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A Remark About the Anomalies of Cyclic Holomorphic Permutation Orbifolds

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Using a result of Longo and Xu, we show that the anomaly arising from a cyclic permutation orbifold of order 3 of a holomorphic conformal net $\mathcal{A}$ with central charge $c = 8k$ depends on the "gravitational anomaly" $k \pmod 3$. In particular, the conjecture that holomorphic permutation orbifolds are non-anomalous and therefore a stronger conjecture of Müger about braided crossed S_n-categories arising from permutation orbifolds of completely rational conformal nets are wrong. More generally, we show that cyclic permutations of order n are non-anomalous if and only if $3 \nmid n$ or $24|c$. We also show that all cyclic permutation gaugings of $\mathrm{Rep}(\mathcal{A})$ arise from conformal nets.

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1. Orbifolds and anomalies

Conformal nets axiomatize chiral conformal field theory in the framwork of algebraic quantum field theory using von Neumann algebras. There is a notion of a completely rational conformal net [14, Corollary 37 & 38] whose representation category $\mathrm{Rep}(\mathcal{A})$ is a modular tensor category. Let $\mathcal{A}$ be a **holomorphic conformal net**, i.e. a completely rational conformal net with trivial representation category $\mathrm{Rep}(\mathcal{A}) \cong \mathrm{Hilb}$. Here we denote by Hilb the trivial unitary modular tensor category of finite-dimensional Hilbert spaces. Let $G \leq \mathrm{Aut}(\mathcal{A})$ be a finite group of automorphisms of the net $\mathcal{A}$, see [22, Definition 3.1] and [17, Definition 2.3]. Then it is well-known [17, Corollary 4.7 and below], see also [14, Section 7], [18, 3.6 Corollary], [1, Remark 3.6], [2, Section 6.1], that there is a unique class $[\omega] \in H^3(G, \mathbb{T})$, such that the category of G-twisted representations of $\mathcal{A}$ denoted by $G\text{–}\mathrm{Rep}^I(\mathcal{A})$ is tensor equivalent to the category Hilb^ω_G of G-graded finite-dimensional Hilbert spaces with associator given by ω. More precisely, for every $g \in G$ there is an irreducible g-twisted representation β_g localized in I, which is unique up to conjugation by a unitary. Then $g \mapsto [\beta_g] \in \mathrm{Out}(\mathcal{A}(I))$ is a *G-kernel* and it follows that $\beta_g\beta_h = \mathrm{Ad}\, u_{g,h}\beta_{gh}$ for unitaries $(u_{g,h})_{g,h\in G}$ and that $\omega\colon G \times G \times G \to \mathbb{T}$ defined by $\omega(g,h,k) \cdot 1 = u_{g,h}u_{gh,k}u^{-1}_{g,hk}\beta_g(u_{h,k})^{-1}$ is a cocycle. The class $[\omega]$ is called the **anomaly** of G and we say that G acts **non-anomalously** if ω is a coboundary. Fur-

thermore, the *orbifold* or *fixed point net* $\mathcal{A}^G$ has a representation category $\mathrm{Rep}(\mathcal{A}^G)$ which is braided equivalent to the Drinfel'd center $\mathcal{Z}(G,\omega) = Z(\mathrm{Hilb}_G^\omega)$.

We denote by S_n the symmetric group on $\{1,\dots,n\}$. Let $\mathcal{A}$ be a holomorphic net and $G \leq S_n$, then G acts by permutation on $\mathcal{A}^{\otimes n}$. It seems to be widely believed that this action should be non-anomalous. But Johnson-Freyd argued that this conjecture is false [10, Example 2.1.1] and we give a counter-example in the framework of conformal nets where the permutation action picks up what can be thought of a **gravitational anomaly**[a] $k \equiv c/8 \pmod 3 \in \mathbb{Z}_3$, where c is the central charge of $\mathcal{A}$.

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2. Cyclic permutations of order 3

2.1. *Twisted doubles of* $\mathbb{Z}_3$.

Recall that a unitary fusion category is called pointed if all simple objects are invertible. Pointed unitary fusion categories with $\mathbb{Z}_3$-fusion rules are classified by $H^3(\mathbb{Z}_3,\mathbb{T}) = \{[\omega_i] : i \in \mathbb{Z}/3\mathbb{Z}\} \cong \mathbb{Z}_3$. Their Drinfel'd centers $\mathcal{Z}(\mathbb{Z}_3,\omega_i) := Z(\mathrm{Hilb}_G^{\omega_i})$ are pointed. Indeed, it can be easily checked that they are braided equivalent to the pointed unitary modular tensor categories $\mathcal{C}(G_i,q_i)$, respectively, where (G_i,q_i) are the metric groups given in Table 1 and $\mathcal{C}(G,q)$ is the braided fusion category associated to (G,q), see Appendix A, in particular Proposition A.1.

Table 1. Twisted doubles of $\mathbb{Z}_3$

i	G_i	$q_i\colon G_i \to \mathbb{Q}/\mathbb{Z}$
0	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$q_0(x,y) \equiv xy/3 \pmod 1$
1	$\mathbb{Z}_9$	$q_1(x) \equiv 4x^2/9 \pmod 1$
2	$\mathbb{Z}_9$	$q_2(x) \equiv 8x^2/9 \pmod 1$

[a]cf. [21, Section 1.4] for how this name might be justified, namely he asks that our k equals 0 (mod 3) in order for the chiral CFT to be dual to quantum gravity.

2.2. *The anomaly*

By a conformal net we mean a diffeomorphism covariant net on the circle, see e.g. [13, Section 2.1. A-G]. Let $\mathcal{A}$ be a holomorphic conformal net. Since $\mathcal{A}$ is diffeomorphism covariant we can assign a central charge $c > 0$. It is conjectured that if $\mathcal{A}$ is holomorphic, then $c \equiv 0 \pmod 8$ and it is a theorem that $c \equiv 0 \pmod 4$ by [12, Lemma 9.7 (2)]. We will from now on assume that the central charge c of $\mathcal{A}$ fulfills $c \in 8\mathbb{N}$. If $\mathcal{A}$ is holomorphic, then any tensor power $\mathcal{A}^{\otimes n}$ is holomorphic [14, Corollary 32]. Let $\sigma \in S_n$ be a permutation. Then there is an element $\sigma \in \mathrm{Aut}(\mathcal{A}^{\otimes n})$ given by

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \tag{2.1}$$

see e.g. [22, Section 6].

Proposition 2.1. *Let $\mathcal{A}$ be a diffeomorphism covariant holomorphic net with $c = 8k$, and let $\mathbb{Z}_3 \cong \langle\tau\rangle \leq \mathrm{Aut}(\mathcal{A}^{\otimes 3})$ be the group generated by the cyclic permutation $\tau = (123)$. Then the anomaly of $\langle\tau\rangle$ is ω_{2k}, i.e. $\langle\tau\rangle$–$\mathrm{Rep}(\mathcal{A}^{\otimes 3})$ is tensor equivalent to $\mathrm{Hilb}_{\mathbb{Z}_3}^{\omega_{2k}}$ and $\mathrm{Rep}((\mathcal{A}^{\otimes 3})^{\langle\tau\rangle})$ is braided equivalent to $\mathcal{Z}(\mathbb{Z}_3, \omega_{2k})$.*

Proof. It is enough to show that $\mathcal{C} := \mathrm{Rep}((\mathcal{A}^{\otimes 3})^{\langle\tau\rangle})$ is braided equivalent to $\mathcal{C}(G_{2k}, q_{2k})$. But this follows from [15, Theorem 6.3e] which gives that the spins in $\mathrm{Rep}((\mathcal{A}^{\otimes 3})^{\langle\tau\rangle})$ coming from twisted sectors α_i are $h_i = i/3 + 8k/9$ for $i = 0, 1, 2$ and then $q([\alpha_i]) \equiv h_i \pmod 1$ by the spin–statistic theorem [9, 3.13 Theorem]. This readily identifies $\mathcal{C}$ to be braided equivalent with $\mathcal{C}(G_{2k}, q_{2k})$. □

Example 2.1. Let $\mathcal{A}_{E_8}$ be the conformal net associated with the even lattice E_8 [4, Section 3]. Then $\langle\tau\rangle$–$\mathrm{Rep}(\mathcal{A}_{E_8}^{\otimes 3})$ is tensor equivalent to $\mathrm{Hilb}_{\mathbb{Z}_3}^{\omega_2}$ with $[\omega_2]$ a generator of $H^3(\mathbb{Z}_3, \mathbb{T})$. Thus $\mathrm{Rep}((\mathcal{A}_{E_8}^{\otimes 3})^{\tau})$ is braided equivalent to $\mathcal{Z}(\mathbb{Z}_3, \omega_2)$.

Example 2.2. Let $\mathcal{A}$ be a holomorphic net with central charge $c = 8k$. Let $S_3 \leq \mathrm{Aut}(\mathcal{A}^{\otimes 3})$ be the group of all permutations. Since $H^3(S_3, \mathbb{T})) \cong H^3(\mathbb{Z}_3, \mathbb{T}) \oplus H^3(\mathbb{Z}_2, \mathbb{T})$, where the isomorphism comes from restriction, it follows that S_3 is anomalous unless $k = 0 \pmod 3$. In particular, $\mathrm{Rep}((\mathcal{A}_{E_8}^{\otimes 3})^{S_3})$ is braided equivalent to $\mathcal{Z}(S_3, \tilde{\omega})$ for some $[\tilde{\omega}] \in H^3(S_3, \mathbb{T})$ of order 3.

In particular, the conjecture by Müger [20, Appendix 5, Conjecture 6.3] that states that for every completely rational conformal net $\mathcal{A}$ the category of S_n-twisted representations S_n–$\mathrm{Rep}(\mathcal{A}^{\otimes n})$ up to tensor equivalence depends only on the modular tensor category $\mathrm{Rep}(\mathcal{A})$ is wrong.

3. Cyclic holomorphic orbifolds

The argument can be generalized to arbitrary cyclic permutations and we get the following result.

Proposition 3.1. *Let $\mathcal{A}$ be a holomorphic net with central charge $c = 8k$ for some $k \in \mathbb{N}$. Let α be a cyclic permutation of order n on $\mathcal{A}^{\otimes m}$. Then the action of $\langle\alpha\rangle \cong \mathbb{Z}_n$ on $\mathcal{A}^{\otimes m}$ is non-anomalous if and only if $3 \nmid n$ or $24 \mid c$.*

3.1. *Cyclic homolorphic twisted orbifolds*

We have the following application of Proposition 3.1.

If $\mathcal{A}$ is holomorphic and $G \leq \mathrm{Aut}(\mathcal{A})$ non-anomalous we can form the so-called **twisted orbifold** $\mathcal{A}^{/\!/G}$ as described in [2, Definition 6.1 and below] by lifting the G-kernel given by G–$\mathrm{Rep}(\mathcal{A})$ to a homorphism $G \hookrightarrow \mathrm{Aut}(\mathcal{A}(I))$ or in other words by choosing a trivilization.

In our concrete case, this can be easier described. Namely, $\mathrm{Rep}((\mathcal{A}^{\otimes m})^{\langle\alpha\rangle})$ is braided equivalent to $\mathcal{C}(\mathbb{Z}_n \times \hat{\mathbb{Z}}_n, q_{\mathrm{st}})$ with the quadratic form $q_{\mathrm{st}}(g,\chi) = \chi(g)$, such that the Lagrangian subgroup $\mathbb{Z}_n \times \{\chi_0\}$ gives $\mathcal{A}^{\otimes m}$. We have a second Lagrangian subgroup $\{0\} \times \hat{\mathbb{Z}}_n$ which gives a new holomorphic net $\mathcal{A}^{\otimes n/\!/\langle\alpha\rangle}$ which is the **twisted orbifold net** $\mathcal{A}^{\otimes m/\!/\langle\alpha\rangle}$ of $\mathcal{A}^{\otimes m}$ with respect to $\langle\alpha\rangle$. Thus we have:

Proposition 3.2. *Let $\mathcal{A}$ be a holomorphic net with central charge $c \in 8\mathbb{N}$. Let α be a cyclic permutation of order n on $\mathcal{A}^{\otimes m}$. If $3 \nmid n$ or $24 \mid c$, we have a holomorphic net given by the twisted orbifold $\mathcal{A}^{\otimes m/\!/\langle\alpha\rangle}$.*

Example 3.1. $(\mathcal{A}_{E_8} \otimes \mathcal{A}_{E_8})^{/\!/\langle\tau_2\rangle}$ is isomorphic to $\mathcal{A}_{D_{16}^+}$.

3.2. *Determining the anomalies*

We now proceed to prove Proposition 3.1. Let $\mathcal{A}$ be a holomorphic net and let τ_n be the cyclic permutation

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto x_2 \otimes x_3 \otimes \cdots \otimes x_1\,. \tag{3.1}$$

Then τ_n yields an inner symmetry $\tau_n \in \mathrm{Aut}(\mathcal{A}^{\otimes n})$, see e.g. [22, Section 6].

For $G \leq \mathrm{Aut}(\mathcal{A})$ and $g \in G$ we denote by $\mathrm{Rep}(\mathcal{A}^G)_g$ the category of representations coming from restrictions of g–$\mathrm{Rep}(\mathcal{A})$.

Lemma 3.1. *Let $\mathcal{A}$ be a holomorphic net with central charge $c = 8k$ for some $k \in \mathbb{N}$.*

(1) For $3 \nmid n$ the spectrum of h_α with $\alpha \in \mathrm{Rep}((\mathcal{A}^{\otimes n})^{\langle\tau_n\rangle})_{\tau_n}$ is $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$ (mod 1).
(2) For $n = 3m$ the spectrum of h_α with $\alpha \in \mathrm{Rep}((\mathcal{A}^{\otimes n})^{\langle\tau_n\rangle})_{\tau_n}$ is $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$ (mod 1) if and only if $c \equiv 0 \pmod{24}$.
(3) For $k = \pm 1 \equiv \frac{c}{8} \pmod 3$ there is an $\alpha \in \mathrm{Rep}((\mathcal{A}^{\otimes n})^{\langle\tau_n\rangle})_{\tau_n}$ with $h_\alpha \equiv -\frac{k}{3n}$ (mod 1).

Proof. Let $n = 3\ell \pm 1$. Then using [15, Theorem 6.3e] we have

$$h_i = \frac{i}{n} + \frac{n^2-1}{24n}c = \frac{i + k\ell(3\ell \pm 2)}{n} \tag{3.2}$$

thus we have (1). Now let $n = 3\ell$, then

$$h_i = \frac{i}{n} + \frac{n^2-1}{24n}c = \frac{i}{n} + \frac{9\ell^2-1}{9\ell}k \tag{3.3}$$

$$= \begin{cases} \frac{i-m}{n} \pmod 1 & k = 3m \\ -\frac{3(m-i)\pm 1}{3n} \pmod 1 & k = 3m \pm 1\,. \end{cases} \tag{3.4}$$

Thus we have (2) and since $3 \nmid 3(m-i) \pm 1$ we get (3). □

Proposition 3.3. *Let $\mathcal{A}$ be a holomorphic net. If $3 \nmid n$ then $\langle\tau_n\rangle$–$\mathrm{Rep}(\mathcal{A})$ is tensor equivalent to $\mathrm{Hilb}_{\mathbb{Z}_n}$.*

Proof. Since there is a τ_n-twisted representation β with $h_\beta = 0 \pmod 1$ from Lemma A.1 it follows that $\mathrm{Rep}((\mathcal{A}^{\otimes n})^{\langle\tau_n\rangle})$ is braided equivalent to $\mathcal{C}(\mathbb{Z}_n \times \hat{\mathbb{Z}}_n, q)$ and because the Lagrangian subgroup lives in the zero graded part we have $\langle\tau_n\rangle$–$\mathrm{Rep}(\mathcal{A})$ is tensor equivalent to $\mathrm{Hilb}_{\mathbb{Z}_n}$ again by Lemma A.1. □

Proposition 3.4. *Let $\mathcal{A}$ be a holomorphic net of central charge $c = 8k$ and $n = 3m$ for some $m, k \in \mathbb{N}$.*

(1) $\mathrm{Rep}((\mathcal{A}^{\otimes n})^{\langle\tau_n\rangle})$ *is braided equivalent to* $\mathcal{C}(\mathbb{Z}_{9m} \otimes \mathbb{Z}_m, q_\mp)$ *with*

$$q_\pm(x,y) = \frac{\pm x^2}{9m} \mp \frac{y^2}{m} \tag{3.5}$$

for $k \equiv \pm 1 \pmod 3$.

(2) $\mathrm{Rep}((\mathcal{A}^{\otimes n})^{\langle\tau_n\rangle})$ *is braided equivalent to* $\mathcal{C}(\mathbb{Z}_{3m} \otimes \hat{\mathbb{Z}}_{3m}, q)$ *for $k \equiv 0 \pmod 3$.*

(3) $\langle\tau_n\rangle$–$\mathrm{Rep}(\mathcal{A})$ *is tensor equivalent to* $\mathrm{Hilb}^{\omega_k}_{\mathbb{Z}_n}$ *with $[\omega_k] = -km[\omega_0]$ for a generator $[\omega_0]$ of $H^3(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_n$.*

In Figure 1, we demonstrate the twisted fusion rules depending on k in an example.

Proof. (2) is proved as before. We note that the cocycle has order three, since $\tau_{3\ell}$ equals τ_ℓ on $(\mathcal{A}^{\otimes 3})^{\otimes \ell}$ since $\mathcal{A}^{\otimes 3}$ has central central charge $c = 24k$. So there are only two choices for the cocycle which are distinguished by the values of h, see Appendix A, which proves (1) and (3). □

Corollary 3.1. *Let $\mathcal{A}$ be a holomorphic net. The action of $\langle\tau_n\rangle \cong \mathbb{Z}_n$ on $\mathcal{A}^{\otimes n}$ is non-anomalous if and only if $3 \nmid n$ or $24 \mid c$.*

In particular, we have proven Proposition 3.1, since any cyclic permutation in $\mathcal{A}^{\otimes m}$ is conjugate to $\tau_n \otimes \mathrm{id}$.

4. All gaugings for cyclic permutation orbifolds

Given a unitary modular tensor category $\mathcal{C}$ we can consider $\mathcal{C}^{\boxtimes n}$ which has a categorical action of any subgroup $G \leq S_n$. Recently, T. Gannon and C. Jones have shown [8, Theorem 1.1, Corollary 1.2] that certain obstructions vanish and that therefore such a symmetry can always be gauged, i.e. there is a G-crossed braided extension $\mathcal{C} \wr G \supset \mathcal{C}^{\boxtimes n}$ which is compatible with the categorical action. The equivariantization $(\mathcal{C} \wr G)^G$ is a new unitary modular tensor category, which corresponds to gauging. If $\mathcal{C} = \mathrm{Rep}(\mathcal{A})$ for a rational conformal net, then G–$\mathrm{Rep}(A^{\otimes n})$ (where G acts by permutations) is a G-crossed braided extension and $\mathrm{Rep}((A^{\otimes n})^G)$ is a special gauging.

Using cyclic orbifolds of rational (not necessarily holomorphic) nets, we show that if a unitary modular tensor category $\mathcal{C}$ is realized by conformal nets, then all $\mathbb{Z}_n$-permutation gaugings of $\mathcal{C}$ are realized.

Proposition 4.1. *Consider the unitary modular tensor category $\mathcal{C} = \mathrm{Rep}(\mathcal{A})$ for a rational conformal net $\mathcal{A}$.*

Then any unitary $\mathbb{Z}_n$-crossed braided extension $\mathcal{C} \wr \mathbb{Z}_n$ of $\mathcal{C}^{\boxtimes n}$ where $\mathbb{Z}_n$ acts by cyclic permutations on $\mathcal{C}^{\boxtimes n}$ is realized as $\mathbb{Z}_n$–$\mathrm{Rep}(\mathcal{B})$ for some conformal net $\mathcal{B}$ and $\mathbb{Z}_n \hookrightarrow \mathrm{Aut}(\mathcal{B})$.

In particular, any gauging of the cyclic permutation on $\mathcal{C}^{\boxtimes n}$ is realized by a conformal net $\mathcal{B}^{\mathbb{Z}_n}$.

Proof. There are n distinguished extensions $\mathcal{C} \wr \mathbb{Z}_n$, see [5, Lemma 2.3]. One is realized by the cyclic permutation orbifold $\mathbb{Z}_n \hookrightarrow \langle \tau_n \rangle \leq \mathrm{Aut}(\mathcal{A}^{\otimes})$. Let $[\omega] \in H^3(\mathbb{Z}_n, \mathbb{T}) \cong \mathbb{Z}_n$. Since $\mathcal{Z}(\mathrm{Hilb}^{\omega}_{\mathbb{Z}_n})$ is pointed, by [2, Theorem 3.6] there is a conformal net associated with a lattice $\mathcal{A}_L$ realizing $\mathcal{Z}(\mathbb{Z}_n, \omega)$. Then there is a $\hat{\mathbb{Z}}_n$-simple current extension $\mathcal{B}_\omega \supset \mathcal{A}_L$ and $\mathbb{Z}_n \hookrightarrow \mathrm{Aut}(\mathcal{B}_\omega)$, such that $\mathbb{Z}_n$–$\mathrm{Rep}(\mathcal{B}_\omega) \cong \mathrm{Hilb}^{\omega}_{\mathbb{Z}_n}$.

Finally, $\Delta(\mathbb{Z}_n)$–$\mathrm{Rep}(\mathcal{A}^{\otimes n} \otimes \mathcal{B}_\omega)$ with $\Delta(\mathbb{Z}_n) \subset \mathbb{Z}_n \times \mathbb{Z}_n$ the diagonal subgroup gives all $\mathbb{Z}_n$-crossed braided extensions by varying the class $[\omega]$ using [2, Proposition 3.4]. □

We note that the reconstruction program asks if for any unitary modular tensor category $\mathcal{C}$ there is a conformal net realizing it. In this perspective, the $H^3(\mathbb{Z}_n, \mathbb{T})$ freedom in gauging of cyclic permutations does not give any obstructions.

Appendix A. Lagrangian extensions

A **premetric group** (A, q) consists of a finite abelian group A which we see as an additive group and a quadratic form $q\colon A \to \mathbb{Q}/\mathbb{Z}$, i.e. $q(na) = n^2 q(a)$ for all $a \in A$ and $n \in \mathbb{Z}$ and $\partial q(a,b) = q(a+b) - q(a) - q(b)$ is a bicharacter. A **metric group** is a premetric group (A, q) with ∂q non-degenerate. A morphism $\tau\colon (A_1, q_1) \to (A_2, q_2)$ is a homorphism $\tau\colon A_1 \to A_2$ with $q_1 = q_2 \circ \tau$.

The following is well-known, see eg. [11, page 47–48] and [6, Theorem 8.4.9].

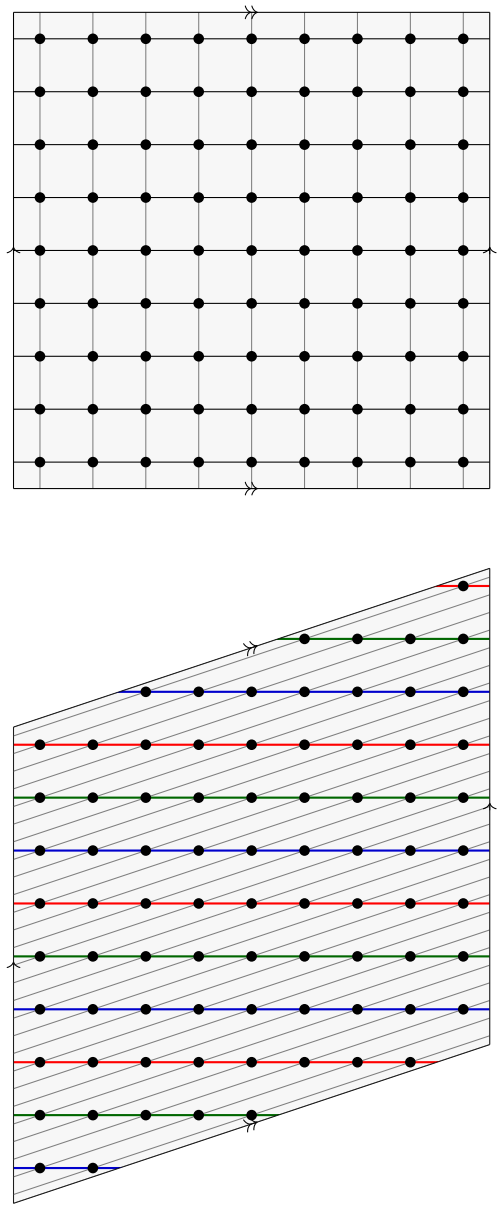

Fig. 1. $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_{27} \oplus \mathbb{Z}_9$ fusion rules for $\mathrm{Rep}((\mathcal{A}^{\otimes 9})^{\langle \tau_9 \rangle})$ for central charge $c \equiv 0 \pmod{24}$ and $c \equiv 8 \pmod{24}$, respectively. The gravitational anomaly $\exp \frac{2i\pi k}{3}$ with $k = 0, 1$, respectively, twists the torus.

Proposition A.1. *Given a metric group (A, q) there is an up to braided equivalence unique unitary modular tensor category denoted by $\mathcal{C}(A, q)$ such that the braiding*

$c_{X_a,X_a} = c_q(g) \cdot 1_{X_a \otimes X_a}$ *and thus the twist* $\theta_{X_a} = \exp(2\pi \mathrm{i} q(a))$ *for all* $a \in A$.

Conversely, given a pointed unitary modular tensor category $\mathcal{C}$*, the finite set* $A = \mathrm{Irr}(\mathcal{C})$ *is an abelian group under the tensor product and the braiding* c *defines a quadratic form* $q(g) \cdot 1_{X_g \otimes X_g} = c_{X_g,X_g}$ *for every* $g = [X_g] \in G$*. Then* $\mathcal{C}$ *is braided equivalent to* $\mathcal{C}(A, q)$*.*

We define $H^3(A, \mathbb{T})_{\mathrm{ab}} = \ker(\psi^*)$, where $\psi^* \colon H^3(G, \mathbb{T}) \to \mathrm{Hom}(\Lambda^3 G, \mathbb{T})$ is given by

$$[\psi^*([\omega])]\,(x, y, z) = \prod_{\omega \in \mathbb{S}_3} \omega(\sigma(x), \sigma(y), \sigma(z))^{\mathrm{sign}(\sigma)}\,. \tag{A.1}$$

The Drinfel'd center $\mathcal{Z}(A, \omega) = Z(\mathrm{Hilb}_A^\omega)$ is pointed if and only if $[\omega] \in H^3(A, \mathbb{T})_{\mathrm{ab}}$ [16, Corollary 3.6], see also [19, Proposition 4.1].

Let $\hat{B}$ be an abelian group. A **Lagrangian extension** of $\hat{B}$ is a triple (A, q, ι) consisting of a metric group (A, q) with $|A| = |\hat{B}|^2$ and a monomorphism $\iota \colon (\hat{B}, 0) \hookrightarrow (A, q)$ of premetric groups. The isomorphism classes of Lagrangian extensions of $\hat{B}$ form an abelian group $\mathrm{Lex}(\hat{B})$ via the multiplication $(A_1, q_1, \iota_1) \boxplus (A_2, q_2, \iota_2)$, see [3, Section 3.3] for details. Given a Lagrangian extension (A, q, ι) of $\hat{B}$ we obtain a Lagrangian algebra $L = \iota(\hat{B})$ in $\mathcal{C}(A, q)$ and $\mathcal{C}(A, q)_L = \mathcal{C}(A, q)_B$ is naturally isomorphic to Hilb_B^ω for some $[\omega] \in H^3(B, \mathbb{T})_{\mathrm{ab}}$ and the map $(A, q, \iota) \to [\omega]$ gives an isomorphism $\mathrm{Lex}(\hat{B}) \to H^3(B, \mathbb{T})_{\mathrm{ab}}$ of abelian groups.

Example A.1. Let A be an abelian group and $\hat{A} = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ the dual group. Then $(A \times \hat{A}, q_{\mathrm{st}}, \iota)$ is an Lagrangian extension of A, where $q_{\mathrm{st}}(a, \chi) = \chi(a)$ and $\iota \colon A \to A \otimes \hat{A}$ is the canonical inclusion. Note that the isomorphism class of $(A \times \hat{A}, q_{\mathrm{st}}, \iota)$ is the unit under $\boxplus$ and thus correspond to the trivial cohomology class in $H^3(A, \mathbb{T})$.

Lemma A.1. *Let* (G, q, ι) *be a Lagrangian extension of* $\mathbb{Z}_n$ *and consider the map* $p \colon G \to G/\mathbb{Z}_n \cong \mathbb{Z}_n$*. If there is a* $x \in G$ *with* $p(x)$ *a generator and* $q(x) = 0$*, then* $(G, q) \cong (\mathbb{Z}_n \times \hat{\mathbb{Z}}_n, q_{\mathrm{st}})$ *and* $\mathcal{C}(G, q)_{\iota(\mathbb{Z}_n)}$ *is tensor equivalent to* $\mathrm{Hilb}_{\mathbb{Z}_n}$*.*

Proof. We claim that the order $\mathrm{ord}(x)$ of x is n. One the one hand, it is a multiple of n. On the other hand, $q(mx) \equiv 0 \pmod 1$ and thus $L' = \langle x \rangle$ is a isotropic subspace of (G, q) and thus $\mathrm{ord}(x) \leq n$. Then $\chi(n) = q(x + \iota(n)) = q(x + \iota(n)) - q(x) - q(\iota(n)) = \partial(\iota(n), x)$ defines a character $\chi \colon \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}$ and $\langle \chi \rangle = \hat{\mathbb{Z}}_n$ because q is non-degenerate. Finally, $\iota(m) + nx \mapsto (m, n\chi)$ gives an isomorphism of metric groups $(G, q) \to (\mathbb{Z}_n \times \hat{\mathbb{Z}}_n, q_{\mathrm{st}})$. □

Example A.2. Let $n = 3m$ and consider the following Lagrangian extension of $\hat{\mathbb{Z}}_{3m} = \{\chi_j\}$ with $\chi_j \colon \mathbb{Z}_{3m} \to \mathbb{Q}/\mathbb{Z}$ with $\chi_j(x) = \frac{jx}{3m}$. We define Lagrangian extensions $(A_\pm, q_\pm, \iota_\pm)$ and (A_0, q_0, ι_0)

$$A_\pm = \mathbb{Z}_{9m} \oplus \mathbb{Z}_m \qquad q_\pm(x, y) = \pm\frac{x^2}{9m} \mp \frac{y^2}{m} \tag{A.2}$$

$$A_0 = \hat{\mathbb{Z}}_{3m} \oplus \mathbb{Z}_{3m} \qquad q_0(\chi, x) = \chi(x) \tag{A.3}$$

with ι_0 the canonical embedding $\iota_0 \colon \hat{\mathbb{Z}}_3 \to \hat{\mathbb{Z}}_3 \oplus \mathbb{Z}_3 = \mathcal{A}_0$. Then $j = (9m-3, 1)$ is a simple current of order $3m$ and we have the short exact sequence

$$\{0\} \longrightarrow \hat{\mathbb{Z}}_{3m} \xrightarrow{\iota_\pm \colon \chi_1 \mapsto j} A_\pm = \mathbb{Z}_{9m} \oplus \mathbb{Z}_m \longrightarrow (\mathbb{Z}_{9m} \oplus \mathbb{Z}_m)/\langle j \rangle \longrightarrow \{0\}\,. \tag{A.4}$$

We have the relations $(A_\pm, q_\pm, \iota_\pm) \boxplus (A_\pm, q_\pm, \iota_\pm) = (A_\mp, q_\mp, \iota_\mp)$ and $(A_+, q_+, \iota_+) \boxplus (A_-, q_-, \iota_-) = (A_0, q_0, \iota_0)$ which gives a subgroup of $\mathrm{Lex}(\hat{\mathbb{Z}}_{3m})$ isomorphic to $\mathbb{Z}_3$. Let $[\omega_\pm]$ be the cohomology class associated with $(A_\pm, q_\pm)$, then $[\omega_\pm] = \pm m[\omega] \in \langle [\omega] \rangle = H^3(\mathbb{Z}_{3m}, \mathbb{T})$. This are the cocycle arising in Proposition 3.4.

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