Potentially Hurwitz Structures: A Characterization of Nests

João Cavalcanti

***Abstract*—We characterize the "structures" defining a class of real matrices—called nests—given by $A \in \mathbb{R}^{n\times n}$ such that, up to relabelling, all of its leading principal submatrices indexed by $\{1, 2, \cdots, k\}$ satisfy $\operatorname{sgn}\det A[\{1, 2, \cdots, k\}] = (-1)^k$, $k \leq n$. By structure we mean elements of $\{-1, 0, 1\}^{n\times n}$, which circumscribe the signs of realizations within a class of real matrices. We frame the problem of characterizing structures in a form amenable to classic control theory, and that generalizes to any class of matrices defined by potentially Hurwitz structures, such as nests.**

I. Overview

Much of modern control theory is realization-centric. A control system realization *typically* feeds algorithms with numerical values that produce binary statements on whether that control system has a property or not, e.g., being controllable. Softer metrics of these properties, e.g., singular values of a controllability Gramian, are also based on a realization. Even when considering unknown realizations, control theory still emphasizes them: for example, by assuming that the admissible realizations of a control system are confined to quantifiable uncertainty in the $\|\cdot\|_\infty$ sense around a nominal model. As an alternative to this conventional view, we pursue a theoretical paradigm favoring *structures* in this paper. We characterize the structures that define a class of real matrices—called nests—that potentially have Hurwitz property.

Structural control theory represents systems as structures, which can have many realizations. In the setting of linear time-invariant (LTI) systems, structures are given by matrices with entries in $\{-1, 0, 1\}$, representing negative and positive numbers plus "hard" zeros. By fixing the signs of each entry in a real matrix, a structure defines a class of realizations. If all realizations of a structure are Hurwitz[1], then the structure is also said to be Hurwitz. The characterization of Hurwitz structures is a pillar of structural control theory [1]. Hurwitz structures satisfy strict conditions that can be checked in polynomial time [2]. The conditions that positive Hurwitz structures must satisfy are even stricter, and are faster to check [3]. Potentially Hurwitz (π-Hurwitz) structures have *at least* one realization that is Hurwitz. Characterizing π-Hurwitz structures is still an open problem [4]. In fact, even for special cases such as stars [5] and trees [6], there is only one characterization of π-Hurwitz structures [7]. Negative results are also scarce [8]. A major challenge is π-Hurwitz structures have relatively few constraints, in contrast to Hurwitz structures. Feedback cycles in Hurwitz structures are negative and limited to two states, which is not true in π-Hurwitz structures [9]. In an effort to put the complexity issue aside, only structures of dimension at most five are considered, and listed, in [6]. But dimensionality *is a curse*, and for dimensions as low as five, *ad hoc* approaches crumble. In fact, the list in [6] is incomplete, and amended in [10]. The structures missing in [6] belong to a set of structures defining *nests*. A nest is a matrix $A \in \mathbb{R}^{n\times n}$ such that, up to relabelling, its leading principal submatrices satisfy $\operatorname{sgn}\det A[\{1, \cdots, k\}] = (-1)^k$. Structures defining nests are π-Hurwitz [10]: there is a positive diagonal D matrix such that AD is Hurwitz. Although nests have been identified as an important class of realizations, the characterization of the π-Hurwitz structures that define them has remained an open problem [10].

In this paper, we characterize the structures defining nests. We do so by rather describing the graphs that correspond to these structures. Informally, a nest is a sequence of supergraphs that increases one vertex at a time. In classic control-theoretic parlance, adding a vertex to a graph is equivalent to adding a pole at the origin to a control system. To stabilize them, graphs grow by means of particular structures called Hamiltonians that act as controllers [11]. Framed in this way, the problem is amenable to classical Root Locus arguments [12]. We use Root Loci to devise conditions on the Hamiltonians themselves, such that successively adding a Hamiltonian stabilizes the extra pole at the origin, while preserving the remaining poles in the complex left-half plane. The solution concept is itself a contribution, because it generalizes to any π-Hurwitz problem, contrary to [7]. A technical reason is the solution in [7] exploits realizations have Metzler property. Following tradition in structural control theory [1], one seeks to recognize extraneous graphical patterns and capture the π-Hurwitz essence in graphical properties the entire (sub)graph satisfies. In view of abundant examples of how a structure can be π-Hurwitz [6], however, we feel this search will prove elusive. Instead, we recognize a π-Hurwitz structure as a sequence of π-Hurwitz superstructures that grow by adding (possibly multiple) poles at the origin. The goal then is devising the rules of how to stabilize the additional poles. Herein, we successfully illustrate the approach to characterize rather general structures, the only restriction being they "add" one pole at the origin at a time.

A structural perspective can complement control theory in many ways. For instance, by analyzing whether the structure of an unstable system is potentially Hurwitz, we can infer whether the instability is a matter of realization or of the "nature" of the system. If a Hurwitz realization exists, perhaps certain parameters of this system can be adjusted to *fix* it.

João Cavalcanti is with the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA. caval@mit.edu.

This work was supported in part by the National Science Foundation under CPS Award No. 1739505, and by NASA under grant number 80NSSC19K1607. João Cavalcanti was also supported by an Aeropuertos Argentina Graduate Fellowship.

[1] In [1], a real matrix is called *sign-stable* if it belongs to an all-Hurwitz matrix class. We explain our decision to introduce new notation in Remark 2.

Otherwise, the very design of the system must be reviewed. Hence, structural results verify whether it makes sense to even try solving a design optimization problem. For example, as in [13], [14], where stabilizability is an assumption, and the role of "structure" is to constrain controller design. Conversely, theoretical results such as ours can inform good practices to *design* systems that are at least potentially Hurwitz. In fact, in proving sufficiency of our characterization, we effectively produce Hurwitz realizations of a structure. That is, structural results can support automated design. Structural theory can also be viewed as yet another approach to robust control. Indeed, models often arise from first principles, and one can confidently infer the signs of the interactions between states. And linear models are also often good enough, thus, e.g., working with $\|\cdot\|_\infty$ and $\|\cdot\|_2$ topologies, as in unstructured robust control [15], effectively discard useful information about the system. On the other end, parametric-style robust control [16] embraces linear models. Uncertainty then lies in a family of polynomials, whose coefficients are assumed to vary independently within intervals of real numbers [16]. In reality, however, coefficients depend on realizations in a substantially more convoluted way, therefore assuming independent coefficients makes the analysis and design much more conservative.

Notation

Uppercase Greek letters are reserved for graphs, defined below. Given a positive integer n, $[n]$ denotes the set $\{1, \cdots, n\}$. For $i, j \in [n]$ and $A \in \mathbb{R}^{n\times n}$, $A[i, j] = A_{i,j}$ denote the entry in i-th row, j-th column. This generalizes to sets: if $\mathcal{I} \subseteq [n]^2$, then $A[\mathcal{I}]$ represents the principal submatrix of A indexed by the elements of $\mathcal{I}$. But for ease of notation, we say $A[n]$ to mean $A[[n]]$. Uppercase I stands for the identity matrix. The *sign function* is defined over the real numbers as

$$\operatorname{sgn} x = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

By complex left half-plane (LHP) we mean the subspace of $\mathbb{C}$ defined by complex numbers with negative real part. Given f a real function defined on a finite domain, by $\operatorname{supp} f$ we mean the set of points over which f is nonzero.

II. Preliminaries

Graphs

A matrix A in $\mathbb{R}^{n\times n}$ maps to a (signed) *graph* $\Gamma = (V, E)$. Elements of $V = [n]$ are *vertices*. Elements of E are *edges*. Edges are triplets $e = (i, j, \operatorname{sgn} A_{j,i})$, and e is said to go from vertex i, the *tail*, to vertex j, the *head*. Vertices that are tails or heads of an edge are *spanned* by Γ. We often omit the word "vertex" in "Γ spans vertex k." A sequence of edges $(i, i+1, \cdot)_{i=1}^{k}$ is a *path* between vertices 1 and k. Likewise, a sequence of edges is a path between two vertices whenever it is a path after appropriately relabeling vertices. Graphs of form $(\{i\}, \{(i, i, \cdot)\})$ are *loops*. If each of the k vertices of a graph Γ is the head of exactly one edge, the tail of exactly one edge, and there is a path between any two vertices, then Γ is a *k-cycle*. For ease of notation, we often write $(i_1 i_2 \cdots i_k)$ instead of $(\{i_1, i_2, \cdots, i_k\}, \{(i_1, i_2, \cdot), (i_2, i_3, \cdot), \cdots (i_k, i_1, \cdot)\})$.

A graph $\Gamma_{sub} = (V_{sub}, E_{sub})$ is a *subgraph* of $\Gamma = (V, E)$ if $V_{sub} \subseteq V$, and $E_{sub} \subseteq E$. With the understanding the matrix A in $\mathbb{R}^{n\times n}$ maps to Γ, $A[\Gamma_{sub}]$ is the submatrix of A indexed by the vertices spanned by Γ_{sub}—a missing edge in Γ_{sub} leads to a zero entry in $A[\Gamma_{sub}]$. A sequence $(\Gamma_i)_{i=1}^{m}$ of subgraphs is *increasing* if $\Gamma_i \subset \Gamma_{i+1}$. The number $|\Gamma_{sub}|$ denotes the cardinality of V_{sub}, also called the *order* of Γ_{sub}. Subgraphs Γ_1 and Γ_2 are *disjoint* if they share no vertices (hence, edges.) A partition of a graph into disjoint subgraphs is also called *decomposition*. In particular, if the subgraphs of a k-decomposition—with k vertices—consist of disjoint cycles, then the k-decomposition is a *k-Hamiltonian*. Likewise, we call a graph of order k a k-graph. Given $\Gamma = (V, E)$, it is convenient to define $\mathcal{H}_k(\Gamma)$, the set of k-Hamiltonians that are subgraphs of Γ. We drop the order prefix and subscript 'k' when refering *Hamiltonians* of the same order of the subgraph, i.e., $\mathcal{H}(\Gamma) = \mathcal{H}_{|\Gamma|}(\Gamma)$. It is also convenient to define $\mathcal{V}(\Gamma) = V$. In addition to partitioning, another way of obtaining a subgraph is by "trimming." Given $\Gamma = (V, E)$, to trim an edge $e \in E$ is to take $\Gamma_{sub} = (V, E\backslash\{e\})$. Similarly, we can trim a subgraph from a graph, by removing subgraph edges from the graph—while preserving edges shared with other cycles. For example, let Γ be a 3-graph given by the union of 2-cycle (12) and 3-cycle (123). Then, removing (12) from Γ results in a graph given solely by (123). Note (12) and (123) share edges, but these common edges are not removed.

Consider a k-cycle $\Psi = (V, E)$. Every vertex spanned by Ψ is the tail and the head of exactly one edge in Ψ. Thus, Ψ maps to a permutation π over V taking heads to tails. A permutation is a bijection from a set onto the same set. The sign of π is determined by its parity: -1 when π is odd; +1 otherwise. π is odd when it is an odd-numbered product of transpositions (permutations of two elements.) For example, $(243) = (24)(23)$ is even. Accordingly, if $(v_i, v_{i+1 \mod k}, s_i)$ are the edges of Ψ, then the sign[2] of Ψ is

$$\operatorname{sgn} \Psi := \operatorname{sgn} \pi \cdot \left(\prod_{i=1}^{k} s_i \right).$$

Analogously, if Hamiltonian Δ is a set of k cycles Ψ_i, then

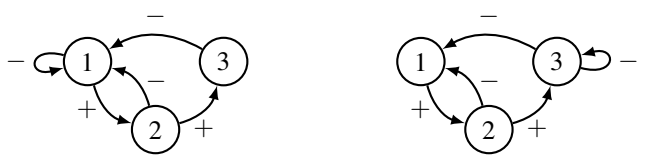
$$\operatorname{sgn} \Delta := \Pi_{i=1}^{k} \operatorname{sgn} \Psi_i.$$

Let $\gamma : A \mapsto \Gamma$ denote the function that gives the graph Γ associated with matrix A. Given $A \in \mathbb{R}^{n\times n}$, let $\rho(A)$ denote its characteristic polynomial $\rho(A)(s) := \det(sI - A)$. With the understanding that $\Gamma = \gamma(A)$, $\rho(\Gamma)$ means $\rho(A)$. Similarly, for $\Gamma_{sub} \subseteq \Gamma$ and $A_{sub} := A[\Gamma_{sub}]$, $\rho(\Gamma_{sub})$ means $\rho(A_{sub})$.

Remark 1. *Given $A \in \mathbb{R}^{n\times n}$, let*

$$\rho(A)(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n.$$

[2]This abuse of notation—using the symbol 'sgn' to refer to functions defined over real numbers and graphs—should cause no confusion.



(a) $|\mathcal{H}_k(\Gamma)| = 1$, $k = 1, 2, 3$. (b) $|\mathcal{H}_3(\Gamma)| = 2$.

Fig. 1. Two signed graphs.

The coefficients of $\rho(A)$ are given by

$$a_k = (-1)^k \sum_{\Delta \in \mathcal{H}_k(\gamma(A))} \det A[\Delta], \qquad k \in [n]. \tag{1}$$

In words, (1) *states the coefficients of $\gamma(A)$ are the "sums" of Hamiltonians in the sense that a_k amounts to the sum of determinants of A indexed by k-Hamiltonians.*

An important consequence of Remark 1 is: *entries that are not spanned by cycles are irrelevant to the eigenvalues of A.* We therefore discard entries/edges not belonging to cycles.

To illustrate Remark 1, consider $A \in \mathbb{R}^{3\times 3}$ given by

$$A = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

The characteristic polynomial of A is

$$\rho(A) = s^3 + s^2 + 2s + 3.$$

Each coefficient amounts to the sum of appropriate Hamiltonians. For example, the second leading coefficient corresponds to 1-Hamiltonians of $\gamma(A)$. Shown in Fig. 1(a), this 1-Hamiltonian is the loop spanning 1 and formula (1) gives $a_1 = (-1)(-1) = 1$. In fact, $\gamma(A)$ also contains exactly one of the other two k-Hamiltonians, $k = 2, 3$. The 2-Hamiltonian is (12). Accordingly, a negative sign multiplies the matrix entries $A[1, 2]$ and $A[2, 1]$ resulting in $a_2 = (-1)^2[-(1)(-2)] = 2$. On the other hand, the permutation associated with the 3-cycle spanning $\gamma(A)$ decomposes as two transpositions, so that $a_3 = (-1)^3[(1)(1)(-3)]$.

Equivalence between cycles and permutations enables more convenient Hamiltonian notation. We denote by $\Delta = \Pi_i \pi_i$ a Hamiltonian Δ containing cycles Ψ_i equivalent to permutations π_i. For example, the graph shown in Fig. 1(b) has two 3-Hamiltonians: (123) and (12). We omit identity permutations when possible, saying (12) instead of $(12)(3)$.

Structures

A matrix A in $\mathbb{R}^{n\times n}$ is *Hurwitz* if all of its eigenvalues have negative real part. We are interested in the relationship between the Hurwitz property and the structure of A. By structure, we mean a matrix taking values in $\{-1, 0, 1\}$, to which realizations adhere. That is, A entries have their signs given by the corresponding structure entry. For example, matrix (2) is an instance of the structure

$$S = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Just as a real matrix A, its structure S produces a graph. Accordingly, the edges of a graph induced by S are triplets $(i, j, S_{i,j})$, retaining only nonzero S entries. In our example, this gives the graph in Fig. 1(a). We then have the following.

Definition 1. *If a graph Γ has a Hurwitz realization A, then Γ is potentially Hurwitz, or π-Hurwitz in short. If every realization A with $\gamma(A) = \Gamma$ is Hurwitz, then Γ is Hurwitz.*

Remark 2. *In this paper, we use slightly unconventional objects and notation in three situations. First, we work with structures, defined in $\{-1, 0, 1\}^{n\times n}$, instead of what are typically called "sign patterns" [17], elements of $\{-, 0, +\}^{n\times n}$. The point is to avoid defining an algebra over $\{-, 0, +\}$. Second, we consider being (potentially) Hurwitz both a graphical and a structural property. Because graphs and structures are equivalent, and because we characterize being potentially Hurwitz graphically, this overloading of the term is more convenient than confusing. Lastly, being potentially Hurwitz is a property typically labelled being "potentially stable" or "potentially sign-stable." Given our previous decision on terminology, choosing the first over the last aims at keeping notation consistent and clear.*

III. Nests: One vertex at a time

In this paper, we characterize the structure of class of real matrices called *nests*. Given a nest $A \in \mathbb{R}^{n\times n}$, there exists a positive diagonal matrix D such that is AD Hurwitz [10].

Definition 2. *Let $A \in \mathbb{R}^{n\times n}$. If up to relabelling indices, $\operatorname{sgn}\det A\,[j] = (-1)^j$, $j \in [n]$, then A is called nest. If $\Gamma = \gamma(A)$, then Γ is called an n-nest, or simply a nest.*

Theorem 1. *[10, Theorem 2.1, p.246] If $A \in \mathbb{R}^{n\times n}$ is a nest, then there exists a positive diagonal matrix D such that, up to relabelling, $AD[k]$ is Hurwitz, $k \in [n]$.*

Remark 3. *Henceforth, all results are a contribution of this paper. Also, moving forward we assume nests $A \in \mathbb{R}^{n\times n}$ have indices relabelled so that $\operatorname{sgn}\det A[j] = (-1)^j$, $j \in [n]$.*

Lemma 1. *Let Γ be a nest. There exist Hamiltonians Δ_i and π-Hurwitz subgraphs $\Gamma_i \subseteq \Gamma$ such that $\Gamma_i \cup \Delta_i \subseteq \Gamma_{i+1}$ and*

$$\Gamma_i \cup \Delta_i \text{ is } \pi\text{-Hurwitz}, \qquad i \in [|\Gamma| - 1]. \tag{3}$$

Proof. Let Γ have order n and let A be a realization of Γ such that $A[j]$ is Hurwitz for every $j \in [n]$. The coefficients of $\rho(A)$ must be positive because A is Hurwitz. In particular, $\rho(A)$ has positive constant coefficient, which collects the Hamiltonians of Γ. Thus, Γ contains a cycle Ψ_n spanning n inducing Hamiltonians that positively add to the constant coefficient of $\rho(A)$. Let Δ_{n-1} be one such Hamiltonian. Then, trim from Γ every cycle spanning n other than Ψ_n. Denote the resulting realization by T_n. Let $\Gamma_{n-1} := \gamma(T_n[n-1])$. Because cycles spanning n have no bearing on $\rho(A_n[j])$ for $j < n$, proper submatrices $T_n\,[j]$ are Hurwitz. It remains to show $\Gamma_{m-1} \cup \Delta_{m-1}$ is π-Hurwitz. Put

$$D_n := T_n - \begin{bmatrix} T_n[n-1] & 0 \\ 0 & 0 \end{bmatrix}, \qquad G_n := T_n - D_n.$$

For α sufficiently small, $n-1$ eigenvalues of $G_n + \alpha D_n$ are arbitrarily close to those of $T_n[n-1]$, all in LHP. Moreover, the constant term of $\rho(G_n + \alpha D_n)$ is identically nonzero[3], hence the remaining eigenvalue of $G_n + \alpha D_n$ is not zero. Furthermore, Ψ_n is the only cycle spanning n, thus the constant term of $\rho(G_n + \alpha D_n)$ is positive. The remaining eigenvalue of $G_n + \alpha D_n$, a realization of $\Gamma_{n-1} \cup \Delta_{n-1}$, is therefore negative. $\Gamma_{n-1} \cup \Delta_{n-1}$ is π-Hurwitz. By arguing the same way for $T_n[n-1]$ as we did for A, we obtain Δ_{n-2} and Γ_{n-2}, and so on, until we have the desired sequence. □

IV. A SEQUENTIAL CHARACTERIZATION OF NESTS

Lemma 1 suggests Hamiltonians are the key to construct π-Hurwitz supergraphs, but having the "right" sign alone is not enough. Consider an increasing sequence of π-Hurwitz graphs $(\Gamma_i)_{i=1}^4$ shown in Fig. 2. Constructing Γ_{i+1} from Γ_i follows the same procedure: add an $(i+1)$-Hamiltonian Δ_i of sign $(-1)^{i+1}$. In Γ_2 and Γ_3, there is only one i-Hamiltonian, and we have a two-part recipe to construct Hurwitz realizations:

G. Let $A_{i+1,\alpha}$ be a realization of $\Gamma_i \cup \Delta_i$ such that its submatrix $A_{i,\alpha}$ corresponding to Γ_i is Hurwitz, whereas the (new) entries corresponding to cycles introduced by Δ_i are given by $\alpha \in (0,1)$;

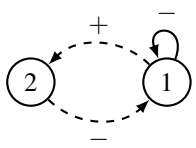
S. For α sufficiently small, $A_{i+1,\alpha}$ has i eigenvalues arbitrarily close to the LHP eigenvalues of $A_{i,\alpha}$, while the positive constant term in $\rho(A_{i+1,\alpha})$ implies the additional eigenvalue is real and negative (see proof of Lemma 1.)

In summary, in part G, we *grow* a previous realization known to verify the nest property by fixing the corresponding previous entries and adding new entries to have values decided on the next part; then, in part S, the new entries are taken *small* enough to preserve previous eigenvalues while moving the new eigenvalue into LHP. When adding a Hamiltonian induces other Hamiltonians of opposite signs, however, it is not immediate the above recipe produces a Hurwitz realization. Because it is not immediate which of the opposing signs dominates which. In other words, the constant term of $\rho(A_{i+1,\alpha})$ could be nonpositive, and part S of the recipe could not apply. The sequence $(\Gamma_i)_{i=1}^4$ illustrates this point. Γ_4 builds upon Γ_3 by adding 4-Hamiltonian $\Delta_3 = (12)(34)$. But adding Δ_3 also induces $\Delta_3' = (34)$, of opposite sign. Therefore, $\rho(A_{4,\alpha})$ need not have a positive constant term. It depends on A_3.

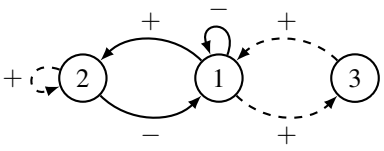
The reason Γ_4 is π-Hurwitz lies in two conditions that warrant recipe G-S will work. One is on every $\Gamma_{sub} \subseteq \Gamma_i \cup \Delta_i$ such that $\mathcal{H}(\Gamma_i[\Gamma_{sub}])$ is empty, all the Hamiltonians in $\mathcal{H}(\Gamma_{sub})$ share the same Δ_i cycles; for example, (34) in $\Gamma_3 \cup \Delta_3$. We call it the *factoring* property.

Definition 3. *Let $(\Gamma_i, \Delta_i)_{i=1}^n$ be a sequence of i-graphs Γ_i and $(i+1)$-Hamiltonians Δ_i such that $\Gamma_{i+1} = \Gamma_i \cup \Delta_i$. Let $\Gamma_{sub} \subseteq \Gamma_i \cup \Delta_i$ and let Ψ be a cycle on Γ_{sub} induced by Δ_i. If $\mathcal{H}(\Gamma_i[\Gamma_{sub}])$ is nonmepty or empty and every Hamiltonian in $\mathcal{H}(\Gamma_{sub})$ contains Ψ, then Ψ is factoring on Γ_{sub}. If every cycle induced by Δ_i is factoring, then $(\Gamma_i, \Delta_i)_{i=1}^n$ is factoring.*

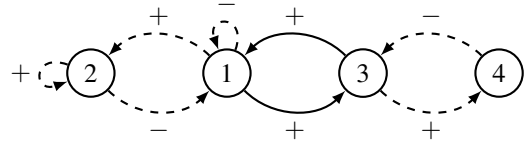
[3]The cycle Ψ_n induces Hamiltonians that add positively to $\rho(A)$.



(a) Γ_2 contains one 2-Hamiltonian Δ_1 (dashed edges.)



(b) Γ_3 contains one 3-Hamiltonian Δ_2 (dashed edges.)



(c) Γ_4 contains 4-Hamiltonians $(12)(34)$ and (34).

Fig. 2. A sequence of π-Hurwitz supergraphs: $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \Gamma_4$.

The second condition that makes Γ_4 a nest is Δ_i are sign-compatible with Γ_i, in the following sense: Δ_i have sign $(-1)^{i+1}$ *and* the Γ_i sub-Hamiltonian Δ_i contains is *dominant*. Concretely, the dominant Hamiltonians of Γ_i are the support of a "dominance" function $\delta_i : D_i \to \{0,1\}$ defined on

$$D_i = \{(\Gamma_{sub}, \Delta) : \Gamma_{sub} \subseteq \Gamma_i \cup \Delta_i, \Delta \in \mathcal{H}(\Gamma_{sub})\}.$$

Hamiltonian Δ_i is sign-compatible with Γ_i if

$$\Gamma_i \cap \Delta_i = \emptyset \quad \text{or} \quad \delta_i(\Gamma_i[\mathcal{V}(\Gamma_i \cap \Delta_i)], \Gamma_i \cap \Delta_i) = 1. \quad (4)$$

The subgraph $\Gamma_i[\mathcal{V}(\Gamma_i \cap \Delta_i)]$ spans the same vertices as $\Gamma_i \cap \Delta_i$ does, and (4) means either Δ_i induces no other[4] Hamiltonian or the Γ_i sub-Hamiltonian Δ_i contains is dominant on $\Gamma_i[\mathcal{V}(\Gamma_i \cap \Delta_i)]$.

To flesh out how δ_i are defined, we turn to our example. Start with δ_1, defined on $D_1 = (\Gamma_1, (1))$. Since Γ_1 contains exactly one Hamiltonian, put $\delta_1(\Gamma_1, (1)) = 1$. Next, extend[5] δ_1 to δ_2:

$$\delta_2(\Gamma_2[1], (1)) = 1, \qquad \delta_2(\Gamma_2, (12)) = 1.$$

Hamiltonian (1) is dominant on $\Gamma_2[1]$ since $\delta_1(\Gamma_1, (1)) = 1$; 2-Hamiltonian (12) is dominant on Γ_2 since it is the only one Γ_2 contains. The procedure we followed to extend δ_i to δ_{i+1} so far consists in the following procedure: until there is a pair $(\Gamma_{sub}, \Delta) \in D_{i+1}$ over which δ_i can be defined, do

Case 1. If $\Gamma_{sub} \subseteq \Gamma_i$, then

$$\delta_{i+1}(\Gamma_{i+1}[\mathcal{V}(\Gamma_{sub})], \Delta) = \delta_i(\Gamma_{sub}, \Delta).$$

[4]By assumption, every $\Gamma_i \cup \Delta_i$ Hamiltonian shares all the cycles Δ_i introduces. Hence, if $\Gamma_i \cap \Delta_i = \emptyset$, then Δ_i is the only such Hamiltonian.

[5]Note vertex 2 has no loops, therefore $\mathcal{H}_1(\Gamma_2[2]) = \emptyset$.

Case 2. Else, if $\mathcal{H}(\Gamma_{sub}) = \Delta$, then

$$\delta_{i+1}(\Gamma_{sub}, \Delta) = 1.$$

Case 1 handles pairs (Γ_{sub}, Δ) such that Δ is dominant on $\Gamma_i[\mathcal{V}(\Gamma_{sub})]$. The Hamiltonian Δ continues to be dominant on Γ_{sub}, which spans the same vertices as $\Gamma_i[\mathcal{V}(\Gamma_{sub})]$ but may have added Δ_i edges[6]. For example, $\delta_2(\Gamma_2[1], (1)) = 1$.

Case 2 handles pairs (Γ_{sub}, Δ) such that Δ is the only Hamiltonian Γ_{sub} contains, defining them dominant[7] on Γ_{sub}. For example, $\delta_2(\Gamma_2, (12)) = 1$.

Extending δ_2 to δ_3 requires another case:

Case 3. Else, if $\delta_i(\Gamma_i[\mathcal{V}(\Gamma_{sub})], \Delta') = 1$, then[8]

$$\delta_{i+1}(\Gamma_{sub}, \Delta) = (1 + (\operatorname{sgn}\Delta)(\operatorname{sgn}\Delta'))/2.$$

Case 3 handles pairs (Γ_{sub}, Δ) such that $\Gamma_i[\mathcal{V}(\Gamma_{sub})]$ has a dominant Hamiltonian. A new Hamiltonian is dominant on Γ_{sub} if and only if it has the same sign of a dominant Hamiltonian on $\Gamma_i[\mathcal{V}(\Gamma_{sub})]$. For example, the subgraph $\Gamma_3[12]$ has two 2-Hamiltonians, (12) and $(1)(2)$. The first, (12), is inherited from Γ_2 thus dominant on $\Gamma_3[12]$, by case 1; the second is induced by positive loop (2), introduced by Δ_2. From $\operatorname{sgn}(12) \neq \operatorname{sgn}(1)(2)$ follows that $\delta_3(\Gamma_3[12], (1)(2)) = 0$. Therefore, δ_3 is defined by

$$\begin{aligned} \delta_3(\Gamma_3[1], (1)) &= 1, & \delta_3(\Gamma_3[2], (2)) &= 1, \\ \delta_3(\Gamma_3[1,2], (12)) &= 1, & \delta_3(\Gamma_3[1,2], (1)(2)) &= 0, \\ \delta_3(\Gamma_3[1,3], (13)) &= 1, & \delta_3(\Gamma_3, (2)(13)) &= 1. \end{aligned}$$

Finally, we add fourth and last case. The last case concerns Hamiltonians induced by Δ_i that are not the only ones on their corresponding subgraphs and that span a subgraph $\Gamma_{sub} \subseteq \Gamma_i \cup \Delta_i$ for which there is no dominant Hamiltonian on $\Gamma_i[\mathcal{V}(\Gamma_{sub})]$, handled by cases 2 and 3, respectively.

Case 4. Else, if $(\Gamma_i[\mathcal{V}(\Gamma_i \cap \Delta_i)], \Delta \cap \Gamma_i) \in D_i$, then

$$\delta_{i+1}(\Gamma_{sub}, \Delta) = \delta_i(\Gamma_i[\mathcal{V}(\Gamma_i \cap \Delta_i)], \Delta \cap \Gamma_i).$$

Case 4 handles pairs (Γ_{sub}, Δ) such that $\Delta \subsetneq \Gamma_i$ but $\Gamma_i \cap \Delta$ is nonempty; Δ is dominant on Γ_{sub} if and only if $\Gamma_i \cap \Delta$ is dominant on $\Gamma_i[\mathcal{V}(\Gamma_i \cap \Delta_i)]$. For example, the 4-Hamiltonian $\Delta_3 = (12)(34)$ introduces the 2-cycle (34), which in turn induces another 4-Hamiltonian $\Delta_3' = (1)(2)(34)$. Because both 4-Hamiltonians contain a cycle Δ_3 introduces, case 1 does not apply; case 2 does not apply because $\mathcal{H}(\Gamma_3 \cup \Delta_3)$ is not a singleton; and, of course, case 3 does not apply either because there is no dominant Γ_3 Hamiltonian on $\Gamma_3 \cup \Delta_3$. Both Δ_3 and Δ_3' intercept Γ_3, however, thus case 4 applies. Accordingly, $\delta_4(\Gamma_4, \Delta_4) = 1$ whereas $\delta_4(\Gamma_4, \Delta_4') = 0$.

Cases 1-4 are mutually exclusive. For example, (1) is a dominant sub-Hamiltonian of $(1)(2)$ on $\Gamma_2[1]$, but $(\Gamma_2, (1)(2))$ does not fall into case 4, it falls into case 3, because Γ_2 has a dominant Hamiltonian. In fact, $\delta_3(\Gamma_3[1,2], (1)(2)) = 0$ since $\operatorname{sgn}(1)(2) \neq \operatorname{sgn}(12)$.

[6]So δ_{i+1} is not precisely an extension because it coincides with δ_i on a subgraph spanning the same vertices, but with possibly more edges.

[7]If a Hamiltonian is dominant in Γ_i *and* it is the only Hamiltonian on its corresponding subgraph, then it is considered on case 1.

[8]If $\operatorname{sgn}\Delta = \operatorname{sgn}\Delta'$, then $(1 + (\operatorname{sgn}\Delta)(\operatorname{sgn}\Delta'))/2 = 1$. Otherwise, $(1 + (\operatorname{sgn}\Delta)(\operatorname{sgn}\Delta'))/2 = 0$.

Algorithm 1: Extension by precedence

input : $\delta_i : D_i \to \{0,1\}$ and (Γ_i, Δ_i)
output: $\delta_{i+1} : D_{i+1} \to \{0,1\}$

```
D ← D_{i+1};
while ∃(Γ_sub, Δ) ∈ D do  // try each (Γ_sub, Δ)
    if Γ_sub ⊆ Γ_i then  // case 1
        δ_{i+1}(Γ_sub, Δ) ← δ_i(Γ_i[V(Γ_sub)], Δ);
    else if H(Γ_sub) = Δ then  // 2
        δ_{i+1}(Γ_sub, Δ) ← 1;
    else if δ_i(Γ_i[V(Γ_sub)], Δ') = 1 then  // 3
        δ_{i+1}(Γ_sub, Δ) ← (1 + (sgn Δ)(sgn Δ'))/2;
    else if (Γ_i[V(Γ_sub)], Γ_i ∩ Δ_i) ∈ D_i then  // 4
        δ_{i+1}(Γ_sub, Δ) ← δ_i(Γ_i[V(Γ_sub)], Γ_i ∩ Δ_i);
    if δ_{i+1}(Γ_sub, Δ) is defined then  // update D
        D ← D \ (Γ_sub, Δ);
    end
end
```

Going through D_{i+1} and checking cases 1-4 until there is no pair (Γ_{sub}, Δ) over which δ_{i+1} can be defined eventually exhausts D_{i+1}, resulting in δ_{i+1} well-defined over D_{i+1}. If $\Gamma_{sub} \subseteq \Gamma_i$, then for each $\Delta \in \mathcal{H}(\Gamma_{sub})$, (Γ_{sub}, Δ) falls into case 1. The remaining subgraphs Γ_{sub} (and their Hamiltonians) all contain a cycle introduced by Δ_i. If $\mathcal{H}(\Gamma_{sub})$ is a singleton, $(\Gamma_{sub}, \mathcal{H}(\Gamma_{sub}))$ falls into case 2. Otherwise, either $\mathcal{H}(\Gamma_{sub})$ contains Hamiltonians from $\mathcal{H}(\Gamma_i[\mathcal{V}(\Gamma_{sub})])$, or it does not. If it does, then (Γ_{sub}, Δ) falls into case 3, for every $\Delta \in \mathcal{H}(\Gamma_{sub}) \setminus \mathcal{H}(\Gamma_i[\mathcal{V}(\Gamma_{sub})])$. Finally, $\mathcal{H}(\Gamma_i[\mathcal{V}(\Gamma_{sub})])$ may be empty, but not $\mathcal{H}(\Gamma_{sub})$. If so, every $\Delta \in \mathcal{H}(\Gamma_{sub})$ must contain [9] a cycle from Γ_i. This is case 4.

We denote the procedure of defining δ_i to δ_{i+1} as *extension by precedence*, formally presented as Algorithm 1.

Definition 4. *If δ_{i+1} is obtained from δ_i by Algorithm 1, then δ_{i+1} extends δ_i by precedence.*

Given $\Gamma_{sub} \subseteq \Gamma_i$, the support and the kernel of δ_i partition $(\Gamma_{sub}, \mathcal{H}(\Gamma_{sub}))$ into two components. Moreover, these two components correspond to disjoint level sets of the sign function. This is trivially true for any Γ_1 a negative loop, so assume it true for $i \leq k$. Case 1 copies δ_k onto δ_{k+1} on the corresponding subgraphs, therefore the claim is true, by assumption. Case 2 trivially satisfies the claim. Case 3 matches dominant/dominated Hamiltonians induced by Δ_i with Γ_i dominant/dominated Hamiltonians by their signs, therefore also satisfies the claim. Finally, consider case 4. By assumption, the support and the kernel of δ_k partition $\mathcal{H}(\Gamma_i[\Gamma_i \cap \Delta_i])$ into two components $\mathcal{S}$ and $\mathcal{K}$ corresponding to disjoint level sets of the sign function. Then, case 4 partitions $\mathcal{H}(\Gamma_{sub})$ into two components, one with dominant

[9]Otherwise, all Hamiltonians in $\mathcal{H}(\Gamma_{sub})$ only have Δ_i cycles. Since they do not intersect, there can be at most one such Hamiltonian on Γ_{sub}, in which case we fall into case 2.

sub-Hamiltonians in $\mathcal{S}$ and the other with dominated sub-Hamiltonians in $\mathcal{K}$. Moreover, since every Hamiltonian in these induced components share the same Δ_k cycles, the induced components are also disjoint level sets of the sign function (possibly with a "flipped" sign in comparison to $\mathcal{S}$.)

To get a sense of why factorability and sign-compatibility are enough for Γ_4 to be a nest, let $A_{i,\alpha}$ be Hurwitz submatrices generated by recipe G-S above, $i \in [3]$. Then, let $A_{4,\alpha}$ be the result of part G of recipe G-S. Every 4-Hamiltonian in $\Gamma_3 \cup \Delta_3$ contains 2-cycle (34). Hence, the constant term in $\rho(A_{4,\alpha})$ is linear in the edges of (34). Thus, to preserve the LHP eigenvalues of $A_{3,\alpha}$, suffices to take the additional edges in $A_{4,\alpha}$ small. Likewise, to preserve the LHP eigenvalues of $A_{2,\alpha}$ in $A_{3,\alpha}$, the positive loop spanning 2 must be small compared to the negative 2-cycle spanning 1 and 2. Hence, Δ_3 dominates Δ_3', $\rho(A_{4,\alpha})$ has positive constant term, and part S of the above recipe applies to $A_{4,\alpha}$.

Definition 5. *Let $(\Gamma_i, \Delta_i)_{i=1}^n$ be as in Definition 3. If*

D. $\operatorname{sgn} \Delta_i = (-1)^{i+1}$ *and* $\delta_i(\Gamma_i[\mathcal{V}(\Gamma_i \cap \Delta_i)], \Gamma_i \cap \Delta_i) = 1$*;*

X. δ_{i+1} *extends* δ_i *by precedence;*

then $(\Gamma_i, \Delta_i)_{i=1}^n$ is sign-compatible.

We are now ready to characterize nests. First, sufficiency.

Theorem 2. *If Γ_1 is a negative loop and $(\Gamma_i, \Delta_i)_{i=1}^n$ is factoring and sign-compatible, then every $\Gamma_i \cup \Delta_i$ is a nest.*

Proof. The proof is a formalization of the main ideas of recipe G-S in light of the procedure of extension by precedence. But we argue part S differently, not by exploiting continuity in the determinant. Instead, we exploit continuity in elementary *Root Locus* to incrementally produce Hurwitz realizations G_i of Γ_i verifying Definition 2, such that $G_i[\Gamma_j]$ are Hurwitz, $j \in [i]$. The point is to emphasize Δ_i acts as a controller stabilizing an extra pole at the origin, while preserving Γ_i poles in LHP.

Let $G_1 = -1$, and let $\alpha_1 \in (0, 1)$. To construct a Hurwitz realization G_2 of $\Gamma_2 = \Gamma_1 \cup \Delta_1$, first fix $G_2[\Gamma_1] = G_1$, then take the negative edge in $G_2[\Delta_1]$ equal $-\alpha_1$. If there is another edge spanning 2 (i.e., if Δ_1 is a 2-cycle), take it equal 1 in $G_2[\Delta_1]$. Letting $\rho(G_i)$ denote the characteristic polynomial of G_i, then $\rho(G_2)(s) = s^2 + c_{1,\alpha} s + \alpha_1$ can be expressed as

$$\rho(G_2)(s) = \alpha_1(s^2 + 1) + (1 - \alpha_1) s \rho_{\alpha_1}(G_1)(s).$$

As $\alpha_1 \to 0$, $\rho_{\alpha_1}(G_1)(s) = s + c_{1,\alpha}(1 - \alpha_1)^{-1}$ converges to $\rho(G_1)(s) = s + 1$, pointwise[10]. The roots of $\rho(G_2)$ verify

$$1 + K_1 \frac{s^2 + 1}{s \rho_{\alpha_1}(G_1)(s)} = 0, \tag{5}$$

an unusual Root Locus [12] in the sense that the poles of (5) as well the gain $K_1 := \alpha_1/(1 - \alpha_1)$ are functions of α_1. That is, K_1 does not vary independently of the poles of (5). As $\alpha_1 \to 0$, the poles of (5) tend to the roots of $s\rho(G_1)(s)$; one is in LHP—the real root of $\rho(G_1)(s) = (s + 1)$—the other is 0. The zeros are two roots of (positive) unit, none of which are on $(0, +\infty)$. That is, the positive real axis is not part of the Root Locus for any fixed α_1. Now, the Root Locus is continuous with respect to its poles and zeros, therefore for α_1 sufficiently small the closed-loop poles of (5) are in LHP. For concreteness, choose $\alpha_1 = 1/2$. If Δ_1 introduces a negative loop (2), the dominance function δ_2 is

$$\delta_2(\Gamma_2[1], (1)) = \delta_2(\Gamma_2[2], (2)) = \delta_2(\Gamma_2, (1)(2)) = 1; \tag{6}$$

otherwise Δ_1 introduces a negative 2-cycle and

$$\delta_2(\Gamma_2[1], (1)) = \delta_2(\Gamma_2, (12)) = 1. \tag{7}$$

In either case, there are only dominant Hamiltonians on Γ_2 and its subgraphs. Also, the support and the kernel of δ_2 partition every nonempty $\mathcal{H}(\Gamma_{sub})$, $\Gamma_{sub} \subseteq \Gamma_2$, into two components. Furthermore, these components coincide with disjoint level sets of the sign function. In particular, the sums of dominant Hamiltonians on their corresponding subgraphs are trivially greater than, in absolute value, the sum of the other Hamiltonians on the same subgraphs.

We repeat this argument inductively to construct a Hurwitz sequence $(G_i)_{i=1}^n$ verifying Definition 2, with $\Gamma_i = \gamma(G_i)$.

Induction hypotheses. There exist Hurwitz $(G_i)_{i=1}^k$ such that G_i are nests, and $\Gamma_i = \gamma(G_i)$, $i \leq k$. Moreover, for every $\Gamma_{sub} \subseteq \Gamma_i$ such that $\mathcal{H}(\Gamma_{sub})$ is nonempty,

$$\left| \sum_{(\Gamma_{sub}, \Delta) \in \operatorname{supp} \delta_i} \det G_i[\Delta] \right| > \left| \sum_{(\Gamma_{sub}, \Delta) \in (\operatorname{supp} \delta_i)^c} \det G_i[\Delta] \right|. \tag{8}$$

Induction step. Fix $G_{k+1}[\Gamma_k] = G_k$. Let $\alpha_{k+1} \in (0, 1)$. Let $\mathcal{C}_k$ denote the set of cycles Δ_k introduces. For each cycle in $\mathcal{C}_k$, fix one edge of G_{k+1} with absolute value α_{k+1}; take remaining new edges with absolute value 1. The factoring property $(\Gamma_i, \Delta_i)_{i=1}^k$ has implies α_{k+1} factors the constant term of $\rho(G_{k+1})$, at least once. So let $h = |\mathcal{C}_k| \geq 1$, that is, the number of cycles Δ_k introduces. Then, α_{k+1}^h factors the constant term of $\rho(G_{k+1})$. Let $\alpha_{k+1}^h c_{k+1}$ denote this term, c_{k+1} a number depending on G_k that will be shown positive. The number c_{k+1} absorbs negative signs any of the $\mathcal{C}_k$ cycles might have. Regardless, $\alpha_{k+1}^h c_{k+1}$ is positive. Indeed, by the induction assumption, the support of δ_k restricted to $\Gamma_k[\mathcal{V}(\Gamma_k \cap \Delta_k)]$ is the 1-level-set of the sign function; the Δ_k sub-Hamiltonian on $\Gamma_k[\mathcal{V}(\Gamma_k \cap \Delta_k)]$ belongs to this level set, which dominates the other in the sense of inequality (8); by assumption, $(\Gamma_i, \Delta_i)_{i=1}^k$ is factoring, that is, all $\Gamma_k \cup \Delta_k$ Hamiltonians share the same Δ_k cycles, therefore the sign of $\det[G_{k+1}]$ is one for every choice of α_{k+1}; this sign matches that of every $\Gamma_k \cup \Delta_k$ Hamiltonian whose sub-Hamiltonian belongs to the 1-level-set of δ_k; Δ_k is sign-compatible, by assumption, therefore one of such $\Gamma_k \cup \Delta_k$ Hamiltonians; finally, again because Δ_k is sign-compatible, this sign produces positive $\det[G_{k+1}]$, by property D of sign-compatibility. Therefore, c_{k+1} is positive. So let

$$\rho(G_{k+1})(s) = s^{k+1} + c_{1,\alpha} s^k + \cdots + \alpha_{k+1}^h c_{k+1}. \tag{9}$$

As $\alpha_{k+1} \to 0$, $c_{j,\alpha}$ converge to c_j, for all $j \leq k$, where

$$\rho(G_k)(s) = s^k + c_1 s^{k-1} + \cdots + c_k.$$

[10] Uniformly: if Δ_1 is a 2-cycle, then $c_{1,\alpha} = 1$; otherwise, $c_{1,\alpha} = 1 + \alpha_1$.

The roots of (9) are defined by characteristic equation

$$1 + K_{k+1}\frac{s^{k+1} + c_{k+1}}{s\rho_{\alpha_{k+1}}(G_1)(s)} = 0, \tag{10}$$

with $K_{k+1} = \alpha_{k+1}^h/(1-\alpha_{k+1}^h)$. As $\alpha_{k+1} \to 0$, the poles of (10) converge to a single pole at 0 plus the roots of $\rho(G_k)$. These roots are in LHP, by the induction hypothesis. The zeros are $k+1$ roots of the (scaled) unit circle, none of which on interval $(0, +\infty)$ because c_{k+1} is positive. That is, the Root Locus of $s\rho(G_1)(s) + K_{k+1}(s^{k+1} + c_{k+1})$ has empty intersection with the positive real axis. But 0 is itself a pole, hence the Root Locus of $s\rho(G_1)(s) + K_{k+1}(s^{k+1} + c_{k+1})$ does contain an interval $(-\epsilon, 0]$, for small $\epsilon > 0$. Since (10) is continuous with respect to poles and zeros, the roots of $\rho(G_{k+1})$ are thus all in LHP for sufficiently small α_{k+1}. In fact, the roots of (9) are all in LHP for *arbitrarily* small α_{k+1}. We exploit this to choose α_{k+1} for G_{k+1} to satisfy the induction hypothesis.

By assumption, δ_{k+1} extends δ_k by precedence. If $\mathcal{H}(\Gamma_{sub})$ is a singleton, any choice of $G_{k+1}[\Gamma_{sub}]$ trivially satisfies the induction hypotheses, including inequality (8). Therefore this adds no constraints to the choice $G_{k+1}[\Delta_k \setminus \Gamma_k]$. The same is true if all Hamiltonians share the same new Δ_{k+1} cycles: (8) holds for $i = k+1$ because (8) holds for $i = k$, by hypothesis. Similarly, (8) holds on $G_{k+1}[\Gamma_{sub} \cap \Gamma_k]$ by hypothesis. Thus, denoting by D and d the left and right-hand side of (8) restricted to $G_{k+1}[\Gamma_{sub} \cap \Gamma_k]$, since $D > d$, by choosing entries in $G_{k+1}[\Gamma_{sub} \setminus \Gamma_k]$ small enough, (8) continues to hold on $G_{k+1}[\Gamma_{sub}]$, by continuity (whether the Γ_{sub} Hamiltonians induced by Δ_k are dominant or not.) □

To establish the sequential characterization of nests is tight, we prove a converse to Theorem 2. In doing so, we make two mild assumptions, in the interest of space. We restrict ourselves to nests that are also minimal graphs. A π-Hurwitz graph is called *minimal* if it is no longer π-Hurwitz after deleting one of its edges. Secondly, given $\Gamma_{sub} \subseteq \Gamma_i \cup \Delta_i$, we assume there is no cycle that is a Γ_{sub} sub-Hamiltonian and whose Δ_i edges are a subset of another cycle that is also a Γ_{sub} sub-Hamiltonian.

Assumption 1. *If Γ is a minimal $(n+1)$-nest, then Γ contains a minimal n-nest, unique up to relabelling.*

Assumption 2. *Suppose $(\Gamma_i, \Delta_i)_{i=1}^{k-1}$ is a factoring and sign-compatible sequence and let $\Gamma_k := \Gamma_{k-1} \cup \Delta_{k-1}$. If Δ_k is a $(k+1)$-Hamiltonian on $\Gamma_k \cup \Delta_k$, then every Δ_k-induced cycle on $\Gamma_k \cup \Delta_k$ that is part of a Hamiltonian on a subgraph $\Gamma_{sub} \subseteq \Gamma_k \cup \Delta_k$ is uniquely identified by a Δ_k edge.*

Theorem 3. *If Γ is a minimal n-nest, then it contains a factoring and sign-compatible sequence $(\Gamma_i, \Delta_i)_{i=1}^{n-1}$.*

Proof. We argue inductively. Since the induction hypotheses below can be readily verified by inspection for the case where $n \leq 2$, we state them directly.

Induction hypotheses.

1. If Γ is a minimal nest of order $n \leq k$, then Γ contains a factoring and sign-compatible sequence $(\Gamma_i, \Delta_i)_{i=1}^{n-1}$, and, in fact, $\Gamma = \Gamma_{n-1} \cup \Delta_{n-1} =: \Gamma_n$.
2. If G is a realization of Γ_n such that $G[j]$ is Hurwitz for all $j \in [n]$, then

$$\left|\sum_{\Delta \in \mathcal{D}} \det G[\Delta]\right| > \left|\sum_{\Delta \in \mathcal{H}(\Gamma[\mathcal{I}]) \setminus \mathcal{D}} \det G[\Delta]\right| \tag{11}$$

holds over

$$\mathcal{D} = \left\{\Delta \in \mathcal{H}(\Gamma[\mathcal{I}]) : \exists (\Gamma_n[\mathcal{I}], \Delta') \in \operatorname{supp} \delta_n : \operatorname{sgn}\Delta = \operatorname{sgn}\Delta'\right\}.$$

3. If $(\Gamma_i, \Delta_i)_{i=1}^{m}$ is a factoring and sign-compatible continuation of $(\Gamma_i, \Delta_i)_{i=1}^{n-1}$, i.e., $m \geq n-1$, and Δ_{m+1} is an $(m+2)$-Hamiltonian spanning $\Gamma_{m+1} := \Gamma_m \cup \Delta_m$ such that every cycle induced by Δ_{m+1} is factoring on Γ_{m+1}, then every cycle induced by Δ_{m+1} on every subgraph of Γ_i, with $i \leq n$, is factoring. In particular, if $m = n-1$, then $(\Gamma_i, \Delta_i)_{i=1}^{n}$ is a factoring and sign-compatible continuation of $(\Gamma_i, \Delta_i)_{i=1}^{n-1}$.

Induction step. Let Γ have order $k+1$. By definition, Γ admits a realization G such that $G[j]$ is Hurwitz for all $j \leq k+1$. The subgraph $\Gamma[k]$ is a nest, too. Assumption 1 implies this nest is minimal as well. In turn, $\Gamma[k]$ contains a factoring and sign-compatible sequence $(\Gamma_i, \Delta_i)_{i=1}^{k-1}$, by induction hypothesis 1. Define $\Gamma_k = \Gamma_{k-1} \cup \Delta_{k-1}$. Then, $\Gamma[k] = \Gamma_k$. Furthermore, for $\mathcal{I} \subseteq [k]$ such that $\mathcal{H}(\Gamma_k[\mathcal{I}])$ is nonempty, inequality (11) holds over $\mathcal{D}$.

Because G is Hurwitz, $\rho(G)$ has positive constant coefficient. Hence, Γ contains a Hamiltonian of sign $(-1)^{k+1}$. In fact, since (11) holds on any subgraph of Γ_k, there must be one such Hamiltonian, Δ_k, that is "sign-compatible" with Γ_k in the sense that if Δ_k contains a Γ_k sub-Hamiltonian, then this sub-Hamiltonian has the same sign as the dominant Γ_k Hamiltonians on the corresponding subgraph[11].

Every cycle induced by Δ_k that is a sub-Hamiltonian on Γ is uniquely identified by a Δ_k edge, by Assumption 2. Hence, by deleting a Δ_k edge, we trim exactly one cycle induced by Δ_k that is a sub-Hamiltonian on $\Gamma_k \cup \Delta_k$. Moreover, Γ_k is a nest, so one "sign-compatible" $(k+1)$-Hamiltonian is enough to produce a Hurwitz realization G of $\Gamma_k \cup \Delta_k$ such that $G[j]$ is Hurwitz for all $j \in [k+1]$. Also, by (11) there is a "sign-compatible" $(k+1)$-Hamiltonian. Furthermore, a cycle induced by Δ_k that is not factoring on $\Gamma_k \cup \Delta_k$ can be replaced by a Γ_k dominant sub-Hamiltonian. But then every cycle induced by Δ_k must be factoring on $\Gamma_k \cup \Delta_k$, because Γ is minimal. In turn, $(\Gamma_i, \Delta_i)_{i=1}^{k}$ is factoring, by hypothesis 3, and we can extend δ_k to δ_{k+1} by precedence. Therefore, dominance function δ_{k+1} is well-defined, and the sequence $(\Gamma_i, \Delta_i)_{i=1}^{k}$ is sign-compatible[12] as well. Then, (11) clearly holds on $\mathcal{D}$, with $n = k+1$, for any $\mathcal{I} \subseteq [k+1]$ such that

[11] Strictly speaking, to say Δ_k is sign-compatible, we need a dominance function δ_{k+1}. To extend δ_k to δ_{k+1}, however, Δ_k must be factoring, which we have yet to prove Δ_k. Hence, at this point Δ_k is "sign-compatible."

[12] Not just "sign-compatible".

$\mathcal{H}(\Gamma[\mathcal{I}])$ is nonempty. That is, induction hypotheses 1 and 2 hold for $n = k + 1$. We now prove the last one also does.

Define $\Gamma_{k+1} := \Gamma_k \cup \Delta_k$, and let Δ_{k+1} be a $(k+2)$-Hamiltonian spanning Γ_{k+1} such that every cycle induced by Δ_{k+1} is factoring on $\Gamma_{k+1} \cup \Delta_{k+1}$. Then, every cycle induced by Δ_{k+1} is factoring on every subgraph of Γ_k, by hypothesis 3. In turn, if a cycle induced by Δ_{k+1} is not factoring on some $\Gamma_{sub} \subseteq \Gamma_{k+1} \cup \Delta_{k+1}$, then Γ_{sub} must contain a cycle induced by Δ_k. In fact, $\mathcal{H}(\Gamma_{sub})$ must contain two Hamiltonians $\Delta_{sub,1}$ and $\Delta_{sub,2}$ such that $\Delta_{sub,1}$ contains Ψ_1 but not Ψ_2, both cycles induced by Δ_{k+1}, and *vice versa*. Now, both $\Delta_{sub,1} \setminus \Psi_1$ and $\Delta_{sub,2} \setminus \Psi_2$ contain cycles Φ_i induced by Δ_k, which are factoring, because $(\Gamma_i, \Delta_i)_{i=1}^k$ is factoring. But then Ψ_1 is not factoring on $\Delta_{sub,1} \setminus (\cup_i \Phi_i)$, i.e., Ψ_1 is not factoring on Γ_k, a contradiction. □

Remark 4. *Routh-Hurwitz theorem counts LHP roots of a polynomial by consecutively producing smaller-order polynomials and comparing their leading coefficients [18]. In an LTI setting, these leading coefficients are functions of the actual entries of the system matrix, carrying intricate interdependencies between them. This hinders exploiting Routh-Hurwitz to gain insight on what makes structures (potentially) Hurwitz.*

On the other hand, decomposing polynomials additively preserves the direct relationship between their coefficients and actual matrix entries. This is a core idea we exploit. For example, consider the system of Fig. 2. Starting with a real pole at -1 and then taking entries with cycle gains given by ϵ_i, we produce polynomials in the form $\rho_{i+1}(s) = s\rho_i(s) + \rho_{res,i+1}(s, \epsilon_{i+1})$, $\rho_{res,i+1}(s, \epsilon_{i+1})$ "residual" polynomials,

$$\begin{aligned}
\rho_1(s) &= s + 1,\\
\rho_2(s) &= s\rho_1(s) + \epsilon_2,\\
\rho_3(s) &= s\rho_2(s) + \epsilon_3 s^2 + 2\epsilon_3 s + \epsilon_3^2,\\
\rho_4(s) &= s\rho_3(s) + \epsilon_4 s^2 + \epsilon_4(1+\epsilon_2)s + (\epsilon_2 - \epsilon_3)\epsilon_4.
\end{aligned}$$

V. Last remarks

We characterize, in graphical terms, the structure of nests. They are sequences of supergraphs, building upon the previous one by adding appropriately signed Hamiltonians. The notion of sign-compatibility we introduce captures the precise meaning of being appropriately signed. Informally, it means Hamiltonians are positive and have the same sign of the first Hamiltonians formed in the sequence, which are dominant. Framing the problem this way makes it amenable to classic control theory; root locus arguments, specifically. From this point of view, the new Hamiltonians act as controllers stabilizing an extra pole at the origin added to the previous π-Hurwitz graph. Our conceptual contribution is precisely framing the problem this way.

A general solution concept

One interpretation of Theorem 2 is "Hamiltonians Δ_i act as controllers stabilizing Γ_i subjected to a *single* integrator." By considering not only single, but *any* integrator, this interpretation turns into a prototype for a general solution concept to characterize potentially Hurwitz structures, beyond nests.

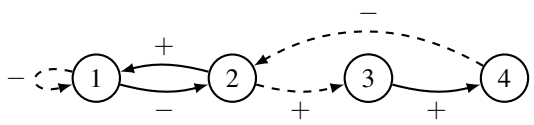


Fig. 3. 4-Hamiltonian (dashed edges) stabilizes two poles at the origin.

For example, the graph Γ shown in Fig. 3 is π-Hurwitz, but not a nest. It is, however, a sequence $(\Gamma_i, \Delta_i)_{i=1}^2$ of π-Hurwitz supergraphs Γ_i and Hamiltonians Δ_i, such that $\Gamma_{i+1} = \Gamma_i \cup \Delta_i$. Namely, Γ_1 is the loop spanning 1, Δ_1 is the 2-cycle spanning 1 and 2, and Δ_2 is the 4-Hamiltonian. That is, Δ_2 increases the order of Γ_2 by 2—rather than by 1, as in a nest.

Root Locus explains why nests, the graph on Fig. 3, and any other graph is π-Hurwitz. The question now is whether this Root Locus argument scales up to "bigger steps" between supergraphs. A question we will pursue moving forward with the structural control theory agenda.

References

[1] C. Jeffries, V. Klee, and P. Van den Driessche, "When is a matrix sign stable?" *Canadian Journal of Mathematics*, no. 2, pp. 315–326, 1977.

[2] V. Klee and P. van den Driessche, "Linear algorithms for testing the sign stability of a matrix and for finding Z-maximum matchings in acyclic graphs," *Numer. Math.*, vol. 28, pp. 273–285, 1977.

[3] C. Briat, "Sign properties of Metzler matrices with applications," *Linear Algebra Appl.*', vol. 515, pp. 53–86, 2017.

[4] M. Catral, D. D. Olesky, and P. van den Driessche, "Allow problems concerning spectral properties of sign pattern matrices: A survey," *Linear Algebra Appl.*, vol. 430, no. 11-12, pp. 3080–3094, 2009.

[5] Y. Gao and J. Li, "On the potential stability of star sign pattern matrices," *Linear Algebra Appl.*, vol. 327, pp. 61–68, 2001.

[6] C. Johnson and T. Summers, "The potentially stable tree sign patterns for dimensions less than 5," *Linear Algebra Appl.*, vol. 126, pp. 1–13, 1989.

[7] J. Cavalcanti and H. Balakrishnan, "Sign-stability of Positive Markov Jump Linear Systems," *Automatica*, vol. 111, 2020.

[8] C. Jeffries and C. Johnson, "Some sign patterns that preclude matrix stability," *SIAM J. Matrix Anal. Appl.*, vol. 9, pp. 19–25, 1988.

[9] T. Bone, "Positive feedback may sometimes promote stability," *Linear Algebra Appl.*, vol. 51, pp. 143–151, 1983.

[10] C. Johnson, "Nested Sequences of Principal Minors and Potential Stability," *Linear Algebra Appl.*, vol. 262, pp. 243–257, 1997.

[11] R. Diestel, *Graph Theory*, 5th ed. Springer-Verlag, Heidelberg, 2017.

[12] K. Ogata, *Modern Control Engineering*, 5th ed. Prentice-Hall, 2010.

[13] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Trans. Automat. Contr.*, vol. 51, no. 2, pp. 274–286, 2006.

[14] Y.-S. Wang, N. Matni, and J. Doyle, "A system level approach to controller synthesis," *IEEE Trans. Automat. Contr.*, vol. 64, no. 10, pp. 4079 – 4093, 2019.

[15] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs, NJ: Prentice Hall, 1996.

[16] S. Bhattacharyya, "Robust control under parametric uncertainty: An overview and recent results," *Annual Reviews in Control*, vol. 44, pp. 45–77, 2017.

[17] D. Grundy, D. Olesky, and P. van den Driessche, "Constructions for potentially stable sign patterns," *Linear Algebra Appl.*, vol. 436, pp. 4473–4488, 2012.

[18] F. Gantmacher, *The Theory of Matrices*. AMS Chelsea, 1959.