

THE TOPOLOGY OF BAUMSLAG-SOLITAR REPRESENTATIONS

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ABSTRACT. Let $\Gamma = \langle a, b \mid ab^p a^{-1} = b^q \rangle$ be a Baumslag–Solitar group and G be a complex reductive algebraic group with maximal compact subgroup $K < G$. We show that, when p and q are relatively prime with distinct absolute values, there is a strong deformation retraction of $\text{Hom}(\Gamma, G)$ onto $\text{Hom}(\Gamma, K)$.

1. INTRODUCTION

Let Γ be a finitely generated group (fix a finite generating set $S \subset \Gamma$) and let G be a (topological) group. Identifying a group homomorphism $\rho: \Gamma \rightarrow G$ with its restriction $\rho|_S$ identifies $\text{Hom}(\Gamma, G)$ with the subset of G^S cut out by the relations defining Γ . In particular, $\text{Hom}(\Gamma, G)$ is a closed subset and we equip it with the relative topology. In the case under consideration, G will be a complex linear algebraic group and this identification also endows $\text{Hom}(\Gamma, G)$ with the structure of a complex affine algebraic variety.

For any topological subgroup $K < G$, the inclusion of K in G induces a topological inclusion $\text{Hom}(\Gamma, K) \hookrightarrow \text{Hom}(\Gamma, G)$. When K is compact, $\text{Hom}(\Gamma, K)$ is itself compact and thus more amenable to topological analysis. In particular, individual elements of $\text{Hom}(\Gamma, K)$ are also often easier to analyze. For example, every unitary representation of Γ is completely reducible whereas this is usually not the case for general linear representations. For this reason, it is remarkable that in some cases the inclusion above is in fact a homotopy equivalence, due to the existence of a strong retraction from $\text{Hom}(\Gamma, G)$ onto $\text{Hom}(\Gamma, K)$. This should be contrasted with the fact that, for general finitely generated groups, rigidity results (e.g. those of Selberg [12]) ensure that this inclusion is not even a bijection of connected components.

Nevertheless, recent success in producing such retractions has been achieved using a mixture of topological and algebraic tools for classical families of finitely generated groups. Notable positive results include the case of free-abelian groups by Pettet–Souto [9], torsion-free expanding nilpotent groups by Silberman–Souto [13] and general nilpotent groups by Bergeron [2].

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In this paper, we turn our attention to a class of groups that has often served as a testing ground for ideas in geometric group theory: the Baumslag-Solitar groups

$$\mathrm{BS}(p, q) := \langle a, b \mid ab^p a^{-1} = b^q \rangle.$$

It is, in some sense, their “bad behaviour” that motivated their discovery in the work of Baumslag–Solitar [1] as the first examples of non-Hopfian residually finite groups. More recently, they also proved to be interesting cases in Gromov’s proposed classification of finitely generated groups up to quasi-isometry as exhibited by Farb–Mosher [4, 5] and Whyte [15].

From our point of view, analyzing the topology of their representation varieties, we think of the Baumslag-Solitar groups as the simplest interesting infinite groups which are not lattices in Lie groups. Our main result establishes the following:

Theorem 1.1. *Let p and q be relatively prime integers with distinct absolute values and consider the Baumslag-Solitar group*

$$\Gamma = \langle a, b \mid ab^p a^{-1} = b^q \rangle.$$

If G is the group of complex points of a reductive algebraic group and $K < G$ is a maximal compact subgroup, then there is a strong deformation retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$. In particular, the two spaces are homotopy equivalent.

We conclude this introduction with a short outline of the proof:

- (1) An application of Kempf–Ness theory provides a retraction of $\mathrm{Hom}(\Gamma, G)$ to a subset $\mathrm{Hom}_{\mathrm{KN}}(\Gamma, G)$ which, in particular, consists of completely reducible representations.
- (2) We produce an integer O (depending only on Γ and G) such that, for every representation $\rho \in \mathrm{Hom}_{\mathrm{KN}}(\Gamma, G)$, the abelian group $\rho(\langle b \rangle)$ is finite and of order at most O .
- (3) Letting $\tilde{\mathcal{F}}$ (respectively $\tilde{\mathcal{F}}_K$) denote the manifold of subgroups of G (respectively of K) of order at most O , we show that

$$\mathrm{Hom}_{\mathrm{KN}}(\Gamma, G) \rightarrow \tilde{\mathcal{F}}, \quad \rho \mapsto \rho(\langle b \rangle)$$

is a fiber bundle onto a union of connected components. Since the inclusion of $\tilde{\mathcal{F}}_K$ into $\tilde{\mathcal{F}}$ is a homotopy equivalence, it follows that $\mathrm{Hom}_{\mathrm{KN}}(\Gamma, G)$ retracts onto the subset consisting of those ρ with $\rho(\langle b \rangle) \subset K$.

- (4) Each fiber of the bundle is the domain of a natural covering map onto a subset of the semisimple elements of a reductive group G' which admits a deformation retraction into a compact subgroup $K' \subset G'$. Lifting this retraction to the fibers, we obtain the desired retraction of $\mathrm{Hom}_{\mathrm{KN}}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$.

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2. BACKGROUND ON REDUCTIVE ALGEBRAIC GROUPS

An (affine) *algebraic group* is, properly, an (affine) variety endowed with a group law for which the group operations are morphisms of varieties, i.e., polynomial maps. For our purposes, we will use the terms “affine variety” to mean an affine algebraic set (the zero locus of a family of complex polynomials in several variables) and “algebraic group” to mean the set of complex points of the affine algebraic group. It turns out that affine algebraic groups are *linear* so we may (and always will) identify them with a Zariski closed subgroup of $\mathrm{SL}_n\mathbb{C}$ (see, for instance, Springer [14]).

Given a compact Lie group K , the Peter-Weyl Theorem provides a faithful embedding $K \hookrightarrow \mathrm{GL}_n\mathbb{R}$ for some n . Identifying K with its image realizes it as a real algebraic subgroup of $\mathrm{GL}_n\mathbb{R}$. We then define the *complexification* $G := K_{\mathbb{C}}$ to be the vanishing locus in $\mathrm{GL}_n\mathbb{C}$ of the ideal defining K . The group G is a complex algebraic group which is independent up to isomorphism of the embedding provided by the Peter-Weyl Theorem. A complex linear algebraic group G is *reductive* if and only if it is the complexification of a compact Lie group K as above. In this case, K is always a maximal compact subgroup G . We summarize some well-known facts in the following (see, for instance, Helgerson [6]):

Proposition 2.1. *Let $\mathcal{S}(G)$ denote the set of maximal compact subgroups of G . This set admits the structure of a connected non-positively curved symmetric space in such a way that the action*

$$G \curvearrowright \mathcal{S}(G), \quad (g, K) \mapsto gKg^{-1}$$

is a smooth isometric action. In particular:

- (1) *any two maximal compact subgroups of G are conjugated by an element in the identity component G° , and*
- (2) *if a compact subgroup $H \subset G$ normalizes a maximal compact subgroup $K \in \mathcal{S}(G)$, then $H \subset K$.* □

Let us denote the centralizer and normalizer of a subgroup $H \subset G$ by

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\}, \text{ and } Z_G(H) := \{g \in G \mid \forall h \in H : ghg^{-1} = h\}.$$

We will also make use of the subset $N_{G^\circ}(H) := G^\circ \cap N_G(H)$. We record the following possibly non-standard facts for later use:

Lemma 2.2 (Pettet-Souto [9], Lemma 4.4). *Let G and K be as above. If $H \subset K$ is a subgroup, then the natural inclusion between normalizers*

$$N_K(H) \hookrightarrow N_G(H)$$

$$N_{K^\circ}(H) \hookrightarrow N_{G^\circ}(H)$$

are homotopy equivalences. $\square$

Corollary 2.3. *For every abelian $A \subset K$, $N_K(A)$ is maximal compact in $N_G(A)$.*

Proof. We first observe that $N_G(A)$ is reductive. Indeed, since A is diagonalizable, the identity component of the normalizer $N_G(A)^\circ$ coincides with the identity component of the centralizer $Z_G(A)^\circ$ which is itself reductive by a result of Richardson [10, Theorem 6.1].

Now, the compact subgroup $N_K(A) \subset N_G(A)$ is contained in some maximal compact subgroup $\kappa \subset N_G(A)$. Here, it follows from Lemma 2.2 and the fact that $N_G(A)$ is reductive that the inclusions $N_K(A) \hookrightarrow N_G(A)$ and $\kappa \hookrightarrow N_G(A)$ are homotopy equivalences. This implies that the inclusion of $N_K(A)$ into κ is also a homotopy equivalence. Since the latter two are closed manifolds, this implies that the inclusion is surjective, meaning that $\kappa = N_K(A)$ after all. $\square$

Finally, let V be a complex vector space and consider a G -variety $X \subset V$, i.e., suppose that G acts on $X \subset V$ via an embedding $G \subset \mathrm{GL}(V)$. If $K \subset G$ is a maximal compact subgroup, we may assume without loss of generality that V is equipped with a K -invariant Hermitian norm $\|\cdot\|$. The restriction of this norm to X provides a gateway to analyze its topology through a package of results known as *Kempf–Ness Theory* (see [2, Section 3] for an introduction and [11, Chapter 5] for the details). To state the results we need, write $\mathcal{M}$ for the *Kempf–Ness Set* of minimal vectors in X :

$$\mathcal{M} := \{x \in X : \forall g \in G, \|x\| \leq \|g \cdot x\|\}.$$

Then:

- (1) there is a strong deformation retraction of X onto $\mathcal{M}$, and
- (2) the G -orbit of every $x \in \mathcal{M}$ is closed.

3. THE MAIN THEOREM

Let p and q be relatively prime integers with $|p| \neq |q|$. We consider the Baumslag–Solitar group

$$\Gamma = \langle a, b \mid ab^p a^{-1} = b^q \rangle.$$

Let us also fix once and for all a complex reductive algebraic group G along with a maximal compact subgroup $K \subset G$.

As explained in the introduction, the image of $\mathrm{Hom}(\Gamma, G)$ in $G \times G$ under the injective map

$$\rho \mapsto (\rho(a), \rho(b))$$

is the closed subset cut out by the relation

$$\rho(a)\rho(b)^p\rho(a)^{-1} = \rho(b)^q.$$

This simultaneously endows $\text{Hom}(\Gamma, G)$ with compatible structures of an affine algebraic variety (when G is viewed as an affine algebraic group) and a Hausdorff topological space (when G is viewed as a Lie group). It is well known (see, for instance, [7]) and otherwise easy to see that, in both points of view, the geometric structures obtained from distinct presentations of the same group Γ are isomorphic. The choice of presentation is thus immaterial.

If $K \subset G$ is a maximal compact subgroup, the inclusion $K \hookrightarrow G$ induces a natural inclusion $\text{Hom}(\Gamma, K) \hookrightarrow \text{Hom}(\Gamma, G)$ and we would like to show that there is a deformation retraction of the latter space into the former.

3.1. Kempf–Ness Theory. In this section we will take the first step of our argument, obtaining an intermediate deformation retract between $\text{Hom}(\Gamma, K)$ and $\text{Hom}(\Gamma, G)$.

For this, observe that the action of G on itself by conjugation induces an action of G on $\text{Hom}(\Gamma, G)$ via $(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$. In terms of our embedding this is simply the restriction of the diagonal action of G on $G \times G$ to the subspace/subvariety $\text{Hom}(\Gamma, G)$. Fixing an embedding of G in $\text{SL}_n\mathbb{C}$, this allows us to view

$$\text{Hom}(\Gamma, G) \subset G^2 \subset (M_n\mathbb{C})^2 \cong \mathbb{C}^{2n^2}$$

as a G -variety. Endowing the ambient vector space with a K -invariant norm (for example, we may assume the embedding maps K to $\text{SU}(n)$ and take the Hilbert–Schmid norm on $M_n(\mathbb{C})$), we obtain a retraction of $\text{Hom}(\Gamma, G)$ onto the associated Kempf–Ness set $\text{Hom}_{\text{KN}}(\Gamma, G)$, given concretely as the set:

$$\left\{ \rho: \Gamma \rightarrow G \mid \forall g \in G : \|g\rho(a)g^{-1}\|_{\text{HS}}^2 + \|g\rho(b)g^{-1}\|_{\text{HS}}^2 \geq \|\rho(a)\|_{\text{HS}}^2 + \|\rho(b)\|_{\text{HS}}^2 \right\}.$$

3.2. Manifolds of Finite Abelian Groups. Keeping the notation as above, let B denote the subgroup of Γ generated by b . Our goal in this section is to realize the Kempf–Ness set $\text{Hom}_{\text{KN}}(\Gamma, G)$ as the total space of a bundle over a manifold of finite abelian subgroups of G (see also Bergeron–Silberman [3, Proposition 2.1] and Pettet–Souto [9, Section 3.4]).

Proposition 3.1. *The set of abelian groups*

$$\mathcal{F} := \{\rho(B) \subset G : \rho \in \text{Hom}_{\text{KN}}(\Gamma, G)\}$$

(clearly invariant under conjugation by G) admits a manifold structure with finitely many connected components such that the G -action is smooth and the projection map

$$\pi: \text{Hom}_{\text{KN}}(\Gamma, G) \rightarrow \mathcal{F}, \quad \pi(\rho) := \rho(B)$$

is a locally trivial fiber bundle. Moreover, if $H \in \mathcal{F}$, the image of a representation $\rho \in \pi^{-1}(H)$ is contained in the normalizer $N_G(H)$.

The following two Lemmas and their proofs are inspired by the work of McLaury [8] on irreducible linear representations of Baumslag–Solitar groups. We quote Theorem 3.2 of his paper as Lemma 3.2 below but give a full proof since there are minor errors in the argument given in [8]. Lemma 3.3 then generalizes Theorems 3.3 and 5.1 of [8] to our context of representations in general reductive groups lying in the Kempf–Ness set. These Lemmas are the only places in the paper where we directly use the hypothesis that p and q are relatively prime.

Lemma 3.2. *Let G be a linear algebraic group and let Γ be a Baumslag–Solitar group with p and q relatively prime. If $\rho \in \text{Hom}(\Gamma, G)$, then $\rho(\Gamma)$ is solvable.*

Proof. Let $L = \overline{\rho(\Gamma)}$ be the Zariski closure of the image of Γ , and let $H = \overline{\rho(B)}$ be the Zariski closure of $\rho(B)$. The subgroups $H_1 = \overline{\langle \rho(b^p) \rangle}$ and $H_2 = \overline{\langle \rho(b^q) \rangle}$ are of finite index in H (the indices divide p, q respectively), and in particular we have an identity $H^\circ = H_1^\circ = H_2^\circ$ of (algebraic) connected components. Since $\rho(a)$ conjugates $\rho(b^p)$ to $\rho(b^q)$ it also conjugates H_1 to H_2 and therefore normalizes H° . H° is also normalized by $\rho(b) \in H$ and we conclude that H° is normalized by $\rho(\Gamma)$ and is hence normal in L .

The two finite subgroups $H_1/H^\circ, H_2/H^\circ < L/H^\circ$ are conjugate by the image of $\rho(a)$ in L/H° . In particular, they have the same order. Now these two subgroups are contained in the finite group H/H° , so they also have the same index there – in other words $[H : H_1] = [H : H_2]$. However, as noted above these indices divide p, q respectively. Since $\gcd(p, q) = 1$ we conclude that $H = H_1 = H_2$ and thence that H itself is normalized by $\rho(a)$. It also contains $\rho(b)$ and it now follows that H is normal in L .

Finally, since $\rho(b) \in H$ the image of $\rho(a)$ in L/H is a (Zariski-) topological generator. Having shown that both H and L/H are abelian we conclude that L and its subgroup $\rho(\Gamma)$ are solvable. $\square$

Lemma 3.3. *If $\rho \in \text{Hom}_{KN}(\Gamma, G)$, then $\rho(\Gamma)$ is virtually abelian and consists of semisimple elements. Furthermore, the abelian group $\rho(B)$ is normal in $\rho(\Gamma)$, finite, and its order is bounded by a constant $O = O(\Gamma, G)$ that is independent of ρ .*

Proof. Recall that, since ρ lies in the Kempf–Ness set, its orbit under conjugation must be closed. A result of Richardson [10, Theorem 3.6] now implies that the Zariski closure L of the image of ρ is reductive. It follows that L° is a connected reductive solvable group, in other words an algebraic torus (possibly zero-dimensional). This makes L and its subgroup $\rho(\Gamma)$ virtually abelian and ensures that every element of L° is semisimple. Since every element of L has a positive power in L° , the Jordan

decomposition shows that every element of L (in particular, every element of $\rho(\Gamma)$) is semisimple as well.

Now $H = \overline{\rho(B)}$ is a commutative subgroup consisting of semisimple elements. By [14, Cor. 3.2.9] the group $N_G(H)/Z_G(H)$ is finite, and it follows that conjugation by $\rho(a)$ is an automorphism of H of finite order. In particular, there is r such that $\rho(a^r)$ commutes with H , in particular with $\rho(b)$. On the other hand, $ab^p a^{-1} = b^q$ implies $ab^k a^{-1} = b^{qk/p}$ the group $H/Z_G(L)$ is finite, and it follows that conjugation by $\rho(a)$ is an automorphism of L of finite order. In particular, there is an r such that $\rho(a^r)$ commutes with L , hence with $\rho(b)$. On the other hand, $ab^p a^{-1} = b^q$ implies $ab^k a^{-1} = b^{qk/p}$ whenever k is divisible by p . Conjugating b^{p^r} by a r times we obtain the identity $a^r b^{p^r} a^{-r} = b^{q^r}$, and upon applying ρ that $\rho(b)^{p^r} = \rho(b)^{q^r}$. Since p and q are distinct, we conclude that $\rho(b)$ has finite order (dividing $|p^r - q^r|$) and that $\rho(B)$ is finite.

Since $\rho(B)$ is finite, it is also Zariski closed and $H = \rho(B)$. It follows that $\rho(B)$ is normalized by L , hence by $\rho(\Gamma)$. As before, $\rho(\Gamma)/\rho(B)$ is cyclic, generated by the image of a .

We now rely on our linear embedding $G \subset \mathrm{SL}_n \mathbb{C}$. Since L is reductive, this linear representation of L decomposes as a direct sum of irreducible representations. Since $\rho(\Gamma)$ is Zariski-dense in L , those subspaces are also irreducible as representations of Γ . The calculations of [8, Thm. 5.1] now show that every eigenvalue of b in this representation is a root of unity of order dividing $|p^k - q^k|$ for some $1 \leq k \leq n$, hence dividing $O = \prod_{k=1}^n |p^k - q^k|$. Since $\rho(b)$ is semisimple it follows that $\rho(b)$ has order dividing O , and in particular that $|\rho(B)| \leq O$. $\square$

Proof of Proposition 3.7. Before we study $\mathcal{F}$, we first consider the slightly larger set

$$\tilde{\mathcal{F}} := \{H < G : H \text{ is abelian of order at most } O\}.$$

Observe that G° (the identity component of G) acts by conjugation on $\tilde{\mathcal{F}}$ with closed stabilizers. As such, we can endow $\tilde{\mathcal{F}}$ with the orbifold structure with respect to which each G° -orbit is a connected homogeneous G° -manifold. Concretely, if we define the “connected normalizer” as $N_{G^\circ}(H) := N_G(H) \cap G^\circ$, then the connected component of $H \in \tilde{\mathcal{F}}$ will be its G° orbit, topologized via the identification with $G^\circ/N_{G^\circ}(H)$. There being only finitely many G -orbits in $\tilde{\mathcal{F}}$, we necessarily give it the topology of the disjoint union of the orbits.

A homomorphism $\rho : B \rightarrow G$ need not extend to the full group Γ so the map

$$\pi : \mathrm{Hom}_{\mathrm{KN}}(\Gamma, G) \rightarrow \tilde{\mathcal{F}}, \quad \pi(\rho) = \rho(B)$$

need not be surjective. Recall, however, that we have defined $\mathcal{F} = \pi(\mathrm{Hom}_{\mathrm{KN}}(\Gamma, G))$ and observe that the G° -equivariance of π implies that $\mathcal{F}$ is the union of those components of $\tilde{\mathcal{F}}$ it intersects, giving it the desired manifold structure.

For the bundle structure, let $\mathcal{Z} \subset \mathcal{F}$ denote the connected component of a finite abelian subgroup $H \in \mathcal{F}$, let $\mathcal{H} := \pi^{-1}(\mathcal{Z}) \subset \text{Hom}_{\text{KN}}(\Gamma, G)$ and let $\mathcal{H}_H := \pi^{-1}(H)$. We can then identify $\mathcal{H}$ and the restriction π there with the twisted product

$$(G^\circ \times \mathcal{H}_H)/N_{G^\circ}(H) \rightarrow G^\circ/N_{G^\circ}(H)$$

where $N_{G^\circ}(H)$ acts on G° (resp. $\mathcal{H}_H$) by right multiplication (resp. conjugation). This shows that π is a locally trivial fibre bundle. $\square$

We will also need to consider the analogous manifold of abelian subgroups in the maximal compact subgroup $K \subset G$. To this end, let

$$\tilde{\mathcal{F}}_K := \{H \in \tilde{\mathcal{F}} \mid H \subset K\}$$

be the manifold of abelian subgroups of K of order at most O . It follows, as before, that the conjugation action of K on $\tilde{\mathcal{F}}_K$ has finitely many orbits and we can endow $\tilde{\mathcal{F}}_K$ with the structure of a homogeneous manifold. Concretely, the connected component of $H \in \tilde{\mathcal{F}}$ is identified with $K^\circ/N_{K^\circ}(H)$. We record the following two facts about manifolds of finite abelian subgroups:

Proposition 3.4. *The manifold $\tilde{\mathcal{F}}_K$ is a deformation retract of $\tilde{\mathcal{F}}$.*

Proof. The argument is analogous to the one in [9, Proposition 4.1]. We first verify that $\tilde{\mathcal{F}}_K \hookrightarrow \tilde{\mathcal{F}}$ induces a bijection at the level of connected components. To see this, recall that any $H \in \mathcal{F}$, being compact, can be conjugated into K by an element of G° , which shows that $\pi_0(\tilde{\mathcal{F}}_K) \rightarrow \pi_0(\tilde{\mathcal{F}})$ is surjective. Conversely, [9, Lemma 3.4] asserts that two subgroups of K which are conjugate by an element G° are also conjugate by an element of K° , in other words that our map of connected components is injective.

Let $\mathcal{Z}$ denote a component of $\tilde{\mathcal{F}}$ and let $\mathcal{Z}_K$ denote the unique component of $\tilde{\mathcal{F}}_K$ contained in $\mathcal{Z}$. To complete the proof, it suffices to show that $\mathcal{Z}_K \hookrightarrow \mathcal{Z}$ is a homotopy equivalence. Fix a subgroup $H \in \mathcal{Z}_K$ and use it to identify $\mathcal{Z}_K$ and $\mathcal{Z}$ with $K^\circ/N_{K^\circ}(H)$ and $G^\circ/N_{G^\circ}(H)$ respectively. We now have a commutative diagram

$$\begin{array}{ccccc} N_{K^\circ}(H) & \longrightarrow & K^\circ & \longrightarrow & \mathcal{Z}_K \\ \downarrow & & \downarrow & & \downarrow \\ N_{G^\circ}(H) & \longrightarrow & G^\circ & \longrightarrow & \mathcal{Z} \end{array}$$

where the two rows are principal bundles. Since the first two columns are homotopy equivalences (that was Lemma 2.2), it follows that the last column is also a homotopy equivalence. $\square$

Proposition 3.5 (Cartan's Lemma). *There is a continuous map $\kappa: \tilde{\mathcal{F}} \rightarrow \mathcal{S}(G)$ with:*

- (1) $H \subset \kappa(H)$ for all $H \in \tilde{\mathcal{F}}$, and
- (2) $\kappa(H) = K$ for all $H \in \tilde{\mathcal{F}}_K$.

Proof. Let $d(\cdot, \cdot)$ denote the Riemannian metric on the simply connected non-positively curved complete metric space $\mathcal{S}(G)$. The strict convexity of $d(\cdot, \cdot)^2$ implies that the function

$$f_H: \mathcal{S}(G) \rightarrow \mathbb{R}_+, \quad f_H(\kappa) = \sum_{g \in H} d(\kappa, gKg^{-1})^2$$

is strictly convex. Since it is also proper, the function f_H attains its minimum at a unique point $\kappa = \kappa(H) \in \mathcal{S}(G)$. It follows from the H -invariance of f_H that this minimum is also H -invariant and, hence, H is contained in the maximal compact subgroup stabilizing $\kappa(H)$. Notice also that if $H \subset K$, then the non-negative function f_H vanishes at K , showing that $\kappa(H) = K$. Finally, it follows from the non-positive curvature that the minimum of f_H depends continuously on H , that is, that

$$\kappa: \tilde{\mathcal{F}} \rightarrow \mathcal{S}(G), \quad H \mapsto \kappa(H)$$

is continuous. □

3.3. The Covering Map. Before going further, we need another bundle. For any finite abelian subgroup $H \subset G$ let

$$G_H := N_G(H)/H$$

and let us denote the corresponding natural projection by

$$p: N_G(H) \rightarrow G_H.$$

Notice, as in the proof of Corollary 2.3, that $N_G(H)$ and thus G_H are reductive.

Recall from the proof of Proposition 3.1 that if $\rho \in \pi^{-1}(H)$ then $\rho(B) = H$ and $\rho(\Gamma) \subset N_G(H)$. Thus, given any such ρ we obtain a representation

$$p \circ \rho: \Gamma \rightarrow G_H$$

with image landing in the subset $(G_H)_s$ of semisimple elements of G_H . By construction, the subgroup B of Γ is contained in the kernel of $p \circ \rho$ so this map is uniquely determined by the image of the element $a \in \Gamma$. Altogether, we have a map

$$p_*: \pi^{-1}(H) \rightarrow (G_H)_s$$

with finite fibers because the projection $N_G(H) \rightarrow G_H$ has finite kernel. In particular, $p_*(\pi^{-1}(H))$ is contained in the set $(G_H)_s$ of semisimple elements of G_H .

The action by conjugation of $N_{G^\circ}(H)$ on $N_G(H)$ induces an action

$$(G_H)_s \curvearrowright N_{G^\circ}(H), \quad (p(g), h) \mapsto p(hgh^{-1}).$$

This can now be used to construct the twisted product

$$\mathcal{G} := (G^\circ \times (G_H)_s) / N_{G^\circ}(H) \xrightarrow{\pi_{\mathcal{G}}} G^\circ / N_{G^\circ}(H) = \mathcal{Z}.$$

Here, as in the previous section, $\mathcal{Z}$ denotes the component of $\mathcal{F}$ containing H . The map p_* is $N_{G^\circ}(H)$ -equivariant and, in particular, it induces a morphism of fiber

bundles from $\mathcal{H}$ to $\mathcal{G}$ where $\mathcal{H} = \pi^{-1}(\mathcal{Z})$ is a component of the bundle defined in Proposition 3.1. In fact, can say a bit more:

Proposition 3.6. *The bundle map $p_* : \mathcal{H} \rightarrow \mathcal{G}$ is a finite cover onto a union of connected components of $\mathcal{G}$.*

Proof. Since p_* is a morphism of fiber bundles, it suffices to show that the map is a covering onto a union of connected components at the level of fibers. In other words, for every $H \in \mathcal{Z}$, we need to verify our claim for the map

$$p_* : \pi^{-1}(H) \rightarrow \pi_G^{-1}(H)$$

where $\pi^{-1}(H)$ consists of homomorphisms $\rho : \Gamma \rightarrow N_G(H)_s$ with $\rho(B) = H$ and $\pi_G^{-1}(H) = (G_H)_s = (N_G(H)/H)_s$. Since all the objects in the game are semi-analytic sets, it suffices to show that p_* is locally injective and has the path-lifting property.

We start by showing that p_* is locally injective. Suppose that ρ and ψ are representations in $\pi^{-1}(H)$ that are very close to each other. Recalling that we identify representations with the image of our generating set for $\Gamma = \langle a, b \rangle$, this means that the pair of elements $\rho(a)$ and $\psi(a)$ (respectively $\rho(b)$ and $\psi(b)$) differ by an element g_a (respectively g_b) of $N_G(H)$ very close to the identity. Now, if $p_*(\rho) = p_*(\psi)$ then g_a and g_b are elements of H . Moreover, since H is discrete and g_a and g_b are very close to the identity, it follows that they must both be the identity. This shows that $\rho = \psi$ and thus p_* is locally injective.

We show that p_* has the path-lifting property. To do this, it is convenient to identify $\pi_G^{-1}(H)$ with the subset $(G_H)_s \times \text{Id} \subset G_H \times G_H$ and to view $\pi^{-1}(H)$ as a subset of $N_G(H) \times N_G(H)$. Let $\rho \in \pi^{-1}(H)$ and consider a path

$$\gamma : [0, 1] \rightarrow (G_H)_s \times \text{Id} \subset G_H \times G_H, \quad \gamma(t) = (\gamma_1(t), \gamma_2(t))$$

with $\gamma(0) = p_*(\rho)$. This path always admits a unique lift $\tilde{\gamma}$ to the finite cover

$$N_G(H) \times N_G(H) \rightarrow G_H \times G_H$$

with $\tilde{\gamma}(0) = (\rho(a), \rho(b))$. To complete the proof we need to show that

$$\tilde{\gamma}(t) \in \pi^{-1}(H) \subset N_G(H) \times N_G(H),$$

i.e., we need to check that the relations of Γ are preserved within the path and that the corresponding homomorphisms map B onto H .

Let w be a word in the letters a and b which is trivial in Γ . Substituting $\gamma_i(t)$ (respectively $\tilde{\gamma}_i(t)$) for a and b in w yields a continuous path $w(t)$ in G_H (respectively $N_G(H)$) such that $w(t)$ is the image of $\tilde{w}(t)$ under the quotient map. By construction, $w(t)$ is the identity element of G_H for all t . It follows that $\tilde{w}(t) \in H$ for every t and, since H is discrete, this is a constant map as well. Given that $\tilde{w}(0)$ is trivial (it corresponds to the homomorphism ρ), we conclude that $\tilde{w}(t)$ is trivial for all t as well. This shows that $\tilde{\gamma}(t)$ is a path of homomorphisms $\rho_t : \Gamma \rightarrow N_G(H)_s$. The image

of these homomorphisms is always contained in $N_G(H)_s$ because $\gamma_i(t)$ is semisimple and the quotient map is finite-to-one.

Finally, we show that the homomorphisms ρ_t determined by $\tilde{\gamma}(t)$ send the subgroup $\langle b \rangle = B \subset \Gamma$ onto H . This follows, once again, because the homomorphisms to G_H determined by $\gamma(t)$ map b to the identity in G_H and, as above, this implies that $\rho_t(b)$ is a fixed element of H . Since $\rho_0(b)$ is a generator of H this must then hold for $\rho_t(b)$. $\square$

3.4. Putting the Puzzle Back Together. Recall that we denote the manifold of maximal compact subgroups of G by $\mathcal{S}(G)$ and, from Proposition 3.5, that we have a continuous map $\kappa : \mathcal{F} \rightarrow \mathcal{S}(G)$. We use this map to define sub-bundles of $\pi : \text{Hom}_{KN}(\Gamma, G) \rightarrow \mathcal{F}$ and $\pi_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{Z}$ which will be intermediate steps in our deformations of $\text{Hom}_{KN}(\Gamma, G)$ onto $\text{Hom}(\Gamma, K)$.

First, consider the set

$$\mathcal{K} := \{\rho \in \text{Hom}_{KN}(\Gamma, G) \mid \rho(\Gamma) \subset \kappa(\pi(\rho))\}.$$

Observe that, by the continuity of κ , the restriction of π to $\mathcal{K}$ is a sub-bundle $\pi_{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{F}$. Since $\kappa(H) = K$ for every $H \in \mathcal{F}_K$, it follows that

$$\text{Hom}(\Gamma, K) = \pi_{\mathcal{K}}^{-1}(\mathcal{F}_K).$$

Recalling from Proposition 3.4 that $\mathcal{F}_K$ is a deformation retract of $\mathcal{F}$, it follows moreover that $\text{Hom}(\Gamma, K)$ is a deformation retract of $\mathcal{K}$.

Second, observe by Corollary 2.3 that $N_{\kappa(H)}(H)$ is a maximal compact subgroup of $N_G(H)$. This ensures that

$$K_H := N_{\kappa(H)}(H)/H$$

is also a maximal compact subgroup of G_H . We can now consider the subset $\mathcal{C} \subset \mathcal{G}$ defined fiber-wise by specifying that

$$\mathcal{C} \cap \pi_{\mathcal{G}}^{-1}(H) = K_H$$

for every $H \in \mathcal{Z}$. It follows, once again, from the continuity of κ that K_H varies continuously so we obtain a sub-bundle

$$\pi_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{Z}$$

of $\pi_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{Z}$. In fact, the following holds:

Lemma 3.7. *Keeping the notation as above, we have $p_*^{-1}(\mathcal{C}) = \mathcal{K} \cap \pi^{-1}(\mathcal{Z})$.*

Proof. Suppose $\rho : \Gamma \rightarrow G$ is a representation with $p_*(\rho) \in \mathcal{C}$. This means that

$$(p \circ \rho)(\Gamma) = p(\rho(\Gamma)) \subset K_{\pi(\rho)}.$$

In particular, we see that $\rho(\Gamma) \subset p^{-1}(K_{\pi(\rho)}) = \kappa(\pi(\rho))$ and therefore $\rho \in \mathcal{K}$. $\square$

In order to alleviate the notation (with the hope that no confusion shall arise), from now on we will use $\mathcal{G}$ (respectively $\mathcal{C}$) to denote the union of the corresponding bundles (respectively sub-bundles) over all connected components of $\mathcal{F}$ rather than restricting our attention to $\mathcal{Z} \subset \mathcal{F}$.

We can then consider the globally induced morphism of fiber bundles

$$p_* : \mathrm{Hom}_{\mathrm{KN}}(\Gamma, G) \rightarrow \mathcal{G}$$

where, applying Lemma 3.7 component by component, we have $p_*^{-1}(\mathcal{C}) = \mathcal{K}$. Similarly, applying Proposition 3.6 component by component, we have that p_* is a covering map onto a union of connected components of $\mathcal{G}$. To summarize, we have:

Proposition 3.8. *The bundle $\pi_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{F}$ contains a sub-bundle $\pi_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{F}$ such that, for every $H \in \mathcal{F}$, we have:*

- (1) $\pi_{\mathcal{G}}^{-1}(H) = (G_H)_s$, and
- (2) $\pi_{\mathcal{C}}^{-1}(H) = K_H$.

Here, K_H is a maximal compact subgroup of the reductive group G_H and $(G_H)_s$ denotes the set of semisimple elements in G_H . Moreover, the bundle morphism

$$p_* : \mathrm{Hom}_{\mathrm{KN}}(\Gamma, G) \rightarrow \mathcal{G}$$

is a finite cover onto a union of connected components of $\mathcal{G}$ with $p_*^{-1}(\mathcal{C}) = \mathcal{K}$. $\square$

3.5. Proof of the Main Theorem. With Proposition 3.8 at our disposition, we are now ready to complete the proof of our main theorem. After our applications of Kempf–Ness theory in Section 3.1, it suffices to show that $\mathrm{Hom}(\Gamma, K)$ is a deformation retract of $\mathrm{Hom}_{\mathrm{KN}}(\Gamma, G)$.

To begin, recall from Propositions 3.1 and 3.8 that we have bundles

$$\pi : \mathrm{Hom}_{\mathrm{KN}}(\Gamma, G) \rightarrow \mathcal{F}, \text{ and } \pi_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{F},$$

along with respective sub-bundles

$$\pi : \mathcal{K} \rightarrow \mathcal{F}, \text{ and } \pi_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{F}.$$

Recall, moreover, that they are related by a covering map $p_* : \mathrm{Hom}_{\mathrm{KN}}(\Gamma, G) \rightarrow \mathcal{G}$. Our deformation retraction now proceed in three steps:

Step one: For every $H \in \mathcal{F}$, we have a retraction of the fiber $\pi_{\mathcal{G}}^{-1}(H) = (G_H)_s$ onto the fiber $\pi_{\mathcal{C}}^{-1}(H) = K_H$. Since everything in sight is a fiber bundle, this implies that there is a fiber-preserving retraction of $\mathcal{G}$ onto $\mathcal{C}$.

Step two: Use the covering map $p_* : \mathrm{Hom}_{\mathrm{KN}}(\Gamma, G) \rightarrow \mathcal{G}$ to lift the retraction from step one to a fiber-preserving retraction of $\mathrm{Hom}_{\mathrm{KN}}(\Gamma, G)$ onto the sub-bundle $\mathcal{K}$.

Step three: Lift the deformation retraction of $\mathcal{F}$ onto $\mathcal{F}_K$ given by Lemma 3.4 to a retraction of $\mathcal{K}$ onto $\mathcal{K} \cap \pi^{-1}(\mathcal{F}_K) = \mathrm{Hom}(\Gamma, K)$.

This completes the proof of our main theorem. $\square$

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