Far-Field Minimum-Fuel Spacecraft Rendezvous using Koopman Operator and ℓ_2/ℓ_1 Optimization

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Abstract— **We propose a method to compute approximate solutions to the minimum-fuel far-field rendezvous problem for thrust-vectoring spacecraft. When the distance between the active and the target spacecraft is significantly greater than the distance between the target spacecraft and the center of gravity of the planet, linearization-based approximations of the nonlinear rendezvous dynamics may not be sufficiently accurate. For this reason, control methods that rely on such linearizations may not be appropriate for far-field rendezvous. In this paper, we address the control design problem based on a nonlinear state space model. To overcome the well-known challenges of nonlinear control design, we utilize a Koopman operator based approach in which the nonlinear spacecraft rendezvous dynamics is lifted into a higher dimensional space over which the nonlinear dynamics can be approximated by a linear system which is more suitable for control design purposes than the original nonlinear model. An Iteratively Recursive Least Squares (IRLS) algorithm from compressive sensing is then used to solve the minimum fuel control problem based on the lifted linear system. Numerical simulations are performed to show the efficacy of the proposed Koopman operator based approach.**

I. Introduction

We propose a Koopman operator based method for the computation of nearly optimal (approximate) solutions to the minimum-fuel far-field rendezvous problem for thrust-vectoring spacecraft. In a typical rendezvous problem, the relative motion of the active chaser spacecraft with respect to a target spacecraft in a circular or elliptical orbit can be described in terms of a system of autonomous nonlinear differential equations. The control design in such problems is based, however, on linearized equations of motion such as the Hill–Clohessy–Wiltshire (H–C–W) equations, which correspond to a time-invariant system of equations, or the Tschauner–Hempel (T–H) equations, which correspond to a periodic linear system. These widely used linearized models are rarely effective to describe the relative motion for far-field rendezvous [1]. Therefore, linearization-based control design techniques cannot guarantee the desired accuracy in far-field rendezvous problems. The Koopman operator approach utilized herein allows one to account for the nonlinearities of the dynamics of the spacecraft rendezvous problem while at the same time linear control design techniques are still applicable. The key idea of the Koopman operator is that the nonlinear dynamics of the rendezvous problem can be approximated by a higher dimensional linear state space model based on which we can compute approximate solutions to the minimum-fuel rendezvous problem for a thrust vectoring spacecraft. The proposed control algorithms rely on tools from compressive sensing [2] and in particular, ℓ_2/ℓ_1 optimization and the Iteratively Reweighted Least Squares algorithm [3], [4], [5].

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Literature review: A rendezvous mission is usually divided into far-field rendezvous, near-field rendezvous, and final approach. Various control approaches have been proposed for near-field and final approach rendezvous operations [6], [7], [8], [9], [10], [11], [12]. However, most of these approaches use linearized rendezvous equations. References [13], [14] consider more general and challenging proximity operation problems under realistic constraints. While these methods are very robust, they are rarely effective for far-field rendezvous [1]. In addition, these methods [8], [9], [10], [15] cannot be used for far-field rendezvous as these linearized equations give inaccurate results, are computationally expensive and do not guarantee any optimality in terms of fuel consumption.

Koopman operator is an infinite dimensional linear operator that describes the evolution of functions of states (referred to as observable functions or just observables). This operator allows one to "convert" a finite-dimensional nonlinear system into a linear system by lifting the state space of the former system to a higher dimensional state space over which it admits a linear, yet infinite-dimensional, state space model representation. However, in practical applications, a finite-dimensional approximation of the Koopman operator can provide a sufficiently accurate description of the evolution of a nonlinear dynamical systems. By applying linear control design techniques to the system on the "lifted" state space, one obtains indirectly a controller that can be applied to the original nonlinear system of interest [16], [17], [18]. Koopman operator methods for state estimation and nonlinear system identification are used in [19], [20]. Recent studies on the computation of finite-dimensional approximations to the Koopman operator that lead to better approximations of nonlinear dynamics can be found in [21]. A systematic process to choose the observable functions that can best approximate the Koopman operator remains, however, an open research problem. Some recent efforts to address the latter problem based on a combination of machine learning and trial and error methods can be found in [22], [23].

Main contributions: In this paper, we use the Koopman operator to lift the nonlinear spacecraft rendezvous dynamics into a higher but finite-dimensional space over which it can be approximated by a linear system. An Iteratively Recursive Least Squares (IRLS) [5] algorithm is then used to compute approximate solutions for control sequences that minimize the fuel consumption for far-field rendezvous of a thrust vectoring spacecraft. Through numerical simulations, it is observed that the Koopman based approach is able to steer the active spacecraft to the desired final states for both near-field and far-field rendezvous with higher accuracy than when the same controller is designed based on the linearized model for rendezvous. The superiority of the Koopman approach over the standard linearization-based approach is more significant in the case of far-field rendezvous, in which the latter often gives significantly large miss-target errors. To the best of our knowledge, this is the first paper which utilizes the Koopman operator for spacecraft rendezvous problems.

Structure of the paper: The organization of the paper is as follows. In Section II, the continuous-time and discrete-time nonlinear state space models for spacecraft rendezvous are introduced. Koopman operator is reviewed in Section III. Section IV introduces the proposed solution approach for the minimum fuel problem based on the IRLS algorithm. Numerical simulations are presented in Section V, and Section VI presents concluding remarks.

II. State space model and problem setup

In this section, we briefly discuss the governing equations and introduce continuous-time and discrete-time state space models for spacecraft rendezvous. Then, we introduce the problem addressed in this paper. Assume that the target spacecraft is in an elliptical orbit with eccentricity e. Consider the Local-Vertical-Local-Horizontal coordinate system $X-Y-Z$ as shown in Fig. 1 where the origin is fixed at the center of mass of the target spacecraft, and the Y axis is normal to the orbital plane $X-Z$. The relative motion of the active chaser spacecraft in the LVLH frame can be captured by the following nonlinear equation [24]:

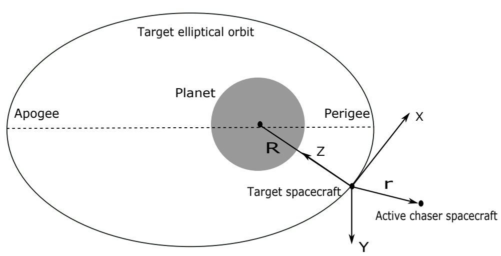


Fig. 1: Local Vertical Local Horizontal (LVLH) coordinate system for spacecraft rendezvous.

$$\ddot{\boldsymbol{r}} = -\mu\left(\boldsymbol{R}+\boldsymbol{r}/|\boldsymbol{R}+\boldsymbol{r}|^3 - \boldsymbol{R}/|\boldsymbol{R}|^3\right) + \boldsymbol{u}, \tag{1}$$

where μ is the gravity constant, $\boldsymbol{u}$ is the control input (acceleration vector due to thrust forces on the active chaser spacecraft), $\boldsymbol{r}$ is the vector from the target spacecraft to the active chaser spacecraft and $\boldsymbol{R}$ is the relative position vector from the center of gravity of the planet to the target spacecraft.

A. Continuous-time nonlinear model

Using the notation $\boldsymbol{r} = \left[\begin{array}{ccc} x & y & z \end{array}\right]^{\mathrm{T}}$, Eq. (1) can be written as [24]

$$\left[\begin{array}{c} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{array}\right] = \left[\begin{array}{c} 2\omega\dot{z} + \dot{\omega}z + \omega^2 x - \frac{\mu x}{|\boldsymbol{R}+\boldsymbol{r}|^3} \\ -\frac{\mu y}{|\boldsymbol{R}+\boldsymbol{r}|^3} \\ \omega^2 z - 2\omega\dot{x} - \dot{\omega}x - \mu\left(\frac{z-R}{|\boldsymbol{R}+\boldsymbol{r}|^3} + \frac{1}{R^2}\right) \end{array}\right] + \boldsymbol{u}, \tag{2}$$

where $R := |\boldsymbol{R}|, r := |\boldsymbol{r}|$, $|\boldsymbol{R}+\boldsymbol{r}|^2 := x^2 + y^2 + (z-R)^2$, and ω is the orbital rate of the rotating coordinate system. Let h be the orbital angular momentum of the target. Then, $R^2\omega = h =$ constant. Let $e \in [0, 1)$ be the eccentricity of the target orbit, ν the true anomaly, $\rho = 1 + e\cos\nu$, and

$$k = \mu/h^{\frac{3}{2}} = \text{constant},\ \omega = h/R^2 = k^2\rho^2. \tag{3}$$

The eccentric anomaly E and the true anomaly ν satisfy the following equations:

$$\sin(E) = \frac{\sqrt{1-e^2}\sin(\nu)}{1+e\cos(\nu)},\ \ \cos(E) = \frac{e+\cos(\nu)}{1+e\cos(\nu)} \tag{4}$$

In addition, the eccentric anomaly E and time t satisfy the well-known Kepler's equation: $t = T_o\big(E - e\sin(E)\big)/2\pi$, where T_o is the time period of the orbit. The nonlinear equations of motion given in (2) can be rewritten in state space form as follows:

$$\dot{\boldsymbol{x}}_c = \boldsymbol{f}(\boldsymbol{x}_c, \boldsymbol{u}), \tag{5}$$

where $\boldsymbol{x}_c = [x\ y\ z\ \dot{x}\ \dot{y}\ \dot{z}]^{\mathrm{T}}$. The vectors $[x\ y\ z]^{\mathrm{T}}$ and $[\dot{x}\ \dot{y}\ \dot{z}]^{\mathrm{T}}$ correspond to, respectively, the position and velocity of the active chaser spacecraft with respect to the target spacecraft in the LVLH frame.

B. Discrete-time nonlinear model

A classical fourth order Runga Kutta discretization method [25] is used to convert the continuous-time nonlinear dynamical system given by Eq. (5) to a discrete-time nonlinear dynamical system as follows:

$$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) + (T/6)\left(\boldsymbol{k}_{|1} + 2\boldsymbol{k}_{|2} + 2\boldsymbol{k}_{|3} + \boldsymbol{k}_{|4}\right), \tag{6}$$

where $k \in [0, N-1]_d$, $t_k = \frac{t_f}{N}k = Tk$, t_f is the final time, $T > 0$ is the sampling period, $\boldsymbol{k}_{|1}$, $\boldsymbol{k}_{|2}$, $\boldsymbol{k}_{|3}$, and $\boldsymbol{k}_{|4}$ are given as follows [25]:

$$\begin{aligned} &\boldsymbol{k}_{|1} = \boldsymbol{f}(\boldsymbol{x}(k), \boldsymbol{u}(k)), \quad \boldsymbol{k}_{|2} = \boldsymbol{f}(\boldsymbol{x}(k) + T\boldsymbol{k}_{|1}/2, \boldsymbol{u}(k)) \\ &\boldsymbol{k}_{|3} = \boldsymbol{f}(\boldsymbol{x}(k) + T\boldsymbol{k}_{|2}/2, \boldsymbol{u}(k)), \\ &\boldsymbol{k}_{|4} = \boldsymbol{f}(\boldsymbol{x}(k) + T\boldsymbol{k}_{|3}, \boldsymbol{u}(k)), \end{aligned}$$

The state of the continuous-time system $\boldsymbol{x}_c$ and the state $\boldsymbol{x}$ of the discrete-time system are related as follows: $\boldsymbol{x}_c(t_k) \approx \boldsymbol{x}(k)$. From Eq. (6), the discrete nonlinear spacecraft rendezvous can be written in compact form as follows.

$$\boldsymbol{x}(k+1) = \boldsymbol{h}(\boldsymbol{x}(k), \boldsymbol{u}(k)). \tag{7}$$

Next, we present the problem that we address in this paper.

Problem 1: Given the discrete-time nonlinear dynamics (7), $N > 0$, the initial state $\boldsymbol{x}_0$ and the final $\boldsymbol{x}_f$, find the control input $\boldsymbol{u}^\star(k)$ for all $k \in [0, N-1]_d$ which will steer the active spacecraft from initial state $\boldsymbol{x}_0$ to final state $\boldsymbol{x}_f$ at $k = N$ while minimizing the following performance index:

$$J_{2,1}\left(\boldsymbol{u}^\star(k)\right) := \sum_{i=0}^{N-1} \|\boldsymbol{u}(i)\|_2. \tag{8}$$

The solution to Problem 1 poses significant challenges and requires the use of computationally expensive and sophisticated optimization algorithms [6], [13], [14]. Instead, we propose the following two-step approach. First, we use a Koopman operator based approach to approximate the discrete nonlinear model (7) to a higher dimensional (lifted) linear state space model. Second, we exploit the linearity of this lifted state space model to solve the minimum-fuel problem for a thrust vectoring spacecraft.

III. Koopman operator

A. Quick review of Koopman operator

The Koopman operator $\mathcal{K} : \mathcal{F} \rightarrow \mathcal{F}$ is an infinite-dimensional operator which operates on the lifted space, $\mathcal{F}$, which is a function space comprised of observable functions (or observables). The lifted space is invariant under the action of the operator $\mathcal{K}$. In addition, for a given collection of observable functions, $\boldsymbol{g} = [g_1, g_2, \dots, g_{N_k}]^{\mathrm{T}}$, where $g_i(\boldsymbol{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$, for $i \in \{1, \dots, N_k\}$ and $N_k \gg n$, ($\boldsymbol{g}$ corresponds to a finite truncation of an infinite collection of basis functions that span the lifted space $\mathcal{F}$), it holds that $(\mathcal{K}\boldsymbol{g})\boldsymbol{x}(k) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}(k))) = \boldsymbol{g}(\boldsymbol{x}(k+1))$, which implies that the evolution of this set of functions is linear. In contrast with linearization-based approximations of nonlinear dynamics around a fixed linearization point which become less accurate as one moves away from the latter point, the Koopman operator describes the exact evolution of the observables of a nonlinear system globally. Finally, we will refer to the vector $\boldsymbol{z}(k) = \boldsymbol{g}(\boldsymbol{x}(k))$ as the lifted state as it corresponds to the state of the system in the lifted state space (in which its evolution is linear). For more details, the reader can refer to [16], [23].

B. Lifted dynamics for rendezvous operations

In this section, we present the main steps for the approximation of the discrete-time nonlinear rendezvous equation (7) with higher dimensional linear state space model using Koopman operator. Consider the discrete-time nonlinear rendezvous equation given in Eq. (7). Our goal is to approximate Eq. (7) as the following linear lifted state space model

$$\boldsymbol{z}(k+1) = A_{\mathrm{koop}}\boldsymbol{z}(k) + B_{\mathrm{koop}}\boldsymbol{u}(k), \tag{9}$$

where N_k is the dimension of the lifted state $\boldsymbol{z}(k)$, $A_{\mathrm{koop}} \in \mathbb{R}^{N_k \times N_k}$, $B_{\mathrm{koop}} \in \mathbb{R}^{N_k \times m}$, $\boldsymbol{z}(k) \in \mathbb{R}^{N_k}$, $\boldsymbol{u}(k) \in \mathbb{R}^m$ and $k \in [0, N-1]_d$. The initial condition $\boldsymbol{z}_0$ is given by

$$\boldsymbol{z}_0 = \boldsymbol{g}(\boldsymbol{x}_0) = [g_1(\boldsymbol{x}_0), \quad g_2(\boldsymbol{x}_0), \dots, g_{N_k}(\boldsymbol{x}_0)]^{\mathrm{T}} \tag{10}$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ is the initial condition for the original discrete nonlinear equation given in Eq. (7). The terminal state $\boldsymbol{z}(N)$ can be written as

$$\boldsymbol{z}(N) = A_{\mathrm{koop}}^{N}\boldsymbol{z}_0 + \sum_{\tau=0}^{N-1} A_{\mathrm{koop}}^{N-1-\tau} B_{\mathrm{koop}}\boldsymbol{u}(\tau), \tag{11}$$

where $\boldsymbol{z}(N) = \boldsymbol{g}(\boldsymbol{x}(N))$ and $\boldsymbol{x}(N)$ denotes the terminal state of the original discrete-time nonlinear equation (7). The terminal state can be rewritten in a compact form as

$$\boldsymbol{z}(N) = \boldsymbol{C}_{N_{\mathrm{koop}}}\boldsymbol{u}_{\mathrm{koop}} + \boldsymbol{\beta}_{\mathrm{koop}}, \tag{12}$$

where $\boldsymbol{C}_{N_{\mathrm{koop}}} \in \mathbb{R}^{N_{\mathrm{k}} \times Nm}$, $\boldsymbol{u}_{\mathrm{koop}} \in \mathbb{R}^{Nm}$ and $\boldsymbol{\beta}_{\mathrm{koop}} \in \mathbb{R}^{N_{\mathrm{k}}}$ are defined as follows:

$$\boldsymbol{u}_{\mathrm{koop}} := [\boldsymbol{u}(0)^{\mathrm{T}},\ \boldsymbol{u}(1)^{\mathrm{T}},\ \dots, \boldsymbol{u}(N-1)^{\mathrm{T}}]^{\mathrm{T}}, \tag{13a}$$

$$\boldsymbol{C}_{N_{\mathrm{koop}}} := [A_{\mathrm{koop}}^{N-1}B_{\mathrm{koop}}, \dots, B_{\mathrm{koop}}], \tag{13b}$$

$$\boldsymbol{\beta}_{\mathrm{koop}} := A_{\mathrm{koop}}^{N}\boldsymbol{z}_0. \tag{13c}$$

C. A data-driven method to compute A_{koop} and B_{koop}

In our problem, the nonlinear spacecraft rendezvous dynamics is described by the discrete-time state space model (7) which is known a priori. We now use a data-driven approach to approximate the matrices A_{koop} and B_{koop} that appear in (9). To this aim, a set of random control inputs and a set of initial states $\boldsymbol{x}_0$ that correspond to a random sample from the uniform distribution $[-1, 1]$ are used. These randomly generated control inputs are applied sequentially to (7) with initial state $\boldsymbol{x}_0$ to get the subsequent states. Let the control input $\boldsymbol{u}(k)$ be applied to take the state of the active spacecraft from $\boldsymbol{x}(k)$ to $\boldsymbol{x}(k+1)$. In this way, we construct the matrices $\boldsymbol{X}, \boldsymbol{U}$, and $\boldsymbol{Y}$ where $\boldsymbol{X} := [\boldsymbol{x}(0), \dots, \boldsymbol{x}(d)]$, $\boldsymbol{U} := [\boldsymbol{u}(0), \dots, \boldsymbol{u}(d)]$, and $\boldsymbol{Y} = [\boldsymbol{x}(1), \dots, \boldsymbol{x}(d+1)]$ where $(d+1)$ is the number of data points. The matrix $\boldsymbol{Y}$ can be expressed as $\boldsymbol{Y}{=}\boldsymbol{f}(\boldsymbol{X}, \boldsymbol{U})$.

Given the data $\boldsymbol{X}, \boldsymbol{Y}$, and $\boldsymbol{U}$, the matrices A_{koop} and B_{koop} in (9) are obtained via the solution to the following least squares optimization problem:

$$\min_{A_{\mathrm{koop}}, B_{\mathrm{koop}}} \|\boldsymbol{Y}_{\mathrm{lift}} - A_{\mathrm{koop}}\boldsymbol{X}_{\mathrm{lift}} - B_{\mathrm{koop}}\boldsymbol{U}\|_F\,, \tag{14}$$

where $\boldsymbol{X}_{\mathrm{lift}}{=}[\boldsymbol{g}(\boldsymbol{x}(0)), \dots, \boldsymbol{g}(\boldsymbol{x}(d))]$ and $\boldsymbol{Y}_{\mathrm{lift}}{=}[\boldsymbol{g}(\boldsymbol{x}(1)), \dots, \boldsymbol{g}(\boldsymbol{x}(d+1))]$ with $\boldsymbol{g}(\boldsymbol{x}){=}[g_1(\boldsymbol{x}), \dots, g_{N_k}(\boldsymbol{x})]^{\mathrm{T}}$, being a given collection of nonlinear observable functions $g_i(\boldsymbol{x})$, for $i \in \{1, \dots, N_k\}$. The symbol $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. The analytical solution to (14) is given by $[A_{\mathrm{koop}}, B_{\mathrm{koop}}] = \boldsymbol{Y}_{\mathrm{lift}}\ [\boldsymbol{X}_{\mathrm{lift}}, \boldsymbol{U}]^{\dagger}$ where $(.)^{\dagger}$ denotes the Moore-Penrose pseudoinverse operator.

IV. Proposed solution approach for the minimum fuel problem based on the IRLS algorithm

Now that we have approximated the matrices A_{koop} and B_{koop} of the lifted space dynamics (9), a modified version of Problem 1 is presented next.

Problem 2: Let $\boldsymbol{x}_0,\ \boldsymbol{x}_f \in \mathbb{R}^6$ and $N > 0$ be given. Find a control sequence $\boldsymbol{u}_{\mathrm{koop}}^{\star}(k) \in \mathbb{R}^3$ for all $k \in [0, N-1]_d$ that will minimize the performance index given in (8) and subject to the following terminal equality constraint:

$$\boldsymbol{C}_{N_{\mathrm{koop}}}\boldsymbol{u}_{\mathrm{koop}} + \boldsymbol{\beta}_{\mathrm{koop}} = \boldsymbol{z}_f, \tag{15}$$

where $\boldsymbol{z}_f = \boldsymbol{g}(\boldsymbol{x}(N))$.
The proposed approach to solve Problem 2 is based on the an iterative approach known as the Iteratively Reweighted Least Squares (IRLS) algorithm. It is a popular tool for the computation of the minimum ℓ_2/ℓ_1 or ℓ_1 norm solution to an under-determined system of linear (algebraic) equations in the literature of compressive sensing [2].

A. IRLS Algorithm

The iterative approach presented here computes an approximate solution to the minimum ℓ_2/ℓ_1 norm problem in closed form via the solution of a corresponding sequence of convex quadratic programs. In particular, at every iteration j, $\boldsymbol{u}_{\mathrm{koop}}^{[j+1]}$ corresponds to the solution of the following convex quadratic program:

$$\text{(QP):}\quad \min_{\boldsymbol{u}} \sum_{i=0}^{N-1}\sum_{k=1}^{m} \boldsymbol{u}(i)^{\mathrm{T}}\boldsymbol{w}^{[j]}(k)\boldsymbol{u}(i) \ \ \text{subject to}\ \ (15)$$

Algorithm 1 IRLS algorithm for solving ℓ_2/ℓ_1 optimization problem

$\boldsymbol{w}^{[0]}(i) = 1 \; \forall \; i \in [1, Nm]_d$
$\epsilon^{[0]} = 1$
for $j = 0$ to $j_{\max}$ **do**
  for $k = 0, \ldots, N-1$ **do**
  $\mathbf{W}^{[j]}(k) = \text{diag}\left(\boldsymbol{w}^{[j]}(km+1) \ldots \boldsymbol{w}^{[j]}(km+m)\right)$
  end for
  $\boldsymbol{\mathcal{W}}^{[j]} = \text{bdiag}\left(\mathbf{W}^{[j]}(0), \ldots, \mathbf{W}^{[j]}(N-1)\right)$
  $\boldsymbol{u}_{\text{koop}}^{[j+1]} = (\boldsymbol{\mathcal{W}}^{[j]})^{-1}(\boldsymbol{C}_{N_{\text{koop}}}^{\mathrm{T}}(\boldsymbol{\mathcal{W}}^{[j]})^{-1})^{\mathrm{T}} (\boldsymbol{C}_{N_{\text{koop}}}^{\mathrm{T}}(\boldsymbol{\mathcal{W}}^{[j]})^{-1} + \mathbf{I})^{-1}(\boldsymbol{C}_{N_{\text{koop}}}^{\mathrm{T}}(\boldsymbol{\mathcal{W}}^{[j]})^{-1})^{\mathrm{T}}\boldsymbol{\beta}_{\text{koop}}$
  $\varepsilon^{[j+1]} = \min\left\{\varepsilon^{[j]}, \|\boldsymbol{u}_{\text{koop}}^{[j+1]}\|_\infty\right\}$
  for $\ell = 1, \ldots, Nm$ **do**
    $\boldsymbol{w}^{[j+1]}(\ell) = \left(\left(\boldsymbol{u}_{\text{koop}}^{[j+1]}(\ell)\right)^2 + \left(\varepsilon^{[j+1]}\right)^2\right)^{-1/4}$
  end for
  if $\epsilon \in [0, \bar{\epsilon}]$ **then**
    report "success″
  end if
end for
if $\epsilon \notin [0, \bar{\epsilon}]$ **then**
  report "failure″
end if

with $\boldsymbol{w}^{[j]} := [w^{[j]}(0)^{\mathrm{T}}, w^{[j]}(1)^{\mathrm{T}}, \ldots, w^{[j]}(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}_{>0}$, where $w^{[j]}(k) \in \mathbb{R}^m$ for all $k \in [0, N-1]_d$ and $m = 3$ for a thrust vectoring spacecraft [7]. First, the input parameters $\boldsymbol{w}^{[0]}(k)$ for all $k \in [1, Nm]$ and $\varepsilon^{[0]}$ are initialized to 1 and j is set to zero. We define the weight matrix

$$\mathbf{W}^{[j]}(k) = \text{diag}\left(\boldsymbol{w}^{[j]}(km+1), \ldots, \boldsymbol{w}^{[j]}(km+m)\right), \quad (16)$$

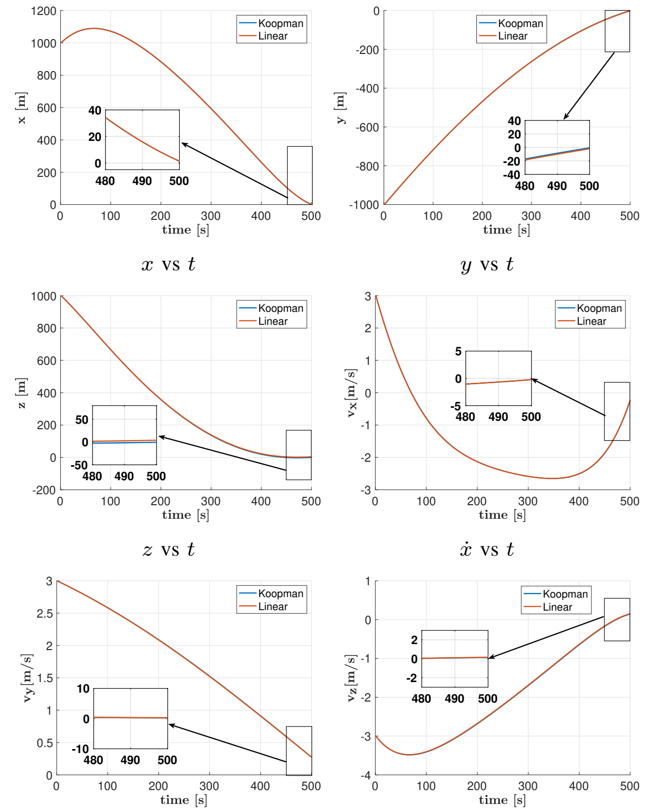
for $k \in [0, N-1]_d$, which is a positive definite matrix provided that $\boldsymbol{w}^{[j]} \geq 0$. Furthermore, let

$$\boldsymbol{\mathcal{W}}^{[j]} = \text{bdiag}\left(\mathbf{W}^{[j]}(0), \ldots, \mathbf{W}^{[j]}(N-1)\right). \quad (17)$$

Then, the solution $\boldsymbol{u}_{\text{koop}}^{[j+1]}$ to the (QP) is given by

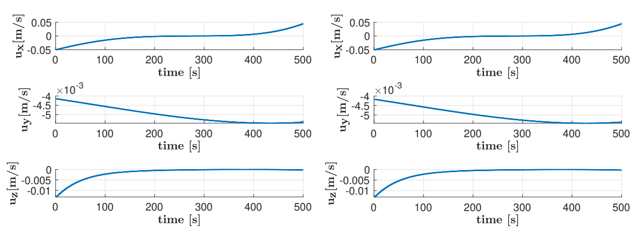
$$\boldsymbol{u}_{\text{koop}}^{[j+1]} = (\boldsymbol{\mathcal{W}}^{[j]})^{-1}(\boldsymbol{C}_{N_{\text{koop}}}^{\mathrm{T}}(\boldsymbol{\mathcal{W}}^{[j]})^{-1})^{\mathrm{T}} (\boldsymbol{C}_{N_{\text{koop}}}^{\mathrm{T}}(\boldsymbol{\mathcal{W}}^{[j]})^{-1} + \mathbf{I})^{-1}(\boldsymbol{C}_{N_{\text{koop}}}^{\mathrm{T}}(\boldsymbol{\mathcal{W}}^{[j]})^{-1})^{\mathrm{T}}\boldsymbol{\beta}_{\text{koop}}, \quad (18)$$

where $\boldsymbol{C}_{N_{\text{koop}}}$ and $\boldsymbol{\beta}_{\text{koop}}$ are given by Eqs. (13b) and (13c) respectively. The weight matrices $\mathbf{W}^{[j]}(k)$ and $\boldsymbol{\mathcal{W}}^{[j]}(k)$ are updated at every iteration and are used to compute the control sequence $\boldsymbol{u}_{\text{koop}}$ at every iteration. This sequence of control sequences ultimately converges to the optimal control sequence $\boldsymbol{u}_{\text{koop}}^{\star}$ after a certain number of iterations that minimizes the ℓ_2/ℓ_1 norm and solves Problem (2). The main steps of the IRLS algorithm, which will generate a control sequence that minimizes the ℓ_2/ℓ_1 control norm given by the performance index in (8) are described next. The value of $\varepsilon^{[j+1]}$ is now updated to $\min\left\{\varepsilon^{[j]}, \|\boldsymbol{u}_{\text{koop}}^{[j+1]}\|_\infty\right\}$, where $\|\boldsymbol{u}_{\text{koop}}^{[j+1]}\|_\infty$ denotes the ℓ_∞-norm of the vector $\boldsymbol{u}_{\text{koop}}^{[j+1]}$. The vector $\boldsymbol{w}^{[j+1]}$ is updated again as follows: $\boldsymbol{w}^{[j+1]}(\ell) = \left(\left(\boldsymbol{u}_{\text{koop}}^{[j+1]}(\ell)\right)^2 + \left(\varepsilon^{[j+1]}\right)^2\right)^{-1/4}$, for all $\ell \in [1, Nm]$, where $\boldsymbol{u}_{\text{koop}}^{[j+1]}(\ell)$ is the ℓ^{th} element of the vector $\boldsymbol{u}_{\text{koop}}^{[j+1]}$ from Eq. (18). The value of j is now set to $j+1$. Consequently, the updated $\boldsymbol{w}^{[j]}(\ell)$ is used to first update $\mathbf{W}^{[j]}(k)$ and next to update the matrices $\boldsymbol{\mathcal{W}}^{[j]}$ and $\boldsymbol{u}_{\text{koop}}^{[j+1]}$ given by Eqns. (17) and



x vs t | y vs t | z vs t | $\dot{x}$ vs t | $\dot{y}$ vs t | $\dot{z}$ vs t

Fig. 2: Evolution of states for near-field rendezvous. In this case, the control inputs $\boldsymbol{u}_{\text{lin}}$ and $\boldsymbol{u}_{\text{koop}}$ generated using the linearized dynamics (24) and the lifted space dynamics (12) respectively are able to steer the active spacecraft from initial state $\boldsymbol{x}_0$ to final state $\boldsymbol{x}_f$ with comparable accuracy. However, as seen from Table I, the Koopman operator based approach gives better performance in terms of the terminal state error.



$\boldsymbol{u}_{\text{koop}}$ for near-field rendezvous | $\boldsymbol{u}_{\text{lin}}$ for near-field rendezvous

Fig. 3: Control inputs for near-field rendezvous

(18). This operation is repeated until the sequence of control sequences $\{\boldsymbol{u}_{\text{koop}}^{[j]}\}$ converges to the optimal control sequence $\boldsymbol{u}_{\text{koop}}^{\star}$. If $\varepsilon^{[j]} \notin [0, \bar{\varepsilon}]$, two cases arise. First, if $j < j_{\max}$, then go to Eq. (16) and if $j = j_{\max}$, then conclude that the algorithm failed to converge. Hence it is suggested to set a larger $j_{\max}$ to increase the chances of success. Else if j is less than or equal to $j_{\max}$ and $\varepsilon^{[j]} \in [0, \bar{\varepsilon}]$, then Algorithm 1 is terminated successfully. The pseudo code for the IRLS algorithm is given in Algorithm 1.

V. Numerical simulations

Simulation studies presented in this section have been carried out using MATLAB R2020a on Intel Core i7 2.2GHz processor. Two cases are considered. First, we consider a near-field rendezvous in which the distance between the target spacecraft and active spacecraft is much less than the distance between the planet and the target spacecraft (i.e. $r \ll R$). Second, we consider the case for far-field rendezvous in which $R \approx r$. The target spacecraft is moving in an elliptical orbit whose semimajor axis is equal to 6763×10^3m and its eccentricity $e = 0.73074$.

Terminal state error	Koopman	Linear
Near-field rendezvous	1.6246	4.7369
Far-field rendezvous	2.9320	605.6255

TABLE I: ℓ_2 norm of the terminal state error

The nonlinear dynamics (2) is discretized using fourth order Runga Kutta method with discretization step T equal to 1s and N equal to 500. For the Koopman operator we consider $N_k = 120$. To generate the sequence of data $\boldsymbol{x}(k)$ for $k \in \{0. \ldots, d\}$, we sample 1000 initial conditions which are taken from the uniform distribution over $[-1, 1]^6$. For each sample, we apply control inputs $\boldsymbol{u}(k)$ which are taken randomly from a uniform distribution over $[-1, 1]^3$. Then, for each sample of randomly generated initial conditions, we use the discrete nonlinear dynamics in Eq. (7) to propagate the dynamics with the given control inputs $\boldsymbol{u}(k)$. For each initial condition, we simulate/propagate 2000 states along each trajectory. This data generation process results in matrices $\boldsymbol{X}, \boldsymbol{U}$ and $\boldsymbol{Y}$ of size $6 \times 2 \cdot 10^6$. Therefore, the total number of data points is equal to 1000×2000 =$2 \cdot 10^6$. The following set of observable functions were used in our simulations:

$$[g_1 \; g_2 \; g_3 \; g_4 \; g_5 \; g_6\,] = [x \; y \; z \; \dot{x} \; \dot{y} \; \dot{z}]$$

$$[g_7, g_8, g_9, g_{10}, g_{11}, g_{12}, g_{13}] = \frac{[1, \dot{x}, \dot{y}, \dot{z}, x, y, z]}{(1 + x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

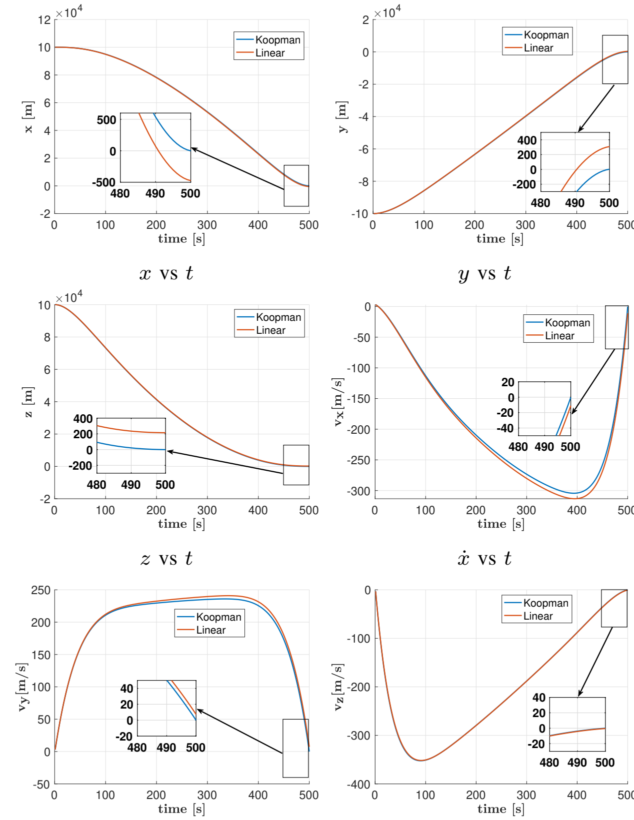
$$[g_{14}, g_{15}, g_{16}] = \frac{[x^2\dot{x}, y^2\dot{y}, z(z - \|\boldsymbol{x}_0\|_2)\dot{z}]}{(x^2 + y^2 + (z - \|\boldsymbol{x}_0\|_2)^2)^{\frac{5}{2}}}$$

$$[g_{17}, g_{18}, g_{19}] = \frac{[x, y, z]}{[x^2 + y^2 + (z - \|\boldsymbol{x}_0\|_2)^2]^{\frac{3}{2}}}$$

where $g_i(\boldsymbol{x}) = 1/\sqrt{1 + \alpha_i^2}$, $\alpha_i = \sum_{j=1}^{6}(\boldsymbol{x}(j)^2 - \boldsymbol{c}_i(j)^2)$ and $\boldsymbol{c}_i$ is a random sample taken from a uniform distribution over $[-1, 1]^6$, for $i \in [20, 120]_d$.

A. *Near-field rendezvous ($r \ll R$)*

Consider a scenario in which the active spacecraft is performing a near-field rendezvous with a target spacecraft. In this case, $10^3 \approx r \ll R \approx 10^6$. We consider the initial state $\boldsymbol{x}_0 = (10^3, -10^3, 10^3, 3, 3, -3)$ and final state $\boldsymbol{x}_f = (0, 0, 0, 0, 0, 0)$. The control inputs $\boldsymbol{u}_{\text{lin}}$ and $\boldsymbol{u}_{\text{koop}}$ are computed by using the linearized and the lifted space linear dynamics respectively. It is observed from Fig. 2 that these control inputs when applied to the nonlinear discrete rendezvous dynamics in Eq. (7) can steer the active spacecraft to the desired final states. This is mainly because for the near-field rendezvous, the linearized rendezvous equations can represent the nonlinear spacecraft rendezvous dynamics relatively well.

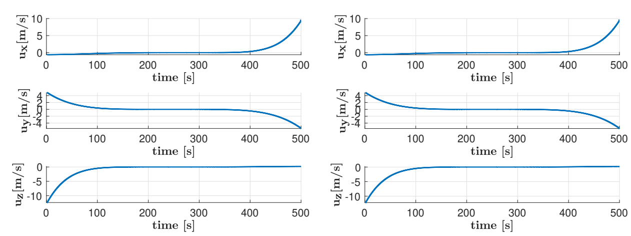


x vs t $\quad$ y vs t

z vs t $\quad$ $\dot{x}$ vs t

$\dot{y}$ vs t $\quad$ $\dot{z}$ vs t

Fig. 4: Evolution of states for far-field rendezvous. The control input $\boldsymbol{u}_{\text{koop}}$ generated using the lifted space dynamics (12) is able to steer the active spacecraft from initial state $\boldsymbol{x}_0$ to final state $\boldsymbol{x}_f$ with better accuracy than the control input $\boldsymbol{u}_{\text{lin}}$ which is generated using the linear dynamics.



$\boldsymbol{u}_{\text{koop}}$ for far-field rendezvous $\quad$ $\boldsymbol{u}_{\text{lin}}$ for far-field rendezvous

Fig. 5: Control inputs for far-field rendezvous

B. *Far-Field rendezvous ($R \approx r$)*

Now, consider the case of far-field rendezvous where $R \approx r \approx 10^6$. We consider the initial state $\boldsymbol{x}_0 = (10^5, -10^5, 10^5, 3, 3, -3)$ and final state $\boldsymbol{x}_f = (0, 0, 0, 0, 0, 0)$. Again, the control inputs $\boldsymbol{u}_{\text{lin}}$ and $\boldsymbol{u}_{\text{koop}}$ are generated using the linearized and lifted space dynamics respectively. It is observed that $\boldsymbol{u}_{\text{lin}}$ is not able to steer the active spacecraft to the desired final states as shown in Fig. 4. However, $\boldsymbol{u}_{\text{koop}}$ is able to steer the active spacecraft to the desired final states. It can also be observed from Table I that the ℓ_2-norm of the terminal state error is orders of magnitude higher for near-field rendezvous than in the case of far-field rendezvous.

VI. Conclusions

We have presented an iterative scheme for the computation of approximate solutions to the minimum-fuel far-field spacecraft rendezvous problem for a thrust vectoring spacecraft. The proposed approach uses tools from Koopman operator theory to associate the nonlinear dynamics of the rendezvous problem with a linear system of higher dimension which evolves in the so-called lifted state space (or space of observables). The latter linear system is subsequently used together with an Iteratively Recursive Least Squares (IRLS) algorithm to generate approximate solutions to the minimum-fuel rendezvous problem for a thrust vectoring spacecraft. Our numerical simulations have shown that the control input computed based on the lifted space dynamics can steer the system to its goal terminal state for both near-field and far-field rendezvous with accuracy which is orders of magnitude higher than the linearization-based methods, which only produced satisfactory results for the short-field rendezvous problem.

VII. Appendix

In this section, we present the Tschauner–Hempel (T–H) linearized equations for spacecraft rendezvous. Consider the following linearized rendezvous equation given by:

$$\dot{\boldsymbol{x}}(t) = A_c(t)\boldsymbol{x}(t) + B_c(t)\boldsymbol{u}(t) \tag{19}$$

where $A_c(t)$ and $B_c(t)$ are given as follows

$$A_c(t) = \begin{bmatrix} \mathbb{O}^{3\times 3} & \mathbb{I}^3 \\ A_1 & A_2 \end{bmatrix}, \quad B_c = [\mathbb{O}^{3\times 3} \quad \mathbb{I}^3]^{\mathrm{T}}$$

and A_1 and A_2 are given as follows:

$$A_1 = \begin{bmatrix} \omega^2 - k\omega^{\frac{3}{2}} & 0 & \dot{\omega} \\ 0 & -k\omega^{3/2} & 0 \\ -\dot{\omega} & 0 & \omega^2 + 2k\omega^{3/2} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 2\omega \\ 0 & 0 & 0 \\ -2\omega & 0 & 0 \end{bmatrix}.$$

If $R \gg r$, then the system in (2) can be linearized about the origin and can be described by the following non-autonomous discrete-time state space model:

$$\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k) + B(k)\boldsymbol{u}(k), \quad k \in [0, N-1]_d \tag{20}$$

where the matrices $A(k)$ and $B(k)$ are defined as follows:

$$A(k) = \Phi(t_{k+1}, t_k), \ B(k) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \sigma) B_c d\sigma,$$

where Φ is the state transition matrix. Using Eq. (20), it follows that the terminal state at $k = N$ is given by

$$\boldsymbol{x}(N) = \Phi_d(N, 0)\boldsymbol{x}(0) + \sum_{\tau=0}^{N-1} \Phi_d(N, \tau+1)B(\tau)\boldsymbol{u}(\tau). \tag{22}$$

where the state transition matrix of the discrete-time system (20), $\Phi_d(k, m)$, is introduced as follows:

$$\Phi_d(k, m) = \begin{cases} A(k-1)\dots A(m), & k > m \geq 0, \\ \mathbf{I}_6, & k = m, \end{cases} \tag{23}$$

where k and m are non negative integers. From Eq. (22), the terminal state $\boldsymbol{x}(N) = \boldsymbol{x}_f$ can be written as follows:

$$\boldsymbol{x}(N) = \boldsymbol{\beta} + \boldsymbol{C}_N \boldsymbol{u}_{\text{lin}}, \tag{24}$$

where $\boldsymbol{u}_{\text{lin}}$, $\boldsymbol{\beta}$ and $\boldsymbol{C}_N$ are given by

$$\boldsymbol{u}_{\text{lin}} = [\boldsymbol{u}(0)^{\mathrm{T}}, \ \dots \boldsymbol{u}(N-1)^{\mathrm{T}}]^{\mathrm{T}}, \ \ \boldsymbol{\beta} = \Phi_d(N, 0)\boldsymbol{x}(0)$$
$$\boldsymbol{C}_N = [\Phi_d(N, 1)B(0), \ \Phi_d(N, 2)B(1), \dots B(N-1)]$$

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