Path-Dependent Controller and Estimator Synthesis with Robustness to Delayed and Missing Data

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ABSTRACT

This paper presents path-dependent feedback controllers and estimators with bounded tracking and estimation error guarantees for discrete-time affine systems with time-varying delayed and missing data, where the set of all temporal patterns for the missing or delayed data is constrained by a fixed-length language. In particular, we propose two controller/estimator synthesis approaches based on output feedback and output error feedback parameterizations such that the tracking or estimation errors satisfy a property known as equalized recovery, where the errors are guaranteed to satisfy a recovery level at the start and the end of a finite time horizon, but may temporarily increase (by a bounded amount) within the horizon. To achieve this, we introduce a mapping of the fixed-length delayed/missing data language onto a reduced event-based language, and present designs with feedback gain matrices that adapt based on the observed path in the reduced language, resulting in improved performance. Furthermore, we propose a word observer that finds the set of words (i.e., the delayed/missing data patterns) in the original fixed-length language that are compatible with the observed path. The effectiveness of the proposed approaches when compared to existing approaches is demonstrated using several illustrative examples.

CCS CONCEPTS

• **Computing methodologies → Computational control theory**.

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1 INTRODUCTION

Cyber-physical systems (CPS), e.g., self-driving vehicles, smart medical devices and autonomous robot swarms, integrate networked computation and physical processes, often with a shared communication channel. This channel is used to send measured data from sensors to controllers that then determine the control input commands to the actuators to operate/regulate physical systems/processes. However, missing data or delayed measurements caused by sensor malfunctions or communication network congestion/losses [28, 35] could degrade the control performance and potentially lead to unsafe system behaviors. Thus, to guarantee the safe operation of safety-critical CPS, there is a need for control and estimation algorithms that are robust to delayed and/or missing data.

Literature review: The controller and estimator synthesis problem for systems subject to missing or delayed data or measurements have been extensively investigated in the context of networked control systems (e.g., [4, 28, 35]) as well as in emerging security problems involving denial of service or false data injection attacks (e.g., [1, 11, 32]). For missing and delayed data or measurements modeled by probability distributions, extensions of the Kalman filter have been proposed (e.g., [3, 17, 28]) to estimate the system state, including the complete in-sequence information method in [34] and the nonlinear Bayes filter in [31] that recalculates the state estimates once the delayed measurement arrives at the current time. Similarly, stabilizing or optimal controllers have been studied in this setting of probabilistic data loss/delay models (e.g., [12, 26, 35]). However, these works primarily aim to achieve the best expected or average estimation/control performance, while safety-critical applications often require worst-case estimation and tracking error guarantees.

Another modeling approach for time-varying missing and delayed data is to characterize the set of all admissible temporal patterns of missing or delayed data, e.g., 'data are delayed by at most 3 time steps,' using automata [19] or fixed-length languages [14, 23–25]. Theoretical analysis of observability, controllability and stabilization for (noiseless) discrete-time linear systems subject to missing and delayed data have been studied in [18, 19], and more recently, finite-horizon controller and estimator design has been considered in [14, 23–25], where the goal is to guarantee a property known as *equalized recovery*, i.e., the tracking and estimation error in the presence of missing and delayed data could have a more relaxed upper bound within the finite horizon, but is guaranteed to recover and return back to the initial bounds by the end of the horizon. The notion of equalized recovery is an extension of *equalized performance* in observer designs (e.g., [8, 9]) and set invariance in control (e.g., [7]), which require that the estimation errors or state bounds are invariant. In particular, [23, 24] developed a prefix-based controller/estimator for systems with missing data, similar to the setting we consider; however, this approach does not directly apply to the delayed data case that we consider in this paper. Moreover, when compared with results in [14] that does consider delayed data, our approach allows for adaptation based on the observed path, resulting in better equalized recovery performance.

A further relevant research area pertains to measurement scheduling in control systems (e.g., [2, 10]). However, this research differs

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from our setting where the data patterns are not scheduled but adversarially chosen from the set of admissible patterns.

Contribution: In this paper, we design path-dependent finite-horizon controllers and estimators that achieve equalized recovery for time-varying affine systems when the output/measurement data is prone to time-varying misses and delays (including out-of-sequence data), where the temporal pattern of the delayed/missing data phenomenon is constrained to a set of all possible patterns using a fixed-length language. To tackle this problem, we propose finite-horizon affine feedback laws based on two parameterization approaches that are commonly used in the optimal control literature, that is with output feedback and output error (or disturbance) feedback, that were found to be equivalent in the absence of missing or delayed data (e.g., [13, 29]). In particular, we extend existing equalized recovery controllers/estimators for both parameterizations (e.g., [14, 23–25]), to allow time-varying and path-dependent intermediate levels and consider more general polytopic sets for describing the tracking or estimation error bounds.

Additionally, we construct a reduced event-based language with unique event sequences and synthesize feedback gains for each of these unique event sequences in the reduced language, instead of each possible event sequence, in a manner that resolves conflicts arising from the ambiguity between event sequences with only partial observations of the sequence from the history of observed data patterns/subsequences up to the current time, i.e., the observed path. This enables our controllers/estimators to adapt based on the observed path, resulting in marked improvements over existing works [14, 25] that can only consider the worst-case missing or delayed data pattern within the language. Moreover, the proposed controllers/estimators are applicable for delayed data patterns, in addition to missing data patterns that was considered in [23, 24]. Further, we design a word observer that can estimate the set of all missing/delayed data patterns that are compatible with the observed path at each time step, which can be useful for fault/attack pattern identification and communication network optimization. These improvements are illustrated in several simulation examples.

2 PROBLEM FORMULATION

2.1 System Dynamics and Delayed Data Language

System Dynamics: We consider a discrete-time affine time-varying system subject to bounded process and measurement noise. The model of the system dynamics is given by:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + W_k w_k + f_k, \\ z_k &= C_k x_k + V_k v_k, \\ Y_k &= \{z_{k-\omega(i)} | i + \omega(i) = k, i \le k\}, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state at time k, $u_k \in \mathbb{R}^m$ is the input to the system, $w_k \in \mathbb{R}^{n_w}$ is the process noise, $v_k \in \mathbb{R}^{n_v}$ is the measurement noise, $z_k \in \mathbb{R}^p$ is the time-stamped observation or measurement that is possibly affected by a time-delay attack or any naturally occurring delay, $Y_k \subset \mathbb{R}^p$ is the set of all measurement data that is received at time step k, $\omega(i)$ is the time delay associated with z_i at the time step i and satisfies $\omega(i) \in \mathbb{N}_0^{\overline{\omega}}$, and $\overline{\omega}$ is an upper bound on the number of time steps that a packet is delayed by. The discrete variable $\omega(i) = 0$ denotes that the i-th measurement is received without delay, while $\omega(i) = a$ implies that the i-th measurement is delayed by a steps. The system matrices A_k, B_k, C_k, W_k, V_k and f_k are all known and of appropriate dimensions. We assume that the process and measurement noises w_k and v_k are unknown but polyhedrally constrained with $w_k \in \mathcal{W} = \{w \in \mathbb{R}^{n_w} | P_w w \le q_w\}$ and $v_k \in \mathcal{V} = \{v \in \mathbb{R}^{n_v} | P_v v \le q_v\}$ for every time step k, respectively. In addition, we suppose that the control input u_k is bounded with $u_k \in \mathcal{U} = \{u \in \mathbb{R}^m \mid P_u u \le q_u\}$. Without loss of generality, we assume that the initial time is $k = 0$.

Delayed Data Language [15]: Given a fixed length T, we consider a delayed data model where delayed data patterns are restricted to a set expressed by fixed-length language specifications, e.g., 'the i-th observation is delayed by at most m time steps' or 'at most m delayed/missing measurements in a fixed interval.' The delayed data patterns can be the result of naturally occurring delays or packet dropouts due to communication network congestion or losses, or caused by deliberate (cyber) attacks by an adversary or hacker. Formally, our delayed data model is a fixed-length language $\mathcal{L} = \{\mathcal{W}_\alpha\}_{\alpha=1}^{|\mathcal{L}|}$ that specifies the set of allowable delay mode sequences $\omega_\alpha(0)\omega_\alpha(1)\omega_\alpha(2)\ldots\omega_\alpha(T-1)$, where the α-th possible sequence is called a word $\mathcal{W}_\alpha$. Note that $i + \omega(i) > T$ implies that the i-th measurement is delayed beyond the horizon T, which is equivalent to the case that the i-th measurement is missing. Thus, in our setting, missing data can be considered as a special case of delayed data.

2.2 Equalized Recovery

One of the main objectives of this paper is to design a path-dependent bounded-error estimator, where the estimation error is guaranteed to return/recover to the same bound that it started with after a fixed number of time steps, as an extension of the notion of *equalized performance* in [8, 9]. Another problem of interest is to synthesize a controller that can ensure that the states of the closed-loop system remain within a certain distance from the origin (i.e., stable in the sense of Lyapunov), while being subjected to input constraints. Moreover, we can pose a similar problem for tracking control to track a given desired state trajectory $x_{d,0}, x_{d,1}, \ldots, x_{d,T}$ (and its corresponding $u_{d,0}, u_{d,1}, \ldots, u_{d,T-1}$ such that $x_{d,k+1} = A_k x_{d,k} + B_k u_{d,k} + f_k$), and the objective is to guarantee that the bounds on the tracking error $x_{\xi,k} \triangleq x_k - x_{d,k}$ recover to the initial error bound, with a potential temporary (bounded) increase within the horizon due to missing/delayed data.

The bounded-error estimator and feedback control synthesis problems can be formulated as a generic equalized recovery problem for a transformed system:

$$\begin{aligned} x_{\xi,k+1} &= A_k x_{\xi,k} + B_{\xi,k} u_{\xi,k} + W_k w_k + f_{\xi,k}, \\ z_{\xi,k} &= C_k x_{\xi,k} + V_k v_k, \\ Y_{\xi,k} &= \{z_{\xi,k-\omega(i)} | i + \omega(i) = k, i \le k\}, \end{aligned} \tag{2}$$

where the transformed states $x_{\xi,k}$, the transformed output $z_{\xi,k}$, the set of available transformed outputs $Y_{\xi,k}$, the transformed input $u_{\xi,k} \in \mathcal{U}_\xi$, $B_{\xi,k}$ and $f_{\xi,k}$ depend on the problem of interest.

Specifically, for the bounded-error estimator design problem, the estimation error system for the state estimation error given by $x_{\xi,k} \triangleq x_k - \hat{x}_k$ can be found to be of the form in (2) with $B_{\xi,k} \triangleq I$, $f_{\xi,k} \triangleq 0$, $\mathcal{U}_\xi \triangleq \mathbb{R}^m$, and the transformed output $z_{\xi,k} \triangleq z_k - C\hat{x}_k$, where $\hat{x}(t)$ is a known signal obtained from the following observer:

$$\hat{x}_{k+1} = A_k\hat{x}_k + B_k u_k - u_{e,k} + f_k, \quad (3)$$

where the injection term $u_{\xi,k} \triangleq u_{e,k}$ is the transformed input.

On the other hand, the constrained feedback controller synthesis problem for the system with delayed and missing data in (1) is one with the system dynamics of the form in (2) with $B_\xi \triangleq B$, $f_{\xi,k} \triangleq f_k$, $u_{\xi,k} \triangleq u_k$, $z_{\xi,k} \triangleq z_k$ and $x_{\xi,k} \triangleq x_k$, as well as $\mathcal{U}_\xi \triangleq \mathcal{U}$. Further, for the tracking control problem with a desired trajectory that satisfies $x_{d,k+1} = A_k x_{d,k} + B_k u_{d,k} + f_k$ over a horizon T, the corresponding tracking error system dynamics takes the form in (2) with $x_{\xi,k} \triangleq x_k - x_{d,k}$, $B_{\xi,k} \triangleq B_k$, $f_\xi \triangleq 0$, $u_{\xi,k} \triangleq u_k - u_{d,k}$, $z_\xi(t) \triangleq z_k - Cx_{d,k}$, and $\mathcal{U}_\xi \triangleq \{u_{\xi,k} \in \mathbb{R}^m \mid u_{\xi,k} + u_{d,k} \in \mathcal{U}\}$.

Formally, we consider the following *equalized recovery* property that we wish to achieve with our proposed feedback controllers and bounded-error estimators, which generalizes the definition in [23] (where the polyhedral sets are chosen as hypercubes and the intermediate level is time-invariant):

Definition 1 (Equalized Recovery). *A controller/estimator is said to achieve an equalized recovery level μ_1 at time $t = 0$ with recovery time T and intermediate levels $\mu_{2,k} \geq \mu_1$ if for any $x_{\xi,0} \in \mathcal{X}_0 \triangleq \{x \in \mathbb{R}^n | Px \leq \mu_1 q\}$, we must have $x_{\xi,k} \in \mathcal{X}_k \triangleq \{x \in \mathbb{R}^n | Px \leq \mu_{2,k} q\}$ for all $k \in [0,T]$ and $x_{\xi,T} \in \mathcal{X}_0$, where μ_1 and $\mu_{2,k}$ for all $k \in [0,T]$ are scalars, and $\mathcal{X}_0$ and $\mathcal{X}_k$ are polyhedral sets.*

2.3 Problem Statement

We aim to design a path-dependent bounded-error estimator and/or synthesize a path-dependent feedback controller, that satisfies *equalized recovery*, which can be stated as follows:

Problem 1 (Controller/Estimator Design). *Given the system dynamics in (2), a desired recovery level μ_1, a recovery time T as a time horizon and a delayed data model specified by a language $\mathcal{L}$ as well as an initial state $x_{\xi,0}$ satisfying $x_{\xi,0} \in \mathcal{X}_0$, find an optimal affine feedback law $u_{\xi,k}$ that minimizes a cost $J(\{\mu_{2,k}\}_{k=0}^T)$ subject to $\mu_{2,k} \geq \mu_1$, $x_{\xi,k} \in \mathcal{X}$, $\forall k \in [0,T]$ and $x_{\xi,T} \in \mathcal{X}_0$ (cf. Definition 1).*

Specifically, we will address Problem 1 by investigating two affine feedback laws in Section 3.2 that are commonly used in the finite-horizon optimal control literature (e.g., [13, 29]), namely with output feedback and output error feedback parameterizations. Moreover, we design a word observer to estimate, at each step, the set of all potential data patterns (i.e., words) from delayed data language that are compatible with observed data patterns. This will enable the identification of delayed data patterns that can, in turn, be useful for communication network optimization or attack mitigation.

Problem 2 (Word Observer Design). *Given a delayed data model described by a fixed-length language $\mathcal{L}$, design a word observer that can estimate the set of all potential words (i.e., delayed data patterns) from the language $\mathcal{L}$ that are compatible with observed data patterns/path at each time step k.*

3 DESIGN APPROACH

In this section, we propose controller/observer design approaches to solve Problems 1 and 2. We first construct an event-based language $\mathcal{L}^E$ from the delayed data language $\mathcal{L}$. Afterwards, path-dependent controllers/estimators based on output feedback and output error feedback will be designed, which will utilize the information from the observed data pattern/path in the event-based language seen so far to adapt their feedback gains. Moreover, for Problem 2, we design an inverse mapping algorithm that returns the set of all possible delayed data patterns that are compatible with observed data patterns up to the current time.

3.1 Event-Based Language

Given a delayed data language $\mathcal{L}$, containing all possible words corresponding to different allowable delayed/missing data patterns, an event-based language $\mathcal{L}^E$ is first constructed to capture the set of indistinguishable event sequences that correspond to the different delayed data patterns in $\mathcal{L}$. To build the event-based language, the following definitions are introduced first (with $\mathbb{N}_a^b$ as the set of natural numbers from a through b). For examples of this construction, the readers are referred to [15, Section III-A].

Definition 2 (Event [15]). *An event $e_{i,j} = d_0 d_1 d_2 \ldots d_i$ of time step $i \in \mathbb{N}_0^{T-1}$ is a finite sequence of binary variables $d_l \in \{0,1\}$ for all $l \in \mathbb{N}_0^i$, where $j \in \mathbb{N}_0^{2^{i+1}-1}$ is an index denoting the j-th potential event at time step i. The binary variable $d_l = 1$ denotes that the data of time step l is available at current time step i (i.e., all received data up until the current step i), while $d_l = 0$ signifies that the data of time step l is not available at current time step i. Moreover, an event can be defined using $e_{i,j} = \mathtt{binary}(j, i+1)$ at time step i, where the function* `binary` *returns a binary representation of the decimal number $j \in \mathbb{N}_0^{2^{i+1}-1}$ with $i+1$ digits.*

Definition 3 (Event Set [15]). *An event set $e_i = \{e_{i,j}\}_{j=0}^{2^{i+1}-1}$ is a set of all potential events at time step $i \in \mathbb{N}_0^{T-1}$.*

Intuitively, an event at time step i represents the information/data that is available up until the time step i. Since any data from previous steps or current step only has two possibilities, i.e., received or not received at the current time $i \in \mathbb{N}_0^{T-1}$, there are totally 2^{i+1} different cases. Therefore, the index j of $e_{i,j}$ varies from 0 to $2^{i+1}-1$.

Definition 4 (Event Sequence [15]). *An event sequence, denoted $\mathcal{E}_\alpha = e_{0,j_0} e_{1,j_1} e_{2,j_2} \ldots e_{T-1,j_{T-1}}$, is a sequence of events corresponding to a word $\mathcal{W}_\alpha = \{\omega(i)\}_{i=0}^{T-1}$ from the fixed length language $\mathcal{L}$, where the subscripts j_i for all $i \in \mathbb{N}_0^{T-1}$ are determined by the word $\mathcal{W}_\alpha$.*

In other words, an event sequence represents the information/data that is available/accessible at each time step. For each allowable delayed/missing data patterns (i.e., word) in a language $\mathcal{L}$, we can find its corresponding event sequence. As a result, the language $\mathcal{L} = \{\mathcal{W}_j\}_{j=1}^{|\mathcal{L}|}$ containing all allowable delayed data patterns can be mapped on a new event-based language $\mathcal{L}^E = \{\mathcal{E}_\alpha\}_{\alpha=1}^{|\mathcal{L}|}$ containing all potential event sequences. In particular, for a word $\mathcal{W}_\alpha = \omega(0)\omega(1)\omega(2)\ldots\omega(T-1)$, the subscript j_k ($k \in \mathbb{N}_0^{T-1}$) in the corresponding event sequence $\mathcal{E}_\alpha = e_{0,j_0} e_{1,j_1} e_{2,j_2} \ldots e_{T-1,j_{T-1}}$ (cf. Definition 4) can be constructed as

$$j_k = \sum_{\ell=0}^{k} 2^\ell \mathbb{1}_{\omega(k-\ell)\leq \ell}, \ \forall k \in \mathbb{N}_0^{T-1}, \quad (4)$$

where $\mathbb{1}_{\omega(k-\ell)}$ denotes an indicator defined as

$$\mathbb{1}_{\omega(k-\ell)\leq\ell} = \begin{cases} 1, & \omega(k-\ell) \leq \ell, \\ 0, & \omega(k-\ell) > \ell. \end{cases} \tag{5}$$

Note that the resulting event-based language $\mathcal{L}^E = \{\mathcal{E}_\alpha\}_{\alpha=1}^{|\mathcal{L}|}$ could have repeated event sequences (i.e., the mapping is surjective). Thus, we will eliminate repeated event sequences in $\mathcal{L}^E$ to obtain a reduced *event-based language* $\mathcal{L}^{E'} = \{\mathcal{E}'_\alpha\}_{\alpha=1}^{|L^{E'}|} \subseteq \mathcal{L}^E$ with unique event sequences $\mathcal{E}'_\alpha$ for $\alpha \in \mathbb{N}_1^{|L^{E'}|}$.

Having defined an event sequence, we next define the prefix of an event sequence, which will be used later to describe our controller/estimator design.

Definition 5 (Prefix of an Event Sequence [15]). *For an event sequence $\mathcal{E}'_\alpha \in \mathcal{L}^{E'}$ and $i \leq |\mathcal{E}'_\alpha|$, the length i prefix of $\mathcal{E}'_\alpha$ is defined as the event subsequence $(\mathcal{E}'_\alpha)^{[1:i]} = e_{0,j_0}e_{1,j_1}e_{2,j_2}\ldots e_{i-1,j_{i-1}}$, where $|\mathcal{E}'_\alpha|$ denotes the number of events in $\mathcal{E}'_\alpha$. The set of all non-empty prefixes of $\mathcal{E}'_\alpha$ is denoted as $Pref(\mathcal{E}'_\alpha)$.*

In addition, we define an observed path based on the history of observed transformed outputs, which we will relate to the event sequences in the reduced event-based language whose prefixes match the observed path in our design in the following section.

Definition 6 (Observed Event and Path). *Given the history of observed transformed outputs up to the current time step k, i.e., $\bigcup_{i=0}^{k} Y_{\xi,i}$, the observed event at each time step i is defined as $e_i^{obs} = d_{i,0}^{obs}d_{i,1}^{obs}d_{i,2}^{obs}\ldots d_{i,i}^{obs}$, which is a finite sequence of binary variables $d_{i,l}^{obs} \in \{0,1\}$ for all $l \in \mathbb{N}_0^i$, where the binary variable $d_{i,l}^{obs} = 1$ denotes that the data of time step l is available at time step i (i.e., $z_{\xi,l} \in \bigcup_{j=0}^{i} Y_{\xi,j}$), while $d_{i,l} = 0$ signifies that the data of time step l is not available at time step i (i.e., $z_{\xi,l} \notin \bigcup_{j=0}^{i} Y_{\xi,j}$). Then, the observed path $\mathcal{E}_k^{obs}$ is defined as the observed event subsequence at time k (that, by construction, must be a prefix of a reduced event sequence), i.e., $\mathcal{E}_k^{obs} = e_0^{obs}e_1^{obs}e_2^{obs}\ldots e_k^{obs} \in Pref(\mathcal{E}'_\alpha)$ for some $\mathcal{E}'_\alpha \in \mathcal{L}^{E'}$.*

3.2 Affine Feedback Designs

To solve Problem 1, we propose affine feedback controller and estimator designs based on two commonly used affine feedback parameterizations that can be found in the finite-horizon optimal control literature, e.g., [13, 29], namely output feedback and output error feedback parameterizations. Note that these two parameterizations have been shown to be equivalent when there is no missing or delayed data [13, Theorem 3.2]; however, it is unclear if this equivalence still holds when there is missing or delayed data.

In addition, in contrast to the designs in [14, 25] where the worst-case singleton language $\mathcal{L}^*$ is used (worst according to the partial ordering defined in [14, 25]), we allow the controller/estimator gains to adapt to the current observed path (cf. Definition 6), i.e., the currently observed partial data patterns or event subsequence. This path-dependent structure (also known as the prefix-based approach in [23, 24]) has been shown to lead to better performance than the worst-case language approach in [25] when some data may be missing, and we will show in this paper that the performance improvement is also applicable in our controller/estimator designs with either delayed or missing data.

3.2.1 Affine Feedback Parameterizations. In particular, we present two path-dependent affine feedback laws based on commonly used affine feedback parameterizations, where the path dependency refers to the adaptation of the controller/estimator design based on the currently observed data patterns or event sequence.

Output Feedback. Similar to the prefix-based output feedback designs in [23, 24], we consider a feedback policy for the transformed input u_ξ that is affine in the observed transformed outputs up to the current time $\{z_{\xi,i}\}_{i=0}^{k} = \bigcup_{i=0}^{k} Y_{\xi,i}$ and dependent on the currently observed path/event subsequence $\mathcal{E}_k^{obs} \triangleq e_0^{obs}e_1^{obs}e_2^{obs}\ldots e_k^{obs}$ (cf. Definition 6), as follows:

$$u_{\xi,k} = g_k^{\mathcal{E}_k^{obs}} + \textstyle\sum_{i=0}^{k} F_{k,i}^{\mathcal{E}_k^{obs}} \check{z}_{\xi,i}, \tag{6}$$

with

$$\check{z}_{\xi,i} = \begin{cases} z_{\xi,i}, & \text{if } z_{\xi,i} \in \bigcup_{j=0}^{k} Y_{\xi,j}, \\ 0, & \text{otherwise,} \end{cases}$$

where $F_{k,i}^{\mathcal{E}_k^{obs}} \in \mathbb{R}^{m\times p}$ and $g_k^{\mathcal{E}_k^{obs}} \in \mathbb{R}^m$ are gain matrices for this output feedback parameterization. It is noteworthy that the output feedback policy in (6) can equivalently be interpreted as

$$u_{\xi,k} = g_k^{\mathcal{E}_k^{obs}} + \textstyle\sum_{i=0}^{k} F_{k,i}^{\mathcal{E}_k^{obs}} z_{\xi,i}, \tag{7}$$

with $F_{k,i}^{\mathcal{E}_k^{obs}} = 0$ when $z_{\xi,i} \notin \bigcup_{j=0}^{k} Y_{\xi,j}$, a fact that we will leverage in our controller/estimator design with appropriate constraints on the gain matrix as will be described in more detail in Section 3.2.2.

Output Error Feedback. Moreover, we propose an extension of the output error feedback parameterization in [13, 14] to allow for path-dependency based on currently observed path/event subsequence $\mathcal{E}_k^{obs} \triangleq e_0^{obs}e_1^{obs}e_2^{obs}\ldots e_k^{obs}$ (cf. Definition 6) when there is missing or delayed data. Specifically, this approach includes the design of a Luenberger-like observer for transformed state as follows:

$$s_{k+1} = A_k s_k + B_{\xi,k}u_{\xi,k} + f_{\xi,k} + L_k^{\mathcal{E}_k^{obs}} \tilde{z}_{\xi,k}, \tag{8}$$

with

$$\tilde{z}_{\xi,k} = \begin{cases} z_{\xi,k} - C_k s_k, & \text{if } z_{\xi,k} \in \bigcup_{j=0}^{k} Y_{\xi,j}, \\ 0, & \text{otherwise,} \end{cases}$$

where $L_k^{\mathcal{E}_k^{obs}} \in \mathbb{R}^{n\times p}$ is the Luenberger-like gain as a function of the observed path $\mathcal{E}_k^{obs}$ and $u_{\xi,k} \in \mathbb{R}^n$ is the transformed input with *causal* output error injection given by the following:

$$u_{\xi,k} = v_k^{\mathcal{E}_k^{obs}} + \textstyle\sum_{i=0}^{k} M_{k,i}^{\mathcal{E}_k^{obs}} \tilde{z}_{\xi,i}, \tag{9}$$

where $M_{k,i}^{\mathcal{E}_k^{obs}} \in \mathbb{R}^{m\times p}$ and $v_k^{\mathcal{E}_k^{obs}} \in \mathbb{R}^m$ are gain matrices associated with the currently observed path/event subsequence $\mathcal{E}_k^{obs}$ at time k for the output error feedback parameterization. Similar to the previous parameterization, the observer and transformed input, (8) and (9), can equivalently be interpreted as

$$s_{k+1} = A_k s_k + B_{\xi,k}u_{\xi,k} + f_{\xi,k} + L_k^{\mathcal{E}_k^{obs}} (z_{\xi,k} - C_k s_k), \tag{10}$$

$$u_{\xi,k} = v_k^{\mathcal{E}_k^{obs}} + \textstyle\sum_{i=0}^{k} M_{k,i}^{\mathcal{E}_k^{obs}} (z_{\xi,i} - C_i s_i), \tag{11}$$

with $L_i^{\mathcal{E}_k^{obs}} = 0$ and $M_{k,i}^{\mathcal{E}_k^{obs}} = 0$ when $z_{\xi,i} \notin \bigcup_{j=0}^{k} Y_{\xi,j}$, as described in greater detail in Section 3.2.2.

3.2.2 *Constraints on Gain Matrices.* In both parameterizations, since we are interested in the *offline* design of path-dependent gain matrices during the design phase, one may consider designing these gain matrices for all possible (observed) paths/event subsequences. However, since the number of event subsequences grows exponentially with time steps k, we propose to instead consider doubles $(F^{\alpha}_{k,i}, g^{\alpha}_{k})$ or triplets $(M^{\alpha}_{k,i}, v^{\alpha}_{k}, L^{\alpha}_{k})$ for each unique event sequence $\mathcal{E}'_{\alpha}$ of the reduced event-based language $\mathcal{L}^{E'}$, which in general is a much smaller subset of all possible paths/event subsequences. In fact, this is one of the main motivations for considering delayed/missing data languages to restrict the set of possible paths/event subsequences.

Then, when the observed path $\mathcal{E}^{obs}_k$ is available at run time, we only need to select the matrix gains $(F^{\alpha}_{k,i}, g^{\alpha}_{k})$ or $(M^{\alpha}_{k,i}, v^{\alpha}_{k}, L^{\alpha}_{k})$ corresponding to the $\mathcal{E}'_{\alpha} \in \mathcal{L}^{E'}$ such that $\mathcal{E}^{obs}_k = Pref(\mathcal{E}'_{\alpha})$, i.e., we will select the gain matrices at run time as follows:

$$(F^{\mathcal{E}^{obs}_k}_{k,i}, g^{\mathcal{E}^{obs}_k}_{k}) \in \{(F^{\alpha}, g^{\alpha}) | \mathcal{E}^{obs}_k = Pref(\mathcal{E}'_{\alpha})\}, \tag{12}$$

$$(M^{\mathcal{E}^{obs}_k}_{k,i}, L^{\mathcal{E}^{obs}_k}_{k,i}, v^{\mathcal{E}^{obs}_k}_{k}) \in \{(M^{\alpha}, L^{\alpha}, v^{\alpha}) | \mathcal{E}^{obs}_k = Pref(\mathcal{E}'_{\alpha})\}, \tag{13}$$

where (F^{α}, g^{α}) and $(M^{\alpha}, L^{\alpha}, v^{\alpha})$ are stacked versions of $(F^{\alpha}_{k,i}, g^{\alpha}_{k})$ and $(M^{\alpha}_{k,i}, L^{\alpha}_{k,i}, v^{\alpha}_{k})$ as follows:

$$\begin{aligned} F^{\alpha} &= \begin{bmatrix} F^{\alpha}_{0,0} & \cdots & 0 \\ \vdots & \ddots & 0 \\ F^{\alpha}_{T-1,0} & \cdots & F^{\alpha}_{T-1,T-1} \end{bmatrix}, g^{\alpha} = \begin{bmatrix} g^{\alpha}_1 \\ \vdots \\ g^{\alpha}_{T-1} \end{bmatrix}, v^{\alpha} = \begin{bmatrix} v^{\alpha}_1 \\ \vdots \\ v^{\alpha}_{T-1} \end{bmatrix}, \\ M^{\alpha} &= \begin{bmatrix} M^{\alpha}_{0,0} & \cdots & 0 \\ \vdots & \ddots & 0 \\ M^{\alpha}_{T-1,0} & \cdots & M^{\alpha}_{T-1,T-1} \end{bmatrix}, L^{\alpha} = \begin{bmatrix} L^{\alpha}_{0} & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & L^{\alpha}_{T-1} \end{bmatrix}. \end{aligned} \tag{14}$$

However, since multiple event sequences in the reduced event-based language $\mathcal{L}^{E'}$ may share the same prefixes, these doubles and triplets cannot be designed independently, as it may result in implementation conflicts. In other words, the same prefixes of length k mean that these event sequences are not distinguishable from the history of observed path/data patterns up to time k, $\mathcal{E}^{obs}_k$, and thus, the corresponding design gains up to time k must be the same to avoid ambiguity in terms of which $(F^{\alpha}_{k,i}, g^{\alpha}_{k})$ or $(M^{\alpha}_{k,i}, v^{\alpha}_{k}, L^{\alpha}_{k})$ should be used, as discussed in detail in [24]. To remedy this, we need to design the gain matrices (F^{α}, g^{α}) and $(M^{\alpha}, L^{\alpha}, v^{\alpha})$ for each event sequence $\mathcal{E}'_{\alpha}$ of the reduced event-based language $\mathcal{L}^{E'}$ such that if two different event sequences are not distinguishable until time $\bar{k}$, then $(F^{\alpha}_{(k)}, g^{\alpha}_{(k)})$ and $(M^{\alpha}_{(k)}, L^{\alpha}_{(k)}, v^{\alpha}_{(k)})$ for both event sequences should be constrained to be the same for all $k \in \mathbb{N}^{\bar{k}-1}_0$, where the subscript (k) denotes the k-th row of the matrix.

Indistinguishability Constraints. To define these indistinguishability constraints, we will adopt the following definition from [24]:

Definition 7 (Principle Block Minor [24]). *The i-th leading principal block minor of a block matrix $M \in \mathbb{R}^{an \times bp}$, written as $\mathcal{BM}_i(M)$, is the $n \times p$ block matrix, $\mathcal{BM}_i(M) = M_{1:in,1:ip}$, for all $i \in [1, \min(a, b)]$.*

Using the above definitions, we impose the following constraint due to indistinguishability of event sequences/trajectories in $\mathcal{L}^{E'}$ for both affine feedback parameterizations:

$$C^{\mathrm{I}}(\mathcal{L}^{E'}) = \left\{ \{(F^{\alpha}, g^{\alpha})\}_{\alpha=1}^{|\mathcal{L}^{E'}|} \;\middle|\; \begin{array}{c} e \in Pref(\mathcal{E}'_{\alpha}) \cap Pref(\mathcal{E}'_{\beta}) \\ \implies \forall \mathcal{E}'_{\alpha}, \mathcal{E}'_{\beta} \in \mathcal{L}^{E'} : \\ (\mathcal{BM}_{|e|}(F^{\alpha}) = \mathcal{BM}_{|e|}(F^{\beta})) \\ \wedge ((g^{\alpha})_{(1:|e|n)} = (g^{\beta})_{(1:|e|n)}) \end{array} \right\}, \tag{15}$$

$$C^{\mathrm{II}}(\mathcal{L}^{E'}) = \left\{ \begin{array}{c} \{(M^{\alpha}, L^{\alpha} \\ v^{\alpha})\}_{\alpha=1}^{|\mathcal{L}^{E'}|} \end{array} \;\middle|\; \begin{array}{c} e \in Pref(\mathcal{E}'_{\alpha}) \cap Pref(\mathcal{E}'_{\beta}) \\ \implies \forall \mathcal{E}'_{\alpha}, \mathcal{E}'_{\beta} \in \mathcal{L}^{E'} : \\ (\mathcal{BM}_{|e|}(M^{\alpha}) = \mathcal{BM}_{|e|}(M^{\beta})) \wedge \\ (\mathcal{BM}_{|e|}(L^{\alpha}) = \mathcal{BM}_{|e|}(L^{\beta})) \wedge \\ ((v^{\alpha})_{(1:|e|n)} = (v^{\beta})_{(1:|e|n)}) \end{array} \right\}, \tag{16}$$

where the former is for the output feedback parameterization and the latter for the output error feedback parameterization.

Intuitively, if any pair of event sequences shares the same prefix of a particular length, they are indistinguishable at the corresponding time step based on the received information. Since they are indistinguishable (and future information is inaccessible in a causal system), their associated submatrices and subvectors must be constrained to be the same to avoid conflicts during implementation.

Delayed/Missing Data Constraints. Moreover, as described in the previous section, instead of using switched feedback laws in (6), (8) and (9) due to the delayed/missing data, we will equivalently employ the non-switched feedback laws in (7), (10) and (11) by imposing appropriate constraints on F^{α}, M^{α} and L^{α} for each event sequence $\mathcal{E}'_{\alpha} \in \mathcal{L}^{E'}$ associated with delayed/missing data patterns, where all the entries in F^{α}, M^{α} and L^{α} corresponding to unavailable data should also be set to zero. To construct this constraint on F^{α}, M^{α} and L^{α}, we first define an *event matrix* associated with the event sequence $\mathcal{E}'_{\alpha} \in \mathcal{L}^{E'}$:

$$E^{\alpha} = \begin{bmatrix} e^{(0)}_{0,j_0} & 0 & \cdots & 0 \\ e^{(0)}_{1,j_1} & e^{(1)}_{1,j_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e^{(0)}_{T-1,j_{T-1}} & e^{(1)}_{T-1,j_{T-1}} & \cdots & e^{(T-1)}_{1,j_{T-1}} \end{bmatrix}, \tag{17}$$

where $e^{(l)}_{i,j_i}$ specifies the $(l+1)$-th digit of event e_{i,j_i}, i.e., d_l (cf. Definition 2). Using this definition, we impose the following constraint due to delayed and missing data:

$$\mathcal{D}^{\mathrm{I}}(\mathcal{L}^{E'}) = \left\{ \{F^{\alpha}\}_{\alpha=1}^{|\mathcal{L}^{E'}|} \;\middle|\; \begin{array}{c} \forall i, j \in \mathbb{N}^T_1 : \\ F^{\alpha}_{\substack{((i-1)n:(i-E^{\alpha}(i,j))n-1, \\ (j-1)p:(j-E^{\alpha}(i,j))p-1)}} = 0, \end{array} \right\}, \tag{18}$$

$$\mathcal{D}^{\mathrm{II}}(\mathcal{L}^{E'}) = \left\{ \begin{array}{c} \{(M^{\alpha}, \\ L^{\alpha})\}_{\alpha=1}^{|\mathcal{L}^{E'}|} \end{array} \;\middle|\; \begin{array}{c} \forall i, j \in \mathbb{N}^T_1 : \\ M^{\alpha}_{\substack{((i-1)n:(i-E^{\alpha}(i,j))n-1, \\ (j-1)p:(j-E^{\alpha}(i,j))p-1)}} = 0, \\ L^{\alpha}_{\substack{((i-1)n:(i-E^{\alpha}(i,j))n-1, \\ (j-1)p:(j-E^{\alpha}(i,j))p-1)}} = 0 \end{array} \right\}, \tag{19}$$

where the superscripts I and II correspond to the output feedback and output error feedback parameterizations, respectively.

3.2.3 *Equalized Recovery Estimator/Controller Designs.* Armed with the description of the feedback laws and their corresponding constraints on the gain matrices, we now present our estimator/controller synthesis designs for both proposed output feedback and output error feedback parameterizations.

Output Feedback. Since the design incorporates a finite horizon (determined by the fixed-length language), we can stack the transformed states, inputs, outputs and noise signals as follows:

$$\begin{aligned} x_\xi &= (x_{\xi,0},\cdots,x_{\xi,T}) \in \mathbb{R}^{n(T+1)}, & u_\xi &= (u_{\xi,0},\cdots,u_{\xi,T-1}) \in \mathbb{R}^{mT}, \\ z_\xi &= (z_{\xi,0},\cdots,z_{\xi,T-1}) \in \mathbb{R}^{pT}, & w &= (w_0,\cdots,w_{T-1}) \in \mathbb{R}^{n_w T}, \\ v &= (v_0,\cdots,v_{T-1}) \in \mathbb{R}^{n_v T}, & \tilde{f} &= (f_{\xi,0},\cdots,f_{\xi,T-1}) \in \mathbb{R}^{nT}, \end{aligned}$$

and rewrite the entire closed-loop trajectory of $x_{\xi,k}$ corresponding to (2) and (7) for each event sequence $\mathcal{E}'_\alpha \in \mathcal{L}^{E'}$ as:

$$\begin{aligned} x_\xi &= P^\alpha_{xw} w + P^\alpha_{xv} v + P^\alpha_{x0} x_{\xi,0} + H\breve{u}^\alpha_\xi, \\ u_\xi &= P^\alpha_{uw} w + P^\alpha_{uv} v + P^\alpha_{u0} x_{\xi,0} + \breve{u}^\alpha_\xi, \end{aligned} \tag{20}$$

where

$$\begin{aligned} P^\alpha_{xw} &= G + HF^\alpha (I - CHF^\alpha)^{-1} CGW, & P^\alpha_{xv} &= HF^\alpha (I - CHF^\alpha)^{-1} V, \\ P^\alpha_{uw} &= F^\alpha (I - CHF^\alpha)^{-1} CGW, & P^\alpha_{uv} &= F^\alpha (I - CHF^\alpha)^{-1} V, \\ P^\alpha_{x0} &= (I + HF^\alpha (I - CHF^\alpha)^{-1} C) A, & P^\alpha_{u0} &= F^\alpha (I - CHF^\alpha)^{-1} CA, \\ \breve{u}^\alpha_\xi &= F^\alpha (I - CHF^\alpha)^{-1} C(G\tilde{f} + Hg^\alpha) + g^\alpha, \end{aligned}$$

$$W = \begin{bmatrix} W_{\xi,0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_{\xi,T-1} \end{bmatrix}, V = \begin{bmatrix} V_{\xi,0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V_{\xi,T-1} \end{bmatrix}, A = \begin{bmatrix} I_n \\ A_0^1 \\ \vdots \\ A_0^T \end{bmatrix},$$

$$C = \begin{bmatrix} C_0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & C_{T-1} & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & \cdots & 0 \\ A_1^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1^T & \cdots & A_T^T \end{bmatrix}, H = \begin{bmatrix} 0 & \cdots & 0 \\ A_1^1 B_{\xi,0} & \cdots & 0 \\ \vdots & \ddots & 0 \\ A_1^T B_{\xi,0} & \cdots & A_T^T B_{\xi,T-1} \end{bmatrix}, \tag{21}$$

with $A_i^k = A_{k-1}A_{k-2}...A_i$ and F^α and g^α in (14). Note that the above formulation is in general not convex in the design variables F^α and g^α. Nonetheless, [29] has shown that a suitable change of variables (i.e., with Q-parametrization) can recast the above formulation as one that is convex in the new variables Q^α and r^α, defined as:

$$\begin{aligned} Q^\alpha &= F^\alpha (I - CHF^\alpha)^{-1}, \\ r^\alpha &= (I + Q^\alpha CH) g^\alpha, \end{aligned} \tag{22}$$

where the original variables can be recovered as follows:

$$\begin{aligned} F^\alpha &= (I + Q^\alpha CH)^{-1} Q^\alpha, \\ g^\alpha &= (I + Q^\alpha CH)^{-1} r^\alpha = (I + F^\alpha CH) r^\alpha. \end{aligned} \tag{23}$$

Furthermore, [20, 22] has shown that gain matrix constraints that satisfy a property known *quadratic invariance* remain invariant under the change of variables (i.e., Q-parameterization).

However, in the context of estimator/controller synthesis with delayed and missing data, there are two sets of gain matrix constraints that are needed, as described in Section 3.2.2, and it is unclear if either set of constraints remains invariant under the Q-parameterization. In fact, even if *quadratic invariance* can be proven, it would only apply to delayed/missing data constraints and not the indistinguishability constraints that must hold for different α's.

Hence, in this paper, we restrict ourselves to a special case where these sets of constraints can be shown to be remain invariant under the change of variance using Q-parameterization:

Assumption 1. *For each event sequence/path $\mathcal{E}'_\alpha \in \mathcal{L}^{E'}$, the corresponding lower triangular event matrix E^α in* (17) *satisfies:*

$$E^\alpha_{(i,j)} = 0, \forall j > i, \quad \text{if } E^\alpha_{(i,i)} = 0.$$

This special case corresponds to the scenario with only missing data patterns or where all delayed data are discarded (i.e., not used for control or estimation) and treated as 'missing.' It is also notable that this assumption is stronger than the skyline matrix structure for which quadratic invariance holds [21]. Even so, while the change of variables via Q-parameterization is invariant for the "sparsity" constraints related to the delayed/missing data constraints in (18), it is unclear if the indistinguishability constraints under this assumption also remain invariant with Q-parameterization. The following lemma essentially answers this question in the affirmative.

Lemma 3.1. *Suppose Assumption 1 holds. Then, any $\tilde{F}^\alpha \in \mathcal{D}^{\mathrm{I}}(\mathcal{L}^{E'})$ (cf.* (18)*) can be factorized as follows:*

$$\tilde{F}^\alpha = F^\alpha(\mathrm{diag}(E^\alpha) \otimes I), \tag{24}$$

where F^α is unconstrained, $\mathrm{diag}(E^\alpha)$ *is a diagonal matrix with only the diagonal elements of E^α and $\otimes$ is the Kronecker product.*

Consequently, the closed-loop trajectory of $x_{\xi,k}$ corresponding to each event sequence $\mathcal{E}'_\alpha \in \mathcal{L}^{E'}$ can be obtained as in (20) *with C and V replaced by $C^\alpha \triangleq (\mathrm{diag}(E^\alpha) \otimes I)C$ and $V^\alpha \triangleq (\mathrm{diag}(E^\alpha) \otimes I)V$, respectively, and (F^α, g^α) has to satisfy $(F^\alpha, g^\alpha) \in \mathcal{C}^{\mathrm{I}}(\mathcal{L}^{E'})$ (cf.* (15)*) but not $\mathcal{D}^{\mathrm{I}}(\mathcal{L}^{E'})$ in* (18)*. Moreover, the constraint $(F^\alpha, g^\alpha) \in \mathcal{C}^{\mathrm{I}}(\mathcal{L}^{E'})$ can be equivalently imposed on $Q^\alpha = F^\alpha(I - C^\alpha HF^\alpha)^{-1}$ and $r^\alpha = (I + Q^\alpha C^\alpha H)g^\alpha$ (cf.* (22) *with C^α instead of C) as $(Q^\alpha, r^\alpha) \in \mathcal{C}^{\mathrm{III}}$, where $\mathcal{C}^{\mathrm{III}}$ is defined as:*

$$\mathcal{C}^{\mathrm{III}}(\mathcal{L}^{E'}) = \left\{ \{(Q^\alpha, r^\alpha)\}_{\alpha=1}^{|\mathcal{L}^{E'}|} \;\middle|\; \begin{array}{c} e \in Pref(\mathcal{E}'_\alpha) \cap Pref(\mathcal{E}'_\beta) \\ \implies \forall \mathcal{E}'_\alpha, \mathcal{E}'_\beta \in \mathcal{L}^{E'} : \\ (\mathcal{BM}_{|e|}(Q^\alpha) = \mathcal{BM}_{|e|}(Q^\beta)) \\ \wedge ((r^\alpha)_{(1:|e|n)} = (r^\beta)_{(1:|e|n)}) \end{array} \right\}. \tag{25}$$

Proof. First, since Assumption 1 holds, the corresponding E^α matrix is lower triangular with some columns being zero. Then, it can be relatively easily shown by basic block matrix multiplication that any $\tilde{F}^\alpha \in \mathcal{D}^{\mathrm{I}}(\mathcal{L}^{E'})$ (cf. (18)) can be exactly factorized as:

$$\tilde{F}^\alpha = F^\alpha(\mathrm{diag}(E^\alpha) \otimes I),$$

where F^α is a full block lower triangular matrix with no sparsity constraints. Then, note that the stacked/time-concatenated control law in (7) is of the form of

$$u_\xi = g^\alpha + \tilde{F}^\alpha(Cx_\xi + Vv),$$

for some $\tilde{F}^\alpha \in \mathcal{D}^{\mathrm{I}}(\mathcal{L}^{E'})$. In the above, since $\tilde{F}^\alpha$ is factorizable,

$$\tilde{F}^\alpha C = F^\alpha(\mathrm{diag}(E^\alpha) \otimes I)C = F^\alpha C^\alpha,$$

$$\tilde{F}^\alpha V = F^\alpha(\mathrm{diag}(E^\alpha) \otimes I)C = F^\alpha V^\alpha,$$

where we defined $C^\alpha \triangleq (\mathrm{diag}(E^\alpha) \otimes I)C$ and $V^\alpha \triangleq (\mathrm{diag}(E^\alpha) \otimes I)V$. Thus, in this case, we could interpret the new estimator/controller synthesis with output feedback as one with output matrix C^α and feedthrough matrix V^α and without the constraint $\mathcal{D}^{\mathrm{I}}(\mathcal{L}^{E'})$.

Finally, we can directly obtain the equivalence of imposing the constraint on $(F^\alpha, g^\alpha) \in \mathcal{C}^{\mathrm{I}}(\mathcal{L}^{E'})$ (cf. (15)) and on $(Q^\alpha, r^\alpha) \in \mathcal{C}^{\mathrm{III}}(\mathcal{L}^{E'})$ (cf. (25)) by applying [23, Propositions 1 & 2]. □

Now, we present the estimator/controller synthesis approach with output feedback parameterization that borrows ideas from Q-parameterization to obtain a tractable optimization problem.

Theorem 3.2 (Equalized Recovery Estimator/Controller Synthesis with Delayed/Missing Data (Output Feedback)). *Suppose Assumption 1 holds. For a system with delayed/missing data patterns defined by a fixed-length language $\mathcal{L}$ given in* (1), *the affine output feedback estimator/controller given in* (6) *and* (12) *solves Problem 1 (with a given recovery level μ_1 and cost function $J(\cdot)$) if the following problem is feasible:*

$$\begin{aligned}
\min_{Q^\alpha, r^\alpha, \mu_2^\alpha} \quad & J(\{\mu_2^\alpha\}_{\alpha=1}^{|\mathcal{L}^{E'}|}) \\
\text{subject to} \quad & \forall ((I_T \otimes P_w)w \leq \mathbb{1}_T \otimes q_w, (I_T \otimes P_v)v \leq \mathbb{1}_T \otimes q_v, \\
& \quad Px_{\xi,0} \leq \mu_1 q, \alpha \in \mathcal{L}^{E'}) : \\
& (I_T \otimes P_u)(u_\xi^\alpha + u_d) \leq \mathbb{1}_T \otimes q_u, \\
& (I_{T+1} \otimes P)x_\xi^\alpha \leq \mu_2^\alpha \otimes q, \ PR_T x_\xi^\alpha \leq \mu_1 q, \\
& x_\xi^\alpha = P_{xw}^\alpha w + P_{xv}^\alpha v + P_{x0}^\alpha x_{\xi,0} + H\breve{u}_\xi^\alpha, \\
& u_\xi^\alpha = P_{uw}^\alpha w + P_{uv}^\alpha v + P_{u0}^\alpha x_{\xi,0} + \breve{u}_\xi^\alpha, \\
& (Q^\alpha, r^\alpha) \in \mathcal{C}^{\text{III}}(\mathcal{L}^{E'}), \ \mu_2^\alpha \geq \mu_1,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
P_{xw}^\alpha &= (I + HQ^\alpha C^\alpha)GW, & P_{xv}^\alpha &= HQ^\alpha V^\alpha, \\
P_{uw}^\alpha &= Q^\alpha C^\alpha GW, & P_{uv}^\alpha &= Q^\alpha V^\alpha, \\
P_{x0}^\alpha &= (I + HQ^\alpha C^\alpha)A, & P_{u0}^\alpha &= Q^\alpha C^\alpha A, \\
\breve{u}_\xi^\alpha &= Q^\alpha C^\alpha G\tilde{f} + r^\alpha, & R_T &= \begin{bmatrix} 0_{n \times nT} & I_n \end{bmatrix},
\end{aligned} \tag{27}$$

as well as $C^\alpha \triangleq (\text{diag}(E^\alpha) \otimes I)C$ and $V^\alpha \triangleq (\text{diag}(E^\alpha) \otimes I)V$, while the gain matrices F^α and g^α can be found from Q^α and r^α via (23) *and $\mu_2^\alpha \triangleq [\mu_{2,0}^\alpha, ..., \mu_{2,T}^\alpha]^\top$. Moreover, we let $\mu_{2,k} \triangleq \max_\alpha \mu_{2,k}^\alpha$.*

Proof. This proof follows similar steps to the derivation in [23]. From the requirements for equalized recovery in Definition (1), we must have $Px_{\xi,k} \leq \mu_{2,k}q$ for all $k \in [0, T]$ and $Px_{\xi,T} \leq \mu_1 q$, for all (worst-case) realizations of noise w_k, v_k and initial state uncertainty $x_{\xi,0}$. Then, using the change of variables in (22) (i.e., Q-parameterization) and the result from Lemma 3.1, we can directly construct the robust optimization problem given in Theorem 3.2, similar to [29, Section III-C]. Moreover, the original gain matrices (F^α, g^α) can be recovered from (23). □

Then, since the problem in Theorem 3.2 involves semi-infinite constraints (i.e., *for all* constraints), we will leverage robust optimization to convert the problem into the following linear program with a finite number of constraints:

Proposition 3.3 (Robustified Equalized Recovery Estimator/Controller Synthesis with Delayed/Missing Data (Output Feedback)). *The semi-infinite optimization problem* (26) *in Theorem 3.2 (that solve Problem 1 with a given recovery level μ_1 and cost function $J(\cdot)$) is equivalent to the following linear program:*

$$\begin{aligned}
&\min_{Q^\alpha, r^\alpha, \mu_2^\alpha, \Pi^\alpha} \quad J(\{\mu_2^\alpha\}_{\alpha=1}^{|\mathcal{L}^{E'}|}) \\
&\text{subject to} \quad \Pi^\alpha \geq 0, \ \mu_2^\alpha \geq \mu_1, \\
&\Pi^\alpha \begin{bmatrix} \mathbb{1}_T \otimes q_w \\ \mathbb{1}_T \otimes q_v \\ \mu_1 q \end{bmatrix} \leq \begin{bmatrix} \mu_2^\alpha \otimes q \\ \mu_1 q \\ \mathbb{1}_T \otimes q_u \end{bmatrix} - \begin{bmatrix} (I_{T+1} \otimes P)H & 0 \\ PR_T H & 0 \\ I_T \otimes P_u & I_{mT} \end{bmatrix} \begin{bmatrix} Q^\alpha C^\alpha G\tilde{f} + r^\alpha \\ u_d \end{bmatrix}, \\
&\Pi^\alpha \begin{bmatrix} I_T \otimes P_w & 0 & 0 \\ 0 & I_T \otimes P_v & 0 \\ 0 & 0 & P \end{bmatrix} = \begin{bmatrix} (I_{T+1} \otimes P)J_x^\alpha \\ PR_T J_x^\alpha \\ (I_T \otimes P_u)J_u{}^\alpha \end{bmatrix}, \\
&(Q^\alpha, r^\alpha) \in \mathcal{C}^{\text{III}}(\mathcal{L}^{E'}),
\end{aligned} \tag{28}$$

with $J_x^\alpha \triangleq \begin{bmatrix} P_{xw}^\alpha & P_{xv}^\alpha & P_{x0}^\alpha \end{bmatrix}$, $J_u^\alpha \triangleq \begin{bmatrix} P_{uw}^\alpha & P_{uv}^\alpha & p_{u0}^\alpha \end{bmatrix}$, $u_d = (u_{d,0}, \cdots, u_{d,T-1})$ and the definitions in (27).

Proof. This proposition can be proven by directly leveraging techniques from robust optimization [5, 6] to find the robust counterpart to the robust optimization problem in Theorem 3.2. □

Remark 1. *Note that even if Assumption 1 does not hold, the replacement of $\tilde{F}^\alpha \in \mathcal{D}^{\text{I}}(\mathcal{L}^{E'})$ with $F^\alpha(\text{diag}(E^\alpha) \otimes I)$ on the right hand side of* (24) *will enable us to synthesize suboptimal equalized recovery estimators/controllers with delayed/missing data using Theorem 3.2 and Proposition 3.3, as demonstrated in Section 4.1.*

In addition to being applicable for feedback control and bounded-error estimation with more general delayed data languages, when compared to a prior work [23], the above results allows for path-dependent and time-varying intermediate levels μ_2^α (i.e., dependent on α and time step k), which can lead to smaller tracking/estimation errors at run time. Further, we can consider more general polytopes than hypercubes in [23]. Moreover, the path-dependent gain matrices can be selected at run time using (12).

Output Error Feedback. Next, we consider the estimator/controller design based on output error feedback parameterization. We can similarly stack the transformed states, inputs, outputs and noise signals and rewrite the entire closed-loop trajectory of $x_{\xi,k}$ corresponding to (2) and (7) for each event sequence $\mathcal{E}'_\alpha \in \mathcal{L}^{E'}$ as:

$$\begin{aligned}
x_\xi &= \breve{P}_{xw}^\alpha w + \breve{P}_{xv}^\alpha v + \breve{P}_{x0}^\alpha x_{\xi,0} + \breve{P}_{xs}^\alpha s_0 + H\nu^\alpha + G\tilde{f}, \\
u_\xi &= \breve{P}_{uw}^\alpha w + \breve{P}_{uv}^\alpha v + \breve{P}_{u0}^\alpha x_{\xi,0} + \breve{P}_{us}^\alpha s_0 + \nu^\alpha,
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
&\breve{P}_{xw}^\alpha = (I + H(M^\alpha + L^\alpha)C)\Gamma^\alpha W, \\
&\breve{P}_{xv}^\alpha = ((HM^\alpha + GL^\alpha)(I - C\Gamma^\alpha L^\alpha) - \Gamma^\alpha L^\alpha)V, \\
&\breve{P}_{x0}^\alpha = (I + (HM^\alpha + GL^\alpha)C)\Phi^\alpha, \quad \breve{P}_{xs}^\alpha = A - \breve{P}_{x0}^\alpha, \\
&\breve{P}_{uw}^\alpha = M^\alpha C\Gamma^\alpha W, \qquad \breve{P}_{uv}^\alpha = M^\alpha(I - C\Gamma^\alpha L^\alpha)V, \\
&\breve{P}_{u0}^\alpha = M^\alpha C\Phi^\alpha, \qquad \breve{P}_{us}^\alpha = -\breve{P}_{u0}^\alpha,
\end{aligned} \tag{30}$$

with A, C, G, H, W and V in (21), M^α, L^α and ν^α in (14), and Φ^α and Γ^α are defined as[1]:

$$\Phi^\alpha = \begin{bmatrix} I_n \\ \Phi_0^{\alpha,1} \\ \vdots \\ \Phi_0^{\alpha,T} \end{bmatrix}, \Gamma^\alpha = \begin{bmatrix} 0 & \cdots & 0 \\ \Phi_1^{\alpha,1} & \cdots & 0 \\ \vdots & \ddots & 0 \\ \Phi_1^{\alpha,T} & \cdots & \Phi_T^{\alpha,T} \end{bmatrix},$$

where $\Phi_i^{\alpha,k} = \Phi_{k-1}^\alpha \Phi_{k-2}^\alpha ... \Phi_i^\alpha$ and $\Phi_i^\alpha = A_i - L_i^\alpha C_i$.

It is noteworthy that, similar to the output error feedback approach in [13], with a fixed s_0 and L^α, the above formulation is convex in our design variables M^α and ν^α. Hence, no new reformulation such as Q-parameterization is necessary and no additional assumption similar to Assumption 1 is required.

Then, we present the estimator/controller design approach with output error parametrization as the solution of the following:

Theorem 3.4 (Equalized Recovery Estimator/Controller Synthesis with Delayed/Missing Data (Output Error Feedback)). *For a system with delayed/missing data patterns defined by a fixed-length language $\mathcal{L}$ given in* (1), *the affine output error feedback estimator/controller given in* (8), (9) *and* (13) *solves Problem 1 (with a*

[1] Note that Φ^α and Γ^α are functions of α, since there are directly dependent on L^α.

given recovery level μ_1 and cost function $J(\cdot)$) if the following problem is feasible:

$$\begin{aligned}
\min_{M^\alpha, L^\alpha, \nu^\alpha, \mu_2^\alpha} \quad & J(\{\mu_2^\alpha\}_{\alpha=1}^{|\mathcal{L}^{E'}|}) \\
\text{subject to} \quad & \forall((I_T \otimes P_w)w \leq \mathbb{1}_T \otimes q_w, (I_T \otimes P_v)v \leq \mathbb{1}_T \otimes q_v, \\
& \quad Px_{\xi,0} \leq \mu_1 q, \alpha \in \mathcal{L}^{E'}) : \\
& (I_T \otimes P_u)(u_\xi^\alpha + u_d) \leq \mathbb{1}_T \otimes q_u, \\
& (I_{T+1} \otimes P)x_\xi^\alpha \leq \mu_2^\alpha \otimes q, \; PR_T x_\xi^\alpha \leq \mu_1 q, \\
& x_\xi^\alpha = \check{P}_{xw}^\alpha w + \check{P}_{xv}^\alpha v + \check{P}_{x0}^\alpha x_{\xi,0} + \check{P}_{xs}^\alpha s_0 + H\nu^\alpha + G\tilde{f}, \\
& u_\xi^\alpha = \check{P}_{uw}^\alpha w + \check{P}_{uv}^\alpha v + \check{P}_{u0}^\alpha x_{\xi,0} + \check{P}_{us}^\alpha s_0 + \nu^\alpha, \\
& (M^\alpha, L^\alpha, \nu^\alpha) \in \mathcal{C}^{\text{II}}(\mathcal{L}^{E'}) \wedge \mathcal{D}^{\text{II}}(\mathcal{L}^{E'}), \; \mu_2^\alpha \geq \mu_1,
\end{aligned} \tag{31}$$

with $R_T = \begin{bmatrix} 0_{n\times nT} & I_n \end{bmatrix}$, $\mu_2^\alpha \triangleq [\mu_{2,0}^\alpha, ..., \mu_{2,T}^\alpha]^\top$ and the definitions in (16), (19) *and* (30). *Moreover, we let $\mu_{2,k} \triangleq \max_\alpha \mu_{2,k}^\alpha$.*

Proof. The estimator/controller design follows similar steps to the design in [14]. It is straightforward to observe that the estimator/controller solves Problem 1 by construction with the additional indistinguishability and delay/missing data constraints in (16) and (19) on the matrix gains, as described in Section 3.2. ◻

Similar to Theorem 3.2, the problem in Theorem 3.4 also involves semi-infinite constraints (i.e., *for all* constraints), and thus, we resort to robust optimization to convert the problem into the following optimization problem with a finite number of constraints:

Proposition 3.5 (Robustified Equalized Recovery Estimator/Controller Synthesis with Delayed/Missing Data (Output Error Feedback)). *The semi-infinite optimization problem* (31) *in Theorem 3.2 (that solve Problem 1 with a given recovery level μ_1) is equivalent to the following linear optimization problem:*

$$\begin{aligned}
&\min_{M^\alpha, L^\alpha, \nu^\alpha, \mu_2^\alpha, \Pi^\alpha} \quad J(\{\mu_2^\alpha\}_{\alpha=1}^{|\mathcal{L}^{E'}|}) \\
&\text{subject to} \quad \Pi^\alpha \geq 0, \; \mu_2^\alpha \geq \mu_1, \\
&\Pi^\alpha \begin{bmatrix} \mathbb{1}_T \otimes q_w \\ \mathbb{1}_T \otimes q_v \\ \mu_1 q \end{bmatrix} \leq \begin{bmatrix} \mu_2^\alpha \otimes q \\ \mu_1 q \\ \mathbb{1}_T \otimes q_u \end{bmatrix} - \begin{bmatrix} I_{T+1}\otimes P & 0 \\ PR_T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H\nu^\alpha + \check{P}_{xs}^\alpha s_0 + G\tilde{f} \\ (I_T \otimes P_u)(\nu^\alpha + \check{P}_{us}^\alpha s_0) + I_{mT} u_d \end{bmatrix}, \\
&\Pi^\alpha \begin{bmatrix} I_T \otimes P_w & 0 & 0 \\ 0 & I_T \otimes P_v & 0 \\ 0 & 0 & P \end{bmatrix} = \begin{bmatrix} (I_{T+1} \otimes P)\check{J}_x^\alpha \\ PR_T \check{J}_x^\alpha \\ (I_T \otimes P_u)\check{J}_u^\alpha \end{bmatrix}, \\
&(M^\alpha, L^\alpha, \nu^\alpha) \in \mathcal{C}^{\text{II}}(\mathcal{L}^{E'}) \wedge \mathcal{D}^{\text{II}}(\mathcal{L}^{E'}),
\end{aligned} \tag{32}$$

with $\check{J}_x^\alpha \triangleq \begin{bmatrix} \check{P}_{xw}^\alpha & \check{P}_{xv}^\alpha & \check{P}_{x0}^\alpha \end{bmatrix}$, $\check{J}_u^\alpha \triangleq \begin{bmatrix} \check{P}_{uw}^\alpha & \check{P}_{uv}^\alpha & \check{P}_{u0}^\alpha \end{bmatrix}$, $u_d = (u_{d,0}, \cdots, u_{d,T-1})$ and the definitions in (30).

Proof. Similar to the proof of Proposition 3.3, this result can be directly obtained by finding the robust counterpart to the robust optimization problem in Theorem 3.4 using tools from robust optimization [5, 6]. ◻

When compared with the output feedback parameterization, this estimator/controller synthesis approach does not require the satisfaction of Assumption 1, hence the resulting estimator/controller is optimal for a more general class of systems with delayed/missing data. Moreover, in comparison with a prior work [14], we consider time-varying and path-dependent intermediate levels $\mu_{2,k}^\alpha$ as well as a prefix-based/path-dependent design that enables adaptation of the gain matrices based on the observed path, $\mathcal{E}_k^{obs}$, that, in turn, leads to improved performance. Moreover, the path-dependent gain matrices can be selected at run time using (13).

Algorithm 1: $\mathcal{W}_k^{obs} = WordObsv(\mathcal{E}_k^{obs}, \mathcal{L})$

Data: Observed Path
$\mathcal{E}_k^{obs} = (d_{0,0}^{obs})(d_{0,1}^{obs} d_{1,1}^{obs}) \dots (d_{0,k}^{obs} \dots d_{k,k}^{obs})$;
Language $\mathcal{L}$;
Output: Set of Compatible Words $\mathcal{W}_k^{obs}$

```
function WordObsv((e_{i,j_i})_{i=0}^{T-1})
    Initialize W_k^obs = {ϵ};    (where ϵω(0) = ω(0))
    for i = 0 to T − 1 do
        for ω̃ ∈ W_k^obs do
            W' = ∅;
            for ℓ = i to i + ω̄ do
                ω(i) = ℓ − i;
                if ℓ ≤ k    d_{i,ℓ}^obs = 1 then
                    W' = W' ∪ {ω̃ω(i)};
                    break;
                else if ℓ > k then
                    W' = W' ∪ {ω̃ω(i)};
        W_k^obs = W';
    W_k^obs = W_k^obs ∩ L;
    return W^obs
```

Note, however, that the optimization problem in Proposition 3.5 still has bilinear terms, but is fortunately relatively sparse, hence off-the-shelf solvers, e.g., IPOPT [30], can return optimal solutions very quickly. Moreover, as assumed in [13] and as discussed in detail in [14, Section IV-C], we can fix L^α and s_0 (by choosing $s_0 = 0$ and L_k such that $A_k - L_k C_k$ for all k are Hurwitz and have eigenvalues with sufficiently small magnitudes) to obtain a computationally tractable linear program without any loss of optimality.

3.3 Word Observer

In addition, given the observed path/event subsequence at each time step k, i.e., $\mathcal{E}_k^{obs} = e_0^{obs} \dots e_k^{obs}$ with $e_i^{obs} = d_{i,0}^{obs} d_{i,1}^{obs} d_{i,2}^{obs} \dots d_{i,i}^{obs}$ (cf. Definition 6), we propose a word observer that will map $\mathcal{E}_k^{obs}$ to the set of all words that are compatible with observed sequence $\mathcal{W}_k^{obs}$, which can be useful for fault or attack pattern identification.

In particular, we want to find $\mathcal{W}_k^{obs} = WordObsv(\mathcal{E}_k^{obs}, \mathcal{L}) \triangleq InvMap(\mathcal{E}_k^{obs}) \cap \mathcal{L}$. It can be shown that the inverse mapping of the observed path finds the set of all words, i.e., $InvMap(\mathcal{E}_k^{obs}) = \omega(0)\omega(1)\dots\omega(T-1)$, where $\omega(i)$ for each $i \in \mathbb{N}_0^{T-1}$ is given by:

$$\omega(i) \in \begin{cases} \mathbb{N}_{k-i+1}^{\overline{\omega}}, & \text{if } d_{i,\ell}^{obs} = 0 \text{ for all } \ell \in \mathbb{N}_i^k, \\ \{\ell_i^* - i\}, & \text{otherwise}, \end{cases} \tag{33}$$

with ℓ_i^* being the minimum $\ell \in \mathbb{N}_i^k$ such that $d_{i,\ell}^{obs} = 1$ and $\overline{\omega}$ is the maximum delay. Intuitively, ℓ_i^* is the earliest time step at each the data from time step i is available in $\bigcup_{i=0}^k Y_i$. Moreover, if the data at time step i is not received by the current time step k, then this data could be delayed by an interval between $k - i + 1$ and the maximum delay $\overline{\omega}$. It should be noted that in (33), the second case results in a set of words, implying that the mapping from $\mathcal{E}_k^{obs}$ to $\mathcal{W}^{obs}$ is one-to-many. This makes sense because if a data is not available at any time step within $\bigcup_{i=0}^{k} Y_i$, it can be considered as delayed by any number of time steps up to the maximum delay $\overline{\omega}$. The algorithm of the word observer is given in Algorithm 1.

3.4 Implementation Strategy

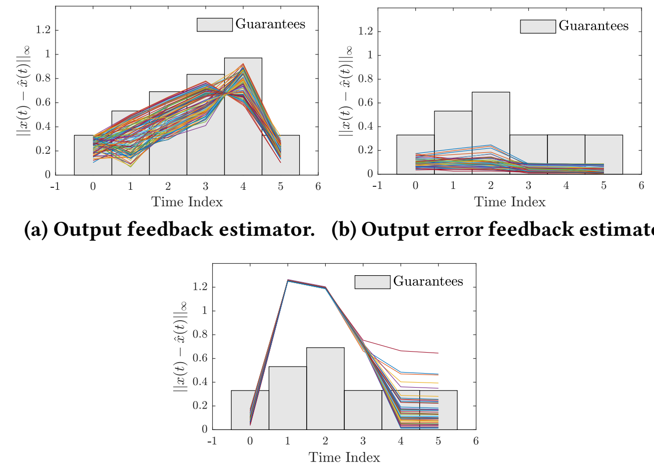
There are multiple different ways, in which the constructed equalized recovery estimators/controllers can be implemented. First, considering the situation that a T-length delayed/missing data pattern occurs periodically, the same gains can be chosen for each period because by construction of the equalized recovery estimators/controllers, the tracking or estimation error bound at the last time step of the period is enforced to be the same at the initial step of the period. In addition, in the case where there is no delayed/missing data, equalized performance (i.e., equalized recovery with $T = 1$ [8]) can be achieved by using the corresponding gains. Then, when a delayed/missing data is detected, we can switch to the equalized recovery estimator/controller associated with a T-length language where the first data is delayed/missing. Subsequently, after the fixed recovery time T, we can switch the equalized recovery estimator/controller to the equalized performance estimator/controller again until another delayed/missing data is observed. Moreover, in the event that the initial tracking or estimation error does not satisfy the equalized recovery/performance level, the proposed estimator/controller can also be combined with any asymptotic estimator/controller (or a modified version of the proposed estimator/controller with a larger initial level), where the latter is used until the desired equalized level is achieved.

4 EXAMPLES AND DISCUSSION

In this section, the performance of the proposed controllers and estimators is validated and compared with the approaches in [24] and [33]. The examples using our proposed estimator are all run using MATLAB 2017a. For the output feedback parameterization in (28) that is a linear program, we use Gurobi [16] as the solver, while for the robustified problem in (32) with output error feedback, the IPOPT solver [30] is used since the optimization problem (32) involves many sparse matrices. Moreover, the parameters of s_0 in (32) will be set to zero for all of the presented examples because it was observed in [14] that the value of s_0 has no effect on the performance of the controller/estimator. Moreover, both measurement and process noises in all of the following examples are bounded by hypercubes, i.e. $\|w\|_\infty \leq \eta_w, \|v\|_\infty \leq \eta_v$.

4.1 Bounded-Error Estimator for Batch Reactor Process (Comparison with [33])

To demonstrate the capability of the estimator proposed in this paper when compared to [33] in the presence of output delays, we consider the batch-reactor process in [27], which is a continuous-time fourth order two-input-two-output system. Using the `c2d` command in MATLAB with a sampling time of $T_s = 0.05$ seconds, the model is discretized, yielding the following system matrices:



(a) Output feedback estimator. (b) Output error feedback estimator.

(c) Estimator from [33].

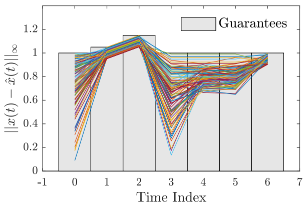
Figure 1: Estimator comparison for batch reactor process example with $\mathcal{W}_{sim} = 21210$.

$$A = \begin{bmatrix} 1.0795 & -0.0045 & 0.2896 & -0.2367 \\ -0.0272 & 0.8101 & -0.0032 & 0.0323 \\ 0.0447 & 0.1886 & 0.7317 & 0.2354 \\ 0.0010 & 0.1888 & 0.0545 & 0.9115 \end{bmatrix},\ B = \begin{bmatrix} 0.0006 & -0.0239 \\ 0.2567 & 0.0002 \\ 0.0837 & -0.1346 \\ 0.0837 & -0.0046 \end{bmatrix},$$

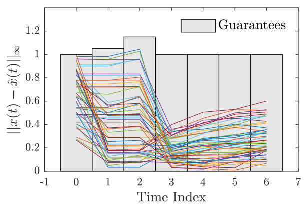
$$C = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top, V = I_p, W = \emptyset.$$

The time horizon of $T = 5$ is considered, with a maximum possible output delay of 2 steps within T, except at the last step, which is always on time. This results in a delayed data model that can be expressed as the fixed-length language containing 3^4 words, i.e. $\mathcal{L} = \{\mathcal{W}_1, \ldots, \mathcal{W}_{81}\}$, and we can find the corresponding event-based language $\mathcal{L}^E$ and reduced language $\mathcal{L}^{E'}$ from Definitions 2–4. The measurement noise bound $\eta_v = 0.05$ is assumed, which corresponds to 5 standard deviations of $\mathcal{N}(0, 0.01^2)$. Solving the robustified problem (32) with the cost function $J(\cdot) = \sum_{k=0}^{T} \sum_{\alpha=1}^{|\mathcal{L}^{E'}|} \mu_{2,k}^{\alpha}$ and $\mu_1 = 0.33$, we obtain the maximum intermediate level $\max_{k,\alpha}(\mu_{2,k}^{\alpha}) = 0.6912$ for $\|\tilde{x}\|_\infty$ (i.e., with (P, q) for describing $\mathcal{X}_0$ and $\mathcal{X}_k$ as hypercubes).

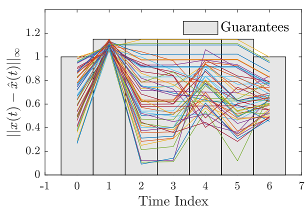
We compare the run-time results of our proposed approaches with a design from [33] that employs Kalman filtering with output delays. We initialize the simulation with $x(0) = [1, 1, 1, 1]^\top$ and randomly generate initial state error and noise signals using truncated zero-mean normal distributions with covariance matrices $P_0 = (\mu_1/5)^2 I_4$, $Q = \emptyset$ and $R = (\eta_v/5)^2 I_2$, where μ_1, η_v correspond to the values that are 5 times their standard deviations. The true delay pattern followed by the plant is $\mathcal{W}_{sim} = 21210$, and the simulation is run 50 times. Figure 1 depicts the trajectories (and their guaranteed error bounds) for the proposed estimators when using output feedback and output error feedback, as well as for the estimator from [33]. The results show the estimation errors from the proposed approaches staying within the guaranteed bounds, as expected. Moreover, they also minimize the error at the end of the horizon in all 50 runs, whereas the estimator from [33] has trajectories that far exceed the guaranteed bounds from the proposed estimators. Note that for the output feedback design, Assumption 1 does hold; hence the obtained estimator is suboptimal, as discussed



(a) Output feedback estimator.



(b) Output error feedback estimator.



(c) Estimator in [24].

Figure 2: Estimator comparison for adaptive cruise control example with missing data at $k = 1$.

in Remark 1. In contrast, the output error feedback estimator does not require the assumption to hold and is optimal, which demonstrates the advantage of the output error feedback design when there is delayed data, whereas the two parameterizations are equivalent when there is no missing or delayed data.

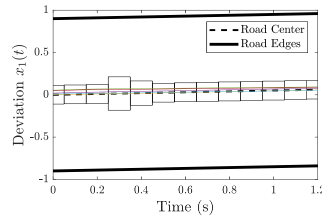
4.2 Bounded-Error Estimator for Adaptive Cruise Control (Comparison with [24])

Missing data patterns in a finite horizon setup can be considered as a special case where the missing output is "delayed" beyond the horizon. Hence, in this example, we compare our approaches with another equalized recovery estimator by [24] which is only applicable for missing data scenarios. Using the same model and simulation parameters for adaptive cruise control in [24], we set $T = 6$, $\mu_1 = 1$ and the language as $\mathcal{L} = \{060000, 006000, 000600, 000060\}$ that is equivalent to the missing data specifications given in [24]. Solving the problems in (28) and (32), we obtain $\max_{k,\alpha} \mu_{2,k}^{\alpha} = 1.1498$ for both designs, which matches the result in [24]. However, due to the time-varying property of $\mu_{2,k}^{\alpha}$ in our approach, the intermediate error bounds do not remain at maximum value, as opposed to the approach in [24], providing a less conservative solution. The simulation is performed with the true output pattern being $\mathcal{W}_{sim} = 060000$, implying that the data is missing at $k = 1$. Figure 2 shows that the estimation errors of our approaches are indeed lower than those from [24], which was presumably made possible by the time-varying recovery levels.

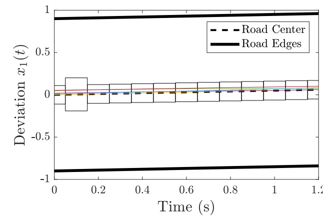
4.3 Controller Synthesis for Lane-Keeping

Next, we demonstrate the use of our design framework for lane keeping. As in [23], we represent the lane-keeping system with a continuous-time double integrator system:

$$\begin{aligned} \dot{x}_{\xi,t} &= \begin{bmatrix} 0 & 1 \\ 0 & -20 \end{bmatrix} x_{\xi,t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{\xi,t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t, \\ z_{\xi,t} &= x_{\xi,t} + v_t, \end{aligned} \tag{34}$$



(a) Data missing at $k = 0, 1$



(b) Data missing at $k = 6, 7$

Figure 3: Tracking control for lane keeping using the output error feedback approach. The boxes represent the maximum bounds of the deviation from the center line.

where $x_{\xi,t} = [x_t, \dot{x}_t]^\top$ includes the deviation x_t from the center of the lane as well as the lateral velocity $\dot{x}_t$, while the outputs of the system are noisy measurements of the state $x_{\xi,t}$. The process and measurement noises are bounded by hypercubes with $\eta_w = 0.05$ and $\eta_v = 0.1$ respectively, whereas the (P, q) pair for $\mathcal{X}_0$ and $\mathcal{X}_k$ is chosen to represent regular hexagonal sets. Using a sampling time of $T_s = 0.1$ seconds, the system in (34) is converted to a discrete-time system. The missing-data language with a fixed horizon $T = 12$ is chosen to represent two consecutive missing outputs within the first 11 steps of the finite horizon T. Using the proposed approach, a tracking controller is designed that tries to follow the center-line of the road. Specifically, the output error feedback controller is applied to two examples for different true missing data patterns—one in which the first two measurements are missing, and second in which 7th and 8th measurements in the horizon are missing. The resulting trajectories of the deviations from the center line for 5 different runs are plotted in Figure 3, where each run corresponds to different values of the random noises and initial states. It can be observed that the system is able to track the reference trajectory when using the proposed output error feedback controller.

5 CONCLUSIONS

In this paper, we proposed path-dependent finite-horizon feedback controllers and bounded-error estimators that achieve equalized recovery for time-varying affine systems subject to delayed observations or missing data. By modeling the delayed/missing data as a fixed-length language and constructing a reduced event-based language with unique event sequences, we synthesized equalized recovery controllers/estimators whose feedback gains can be adapted based on the observed path, i.e., the history of observed data patterns up to the current time step. The proposed controller/estimator is an extension of existing works in [14, 23–25] and can cater to more generic delay/missing data patterns as well as allows for path-dependent and time-varying intermediate recovery levels and more general polytopic sets for describing the tracking or estimation error bounds. Moreover, we designed a word observer that can return the set of all words that are compatible with observed data patterns/path, and demonstrated the effectiveness of our approach via several illustrative examples.

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