} \cdots) \end{aligned} \quad (2.23)$$

where the sum on the right uses entries in the sequence (2.22) and continues as long as we get normal-ordered wedges. Here i is the unique value with $0 < i < n$ and $m - i \equiv l \pmod{n}$. This fact follows by applying the second line in equation (2.20) repeatedly until we obtain a formula for $u_l \wedge u_m$ in terms of normal-ordered wedges only.

In the special case when $\alpha_{ij} = 1$ we get back equation (45) in [32]. The specialization we need may be described as follows. Let g be a function of integers modulo n that satisfies the following Assumption.

Assumption 2.12. Let v denote q^2 . The function g satisfies $g(0) = -v$, and if a is not congruent to 0 modulo n , then $g(a)g(-a) = v$.

Now let us take $\alpha_{ij} = -q^{-1}g(i-j)$ when $i \neq j$ (we always want $\alpha_{ii} = 1$.) We obtain the formula, valid if $m > l$:

$$u_l \wedge u_m = \begin{cases} -u_m \wedge u_l & \text{if } l \equiv m \pmod{n}, \\ g(l-m)u_m \wedge u_l + (q^2-1)(u_{m-i} \wedge u_{l+i} + g(l-m)u_{m-n} \wedge u_{l+n} \\ \quad + q^2u_{m-n-i} \wedge u_{l+n+i} + q^2g(l-m)u_{m-2n} \wedge u_{l+2n} + \cdots) & \text{otherwise.} \end{cases} \quad (2.24)$$

As with (2.23), i is the unique value with $0 < i < n$ and $m-i \equiv l$ modulo n . And as with (2.23) the summation continues as long as the terms are of the form $u_a \wedge u_b$ with $a > b$; this is a finite sum.

Let $\Lambda^N V_n(z)$ be the quotient $V_n(z)^{\otimes N} / \text{Ker}(A^{(N)})$. The definition of a normal-ordered wedge extends to $\Lambda^N V_n(z)$. By identical arguments to the one in Proposition 1.3 of [32], one can show:

Proposition 2.13. $\Lambda^N V_n(z)$ is the quotient of $V_n(z)^{\otimes N}$ by the relations (2.21) and (2.23) in each pair of adjacent factors; the elements

$$u_{m_1} \wedge \cdots \wedge u_{m_N}$$

where $m_1 > m_2 > \cdots > m_N$, form a basis for $\Lambda^N V_n(z)$.

Remark 2.14. Note that for $n = 1$, $m-l$ is always congruent to 0 mod n . Therefore the quantum wedge is defined only using relation (2.21). In this case the definition of the quantum wedge is the same as the definition of the classical ($q = 1$) wedge for $\widehat{\mathfrak{sl}}_m$ for all m .

2.5. The fermionic Fock space. Let S_∞ be the infinite symmetric group generated by simple reflections s_i , $i \in \mathbb{N}$. Let $\widehat{H}_\infty$ be the infinite affine Hecke algebra, with generators T_i , $y_i^\pm$, $i = 1, 2, 3, \dots$ subject to the relations (2.7). It acts on $V_n(z) \otimes V_n(z) \otimes V_n(z) \cdots$ via (2.19); the action is well-defined because each T_i acts only on a pair of adjacent factors.

Let $\mathfrak{U}_m$ be the linear span of vectors of the form

$$u_{i_m} \otimes u_{i_{m-1}} \otimes u_{i_{m-2}} \otimes \cdots$$

such that $i_k = k$ for $k \ll 0$. The Fock space of level m is denoted by $\mathfrak{F}_m$; it is the quotient of $\mathfrak{U}_m$ by the space $\sum_i \text{Ker}(T_i + 1)$, or equivalently, by the relations (2.21) and (2.23) in each pair of adjacent factors.

There is a “formal” action of the quantum group $U_q^\alpha(\widehat{\mathfrak{sl}}_n)$ on the space $\mathfrak{U}_m$ via the coproduct (2.3) which descends to genuine action on $\mathfrak{F}_m$. A basis of $\mathfrak{F}_m$ is given by elements of the form

$$u_{i_m} \wedge u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \cdots$$

where $i_m > i_{m-1} > \cdots$ and $i_k = k$ for $k \ll 0$. Define

$$|m\rangle := u_m \wedge u_{m-1} \wedge u_{m-2} \wedge \cdots \in \mathfrak{F}_m,$$

which we call the vacuum in $\mathfrak{F}_m$. The Fock space $\mathfrak{F}$ is defined as

$$\mathfrak{F} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{F}_m.$$

Remark 2.15. We caution the reader that due to the “correction terms” in (2.24) there may be unexpected terms in many calculations. For example, if $n = 2$, (2.24) shows that

$$u_1 \wedge u_4 \wedge u_1 \wedge u_0 \wedge \cdots = g(-3)u_3 \wedge u_2 \wedge u_1 \wedge u_0 \wedge \cdots.$$

In the usual wedge, the left-hand side would be zero due to the repeated factor u_1 ; however we see that this is not true in the quantum Fock space.

Now let us introduce operators J_k on $\mathfrak{F}$. These operators are $U_q(\widehat{\mathfrak{sl}}_n)$ -module endomorphisms that are denoted B_k in [32]. Let $u_{i_m} \wedge u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \cdots \in \mathfrak{F}_m$ and for a non-zero $k \in \mathbb{Z}$ define the displacement operator $J_k : \mathfrak{F}_m \rightarrow \mathfrak{F}_m$ by

$$\begin{aligned} J_k(u_{i_m} \wedge u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \cdots) &= (u_{i_m-nk} \wedge u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \cdots) \\ &+ (u_{i_m} \wedge u_{i_{m-2-nk}} \wedge u_{i_{m-2}} \wedge \cdots) + \cdots. \end{aligned} \quad (2.25)$$

That this is indeed an action on the quantum Fock space consistent with the quantum wedge resulting in a finite sum of wedges is shown in [32, Lemma 2.1]. For $u_i \wedge \eta \in \mathfrak{F}_m$ we note that

$$J_k(u_i \wedge \eta) = u_{i-nk} \wedge \eta + u_i \wedge J_k(\eta). \quad (2.26)$$

The following commutation relation holds:

$$[J_k, J_l] = k \frac{1 - v^{n|k|}}{1 - v^{|k|}} \delta_{k,-l}. \quad (2.27)$$

This is Proposition 2.6 in [32]. This commutator is not affected by the Drinfeld twisting.

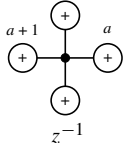
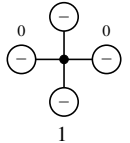
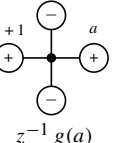
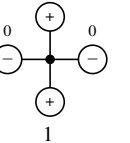
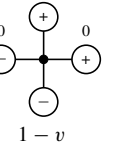
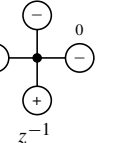
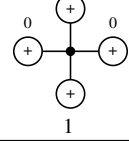
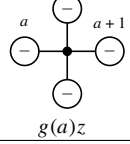
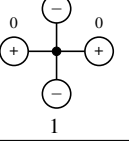
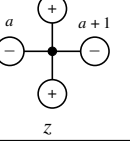
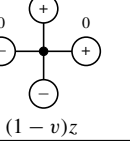
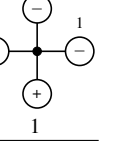
3. The Main Theorem

We recall two types of solvable lattice models called Gamma and Delta ice. These first appeared in [5] in the context of metaplectic Whittaker functions, but as we will exhibit later, they have surprising connections to symmetric functions beyond this particular application.

Let us begin with a planar grid having a finite number r of rows. The grid may either have finitely many or infinitely many columns. We will number the rows $1, \dots, r$; for Delta ice, the row numbers increase from the bottom up, and for Gamma ice, they increase from the top down. We will also number the columns by integers, in decreasing order. The column numbers may be all integers in the case of infinitely many columns or a finite interval, say $0, 1, 2, \dots, N$, in the case of finitely many columns. We will fix nonzero complex numbers $z_1, \dots, z_r$ and associate z_i to the row numbered i . There are *vertices* at every intersection of a row and column, and four *edges* adjacent to each vertex as in Table 1. A *boundary edge* is an edge that is adjacent to a single vertex.

A *state* of Gamma or Delta ice is given by the assignment of a *spin* $\pm$ to each edge of the grid with certain restrictions. To each horizontally oriented edge, we will also associate a *charge* which will be an integer a modulo n . The combination of the spin

Table 1. The Boltzmann weights for Γ and Δ vertices associated to a row parameter $z \in \mathbb{C}^\times$. The charge a above an edge indicates any choice of charge mod n and gives the indicated weight. The weights depend on a parameter v and any function g with $g(0) = -v$ and $g(n-a)g(a) = v$ if $a \not\equiv 0 \pmod n$. If a configuration does not appear in this table, its weight is zero. We take $z = z_i$ in the i -th row (from the top for Gamma ice, or from the bottom for Delta ice). For Gamma ice, the Boltzmann weights used in [5] and [7] are multiplied by z . This change from those papers only multiplies the partition function by a constant power of $z_1 \cdots z_r$.

	a ₁	a ₂	b ₁	b ₂	c ₁	c ₂
Γ -ice						
Δ -ice						

and charge will be called a *decorated spin* and will be denoted $\pm^a$. For Delta ice, we only allow the spin $+^a$ when a is 0 modulo n ; for Gamma ice, we only allow $-^a$ when a is 0 modulo n . Thus in either cases, there are $n+1$ allowed decorated spins.

For the boundary edges, the spins and (for horizontal edges, the charges) will be fixed. Their specification, together with a set of Boltzmann weights associated to each vertex according to 1, will define what we call the *system*. In this section, we will consider systems of infinite width, whose columns are labeled by all integers. In Section 6 we will consider finite systems.

Thus let us describe the boundary conditions when the grid is infinite. The boundary edges are all therefore vertically oriented. Let us fix an integer m and consider two strictly decreasing sequences of integers,

$$\mathbf{i} = (i_m, i_{m-1}, \dots), \quad \mathbf{j} = (j_m, j_{m-1}, \dots) \quad (3.1)$$

such that $i_k = j_k = k$ if $0 \gg k$. The associated boundary spins along the top edge are $-$ for the edges in columns $i_m, i_{m-1}, \dots$ and $+$ for the edges in columns $i_m, i_{m-1}, \dots$. We similarly fix the spins along the bottom boundary to be $-$ in columns $j_m, j_{m-1}, \dots$ and $+$ in the others. With these data we may associate the following vectors in $\mathfrak{F}_m$:

$$\xi = u_{\mathbf{i}} = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \quad \eta = u_{\mathbf{j}} = u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots.$$

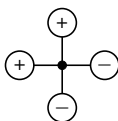
A state of this infinite system thus requires assigning spins to the internal vertical edges and decorated spins for the horizontal internal ones. For Delta ice (resp. Gamma ice), we require that all but finitely many horizontal edges have spins $+^0$ (resp. $-^0$). Regardless of whether the grid is finite or infinite, a state $\mathfrak{s}$ of the system will be called *admissible* if the configuration of spins at the adjacent edges of every vertex is one of the configurations in a fixed row of Table 1. Let $\mathfrak{S}$ denote the set of all admissible states s of the system, determined by the boundary conditions and Boltzmann weights. When no confusion may arise, we sometimes use the same notation $\mathfrak{S}$ to denote either the system or its set of admissible states. The two systems we consider will thus be denoted $\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Delta$ or $\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Gamma$ according to the weights in row one and row two of Table 1, respectively.

Lemma 3.1. *Let $r = 1$ and let $\xi = u_i, \eta = u_j \in \mathfrak{F}_m$. For either $\mathfrak{S}^\Gamma$ or $\mathfrak{S}^\Delta$, there exists at most one admissible state for the system $\mathfrak{S}_{\mathbf{z}, \xi, \eta, 1}$, so $\langle \eta | T(\mathbf{z}) | \xi \rangle$ is the Boltzmann weight of this state (or zero if no admissible state exists). If such a state exists, then, for $\mathfrak{S}_{\mathbf{z}, \xi, \eta, 1}^\Delta$*

$$i_m \geq j_m \geq i_{m-1} \geq j_{m-1} \geq \cdots. \quad (3.2)$$

For $\mathfrak{S}_{\mathbf{z}, \xi, \eta, 1}^\Gamma$, we have instead $j_m \geq i_m \geq \cdots$.

Proof. Let us consider the case of Delta ice. To see that the state (if it exists) is unique, observe that every vertex must have an even number of $-$ signs on its adjoining edges. We have required all but finitely many horizontal edges to have configuration $+^0$. Suppose that $j_m > i_m$. In that case, this observation shows that the spin to the left of the j_m column is $+^0$; so at the vertex in the j_m column the configuration would be



which is an illegal pattern. Thus $i_m \geq j_m$, and continuing this way gives (3.2). The case of Gamma ice is similar. Compare [10] Proposition 19.1 or [2] Section 8.2. $\square$

The *Boltzmann weight of the state* is the product of the Boltzmann weights at the vertices. The *partition function* $Z(\mathfrak{S})$ is the sum of the Boltzmann weights over all states. These definitions make sense by the following result.

Proposition 3.2. *In the case where the grid is infinite, there are only a finite number of states for $\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Delta$ or $\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Gamma$. For each state, all but finitely many vertices have Boltzmann weight 1, so the Boltzmann weight of the state is a finite product.*

Proof. The fact that there are only finitely many states is a consequence of Lemma 3.1. With our assumption that all but finitely many horizontal edges have decorated spin $+^0$ for Delta ice or $-^0$ for Gamma ice, it is not hard to see that for any state $\mathfrak{s}$ all but finitely many vertices are in configuration a_1 or b_1 for Delta ice, or a_2 or b_2 for Gamma ice. Since those vertices have Boltzmann weight 1, the Boltzmann weight of a state is a finite product. $\square$

We will sometimes use the Dirac notation $\xi = |\xi\rangle$ for elements of $\mathfrak{F}$. Let us define an inner product on $\mathfrak{F}$ in which the normal-ordered monomials

$$\xi = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots, \quad i_m > i_{m-1} > \cdots$$

is an orthonormal basis. There is a unique involution on $\mathfrak{F}$ which is conjugate-antilinear and which is the identity on the real vector space spanned by the normal-ordered monomials. If $\xi = |\xi\rangle$ is an element of $\mathfrak{F}$ we will denote by $\langle \xi |$ its image under the involution. Then $\langle \eta | \xi \rangle$ will denote the inner product of ξ and η . This inner product is linear in ξ and conjugate-linear in η .

Now let us specialize to Delta ice. We may define an operator $T_\Delta(\mathbf{z})$ on $\mathfrak{F}_m$ by

$$T_\Delta(\mathbf{z}) \xi = T_\Delta(\mathbf{z}) |\xi\rangle = \sum_{\eta} Z(\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Delta) |\eta\rangle. \quad (3.3)$$

It is a consequence of Lemma 3.1 that there are only finitely many terms in the right-hand side. (This would fail for $\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Gamma$.)

In the same notation we may write

$$Z(\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Delta) = \langle \eta | T_\Delta(\mathbf{z}) | \xi \rangle. \quad (3.4)$$

In the special case where $r = 1$, we will use the notation $T_\Delta(z)$ with $\mathbf{z} = (z)$. We call the operator the *row transfer matrix*. We have $T_\Delta(\mathbf{z}) = T_\Delta(z_1) \cdots T_\Delta(z_r)$.

Remark 3.3. In (3.4) we have specialized to the case of Delta ice. For $\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Gamma$ the sum (3.3) would fail to be finite. Nevertheless we could similarly define $T_\Gamma(\mathbf{z})$ for Gamma ice as an operator on “bras” $\langle \eta |$ instead of “kets” $| \xi \rangle$ by the formula

$$\langle \eta | T_\Gamma(\mathbf{z}) = \sum_{\xi} Z(\mathfrak{S}_{\mathbf{z}, \xi, \eta, r}^\Gamma) \langle \xi |,$$

which is a finite sum. Then (3.4) would still be correct.

We specialize now to the case $r = 1$ and denote $z = z_1$. As in (1.5), we define operators $H_+(z)$ and $H_-(z)$ on $\mathfrak{F}_m$ by

$$H_\pm(z) := \sum_{k=1}^{\infty} \frac{1}{k} (1 - v^k) z^{\pm nk} J_{\pm k}. \quad (3.5)$$

If $\xi \in \mathfrak{F}$ then $H_+(z)\xi = H_+(z)|\xi\rangle$ is a finite sum. For $H_-(z)$, this fails, but as with $T_\Gamma(z)$, we may interpret $H_-(z)$ as an operator by the formula

$$\langle \eta | H_-(z) = \sum_{\xi} \langle \eta | H_-(z) | \xi \rangle \langle \xi |,$$

and this is a finite sum.

Our main theorem (Theorem A), states that

$$e^{H_+(z)} = T_\Delta(z), \quad e^{H_-(z)} = T_\Gamma(z). \quad (3.6)$$

We will prove this in the next section. As an immediate consequence, the row transfer matrices $T_\Delta(z)$ and $T_\Gamma(z)$ are $U_q(\widehat{\mathfrak{sl}}_n)$ -module homomorphisms, because the operators J_k are.

4. Proof of the Main Theorem

The proof is structured as follows. We will first prove the statement for Delta ice using induction to reduce the proof to an identity for two finite subsystems where we get a finite number of cases that are checked in Tables 2 and 3. One reason for starting with Delta ice is because of Remark 3.3 together with normal-ordering issues. The transfer matrix for Gamma ice is then related to the adjoint of the Delta ice transfer matrix in Subsection 4.2. Therefore, until Subsection 4.2 we will consider Delta ice. We will fix z , and let $T = T(z)$ be the transfer matrix (3.3) of the one-rowed system, and $H = H_+(z)$.

We pause to refine the criterion in Lemma 3.1 for an admissible state to exist in the one-row system $\mathfrak{S}_{\mathbf{z}, \xi, \eta, 1}$. For even if (3.2) is satisfied, there may not be an admissible state $\mathfrak{s}$. Let us describe a further condition that must be satisfied.

We may write $\xi = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots$ and $\eta = u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots$. By (3.2) $i_m \geq j_m \geq i_{m-1} \geq \cdots$, and if r is sufficiently negative, then $i_r = r$ and $j_r = r$. The

substance of the lemma that we will now state is that there is a bijection between the two sequences $\mathbf{i} = (i_m, i_{m-1}, \dots)$ and $\mathbf{j} = (j_m, j_{m-1}, \dots)$, and that corresponding elements are congruent modulo n .

Since the elements of $\mathbf{j}$ are distinct, each i_a can be equal to a unique j_b , which must be either j_a or j_{a+1} . In this case we say that i_a and j_b are *paired*. It remains for the bijection to be defined on those elements of $\mathbf{i}$ (resp. $\mathbf{j}$) that are not equal to any element of the other sequence. Thus we say that the index i_a is *isolated* for the pair ξ, η if $j_{a+1} > i_a > j_a$, and similarly we say that the index j_b is *isolated* if $i_b > j_b > i_{b-1}$. The isolated indices i_a and j_b are *paired* if

$$j_{a+1} > i_a > j_a = i_{a-1} > j_{a-1} = i_{a-2} > \dots j_{b+1} = i_b > j_b > i_{b-1}. \quad (4.1)$$

(We omit the condition $j_{a+1} > i_a$ if $a = m$.) The condition (4.1) means there are no isolated indices between i_a and j_b , though there may be many indices that are not isolated. If i_a is not isolated, then either $i_a = j_a$ or $i_a = j_{a+1}$. In this case, we consider i_a to be paired with j_a or j_{a+1} .

Lemma 4.1. *For any admissible state $\mathfrak{s}$, every isolated i_a is paired with a unique isolated j_b . The pairing relationship is a bijection between the i_a and the j_b , and if i_a and j_b are paired, then $i_a \equiv j_b$ modulo n .*

Proof. It is obvious that if i_a (resp. j_b) is not isolated, then it is paired with a unique j_b (resp. i_a). Since these are equal, they are $\equiv 0 \pmod{n}$. Therefore we have to consider the isolated vertices. Here we make use of the hypothesis $\langle \eta | T | \xi \rangle \neq 0$. Consider the state of the model, with the columns labeled:

$$\begin{array}{cccccccc} & i_a & & & j_a & & & j_b \\ & - & + & \dots & - & + & \dots & + \\ + & - & - & & - & - & - & - \\ & + & + & \dots & - & + & \dots & - \end{array}$$

The charges at the two horizontal edges labeled $+$ must both be $\equiv 0$ modulo n . This implies that $i_a \equiv j_b$ modulo n . $\square$

Let $\psi_k^* : \mathfrak{F}_m \rightarrow \mathfrak{F}_{m+1}$ denote the creation operator defined by

$$\psi_j^*(u_{i_m} \wedge u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \dots) = u_j \wedge u_{i_m} \wedge u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \dots \quad (4.2)$$

and introduce the generating function

$$\psi^*(x) = \sum_{j \in \mathbb{Z}} \psi_j^* x^j, \quad (4.3)$$

as well as the operator $\rho_k^*(z) : \mathfrak{F}_m \rightarrow \mathfrak{F}_{m+1}$

$$\rho_k^*(z) = \psi_k^* - z \psi_{k-n}^*. \quad (4.4)$$

We will use the following consequences of the Baker-Campbell-Hausdorff formula. If A and B are elements of a Lie algebra such that $[A, B]$ commutes with both A and B , then

$$e^A e^B = e^{[A, B]} e^B e^A.$$

If $[A, B] = cB$ where c is a constant, then

$$e^A B e^{-A} = e^c B. \quad (4.5)$$

Proposition 4.2. With $H = H_+(z) = \sum_{k=1}^{\infty} \frac{1}{k} (1 - v^k) z^{nk} J_k$ and $\psi^*(x)$ as defined above, we have that

$$e^H \psi^*(x) e^{-H} = \frac{1 - x^n v z^n}{1 - x^n z^n} \psi^*(x) \quad (4.6)$$

or equivalently that

$$e^H \rho_k^*(z^n) = \rho_k^*(v z^n) e^H. \quad (4.7)$$

Proof. For any $\xi \in \mathfrak{F}_m$ we have from (2.26) that $[J_k, \psi_j^*](\xi) = J_k(\psi_j^*(\xi)) - \psi_j^*(J_k(\xi)) = J_k(u_j \wedge \xi) - u_j \wedge J_k(\xi) = J_k(u_j) \wedge \xi = u_{j-nk} \wedge \xi = \psi_{j-nk}^*(\xi)$ which implies

$$[J_k, \psi_j^*] = \psi_{j-nk}^*. \quad (4.8)$$

Then,

$$\begin{aligned} [H, \psi^*(x)] &= \sum_{k \geq 1} \sum_{j \in \mathbb{Z}} \frac{z^{nk} - v^k z^{nk}}{k} x^j \psi_{j-nk}^*(x) \\ &= \sum_{k \geq 1} \frac{z^{nk} - v^k z^{nk}}{k} x^{nk} \psi^*(x) = \log \left(\frac{1 - x^n v z^n}{1 - x^n z^n} \right) \psi^*(x) \end{aligned}$$

from which we obtain (4.6) using the Baker-Campbell-Hausdorff formula. The equivalence of (4.6) and (4.7) follows by comparing coefficients for different powers of x . $\square$

We will work now with finite-dimensional wedge spaces $\mathfrak{F}(k, n - k, r)$ spanned by vectors

$$\xi = u_{i_1} \wedge \cdots \wedge u_{i_r} \quad (4.9)$$

where $k \geq i_1 > \cdots > i_r \geq i_{k-n}$. Let $\mathfrak{F}(k, n - k) = \bigoplus_r \mathfrak{F}(k, n - k, r)$. We will define operators ψ_k^* and $\psi_{k-n}^* : \mathfrak{F}(k, n - k, r) \rightarrow \mathfrak{F}(k, n - k, r + 1)$ by

$$\psi_k^*(\xi) = u_k \wedge \xi, \quad \psi_{k-n}^*(\xi) = u_{k-n} \wedge \xi.$$

These operators are analogous to the operators $\psi_k^*, \psi_{k-n}^* : \mathfrak{F}_m \rightarrow \mathfrak{F}_{m+1}$ already defined, and indeed if $\zeta = u_{j_{m-r}} \wedge u_{j_{m-r-1}} \wedge \cdots \in \mathfrak{F}_{m-r}$ is such that $k - n > j_{m-r} > j_{m-r-1} > \cdots$ then $\xi \wedge \zeta$ is naturally in $\mathfrak{F}_m$ and $\psi_k^*(\xi \wedge \zeta) = \psi_k^*(\xi) \wedge \zeta$ and similarly for ψ_{k-n}^* . We also define $\rho_k^*(z) = \psi_k^* - z \psi_{k-n}^*$ as before.

Finally, we define an operator $\hat{T}$ on $\mathfrak{F}(k, n - k)$. It is enough to define constants $\langle \eta | \hat{T} | \xi \rangle$ where $\xi \in \mathfrak{F}(k, n - k, r)$ and $\eta \in \mathfrak{F}(k, n - k, r')$. Let us write $\varepsilon = \varepsilon(\xi) = (\varepsilon_k, \dots, \varepsilon_{k-n})$ where the spins $\varepsilon_i = \pm$ and $i = i_1, \dots, i_r$ in (4.9) are precisely the values where $\varepsilon_i = -$. Similarly let $\delta = \delta(\eta) = (\delta_k, \dots, \delta_{k-n})$ be spins corresponding to η . Let

$$\eta = u_{j_1} \wedge u_{j_2} \wedge \cdots \wedge u_{j_{r'}}.$$

We require $i_1 \geq j_1 \geq i_2 \geq \cdots$ and for this reason either $r' = r$ or $r' = r - 1$.

Now we define a finite system as follows. We make a grid with $n + 1$ columns labeled $k, k - 1, \dots, k - n$ in decreasing order.

$$\begin{array}{ccccccc} & \varepsilon_k & \varepsilon_{k-1} & \cdots & \varepsilon_{k-n} & & \\ +^0 & & & & & \pm^a & \\ & \delta_k & \delta_{k-1} & \cdots & \delta_{k-n} & & \end{array} \quad (4.10)$$

The boundary conditions at the left and right edge are as follows. At the left boundary, we always put $+^0$. At the right boundary, there will, for each row, be a unique decorated spin $\pm^a$ such that the partition function of this system can have nonzero value. The sign $+$ or $-$ is determined by the condition that the total number of $-$ spins around the whole boundary is even. Thus it is $+$ if $r' = r$ and $-$ if $r' = r - 1$. The charge is also determined by the requirement that there be a (uniquely determined) state $\mathfrak{s}$ with the given boundary conditions. Then we define $\langle \eta | \hat{T} | \xi \rangle$ to be the Boltzmann weight of this state, using the weights in Table 1.

Now the operator $\hat{T} : \mathfrak{F}(k, n-k, r) \longrightarrow \mathfrak{F}(k, n-k, r) \oplus \mathfrak{F}(k, n-k, r-1)$ is defined by

$$\hat{T}(\xi) = \sum_{\eta} \langle \eta | \hat{T} | \xi \rangle \eta.$$

Proposition 4.3. *Let ξ and η be basis vectors of $\mathfrak{F}(k, n-k)$ as above. Then*

$$\langle \eta | \hat{T} \rho_k^*(z^n) | \xi \rangle = \langle \eta | \rho_k^*(vz^{n-1}) \hat{T} | \xi \rangle. \quad (4.11)$$

Moreover, the spins $\pm a$ that appear on the left- and right-hand sides of this calculation are the same (with a determined modulo n).

We will prove this in Section 4.1. The meaning of the second assertion is as follows. Suppose we compute

$$\langle \eta | \hat{T} \rho_k^*(z^n) | \xi \rangle.$$

This equals $\langle \eta | \hat{T} | \psi_k^* \xi \rangle - z^n \langle \eta | \hat{T} | \psi_{k-n}^* \xi \rangle$ and in this computation two right edge spins $\pm a$ and $\pm b$ will appear. (See (4.10).) Similarly on the other side of the computation, two right edge spins $\pm c$ and $\pm d$ will appear. The assertion is that these four spins are equal in sign, and $a \equiv b \equiv c \equiv d$ modulo n .

Proposition 4.4. *Let $\xi = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \in \mathfrak{F}_m$ with $i_m > i_{m-1} > \cdots$ and let $k > i_m$. Then,*

$$T \rho_k^*(z^n) | \xi \rangle = \rho_k^*(vz^n) T | \xi \rangle. \quad (4.12)$$

Proof. Let $\eta \in \mathfrak{F}_m$. We write $\eta = u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots$ with $j_m > j_{m-1} > \cdots$. Unless $k \geq j_m$ it is easy to deduce that $\langle \eta | T \psi_k^* | \xi \rangle$, $\langle \eta | T \psi_{k-n}^* | \xi \rangle$, $\langle \eta | \psi_k^* T | \xi \rangle$ and $\langle \eta | \psi_{k-n}^* T | \xi \rangle$ are all zero from Lemma 3.1, and from the fact that if ξ' does not involve any u_m with $m > k$ then neither does $\psi_k^* \xi'$ or $\psi_{k-n}^* \xi'$. Therefore it is enough to prove that

$$\langle \eta | T \rho_k^*(z^n) | \xi \rangle = \langle \eta | \rho_k^*(vz^n) T | \xi \rangle$$

under the assumption that $k \geq j_m$.

Let us find r such that $i_r \geq k - n > i_{r-1}$ and write $\xi = \xi_1 \wedge \xi_2$ with

$$\xi_1 = u_{i_m} \wedge \cdots \wedge u_{i_r}, \quad \xi_2 = u_{i_{r-1}} \wedge u_{i_{r-2}} \wedge \cdots.$$

Similarly we write $\eta = \eta_1 \wedge \eta_2$ where

$$\eta_1 = u_{j_m} \wedge \cdots \wedge u_{j_{r'}}, \quad \eta_2 = u_{j_{r'-1}} \wedge u_{j_{r'-2}} \wedge \cdots$$

and r' is such that $j_{r'} \geq k - n > j_{r'-1}$.

Now let $\mathfrak{s}$ be the unique state associated with $\langle \eta | T | \psi_k^* \xi \rangle$. We will cut the partition function to the right of the $k - n$ column. Thus we partition the Boltzmann weights into those from columns numbered $\geq k - n$, and those from columns $< k - n$. Since $k \geq i_m, j_m$ the spin in the horizontal edge to the left of the k -th column must be $+^0$. Depending on ξ and η , let $\pm^a$ be the decorated spin attached to the horizontal edge to the right of the $(k - n)$ -th column. We obtain

$$\langle \eta | T | \psi_k^* \xi \rangle = \langle \eta_1 | \hat{T} | \psi_k^* \xi_1 \rangle \cdot C$$

where C is the Boltzmann weight of the following state of an (infinite) truncated system:

$$\begin{array}{cccc} & \varepsilon_{k-n-1} & \varepsilon_{k-n-2} & \cdots \\ \pm^a & & & \\ & \delta_{k-n-1} & \delta_{k-n-2} & \cdots \end{array}$$

where $\varepsilon_i = -$ if i is among the indices $i_{r-1}, i_{r-2}, \dots$ in ξ_2 and δ_i is similarly derived from η_2 .

Now we similarly have

$$\langle \eta | T | \psi_{k-n}^* \xi \rangle = \langle \eta_1 | \hat{T} | \psi_{k-n}^* \xi_1 \rangle \cdot C, \quad \langle \eta | \psi_k^* T | \xi \rangle = \langle \eta_1 | \psi_k^* \hat{T} | \xi_1 \rangle \cdot C,$$

and

$$\langle \eta | \psi_{k-n}^* T | \xi \rangle = \langle \eta_1 | \psi_{k-n}^* \hat{T} | \xi_1 \rangle \cdot C,$$

with the *same* constant C in every case. The fact that the constant C is the same in every case follows from the last assertion in Proposition 4.3. Hence we can pull out the constant and the identity needed follows from (4.11). $\square$

For an element $\xi = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \in \mathfrak{F}_m$ with $i_m > i_{m-1} > \cdots$ we define the degree $\deg(\xi)$ of ξ as follows

$$\deg(\xi) = \sum_{r \leq m} (i_r - r) \quad (4.13)$$

which we note is positive since $i_r \geq r$ for all r , and finite since $i_r = r$ for $r \ll 0$. If $\deg(\xi) = 0$, then ξ is the vacuum $|m\rangle$ in $\mathfrak{F}_m$.

Using the following lemma we can similarly define the degree of any $\xi = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \in \mathfrak{F}_m$ even if it is not normal-ordered.

Lemma 4.5. *The degree defined above has the following properties:*

- (1) Suppose $\xi = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \in \mathfrak{F}_m$ is not normal-ordered, that is $i_r < i_{r-1}$ for some $r \leq m$. Then writing ξ in terms of the basis of $\mathfrak{F}_m$ of normal-ordered wedges, each term has the same degree, which equals $\sum_{r \leq m} (i_r - r)$.
- (2) Let $\xi' = u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \cdots \in \mathfrak{F}_{m-1}$ with $i_{m-1} > i_{m-2} > \cdots$. For any k , let $\xi = u_k \wedge \xi' \in \mathfrak{F}_m$ which is not necessarily normal-ordered. If $\xi \neq 0$, then,

$$\deg(\xi) = (k - m) + \deg(\xi'). \quad (4.14)$$

Note that, even for the quantum wedge, if $i_r = i_{r-1}$ for some r , then $\xi = 0$. However, because of the extra terms in (2.24) compared to the classical ($q = 1$) wedge, if $i_r = i_{r-2}$ for example, then ξ is not necessarily zero.

Proof. For the first statement we notice that in the right-hand side of the quantum wedge (2.23) for $u_j \wedge u_i$ with $j < i$, each term is of the form $u_a \wedge u_b$ with $a + b = i + j$. Since ξ can be normal-ordered by repeated use of (2.23) this proves the first assertion.

The second statement follows from the first by letting $i_m = k$:

$$\deg(\xi) = \sum_{r \leq m} (i_r - r) = (i_m - m) + \sum_{r \leq m-1} (i_r - r) = (k - m) + \deg(\xi').$$

□

Proof of Theorem A (Delta ice). We will show, for an arbitrary $\xi = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \in \mathfrak{F}_m$ with $i_m > i_{m-1} > \cdots$ that $e^H \xi = T\xi$ using induction over the degree of ξ .

The base case, $\deg(\xi) = 0$, is when $|\xi\rangle$ is the vacuum $|m\rangle$, for which we have that $J_k|m\rangle = 0$. Thus $e^H|m\rangle = |m\rangle$. It is easy to check that $T|m\rangle = |m\rangle$ also, as required.

From now on, assume that ξ is not a vacuum, which means that $i_m > m$. Let $\xi' = u_{i_{m-1}} \wedge u_{i_{m-2}} \wedge \cdots \in \mathfrak{F}_{m-1}$ and $\xi'' = u_{i_m-n} \wedge \xi' \in \mathfrak{F}_m$. Then $\xi = u_{i_m} \wedge \xi' = \rho_{i_m}^*(z^n)\xi' + z^n\xi''$. Note that $u_{i_m-n} \wedge \xi'$ is not necessarily normal-ordered or nonzero. Using Lemma 4.5 we have that $\deg(\xi') = \deg(\xi) - (i_m - m) < \deg(\xi)$ and, if $\xi'' \neq 0$, $\deg(\xi'') = \deg(\xi) - n < \deg(\xi)$.

We assume, for $\eta \in \mathfrak{F}$ with $\deg(\eta) < \deg(\xi)$, that $e^H \eta = T\eta$ (which also holds for $\eta = \xi'' = 0$). Then, for the induction step we have that

$$T\xi = T\rho_{i_m}^*(z^n)\xi' + z^nT\xi'' = T\rho_{i_m}^*(z^n)\xi' + z^n e^H \xi''.$$

Using Proposition 4.4 together with the induction hypothesis, we have that

$$T\rho_{i_m}^*(z^n)\xi' = \rho_{i_m}^*(vz^n)T\xi' = \rho_{i_m}^*(vz^n)e^H \xi' = e^H \rho_{i_m}^*(z^n)\xi',$$

where, in the last step we have also used (4.7) of Proposition 4.2. Thus,

$$T\xi = e^H (\rho_{i_m}^*(z^n)\xi' + z^n\xi'') = e^H \xi.$$

The statement for Gamma ice is proved in Subsection 4.2. □

4.1. Proof of Proposition 4.3. Let $\xi = u_{i_1} \wedge \cdots \wedge u_{i_r}$ and $\eta = u_{j_1} \wedge \cdots \wedge u_{j_{r'}}$ be elements of $\mathfrak{F}(k, n-k)$ with $i_1 > i_2 > \cdots > i_r$ and $j_1 > \cdots > j_{r'}$. We must show

$$\langle \eta | \hat{T} \psi_k^* | \xi \rangle - z^n \langle \eta | \hat{T} \psi_{k-n}^* | \xi \rangle = \langle \eta | \psi_k^* \hat{T} | \xi \rangle - v z^n \langle \eta | \psi_{k-n}^* \hat{T} | \xi \rangle. \quad (4.15)$$

Let ε_i and δ_i with $k \geq i \geq k-n$ be the spins associated with ξ and η , so that $\varepsilon_i = -$ if $i = i_j$ for some j , and $\varepsilon_i = +$ otherwise, and similarly for δ_i .

Proposition 4.6. *Suppose that any one of the four terms in (4.15) is nonzero. Then either:*

- (i) *We have $\varepsilon_i = \delta_i$ for $k > i > k-n$; or*
- (ii) *There is a unique value s with $k > s > k-n$ such that $\varepsilon_s = -$ and $\delta_s = +$, and $\varepsilon_i = \delta_i$ for $k > i > k-n$, $i \neq s$.*

Proof. Note that applying ψ_k^* or ψ_{k-n}^* to ξ cannot affect ε_i with $k > i > k - n$. In particular, $\psi_{k-n}^*(\xi) = u_{k-n} \wedge u_{i_1} \wedge \cdots \wedge u_{i_r}$ is not normal-ordered. However when we use (2.24) to put it in normal order, we get

$$u_{k-n} \wedge u_{i_1} \wedge \cdots \wedge u_{i_r} = \pm \left(\prod_{\substack{\varepsilon_i = - \\ k > i > k-n}} g(k-i) \right) u_{i_1} \wedge \cdots \wedge u_{i_r} \wedge u_{k-n} \quad (4.16)$$

where the sign is $+$ if $\varepsilon_k = +$ and $-$ if $\varepsilon_k = -$. For this, there are no correction terms because the interchanged vectors are of the form $u_a \wedge u_b$ with $|a - b| \leq n$.

Therefore each of the four terms in (4.15) is (possibly up to a constant such as the one in (4.16)) of the form $\langle \eta' | \hat{T} | \xi' \rangle$ where ξ' and η' correspond to sequences ε'_i and δ'_i of spins and (for the two terms on the left-hand side) $\delta'_i = \delta_i$ for all $k \geq i \geq k - n$ and also $\varepsilon'_i = \varepsilon_i$ except for one of the two cases $i = k$ or $i = k - n$. Similarly for the two terms on the right-hand side, $\varepsilon'_i = \varepsilon_i$ for all $k \geq i \geq k - n$ and $\delta'_i = \delta_i$ except when $i = k$ or $k - n$. Since $\varepsilon_i = \varepsilon'_i$ and $\delta_i = \delta'_i$ for $k > i > k - n$, we may replace ε_i and δ_i by ε'_i and δ'_i in the statement of the proposition.

Fixing one of these four cases, let $\xi' = u_{i'_1} \wedge u_{i'_2} \wedge \cdots$ and $\eta' = u_{j'_1} \wedge u_{j'_2} \wedge \cdots$. Under the assumption that $\langle \eta' | \hat{T} | \xi' \rangle \neq 0$, analogs of Lemmas 3.1 and 4.1 are true. The analog of Lemma 3.1 means that $i'_1 \geq j'_1 \geq i'_2 \geq j'_2 \geq \cdots$.

Moreover, the proof of Lemma 4.1 will show that there is at most one isolated index in the interval $k > i > k - n$. We recall that an index s is *isolated* if $\varepsilon_s \neq \delta_s$. If $k > s > k - n$, this is clearly equivalent to $\varepsilon'_s \neq \delta'_s$. As in Lemma 4.1 isolated indices come in pairs separated by a multiple of n . Thus if there are isolated indices, we must have $i'_1 = j'_1$, $i'_2 = j'_2$, up to the first isolated index, $i'_m > j'_m$. Then the next isolated index would have to be $\leq i'_s - n$, but this is outside of the considered interval. Let $s = i'_m$. Then $\varepsilon_s = \varepsilon'_s = -$, while $\delta_s = \delta'_s = +$, and there are no other isolated indices. $\square$

So there are two types of cases we have to consider, depending on whether we are in Case (i) or Case (ii) of Proposition 4.6. With each of these cases we have 16 subcases depending on the values of $\varepsilon_k, \delta_k, \varepsilon_{k-n}$ and δ_{k-n} .

Remark 4.7. It is possible to argue more efficiently and only check half these 32 cases, namely those in which $\varepsilon_k = +$. This is because in Proposition 4.4 we have $k > i_m$, and i_m denotes the first minus sign of ξ . For completeness we included all 32 cases in Tables 2 and 3.

For Case (i), let us denote

$$G = \prod_{\substack{\varepsilon_i = - \\ k > i > k-n}} g(k-i).$$

Case (i), subcase: $(\varepsilon_k, \delta_k, \varepsilon_{k-n}, \delta_{k-n}) = (+, +, +, +)$. We observe that $\langle \eta | \rho_k^*(vz^n) \hat{T} | \xi \rangle = 0$ since there is no way a component η of $\rho_k^*(vz^n) \hat{T} | \xi \rangle$ can have both $\delta_k = \delta_{k-n} = +$. So we must show that $\langle \eta | \hat{T} \rho_k^*(z^n) | \xi \rangle = 0$. This has two terms, which will cancel. First $\langle \eta | \hat{T} \psi_k^* | \xi \rangle$ is the Boltzmann weight of the state

$-$	ε_{k-1}	$\cdots$	ε_{k-n+1}	$+$
$+$	$-$	$-$	$-$	-0
$+$	ε_{k-1}	$\cdots$	ε_{k-n+1}	$+$

Table 2. Case (i) subcases, confirming (4.15)

$(\varepsilon_k, \varepsilon_{k-n}, \delta_k, \delta_{k-n})$	$\langle \eta \hat{T} \psi_k^* \xi \rangle$	$-z^n \langle \eta \hat{T} \psi_{k-n}^* \xi \rangle$	$\langle \eta \psi_k^* \hat{T} \xi \rangle$	$-v z^n \langle \eta \psi_{k-n}^* \hat{T} \xi \rangle$
(+ + + +)	$z^n G$	$-z^n G$	0	0
(+ + + -)	$G z^n (1 - v) z$	$-z^n G$	0	$-v z^{n+1} G$
(+ + - +)	1	0	1	0
(+ - + -)	$-G z^{n-1}$	0	0	$-G z^{n-1}$
(+ + - +)	1	0	1	0
(+ + - -)	0	0	0	0
(+ - - +)	1	0	1	0
(+ - - -)	1	0	1	0
(- + + +)	0	0	0	0
(- + + -)	0	$-v z^{2n} G^2$	0	$-v z^{2n} G^2$
(- - + +)	0	0	0	0
(- - + -)	0	0	0	0
(- + - +)	0	$z^n G$	$z^n G$	0
(- + - -)	0	$z^n G$	$(1 - v) z^n G$	$v z^n G$
(- - - +)	0	0	0	0
(- - - -)	0	0	$-v z^n G$	$v z^n G$

that is, $G z^n$, where the product is over r patterns of type a_2 and $n - r$ of type b_2 . The second term is $-z^n \langle \eta | \hat{T} \psi_{k-n}^* | \xi \rangle$. This equals $-z^n G$ times the Boltzmann weight of the state

$$\begin{array}{cccccc} + & \varepsilon_{k-1} & \cdots & \varepsilon_{k-n+1} & - & \\ + & + & & + & +^0 & -1 \\ + & \varepsilon_{k-1} & \cdots & \varepsilon_{k-n+1} & + & \end{array}.$$

Here the factor of G comes from (4.16). The Boltzmann weight of the last state is 1, so the two terms cancel and the proposition is true in this case.

To summarize, there are two ways that a factor of G can appear. One is through (4.16), and the other is through the Boltzmann weight of a state. There are 16 subcases for Case (i) and these are summarized in Table 2. It is easy to see that in all these cases the last assertion of Proposition 4.3 (about the identity of the decorated spins appearing at the right edges of the states contributing to the nonzero terms in any subcase) is satisfied.

We now turn to Case (ii). Let us again do one subcase completely, then summarize all cases in a table. Let us consider the subcase where $(\varepsilon_k, \varepsilon_{k-n}, \delta_k, \delta_{k-n}) = (+, +, +, -)$. We do not need to consider the contributions of ψ_k^* to either the left- or the right-hand side since these would involve an illegal pattern in the s column. On the other hand

$$-z^n \langle \eta | \hat{T} \psi_{k-n}^* | \xi \rangle = (-z^n) \left[\prod_{\substack{k-n+1 \leq i \leq k-1 \\ \varepsilon_i = -1}} g(k-i) \right] Z$$

where Z is the Boltzmann weight of the state

$$\begin{array}{cccccc} + & \varepsilon_{k-1} & \cdots & - & \cdots & - \\ + & + & & + (s) & - & - \\ + & \varepsilon_{k-1} & \cdots & + & \cdots & - \end{array}.$$

The product in brackets comes from (4.16). We have

$$Z = \left[\prod_{\substack{k-n+1 \leq i \leq s-1 \\ \varepsilon_i = -1}} g(s-i) \right] z^{s-k+n} g(s-k).$$

We may combine two factors using the identity $g(k-s)g(s-k) = v$ and so

$$-z^n \langle \eta | \hat{T} \psi_{k-n}^* | \xi \rangle = - \left[\prod_{\substack{k-n+1 \leq i \leq k-1 \\ \varepsilon_i = -1 \\ i \neq s}} g(k-i) \right] \left[\prod_{\substack{k-n+1 \leq i \leq s-1 \\ \varepsilon_i = -1}} g(s-i) \right] v z^{s-k+2n}.$$

On the other side of the equation,

$$-v z^n \langle \eta | \psi_{k-n}^* \hat{T} | \xi \rangle = (-v z^n) \left[\prod_{\substack{k-n+1 \leq i \leq k-1 \\ \varepsilon_i = -1 \\ i \neq s}} g(k-i) \right] Z'$$

where Z' is the Boltzmann weight of the state

$$\begin{array}{cccccc} + & \varepsilon_{k-1} & \cdots & - & \cdots & + \\ + & + & & + (s) & - & -_{s+1-k+n} \\ + & \varepsilon_{k-1} & \cdots & + & \cdots & + \end{array}.$$

That is,

$$Z' = \left[\prod_{\substack{k-n+1 \leq i \leq s-1 \\ \varepsilon_i = -1}} g(s-i) \right] z^{s-k+n}.$$

We see that in this case:

$$\langle \eta | \hat{T} \rho_k^*(z^n) | \xi \rangle = -z^n \langle \eta | \hat{T} \psi_{k-n}^* | \xi \rangle = -v z^n \langle \eta | \psi_{k-n}^* \hat{T} | \xi \rangle = \langle \eta | \rho_k^*(v z^n) \hat{T} | \xi \rangle.$$

Now let us define

$$G' = \prod_{\substack{k-n+1 \leq i \leq k-1 \\ \varepsilon_i = -1 \\ i \neq s}} g(k-i), \quad G'' = \prod_{\substack{k-n+1 \leq i \leq s-1 \\ \varepsilon_i = -1}} g(s-i).$$

We summarize the Case (ii) subcases in Table 3. As in Case (i) it is easy to verify the last assertion of Proposition 4.3 regarding the decorated spins at the right edge, and the first assertion is verified in every subcase by Table 3. Thus Proposition 4.3 is now proved.

Table 3. Case (ii) subcases, confirming (4.15)

$(\varepsilon_k, \varepsilon_{k-n}, \delta_k, \delta_{k-n})$	$\langle \eta \hat{T} \psi_k^* \xi \rangle$	$-z^n \langle \eta \hat{T} \psi_{k-n}^* \xi \rangle$	$\langle \eta \psi_k^* \hat{T} \xi \rangle$	$-v z^n \langle \eta \psi_{k-n}^* \hat{T} \xi \rangle$
(+ + + +)	0	0	0	0
(+ + + -)	0	$-G' G'' v z^{s-k+2n}$	0	$-G' G'' v z^{s-k+2n}$
(+ + - +)	$G'' z^{s-k+n}$	0	$G'' z^{s-k+n}$	0
(+ - + -)	0	0	0	0
(+ + - -)	$G'' z^{s-k+n}$	0	$G'' z^{s-k+n}$	0
(+ + - +)	$G'' (1-v) z^{s-k+n}$	0	$G'' (1-v) z^{s-k+n}$	0
(+ - - +)	0	0	0	0
(+ - - -)	$G'' z^{s-k+n} g(s-k)$	0	$G'' z^{s-k+n} g(s-k)$	0
(- + + +)	0	0	0	0
(- + + -)	0	0	0	0
(- - + +)	0	0	0	0
(- - + -)	0	0	0	0
(- + - +)	0	0	0	0
(- + - -)	0	$G' G'' v z^{s-k+2n}$	0	$G' G'' v z^{s-k+2n}$
(- - - +)	0	0	0	0
(- - - -)	0	0	0	0

4.2. Gamma ice. We will deduce the second identity in (1.6) for Gamma ice from the first, which is already proved. If T is an operator on $\mathfrak{F}$ we define its adjoint $T^\dagger$ by the formula

$$\langle T^\dagger \eta | \xi \rangle = \langle \eta | T \xi \rangle.$$

In the following proof, we will assume that the parameter v is real, and moreover we will assume that the conjugate of $g(a)$ is $g(-a)$. In our applications to Whittaker functions, g is a Gauss sum, $|g(a)| = \sqrt{v}$, the reciprocal of the square root of the residue cardinality. (See Remarks 1 and 2 in [5].) Then $g(a)$ and $g(-a)$ are complex conjugates since $g(a)g(-a) = v$.

Since our result is essentially an algebraic identity, if we prove it under the restriction that v is real and $g(a)$, $g(-a)$ are complex conjugates, it will follow in general. Alternatively, we could take the $g(a)$ to be indeterminates in an algebra over $\mathbb{C}$, with an involution that maps $g(a)$ to $g(-a)$.

Proposition 4.8. *The adjoint of $T_\Delta(z)$ is $T_\Gamma(1/\bar{z})$.*

Proof. We must check the identity

$$\langle T_\Gamma(1/\bar{z}) \eta | \xi \rangle = \langle \eta | T_\Delta(z) \xi \rangle.$$

We will write this

$$\overline{\langle \xi | T_\Gamma(1/\bar{z}) \eta \rangle} = \langle \eta | T_\Delta(z) \xi \rangle.$$

We may check this for normal-ordered $\xi, \eta \in \mathfrak{F}_m$. Let

$$\xi = u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots, \quad \eta = u_{j_m} \wedge u_{j_{m-1}} \wedge \cdots.$$

Both sides vanish unless

$$i_m \geq j_m \geq i_{m-1} \geq j_{m-1} \geq \cdots,$$

which we assume. Now $\langle \eta | T_\Delta(z) \xi \rangle$ is the partition function of a system with a unique state, with $-$ in the top (resp. bottom) vertical edges in the columns i_m (resp. j_m) and

+ elsewhere. Similarly $\langle \xi | T_{\Gamma}(z^{-1}) \eta \rangle$ is the partition function of a system with the top and bottom vertical edges reversed. We may obtain its unique state by taking the state of the $\langle \eta | T_{\Delta}(z) \xi \rangle$ system, and replacing each horizontal decorated spin $+^0$ by $-^0$, or $-^a$ by $+^{-a}$. Now an examination of Table 1 shows that this operation interchanges a_1 patterns with b_2 patterns, and similarly $a_2 \Leftrightarrow b_1$, $c_1 \Leftrightarrow c_2$. Remembering that the c_1 and c_2 patterns occur in pairs, we see that $\langle \xi | T_{\Gamma}(z^{-1}) \eta \rangle$ is obtained from $\langle \eta | T_{\Delta}(z) \xi \rangle$ by replacing $g(a)$ by $g(-a)$. If we further replace z by its complex conjugate, we see that $\langle \xi | T_{\Gamma}(1/\bar{z}) \eta \rangle$ and $\langle \eta | T_{\Delta}(z) \xi \rangle$ are complex conjugates, as required. $\square$

Proposition 4.9. *The adjoint of J_k is J_{-k} if $k \neq 0$.*

Proof. See [42], remark after (21) on page 1055. This point is explained in more detail in [39], Section 3.3 (where the inner product is introduced) and Section 4.1.1, making use of results of both [32] and [42] in the context of a general Boson-Fermion correspondence. $\square$

Proof of Theorem A (Gamma ice). We will prove that $T_{\Gamma}(z) = e^{H_{-}(z)}$. Because J_k and J_{-k} are adjoints, by (3.5)

$$H_{-}(z) = H_{+}(1/\bar{z})^{\dagger}.$$

Exponentiating then gives

$$\exp(H_{-}(z)) = \exp(H_{+}(1/\bar{z}))^{\dagger} = T_{\Delta}(1/\bar{z})^{\dagger} = T_{\Gamma}(z).$$

$\square$

5. LLT and Metaplectic Symmetric Functions

The quantum Fock space of Kashiwara, Miwa and Stern, which underlies our results, is also fundamental in the theory of LLT [42] or ribbon symmetric functions. In this section, inspired by ideas from Lam [38], we will show how the LLT polynomials can be written in the form

$$\mathcal{G}_{\lambda/\mu}^n(\mathbf{z}) = \langle \mu | e^{L_{+}(\mathbf{z})} | \lambda \rangle, \quad (5.1)$$

where $\mathbf{z} = (z_1, \dots, z_r)$ and

$$L_{+}(\mathbf{z}) = \sum_{k=1}^{\infty} \frac{1}{k} p_k(\mathbf{z}) J_k. \quad (5.2)$$

(We are using the notation (1.4) for basis vectors of $\mathfrak{F}_m$, and we may fix $m = 0$ in this section.)

Remark 5.1. As we will prove, the polynomials (5.1) coincide with the LLT or ribbon symmetric polynomials provided we take

$$g(a) = \begin{cases} -v & \text{if } a \equiv 0 \pmod{n}; \\ -\sqrt{v} & \text{otherwise.} \end{cases} \quad (5.3)$$

If g is a more general function satisfying Assumption 2.12, then the results of this section will remain valid, but $\mathcal{G}_{\lambda/\mu}$ will be a *generalization* of the LLT polynomials that are in the literature.

The operator $L_+(\mathbf{z})$ is similar to the operator H_+ defined in (3.5), that appears in our main theorem. Indeed, in Definition 29 of [37], Lam defined a super generalization $\mathcal{G}_{\lambda/\mu}^n(\mathbf{z}|\mathbf{w})$ of the LLT polynomials, and we will prove that

$$\langle \mu | e^{H_+(\mathbf{z})} | \lambda \rangle = \mathcal{G}_{\lambda/\mu}^n(\mathbf{z}^n | v \mathbf{z}^n),$$

where $H_+(\mathbf{z}) = \sum_{i=1}^r H_+(z_i)$. For this statement we are omitting the Drinfeld twisting which introduces the “Gauss sums” $g(a)$ into the definition of the Fock space. If we include the Drinfeld twisting, then we would obtain a generalization of the LLT polynomials, and a similar statement would be true.

Remark 5.2. We do not know a statement generalizing our Theorem A that would express the half-vertex operator e^{L_+} that appears in (5.1) as a row transfer matrix. This is available only in the special case of the supersymmetric LLT polynomials with $\mathbf{w} = v\mathbf{z}$.

Let $J_1, J_2, \dots$ be independent commuting variables. (Eventually we will specialize them to operators on $\mathfrak{F}_0$ as before, but for our first result this is not needed.) Following the definitions in Section 3 of [38], let

$$u_k = \sum_{\lambda \vdash k} z_\lambda^{-1} J_\lambda,$$

where if $\lambda = (1^{m_1} 2^{m_2} \dots)$ is a partition of k then $z_\lambda = \prod_i (i^{m_i} m_i!)$ and $J_\lambda = J_{\lambda_1} J_{\lambda_2} \dots$. Also, if λ is a partition define $\varepsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}$ and define

$$\tilde{u}_k = \sum_{\lambda \vdash k} z_\lambda^{-1} \varepsilon_\lambda J_\lambda.$$

Proposition 5.3. *We have*

$$e^{L_+(\mathbf{z})} = \sum_{v_1=0}^{\infty} \dots \sum_{v_r=0}^{\infty} z_1^{v_1} \dots z_r^{v_r} u_{v_r} \dots u_{v_1} \quad (5.4)$$

and

$$e^{-L_+(-\mathbf{z})} = \sum_{v_1=0}^{\infty} \dots \sum_{v_r=0}^{\infty} z_1^{v_1} \dots z_r^{v_r} \tilde{u}_{v_r} \dots \tilde{u}_{v_1}. \quad (5.5)$$

Proof. Let $\Lambda = \Lambda(\mathbf{z})$ be the ring of symmetric functions in variables $z_1, z_2, \dots$ over $\mathbb{Q}$. Let $\Lambda(\mathbf{w})$ be another copy of Λ , in variables $w_1, w_2, \dots$. We will use the notation of [44] for symmetric functions: $p_k(\mathbf{z}), h_k(\mathbf{z}), e_k(\mathbf{z})$ will denote the power sum, complete and elementary symmetric functions, with $p_\lambda(\mathbf{z}) = \prod p_{\lambda_i}(\mathbf{z})$, $h_\lambda(\mathbf{z}) = \prod h_{\lambda_i}(\mathbf{z})$, and m_λ will be the monomial symmetric functions.

Remembering that the u_k commute, we may rearrange the factors $u_{v_r}, \dots, u_{v_1}$ so that $v_r \geq v_{r-1} \geq \dots \geq v_1$ and rewrite the right-hand side of (5.4) as

$$\sum_v m_v(\mathbf{z}) u_{v_r} \dots u_{v_1} \quad (5.6)$$

where now the sum is over partitions (of length $\leq r$).

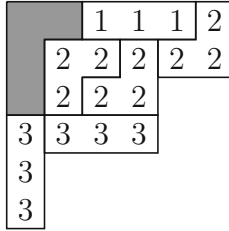


Fig. 1. A 3-ribbon tableau with spin 5 and weight $(1, 3, 2)$

In the ring $\Lambda(\mathbf{z}) \otimes \Lambda(\mathbf{w})$, we have the identity

$$\sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(\mathbf{z}) p_{\lambda}(\mathbf{w}) = \prod_{i,j} (1 - z_i w_j)^{-1} = \sum_{\nu} m_{\nu}(\mathbf{z}) h_{\nu}(\mathbf{w}).$$

This is proved in Macdonald [44] Section I.4. Now

$$\exp \left(\sum_{k=1}^{\infty} \frac{1}{k} p_k(\mathbf{z}) p_k(\mathbf{w}) \right) = \prod_{i,j} (1 - z_i w_j)^{-1} = \sum_{\lambda} m_{\lambda}(\mathbf{z}) h_{\lambda}(\mathbf{w}). \quad (5.7)$$

Indeed,

$$-\log(1 - z_i w_j) = \sum_{k=1}^{\infty} \frac{(z_i w_j)^k}{k}.$$

Summing over i, j and exponentiating gives (5.7). Now we specialize $p_k(\mathbf{w}) \mapsto J_k$. Then $h_k \mapsto u_k$ since by Macdonald [44] (I.2.14)

$$h_k = \sum_{\lambda \vdash k} z_{\lambda}^{-1} p_{\lambda}. \quad (5.8)$$

Thus specializing (5.7) gives (5.4). The identity (5.5) follows similarly from the identity

$$\exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} p_k(\mathbf{z}) p_k(-\mathbf{w}) \right) = \prod_{i,j} (1 + z_i w_j) = \sum_{\lambda} m_{\lambda}(\mathbf{z}) e_{\lambda}(\mathbf{w})$$

which follows from (5.7) on applying the involution in $\Lambda(\mathbf{w})$. See [44] Section I.2. Under the specialization $p_k(\mathbf{w}) \mapsto J_k$ we get $e_k \mapsto \tilde{u}_k$ because

$$e_k = \sum_{\lambda \vdash k} z_{\lambda}^{-1} \varepsilon_{\lambda} p_{\lambda}.$$

This follows from (5.8) by applying the involution using [44] equation (I.2.13). $\square$

We recall from [37,38,42] that an n -ribbon is a skew partition λ/μ of size n that is connected and does not contain any 2×2 block. (Here we are identifying the skew partition with its Young diagram.) The *spin* of an n -ribbon is its height in columns, minus 1. A *horizontal n -ribbon strip* is a skew shape λ/μ that can be decomposed into disjoint n -ribbons, each of which has its top-right most box adjacent to μ , or else its top-right most box lies in the first line. The *spin* $s(\lambda/\mu)$ of λ/μ is then the sum of the

spins of its constituent n -ribbons. Thus we are following [37] in our definition of spin, not [42] who define the spin to be half $s(\lambda/\mu)$. See Figure 1 for an example illustrating the concepts of n -ribbon and horizontal n -ribbon strip.

An n -ribbon skew tableau T of shape λ/μ is a sequence of partitions

$$\mu = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda, \quad (5.9)$$

where α^{i+1}/α^i is a horizontal n -ribbon strip. We may associate with such data a tableau in which the strip α^{i+1}/α^i is filled with i 's. The weight $\nu = \text{wt}(T)$ will then be $(\nu_1, \dots, \nu_r)$ where ν_i is α^{i+1}/α^i divided by n .

Now we define the *LLT* or *ribbon symmetric function*

$$\mathcal{G}_{\lambda/\mu}^n(\mathbf{z}) = \mathcal{G}_{\lambda/\mu}^n(\mathbf{z}; q) = \sum_T q^{s(T)} \mathbf{z}^{\text{wt}(T)},$$

where the sum is over n -ribbon skew tableaux of shape λ/μ . (Here $v = q^2$.) This is consistent with the notation in [37] but differs from the notation in [42].

Let us regard J_k as in prior sections to be an operator on the quantum Fock space $\mathfrak{F}_0$. If λ is a partition, let $|\lambda\rangle$ denote the element $u_{\lambda_1} \wedge u_{\lambda_2-1} \wedge \cdots$ of $\mathfrak{F}_0$. If λ is the empty partition, we will instead use $|0\rangle$ to denote the vacuum. Consistent with our earlier notation, we will denote by $\langle \mu | e^{L_+(\mathbf{z})} | \lambda \rangle$ the coefficient of $|\mu\rangle$ in $e^{L_+(\mathbf{z})} |\lambda\rangle$, where we now regard $e^{L_+(\mathbf{z})}$ as an operator on $\mathfrak{F}_0$.

Following [37, 42] we define an operator $\mathcal{U}_k$ on $\mathfrak{F}_0$ by

$$\mathcal{U}_k |\lambda\rangle = \sum_{\substack{\lambda/\mu \text{ a horizontal } n\text{-ribbon strip} \\ |\lambda/\mu| = nk}} q^{s(\lambda/\mu)} |\mu\rangle,$$

where the sum is over $\mu \subset \lambda$ such that λ/μ is a horizontal n -ribbon strip of size nk . Similarly let

$$\tilde{\mathcal{U}}_k |\lambda\rangle = \sum_{\substack{\lambda/\mu \text{ a vertical } n\text{-ribbon strip} \\ |\lambda/\mu| = nk}} q^{s(\lambda/\mu)} |\mu\rangle.$$

(Vertical n -ribbon strips are defined similarly to horizontal ones.)

We note that the notation in [42] differs from that in [37] (and also [32]) by the transformation $q \mapsto -q^{-1}$. Our notation is consistent with [37].

There is a homomorphism ψ from the ring Λ of symmetric functions to the ring of $U_q(\widehat{\mathfrak{sl}}_n)$ -module endomorphisms of $\mathfrak{F}_0$. This is the map that sends a symmetric polynomial f to the endomorphism $f(y_1^{-1}, y_2^{-1}, \dots)$ where the y_i are as in Section 2.2. If s_λ is a Schur polynomial, the endomorphisms $\psi(s_\lambda)$ were used in [43] in an analog of the Steinberg tensor product theorem for $\mathfrak{F}$. See also [40].

By Theorems 3 and 5 of [37] (following Leclerc and Thibon [43])

$$\psi(h_k) = \mathcal{U}_k, \quad \psi(e_k) = \tilde{\mathcal{U}}_k, \quad \psi(p_k) = J_k.$$

Thus u_k is an element of the abstract polynomial ring generated by $J_1, J_2, \dots$, while $\mathcal{U}_k$ is an endomorphism of $\mathfrak{F}_0$ that corresponds to u_k under the action of the J_k on $\mathfrak{F}_0$.

Theorem 5.4. *The polynomial $\mathcal{G}_{\lambda/\mu}^n$ is symmetric and*

$$\mathcal{G}_{\lambda/\mu}^n(\mathbf{z}; q) = \langle \mu | e^{L_+(\mathbf{z})} | \lambda \rangle. \quad (5.10)$$

Proof. By Proposition 5.3 the right-hand side equals

$$\sum_{v_1=0}^{\infty} \cdots \sum_{v_r=0}^{\infty} z_1^{v_1} \cdots z_r^{v_r} \langle \mu | \mathcal{U}_{v_r} \cdots \mathcal{U}_{v_1} | \lambda \rangle.$$

Now the right-hand side enumerates n -ribbon tableaux in the definition (5.9) and so we obtain (5.10). The symmetry of $\mathcal{G}_{\lambda/\mu}^n$ is due to Lascoux, Leclerc and Thibon. It follows from the fact that the operators $\mathcal{U}_k$ commute. $\square$

A similar result for Hall-Littlewood polynomials was found by Jing [28]. Another vertex operator realization of Hall-Littlewood polynomials may be found in Tsilevich [48]. Hall-Littlewood polynomials are limits of LLT polynomials by [42], Theorem VI.6.

As an application of Theorem 5.4 we will deduce the Cauchy identity for LLT polynomials, a result that is due to Lam [37–39], proved also by van Leeuwen [49]. We will work with two sets of variables, $z_1, \dots, z_r$ and $w_1, \dots, w_r$. Let

$$L_+(\mathbf{z})^* = \sum_{k=1}^{\infty} \frac{p_k(\mathbf{z})}{k} J_{-k}.$$

If the z_i are real, then $L_+(\mathbf{z})$ and $L_+(\mathbf{z})^*$ are adjoints by Proposition 4.9.

We will denote $\mathcal{G}_\lambda = \mathcal{G}_{\lambda/\emptyset}^n$ where $\emptyset$ is the empty partition. We have

$$\langle \lambda | L_+(\mathbf{z})^* | 0 \rangle = \mathcal{G}_\lambda(\mathbf{z}). \quad (5.11)$$

Indeed, since this is a purely algebraic identity, it is sufficient to prove this if z_i are real. Then since $L_+(\mathbf{z})$ and $L_+(\mathbf{z})^*$ are adjoints, this follows by taking the the conjugate of (5.10).

Lam [37] proved a version of the Cauchy identity for LLT polynomials. We will show how this can be deduced from Theorem 5.4.

Proposition 5.5. *We have*

$$\exp(L_+(\mathbf{z})) \exp(L_+(\mathbf{w})^*) = \Omega(\mathbf{z}, \mathbf{w}) \exp(L_+(\mathbf{w})^*) \exp(L_+(\mathbf{z}))$$

where

$$\Omega(\mathbf{z}, \mathbf{w}) = \prod_{i=0}^{n-1} \prod_{i,j} (1 - v^i z_i w_j)^{-1}.$$

Proof. Using (2.27) we have

$$[L_+(\mathbf{z}), L_+(\mathbf{w})^*] = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{v^{nk} - 1}{v^k - 1} \right) z_i^k w_j^k = \log \Omega(\mathbf{z}, \mathbf{w}).$$

The statement then follows from the Baker-Campbell-Hausdorff formula. $\square$

We recall that if λ is a partition, there is a unique smallest partition δ that can be obtained by removing ribbon n -strips from λ . The partition δ is called the n -core of λ . If $\delta = \lambda$ then λ is called an n -core partition. See [44], Example I.1.8.

Lemma 5.6. *If δ is an n -core then $J_k |\delta\rangle = 0$ for all $k > 0$.*

Proof. Clearly $\mathcal{U}_k|\delta\rangle = 0$ for $k > 0$, and so by Proposition 5.3 $e^{L_+(\mathbf{z})}|\delta\rangle = |\delta\rangle$. This means $J_k|\delta\rangle = 0$. $\square$

Theorem 5.7 (Lam). *Let δ be an n -core. Then*

$$\sum_{\lambda} \mathcal{G}_{\lambda/\delta}(\mathbf{z}) \mathcal{G}_{\lambda/\delta}(\mathbf{w}) = \Omega(\mathbf{z}, \mathbf{w}), \quad (5.12)$$

where the sum is over all partitions with n -core δ .

Proof. We will prove this under the assumption that $\mathbf{w}$ is real. Since this is a purely algebraic identity, that is sufficient. We evaluate

$$\langle \delta | \exp(L_+(\mathbf{z})) \exp(L_+^*(\mathbf{w})) | \delta \rangle \quad (5.13)$$

in two different ways. First, by Proposition 5.5, it equals

$$\Omega(\mathbf{z}, \mathbf{w}) \langle \delta | \exp(L_+^*(\mathbf{w})) \exp(L_+(\mathbf{z})) | \delta \rangle = \Omega(\mathbf{z}, \mathbf{w}),$$

since if $k > 0$ we have $J_k|\delta\rangle = \langle \delta | J_{-k} = 0$, so $\exp(L_+(\mathbf{z}))|\delta\rangle = |\delta\rangle$, etc. On the other hand, using Theorem 5.4 and (5.11) the coefficient (5.13) equals

$$\sum_{\lambda} \langle \delta | \exp(L_+(\mathbf{z})) | \lambda \rangle \langle \lambda | \exp(L_+(\mathbf{w}))^* | \delta \rangle = \sum_{\lambda} \mathcal{G}_{\lambda/\delta}(\mathbf{z}) \mathcal{G}_{\lambda/\delta}(\mathbf{w}).$$

$\square$

Now we recall the definition of the *super ribbon function* $\mathcal{G}_{\lambda/\mu}^n(\mathbf{z}|\mathbf{w}; q)$ defined in [37], Definition 29. For this we require a double alphabet $1 < 1' < 2 < 2' < \dots < r < r'$. A *super ribbon tableau* T is a sequence of partitions

$$\mu = \lambda_{r+1} \subset \lambda_{r'} \subset \lambda_r \subset \dots \subset \lambda_{1'} \subset \lambda_1 = \lambda.$$

It is assumed that $\lambda_i/\lambda_{i'}$ is a horizontal n -ribbon strip, and that $\lambda_{i'}/\lambda_{i+1}$ is a vertical n -ribbon strip. We can label the tableaux by labeling the boxes in $\lambda_i/\lambda_{i'}$ with i , and the boxes in $\lambda_{i'}/\lambda_{i+1}$ with i' . Let $\text{wt}(T) = (v_1, \dots, v_r)$ where v_i is the number of i in the tableau, and $\text{wt}'(T) = (v'_1, \dots, v'_r)$ where v_i is the number of i' . Then we define the super ribbon function

$$\mathcal{G}_{\lambda/\mu}^n(\mathbf{z}|\mathbf{w}; q) = \sum_T q^{s(T)} \mathbf{z}^{\text{wt}(T)} (-\mathbf{w})^{\text{wt}'(T)}$$

where the sum is over super ribbon tableaux.

Theorem 5.8. *For any pair of partitions $\mu \subseteq \lambda$,*

$$\langle \mu | e^{L_+(\mathbf{z})} e^{-L_+(\mathbf{w})} | \lambda \rangle = \mathcal{G}_{\lambda/\mu}^n(\mathbf{z}|\mathbf{w}; q). \quad (5.14)$$

$\mathcal{G}_{\lambda/\mu}^n$ vanishes unless λ and μ have the same n -core.

Proof. Since the operators $\mathcal{U}_v$ and $\tilde{\mathcal{U}}_{v'}$ commute, we may apply Proposition 5.3 and rearrange to obtain

$$\langle \mu | e^{L_+(\mathbf{z})} e^{-L_+(\mathbf{w})} | \lambda \rangle = \sum_{v, v'} \mathbf{z}^{\text{wt}(v)} (-\mathbf{w})^{\text{wt}(v')} \langle \mu | \tilde{\mathcal{U}}_{v'} \mathcal{U}_{v_r} \cdots \tilde{\mathcal{U}}_{v_1} \mathcal{U}_{v_1} | \lambda \rangle.$$

Each operator on the right-hand side subtracts either a vertical or horizontal n -ribbon strip, and (5.14) follows. The second statement is clear since removing a vertical or horizontal n -ribbon strip from a partition does not change its n -core. $\square$

Corollary 5.9. *In the notation of (1.9),*

$$\mathcal{M}_{\lambda/\mu}^n(\mathbf{z}) := \langle \mu | T_\Delta(\mathbf{z}) | \lambda \rangle = \mathcal{G}_{\lambda, \mu}^n(\mathbf{z}^n | v \mathbf{z}^n).$$

$\mathcal{M}_{\lambda/\mu}^n$ vanishes unless λ and μ have the same n -core.

Proof. This follows from Theorems A and 5.8 because $H_+(z) = L_+(z^n) - L_+(vz^n)$. $\square$

We may prove a similar Cauchy identity for the metaplectic symmetric functions. The following Theorem holds for any values of $g(a)$ satisfying Assumption 2.12.

Theorem 5.10. *Let δ be an n -core partition. Then*

$$\sum_{\lambda} \mathcal{M}_{\lambda/\delta}^n(\mathbf{z}) \mathcal{M}_{\lambda/\delta}^n(\mathbf{w}) = \Theta(\mathbf{z}, \mathbf{w}), \quad \Theta(\mathbf{z}, \mathbf{w}) := \prod_{i,j} \frac{(1 - v z_i^n w_j^n)(1 - v^n z_i^n w_j^n)}{(1 - z_i^n w_j^n)(1 - v^{n+1} z_i^n w_j^n)}. \quad (5.15)$$

Proof. This is similar to Theorem 5.7. We must first generalize the calculation in Proposition 5.5. Now we work with

$$H_+(\mathbf{z}) = \sum_{k=0}^{\infty} \frac{p_{nk}(\mathbf{z})}{k} (1 - v^k) J_k, \quad H_+(\mathbf{z})^* = \sum_{k=0}^{\infty} \frac{p_{nk}(\mathbf{z})}{k} (1 - v^k) J_{-k}.$$

We see that

$$\begin{aligned} [H_+(\mathbf{z}), H_+(\mathbf{w})^*] &= \sum_{k=0}^{\infty} p_{nk}(\mathbf{z}) p_{nk}(\mathbf{w}) \frac{1}{k} (1 - v^k) (1 - v^{nk}) \\ &= \sum_{i,j} \sum_{k=0}^{\infty} \frac{1}{k} (z_i w_j)^{nk} (1 - v^k - v^{nk} + v^{(n+1)k}) = \log \Theta(\mathbf{z}, \mathbf{w}). \end{aligned}$$

Therefore we have

$$\exp(H_+(\mathbf{z})) \exp(H_+(\mathbf{w})^*) = \Theta(\mathbf{z}, \mathbf{w}) \exp(H_+(\mathbf{w})^*) \exp(H_+(\mathbf{z})).$$

The remainder of the calculation is similar to the proof of Theorem 5.7. $\square$

6. Metaplectic Whittaker Functions

This work originated in the theory of Whittaker functions for the metaplectic n -fold cover of GL_r . These were represented by Gamma and Delta ice partition functions for finite systems in [5, 7]. In this section we will show that metaplectic Whittaker functions can also be expressed as partition functions for our infinite-dimensional systems. More precisely, in (1.9) we defined what we are calling *metaplectic symmetric functions*. Like metaplectic Whittaker functions, they are partition functions of metaplectic ice, but unlike metaplectic Whittaker functions, the $\mathcal{M}_{\lambda, \mu}^n$ are symmetric functions. What we will now show is a way of expressing metaplectic Whittaker functions in terms of the $\mathcal{M}_{\lambda, \mu}^n$.

Let us review the relationship between the metaplectic ice partition functions and metaplectic Whittaker functions, relying on [5–7] for details. Let F be a nonarchimedean local field. Assume that the group μ_{2n} of $2n$ -th roots of unity in F has cardinality $2n$, and that the residue cardinality v^{-1} is prime to n . Let ϖ be a prime element in the ring $\mathfrak{o}$ of integers and let ψ be a fixed additive character of F that is trivial on the ring of integers but no larger fractional ideal. Let

$$g(a) = \frac{1}{v^{-1}} \sum_{t \in (\mathfrak{o}/(\varpi))^{\times}} (\varpi, t)^a \psi\left(\frac{t}{\varpi}\right),$$

where $(\cdot, \cdot)$ is the n -th order Hilbert symbol. (We are calling the residue cardinality v^{-1} instead of v or q since it is the reciprocal of the residue cardinality that will appear in our formulas. We will use q to denote a square root of v .) This function $g(a)$ satisfies Assumption 2.12.

There is a central extension

$$1 \longrightarrow \mu_{2n} \longrightarrow \widetilde{\mathrm{GL}}_r(F) \longrightarrow \mathrm{GL}_r(F) \longrightarrow 1$$

that is essentially an n -fold cover, described in [5]. We will refer to this as the *metaplectic group*.

If $\mathbf{z} \in (\mathbb{C}^{\times})^r$ then there is a principal series representation $\pi_{\mathbf{z}}$ defined in [5]. Associated with $\pi_{\mathbf{z}}$ there are n^r linearly independent *spherical Whittaker functions* on $\widetilde{\mathrm{GL}}_r(F)$. Let $\mathcal{W}_{\mathbf{z}}$ denote the space of functions spanned by these. If $W \in \mathcal{W}_{\mathbf{z}}$, we are interested in the values of W evaluated at

$$\varpi^{\lambda} := \mathbf{s} \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_r} \end{pmatrix},$$

where $\mathbf{s} : \mathrm{GL}_r(F) \rightarrow \widetilde{\mathrm{GL}}_r(F)$ is a standard section (see [5]) and λ is a partition of length $\leq r$. These are combinatorially interesting sums of products of Gauss sums and polynomials in v whose study goes back to Kazhdan and Patterson [34]. In [5, 7] we showed how to represent such Whittaker functions in terms of finite systems of Gamma and Delta ice. In this section we will show that metaplectic Whittaker functions can also be described as partition functions of infinite systems, and thereby relate them to the metaplectic symmetric functions, and to vertex operators.

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of length $\leq r$, let $\mathbf{z} = (z_1, \dots, z_r) \in (\mathbb{C}^{\times})^r$, and let $\sigma = (\sigma_1, \dots, \sigma_r) \in (\mathbb{Z}/n\mathbb{Z})^r$. We will now describe the finite systems $\mathfrak{S}_{\lambda, \sigma}^{\Gamma}$

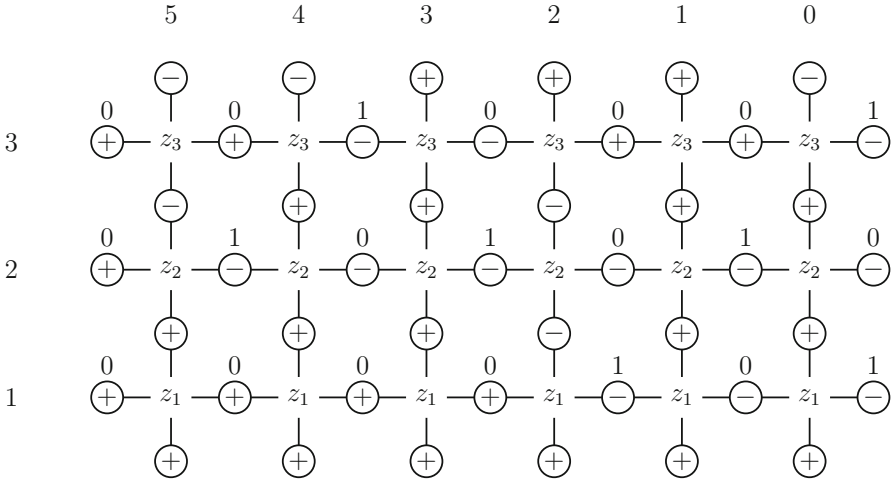


Fig. 2. A state of Delta ice. In this example $n = 2$. The charges (written above the horizontal edges) are integers modulo n that change at the $-$ spins in accordance with Table 1. The charges at $+$ edges must be $\equiv 0$ modulo n . For Gamma ice, the system is similar, but the rows are numbered increasing from top to bottom, and the left edges have variable charge, while the right edges all have charge 0, since in Gamma ice $-^a$ is only allowed with a equal to 0 modulo n

and $\mathfrak{S}_{\lambda, \sigma}^{\Delta}$ depending on these data. These were considered previously in [5] (Gamma ice only) and in [7] (both systems).

Let $\rho = (r - 1, r - 2, \dots, 0)$ so that

$$\lambda + \rho = (\lambda_1 + r - 1, \lambda_2 + r - 2, \dots, \lambda_r)$$

is a strict partition. We consider a grid with r rows and N columns, where N is any positive integer such that $N \geq \lambda_1 + r - 1$. The columns are labeled $N, N - 1, \dots, 0$ in decreasing order from left to right and the rows are labeled $1, \dots, r$ from the top down for Delta ice and from the bottom up for Gamma ice.

On the vertical edges along the top boundary, we put $-$ in the k -th column if k is an entry in $\lambda + \rho$; otherwise we put $+$. On the vertical edges at the bottom, we put $+$ in every column. On the horizontal edges along the left boundary we put the decorated spin $+^0$ (Delta ice) or $+^{\sigma_i}$ in the i -th row (Gamma ice). On the horizontal edges on the right boundary we put the decorated spin $-^{\sigma_i}$ in the i -th column (Delta ice) or $-^0$ (Gamma ice). We use the Gamma Boltzmann weights in the i -th row for Gamma ice and Delta Boltzmann weights for Delta ice. See Figure 2 for an example of the system $\mathfrak{S}_{\lambda, \sigma}^{\Delta}$.

Let δ denote the modular quasicharacter of the Borel subgroup on $\mathrm{GL}_r(F)$, lifted to a function on $\widetilde{\mathrm{GL}}_r(F)$.

Proposition 6.1. *Let $\sigma \in (\mathbb{Z}/n\mathbb{Z})^r$ and $\mathbf{z} \in (\mathbb{C}^\times)^r$. Then there exists a spherical Whittaker function $W_\sigma^\Delta \in \mathcal{W}_{\mathbf{z}}$ such that for λ a partition of length $\leq r$, we have*

$$Z(\mathfrak{S}_{\lambda, \sigma}^{\Delta}) = \delta^{-1/2}(\varpi^\lambda) W_\sigma^\Delta(\varpi^\lambda). \quad (6.1)$$

Proof. By Theorem 6.3 of [5], there exists a spherical Whittaker function W_σ^Γ such that

$$Z(\mathfrak{S}_{\lambda, \sigma}^\Gamma) = \delta^{-1/2}(\varpi^\lambda) W_\sigma^\Gamma(\varpi^\lambda). \quad (6.2)$$

Here we have absorbed the factor $\mathbf{z}^{w_0\rho+\gamma}$ that appears in that theorem into the Whittaker function. It is proved in Theorem 2.3 of [7] that

$$\sum_{\sigma \in (\mathbb{Z}/n\mathbb{Z})^r} Z(\mathfrak{S}_{\lambda, \sigma}^{\Delta}) = (z_1 \cdots z_r)^N \sum_{\tau \in (\mathbb{Z}/n\mathbb{Z})^r} Z(\mathfrak{S}_{\lambda, \tau}^{\Gamma}).$$

The constant $(z_1 \cdots z_r)^N$ does not appear in [7], but this is because in this paper we have changed the Boltzmann weights for Gamma ice, in order that the partition function be convergent for infinite grids. The arguments there are easily refined to show that, for suitable constants $S_{\sigma, \tau}(\mathbf{z})$ independent of λ , we have

$$Z(\mathfrak{S}_{\lambda, \sigma}^{\Delta}) = \sum_{\tau \in (\mathbb{Z}/n\mathbb{Z})^r} S_{\sigma, \tau}(\mathbf{z}) Z(\mathfrak{S}_{\lambda, \tau}^{\Gamma}).$$

Substituting (6.2) into this identity gives us (6.1) with $W_{\sigma}^{\Delta} = \sum_{\tau} S_{\sigma, \tau} W_{\tau}^{\Gamma}$. $\square$

Now we wish to relate the partition functions of these finite systems to the infinite systems defined in Section 3. Let us choose a partition ξ whose first part $\xi_1 \leq r - 1$. Now if λ is a partition of length $\leq r$ let $\lambda \star \xi$ denote the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ where

$$\lambda'_j = \begin{cases} \lambda_j + r - 1 & \text{if } j \leq r, \\ \xi_{j-r} & \text{if } j > r. \end{cases}$$

Note that since we have assumed that $\xi_1 \leq r - 1$ these entries are weakly decreasing, so $\lambda \star \xi$ is a partition. In Frobenius notation,

$$\lambda \star \xi = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_r \\ \xi'_1 + 1 & \xi'_2 + 1 & \xi'_3 + 1 & \cdots & \xi'_r + 1 \end{pmatrix},$$

where ξ' is the conjugate partition of ξ .

Proposition 6.2. *Let ξ be a partition such that $\xi_1 \leq r - 1$ and let $\sigma \in (\mathbb{Z}/n\mathbb{Z})^r$. Then there exist constants $c(\xi, \sigma; \mathbf{z})$ depending on ξ and σ such that if $\lambda = (\lambda_1, \dots, \lambda_r)$ is a partition of length $\leq r$, then*

$$\langle 0 | T_{\mathbf{z}} | \lambda \star \xi \rangle = \sum_{\sigma \in (\mathbb{Z}/n\mathbb{Z})^r} c(\xi, \sigma; \mathbf{z}) Z(\mathfrak{S}_{\lambda, \sigma}^{\Delta}). \quad (6.3)$$

Here in the notation (1.4) both vectors $|0\rangle = |0; 0\rangle$ and $|\lambda \star \xi\rangle = |\lambda \star \xi; 0\rangle$ are in $\mathfrak{F}_0$.

Proof. Let us define an invariant $N : \mathfrak{F}_0 \rightarrow \mathbb{N}$. Suppose $\xi = u_{\mathbf{i}}$, where $\mathbf{i}$ is as in (3.1), with $m = 0$. If $0 > i_0$ then we define $N(\xi) = 0$; otherwise, $N(\xi) = t$ where t is such that $i_{-t} \geq 0 > i_{-t-1}$. If ξ is interpreted as an assignment of spins to a sequence of vertical edges, then $N(\xi)$ is the number of $-$ spins to the left of the 0-th column (inclusively) or equivalently (since $\xi \in \mathfrak{F}_0$), the number of $+$ spins strictly to the right of the 0-th column.

Consider a state of the infinite system $\mathfrak{S}_{\mathbf{z}, |\lambda \star \xi\rangle, |0\rangle, r}^{\Delta}$ of Section 3. For $0 \leq r$ let $\mathbf{i}^{(k)}$ be the decreasing sequence such that in the notation (1.3), $u_{\mathbf{i}^{(k)}}$ is the element of $\mathfrak{F}_0$ corresponding to the configuration of spins below the k -th row, and $u_{\mathbf{i}^{(k-1)}}$ is the element corresponding to the configuration above it. Thus $u_{\mathbf{i}^{(0)}} = |\lambda \star \xi\rangle$ and $u_{\mathbf{i}^{(r)}} = |0\rangle$.

We will show that the spins of the horizontal edges connecting vertices of the 0-th column to those of the -1 -st column are all $-$. Indeed, it follows from Lemma 3.1 that

either $N(u_{\mathbf{i}(k+1)}) = N(u_{\mathbf{i}(k)})$ or $N(u_{\mathbf{i}(k+1)}) = N(u_{\mathbf{i}(k)}) - 1$. But since $N(u_{\mathbf{i}(r)}) = 0$ and $N(u_{\mathbf{i}(0)}) = N(|\lambda \star \xi|) = r$, we must have $N(u_{\mathbf{i}(k)}) = k$ for all k . Now the fact that $N(u_{\mathbf{i}(k)})$ and $N(u_{\mathbf{i}(k-1)})$ have opposite parity implies that the spin in the k -th row on the horizontal edge to the right of the 0-th column is $-$, as required.

Now to complete the proof, we fix spins $\sigma = (\sigma_1, \dots, \sigma_r)$ and collect together the states whose decorated spin on the edge in the k -th row to the right of the 0-th column is $-\sigma_k$. The product of the Boltzmann weights to the left of the 0-th column is the Boltzmann weight of a state of $\mathfrak{S}_{\lambda, \sigma}^\Delta$, and so clearly the sum of such Boltzmann weights equals $Z(\mathfrak{S}_{\lambda, \sigma}^\Delta)$ times a factor that is independent of λ . $\square$

Let us reformulate this result as expressing a metaplectic Whittaker function in terms of the metaplectic symmetric functions.

Theorem 6.3. *Let ξ be a partition (of any length) such that $\xi_1 \leq r - 1$. Then*

$$\sum_{\sigma \in (\mathbb{Z}/n\mathbb{Z})^r} c(\xi, \sigma; \mathbf{z}) W_\sigma^\Delta(\varpi^\lambda) = \delta^{1/2}(\varpi^\lambda) \mathcal{M}_{\lambda \star \xi}^n(\mathbf{z}).$$

Proof. This follows from combining Proposition 6.2 with Proposition 6.1 and the definition (1.9) of the metaplectic symmetric function $\mathcal{M}_\lambda^n$. $\square$

Remark 6.4. By Corollary 5.9, this particular Whittaker function vanishes at ϖ^λ unless $\lambda \star \xi$ has empty n -core. Although $\mathcal{M}_{\lambda \star \xi}^n$ is a symmetric function, this does not imply that the Whittaker function is symmetric in $\mathbf{z}$ because of the factor $c(\xi, \sigma; \mathbf{z})$. These coefficients may be of interest for their own sake.

Remark 6.5. It seems probable that the Whittaker functions on the left-hand side (varying ξ) span the space of Whittaker functions. Such a result would give a two-way connection between metaplectic Whittaker functions and metaplectic symmetric functions.

7. Vertex Operators

So far we have put a lot of focus on operators of the form either:

$$V_+(z) = \exp(H_+[a](z)) \quad \text{or} \quad V_-(z) = \exp(H_-[a](z)), \quad (7.1)$$

which we call *half-vertex operators*, where $H_+[a](z)$ and $H_-[a](z)$ are formal power series in z and z^{-1} respectively defined by (1.7) and (1.8). Recall that H_+ involves the right-moving operators J_k and H_- involves the left-moving operators J_{-k} with $k > 0$. We have proved that the operators $T_\Delta(z)$ and $T_\Gamma(z)$ are of this type.

In this section we will consider their products $V_-(z)V_+(z)$, such as the operator $T_\Gamma(z)T_\Delta(z)$, and investigate if they satisfy the properties of a *vertex operator*. There is one reason to believe that $T_\Gamma(z)T_\Delta(z)$ is a natural entity: in symplectic ice ([8, 21, 24]) one represents the Whittaker function on the n -fold cover of $\mathrm{Sp}(2r)$ with Langlands parameters $z_1, z_1^{-1}, \dots, z_r, z_r^{-1}$ by the partition function of a system having alternating layers of Gamma and Delta ice. The two adjacent layers are joined by a “cap” vertex which does not have an obvious analog in our current setup.

Gamma and Delta ice occur together in another context, namely the equality of the partition functions for Gamma and Delta ice. In [7] this result (established earlier with greater difficulty in [10]) is proved using Yang-Baxter equations. In that context $T_\Gamma(z)T_\Delta(w)$ only appear there with z and w distinct. For this, our Theorem 7.3 below

is relevant, taking the place of the Yang-Baxter equations in our current setup. Still, in this section we are mainly interested in $V(z) = T_\Gamma(z)T_\Delta(z)$ with the parameters equal.

Vertex operators exhibit a property called *locality*. This is a generalization of commutativity that was emphasized in [12, 16, 30]. It is explained in Chapter 1 of [30] that for the vertex operators arising in conformal field theory, locality is a reflection of the locality in the Wightman axioms for a quantum field theory: two fields with disjoint support having spacelike separation commute as operators.

If F is a field, let $F[[z]]$ be the ring of formal power series $\sum_{n \geq 0} a_n z^n$ with $a_n \in F$, and let $F((z))$ be the fraction field of $F[[z]]$, consisting of Laurent series $\sum_{n=-N}^{\infty} a_n z^n$ with only finitely many negative coefficients. Let $\mathcal{H}H((z))$ denote $\mathbb{C}((z)) \otimes \mathcal{H}H$, the space of Laurent series with coefficients in $\mathcal{H}H$.

In vertex algebras a *field* is represented by a formal power series

$$A(z) = \sum_{k=-\infty}^{\infty} A_k z^{-k-1}$$

where A_k is an operator on a Hilbert space $\mathcal{H}$ such that for any vector $|v\rangle \in \mathcal{H}$, $A_k|v\rangle = 0$ for $k \gg 0$. A field gives rise to a map $\mathcal{H} \rightarrow \mathcal{H}((z))$.

Let $B(w) = \sum_{k=-\infty}^{\infty} B_k w^{-k-1}$ similarly be a field. Locality is a generalization of commutativity in the sense that two fields $A(z)$ and $B(w)$ are called *mutually local* if $[A(z), B(w)] = A(z)B(w) - B(w)A(z)$ is a formal distribution concentrated on the diagonal $z = w$. We will explain more precisely what this means.

Note that the matrix elements of $A(z)B(w)$ are elements in $\mathbb{C}((z))((w))$, that is, in $F((w))$ where $F = \mathbb{C}((z))$. Similarly the matrix elements of $B(w)A(z)$ are elements in $\mathbb{C}((w))((z))$. The difference between $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ is illustrated by image of the rational function $1/(z-w)$ embedded into the two spaces as $z^{-1} \sum_{k=0}^{\infty} (w/z)^k$ and $-w^{-1} \sum_{k=0}^{\infty} (z/w)^k$ respectively. Requiring that the matrix elements of $[A(z), B(w)]$ should vanish identically would restrict us to elements in the intersection of $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ in $\mathbb{C}[[z^\pm, w^\pm]]$ which is the space $\mathbb{C}[[z, w]][[z^{-1}, w^{-1}]]$ giving a too strong condition [16]. Instead, we use the more relaxed condition that

$$(z-w)^N [A(z), B(w)] = 0 \quad (7.2)$$

as a formal power series for some positive integer N . In this case we say the fields $A(z)$ and $B(w)$ are *mutually local*.

Let us give another explanation of this notion. We assume that A_k and B_l commute if k and l are either both positive or both negative, that A_0 and B_0 commute with all A_k and B_l . Moreover let us assume that the normal-ordered product

$$:A(z)B(w): = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} z^{-k-1} w^{-l-1} :A_k B_l:$$

is a bounded operator on $\mathcal{H}$, where

$$:A_k B_l: = \begin{cases} B_l A_k & \text{if } k > 0, \\ A_k B_l & \text{otherwise.} \end{cases}$$

Our assumptions imply that $:A(z)B(w): = :B(w)A(z):$. Now consider:

$$A(z)B(w) - :A(z)B(w):.$$

Very often this operator will be given by a power series that is convergent when $|w| < |z|$. Let us denote this as $\phi(z, w)$. Furthermore it may be that $B(w)A(z) - :A(z)B(w):$ is also given by a power series, convergent when $|z| < |w|$, and that this represents the same rational function $\phi(z, w)$. In this case the fields $A(z)$ and $B(w)$ are mutually local.

To clarify this with an example, let us work with a Heisenberg Lie algebra having generators J_k ($k \in \mathbb{Z}$) with J_0 central having the commutator relations

$$[J_k, J_l] = \delta_{k, -l} k \cdot c$$

where c is another central element. This is the special case $n = 1$ of (2.27). The Hilbert space $\mathcal{H}$ is to be generated by a vacuum $|0\rangle$ such that $J_k|0\rangle = 0$ if $k > 0$ and c acts by the identity. Now consider the field

$$J(z) = \sum_{k=-\infty}^{\infty} z^{-k-1} J_k.$$

We have

$$J(z)J(w) - :J(z)J(w): = \sum_{k=1}^{\infty} [J_k, J_{-k}] z^{-1-k} w^{-1+l} \cdot c = \frac{1}{(z-w)^2} \cdot c,$$

the series being convergent when $|w| < |z|$. Since $J(w)J(z) - :J(w)J(z):$ gives the same expression in the complementary domain $|z| < |w|$, the fields $J(z)$ and $J(w)$ are mutually local.

Thus the locality is a generalization of the condition that $A(z)B(w) = B(w)A(z)$. Dong and Lepowsky [12] considered a similar generalization of the condition that

$$A(z)B(w) = e^{i\pi\tau} B(w)A(z), \quad (7.3)$$

for a phase shift $e^{i\pi\tau}$. Our Proposition 7.5 below shows that we need such a generalization of locality. Frenkel and Reshetikhin [17] considered even more generally the case where the phase shift is replaced by an operator $S(w/z)$ that depends analytically only on z and w . For consistency it is necessary that $S(w/z)$ satisfies a parametrized Yang-Baxter equation. This is automatic if $S(w/z)$ is a scalar, in which case this identity is similar to (7.3).

There is another respect in which the framework of [17] is more general than the usual locality, and this is that they allow $S(w/z)$ to have poles not only on the diagonal $z = w$ but on shifted diagonals $z = \gamma w$ where γ lies in a discrete subgroup of $\mathbb{C}^\times$. This concept of locality in [17] is what we see in our examples with the set of lines $z = v^j w$ and $S(w/z)$ being a scalar.

We require that S is a meromorphic function, with poles only along the lines $w = v^j z$ for a finite number of integer values of j , such that

$$A(z)B(w) = S(w/z)B(w)A(z). \quad (7.4)$$

Let us first consider the meaning of this when $A(z) = B(z) = V(z) = V_-(z)V_+(z)$ where $V_\pm(z)$ are defined in (7.1). Suppose that we can find a rational function $\phi(x)$ such that (formally)

$$V_+(z)V_-(w) = \phi\left(\frac{z}{w}\right) V_-(w)V_+(z).$$

Then since $V_-(z)$ commutes with $V_-(w)$ and $V_+(z)$ commutes with $V_+(w)$ we have

$$V(z)V(w) = \phi\left(\frac{z}{w}\right) :V(z)V(w):$$

where the normal-ordered product is

$$:V(z)V(w): = V_-(z)V_-(w)V_+(z)V_+(w). \quad (7.5)$$

Then

$$V(z)V(w) = S(w/z)V(w)V(z), \quad S(x) = \phi(x^{-1})/\phi(x). \quad (7.6)$$

Remark 7.1. Strictly speaking $V_-(z)V_+(z)$ is not an operator on $\mathfrak{F}_m$ since $V_-(z)|\lambda\rangle$ produces an infinite number of terms. However $\langle\mu|V_-(z)V_+(z)|\lambda\rangle$ is a finite sum. Moreover, the normal-ordered product $:V(z)V(w):$ defined by (7.5) is such that $\langle\mu|:V(z)V(w):|\lambda\rangle$ is a finite sum. The normal-ordered product has the advantage of being unchanged if z and w are switched.

We now take the Heisenberg generators J_k to satisfy the commutator relation (2.27), and $\mathcal{H} = \mathfrak{F}_m$ for some fixed m . We supplement the J_k ($k \neq 0$) by J_0 which acts on $\mathfrak{F}_m$ by the scalar m . Define the shift operator $Q : \mathfrak{F}_m \longrightarrow \mathfrak{F}_{m+n}$ by

$$Q(u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots) = u_{i_m+n} \wedge u_{i_{m-1}+n} \wedge \cdots.$$

We will use the notation $|\lambda\rangle = |\lambda; m\rangle$ introduced in (1.4) for basis vectors.

We may regard $V_-(z)V_+(z)$ as a map from $\mathfrak{F}$ into a suitable completion. Depending on the coefficients a it may be useful to supplement $V_-(z)V_+(z)$ by a factor such as $Q^r z^{a_0 J_0}$.

Example 1. The first case we wish to consider is $V_+(z) = T_\Delta(z)$, $V_-(z) = T_\Gamma(z)$. The operators $T_\Delta(z)$ and $T_\Delta(w)$ commute as follows from our main theorem (or by a Yang-Baxter equation argument). Similarly the $T_\Gamma(z)$ mutually commute for varying z . But $T_\Delta(z)$ does not commute with $T_\Gamma(w)$. Moreover, we must be cautious about composing these. Consider

$$\langle\eta|T_\Delta(z)T_\Gamma(w)|\xi\rangle = \sum_{\zeta} \langle\eta|T_\Delta(z)|\zeta\rangle \langle\zeta|T_\Gamma(w)|\xi\rangle. \quad (7.7)$$

There are an infinite number of terms on the right-hand, side. The sum converges provided $|z| < c_1|w|$ where $c_1 = \min(1, |v|^{-1-1/n})$.

Remark 7.2. There are no such convergence issues if we compose in the other (normal-ordered) way: because $T_\Gamma(w)T_\Delta(z)$ does the right-moving modes first, the sum corresponding to (7.7) is a finite sum.

Theorem 7.3. *Suppose that $|z| < c_1|w|$. Then*

$$T_\Delta(z)T_\Gamma(w) = \frac{(1 - vz^n w^{-n})(1 - v^n z^n w^{-n})}{(1 - z^n w^{-n})(1 - v^{n+1} z^n w^{-n})} T_\Gamma(w)T_\Delta(z). \quad (7.8)$$

This, together with Remark 7.2 allows us to analytically continue the conditionally convergent composition $T_\Delta(z)T_\Gamma(w)$, except to the poles of the denominator in (7.8).

Proof of Theorem 7.3. By a computation very similar to the proof of Theorem 5.10, we have

$$e^{H_+(z)} e^{H_-(w)} e^{-H_+(z)} e^{-H_-(w)} = \frac{(1 - vz^n w^{-n})(1 - v^n z^n w^{-n})}{(1 - z^n w^{-n})(1 - v^{n+1} z^n w^{-n})}, \quad (7.9)$$

and (7.8) follows from our Main Theorem (Theorem A). $\square$

In view of our previous discussion, this means that if we define $V(z) = T_\Gamma(z)T_\Delta(z)$, then the operators $V(z)$, $V(w)$ are mutually local in the generalized sense of (7.6) with $\phi(z/w)$ being the right hand side of (7.9).

Example 2. For our next example, we work with the operators

$$L_+(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^k J_k, \quad L_-(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{-k} J_{-k}.$$

The operator $L_+(z)$ appeared in Section 5, and the operator $L_-(z)$ resembles the operator $L_+(z)^*$ that we used there, except that z is replaced by z^{-1} . Now we will make use of the shift operator, and J_0 . Define

$$U_{\pm}(z) = \exp(L_{\pm}(z)), \quad U^{\diamond}(z) = U_-(z)U_+(z), \quad U(z) = Qz^{J_0}U^{\diamond}(z). \quad (7.10)$$

Now let us define the normal-ordered product

$$:U^{\diamond}(z)U^{\diamond}(w): = U_-(z)U_-(w)U_+(z)U_+(w).$$

This is meaningful for all z and w in the sense that if μ, λ are given, then $\langle \mu | :U^{\diamond}(z)U^{\diamond}(w): | \lambda \rangle$ is always a finite sum.

Proposition 7.4. *If $|z|/|w|$ is sufficiently small, then*

$$U^{\diamond}(z)U^{\diamond}(w) = \prod_{j=0}^{n-1} \frac{1}{1 - v^j z/w} :U^{\diamond}(z)U^{\diamond}(w):. \quad (7.11)$$

Proof. We have

$$[L_+(z), L_-(w)] = \sum_{k=1}^{\infty} \frac{1}{k^2} k \frac{v^{nk} - 1}{v^k - 1} \left(\frac{z}{w}\right)^k = - \sum_{j=0}^{n-1} \log(1 - v^j z/w),$$

so by the Baker-Campbell-Hausdorff formula we have

$$U^{\diamond}(z)U^{\diamond}(w) = \prod_{j=0}^{n-1} \frac{1}{1 - v^j z/w} :U^{\diamond}(z)U^{\diamond}(w):.$$

$\square$

We may take (7.11) as giving meaning to $U^{\diamond}(z)U^{\diamond}(w)$ for all z, w except at the poles of the denominator. Then naturally $U(z)U(w)$ may be defined to be $U_0(z)U_0(w)U^{\diamond}(z)U^{\diamond}(w)$ where $U_0(z) = Qz^{J_0}$. (Note that $U_0(w)$ commutes with $U^{\diamond}(z)$.)

Proposition 7.5. *We have*

$$U(z)U(w) = \left(\prod_{j=0}^{n-1} \frac{z - v^j w}{w - v^j z} \right) U(w)U(z).$$

Proof. By Proposition 7.4,

$$U^\diamond(z)U^\diamond(w) = \left(\frac{w}{z} \right)^n \prod_{j=0}^{n-1} \frac{z - v^j w}{w - v^j z} U^\diamond(w)U^\diamond(z). \quad (7.12)$$

On the other hand J_0 and Q commute with J_k if $k \neq 0$ while $[J_0, Q] = nQ$. We have

$$z^{J_0} Q z^{-J_0} = e^{\log(z)J_0} Q e^{-\log(z)J_0} = z^n Q,$$

so

$$U_0(z)U_0(w) = z^n Q^2 z^{J_0} w^{J_0} = \left(\frac{z}{w} \right)^n U_0(w)U_0(z).$$

The statement follows. $\square$

We may now discuss the effect of the factor $U_0(z)$ in this definition. If we had omitted it we would have had locality relation, but the factor $S(w/z)$ would have had to include the $(w/z)^n$ that appears in (7.12). By including the factor $U_0(z)$ in the definition of $U(z)$, we are able to eliminate the pole at $z = 0$. The resulting $\phi(z/w)$ in the locality property (7.6) for $U(z)$ is then $\prod_{j=0}^{n-1} (z - v^j w)/(w - v^j z)$.

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