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ORIGINAL PAPER

Sweeping Area across Physical and Virtual Environments

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Abstract

The concept of area measure is widely viewed as fundamental. It is both intrinsically important and also structurally significant, as a representative of the class of continuous multiplicative quantities. Operating flexibly in this context with meaning and conceptual understanding is thus a critical objective for elementary education. Meanwhile, the difficulties learners encounter with area are well documented in the literature and include challenges with visualizing structured 2D space and conceptualizing the referent-transforming action that converts two length measures into area measure. Responding to these challenges, we present an analysis of their activity where third-grade learners generated figures with area by sweeping one length (a 'squeegee') through another length. We describe a 'duo' of physical and virtual learning environments that we developed to enable this 'sweeping' approach to area – and we show how this duo supported a classroom group of students in engaging with the two challenges mentioned above. In our analysis, we draw upon Charles Goodwin's framework of co-operative action, showing how, at both individual and group levels, learners began to build professional vision around area measure and how the shared gestures they developed pointed toward emerging collective understanding of area as a dynamic quantity.

Keywords Area · Measurement · Sweeping · Co-operative action · Duo of physical and virtual environments

Area is a fundamental construct in the mathematics of measurable quantities. It is important both *intrinsically*, as a core component in modeling many real-world situations, and *structurally*, as an instance of a quantity formed through a multiplicative relation of other quantities. Such quantities are ubiquitous across STEM disciplines as outcomes of

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continuous processes (e.g. 'work' in physics, and all accumulative quantities treated through integral calculus). Thus, being able to reason flexibly about the variation of such quantities in continuous terms is an extremely powerful resource. Our goals for the conceptual teaching and learning of area, then, should include a robust treatment of how area and area units can be created dynamically and continuously from two elements of length.

However, supporting students in developing flexible, multiplicative conceptions of area is a long-standing challenge in mathematics education(Battista 2007; Outhred and Mitchelmore 2000). Indeed, procedural fluency in area measure often develops without firm conceptual backing (Simon and Blume 1994). Problems involving calculating area scan cue students to multiply dimensions or apply formulae, often without the means to assess the reasonableness of the results of these calculations. The conceptual foundations of area measure are, in fact, quite deep, as can be seen in its treatment in differential geometry (see do Carmo 2016; Henderson 2013). In particular, the mathematical view of area shows its dependence on structuring the ambient space with co-ordinates. Moreover, young learners' *challenges* with area (and volume) measure can be linked to challenges with the related operations of structuring space, co-ordinatization and visualization (e.g. Clementset al., 2018).

Meanwhile, in relating area measure to constituent length measures, students must contend with a referent-transforming operation (Schwartz 1988), another concept demanding flexibility in multiplicative reasoning. In particular, the model of multiplication as repeated addition 'breaks' for learners in making sense ofarea, in that it reduces area to a one-dimensional quantity(see Thompson and Saldanha 2003). In this connection, multiplicative reasoning poses a variety of conceptual and representational challenges (Beckmann and Izsák 2015; Izsák and Beckmann 2019).

Literature Review

The study documented in this article contributes to understanding and supporting students' thinking about area. It is a design study (Cobb et al. 2003), involving the creation of a 'duo' of a physical and virtual learning environment (Maschietto and Soury-Lavergne 2013).We extend an innovative perspective on conceptualizing and producing area – sweeping – that provides an entry point into both of the 'deep' conceptual terrains mentioned above: structuring space and transforming units.

A variety of authors have identified sweeping as a promising approach to area measure (Kobiela and Lehrer 2019; Kobiela et al. 2010; Lehrer and Slovin 2014; Panorkou 2018, 2021; Smith et al. 2016; Vishnubhotla and Panorkou 2017). We build on this literature by describing a classroom's engagement with these conceptual challenges through a sequence of sweeping tasks that blended the complementary affordances of physical and virtual environments. This extended exposure to sweeping allowed students to design, generate and analyze figures, providing new perspectives on and provocations to their thinking about area.

Area as a Continuous Quantity

Many existing approaches follow a 'cover and count' approach to partitioning figures and calculating their areas(e.g. Battista et al. 1998; Izsák 2005). In this approach, a figure is conceived of as being subjected to ‘covering’ by individual unit-squares or by a unit-square grid. After covering, the area is calculated by counting the squares used. This conceptualization may be useful for determining the *value of the result* (see Steffe 1988), but it does not offer a way of conceptualizing the *production* of area of a figure as a quantity determined through continuous variation of the linear measures of its sides(Simon and Blume 1996; Thompson and Carlson 2017; Panorkou 2021). In contrast, sweeping activity can reveal: the challenges of generating 2D space by co-ordinating measures of lengths (Panorkou 2021); the value of strategically dissecting a figure and rearranging its components; the richness of students’ flexible ways of composing the area of shapes from dissected components whose individual areas they can reason about and calculate (Kobiela and Lehrer 2019).

Learners as Designers of Figures

Many existing approaches to area measure focus students on analyzing static, pre-created, shapes (see Clements et al. 2018). Sweeping, in contrast, engages students kinestheticallyin creating shapes, which supports conceptualizing area and building resources for reasoning about figures (see Panorkou and Pratt 2016). Additionally, positioning classroom students as creators leverages the diversity of their thinking. Their constructions can be resources for other students and for the class as a group.

To investigate these resources, we designed and integrated two learning environments – one physical, the other virtual – to supporta third-grade class in creating and interpreting shapes by sweeping one length through another. Below, we illustrate how tasks invoking the physical environment, supported by teacher facilitation, fostered seeds of a dynamic construction of area as a variational quantity – illuminating students’ thinking and supporting them in engaging key conceptual challenges. We next turn to discussions showing classroom-level meaning-making based on these experiences, indicating how the group discourse pointed toward more refined ways of thinking about area. We then describe connections between these conceptual seeds and key features of our *Sweeping Area* app,[1]created as a virtual environment to extend students’ individual and collective experiences with physical sweeping. Finally, we outline students’ work with the app as they leveraged its features in their area investigations.

Analytical Framework

This Special Issue explores possibilities for generative relations between physical and virtual learning environments. Several researchers have articulated learning designs that purposefully integrate these modalities. Instead of assessing modalities as ‘competing’ routes to engage a mathematical concept (as in Moyer-Packenham and Westenskow 2013), a ‘duo’ (Maschietto and Soury-Lavergne 2013) of physical and virtual tools can be designed to operate together. While Maschietto and Soury-Lavergne (2013) and Voltolini (2018) use duos in their studies to highlight processes of *instrumental genesis*

[1] Available in the Apple AppStore as a free download for iPads and as a web application for all devices at: http://modelsandmodeling.org/sweepingarea

(Verillon and Rabardel 1995), our study focuses on paired physical/virtual sweeping environments promoting *co-operative action* (Goodwin 2018).

Accordingly, we are interested in the role our physical/virtual duo as an interacting pair of cognitive artifacts (see Bartolini Bussi and Mariotti 2008) mediating the classroom group's engagement with the production and measure of area in distinctive ways. We describe how this pair was used to structure a sequence of actions in which the classroom group created physical and virtual representations of area and negotiated interpretations of those representations. Re-mediation of representational features (Arzarello and Robutti 2010) across physical and virtual modes enabled students to build on varying understandings of area and its measure as they stabilized among the group.

Pursuing the notion that visualizing and structuring area is a challenge (see Clements et al. 2018), we study how students –individually and as a class – began to develop a form of *professional vision* (Goodwin 1994, 2018) related to area measure. Our use of this construct arises from considering the classroom group as a community of learners (Lave and Wenger 1991) grappling with the area construct and with forms of reasoning about and measuring it. Here, professional vision captures patterns of change in students' relations to disciplinary practices (see Ford and Forman 2006; Vossoughi et al. 2020).

To track this emergence, we describe challenges that students encountered with sweeping, and we analyze the initial whole-class discussions that students had about sweeping work. In these interactions, we trace the development of a shared means of describing fundamental features of area. In particular, we identify an expressive *gesture* that stabilizes collective interpretation of area as dynamic. Shared gesture is an appropriate locus for analyzing students' early engagement with sweeping, as expressive gestures at conceptual 'growth points' (McNeill 1992, 2008) can emerge in advance of verbal articulation (e.g. Church and Goldin-Meadow 1986). Moreover, in building a *collective* gesture, the children's interactions paralleled accounts of communities of scientists constructing shared understanding (e.g. Becvar et al. 2005; Ochs et al. 1994) to develop professional vision in their own contexts.

Goodwin's co-operative action framework treats communicative interaction as distributed semiosis, where multiple actors build meaningful actions through innovations and improvisations upon an evolving shared *substrate*. Substrate captures spatial and temporal context as an active resource, out of which actors construct meaningful action and build interpretations of each other's actions. As co-operative action proceeds, the collective substrate is enriched: successive actions add resources through *sedimentation*.

Further, in subsequent work, actors can enlist elements of shared substrate in a variety of ways, creating multi-modal, *laminated* actions that reference and re-use resources with creative transformation. In this article, we use co-operative action to focus on whole-class interactions in a third-grade class occurring with reference to the public display space afforded by the teacher's document camera and smartboard. The size of the projection and its accessibility to the students makes this a quasi-theatrical space, where students present, respond to, interpret and extend each other's sweeping constructions. Goodwin's treatment of *environmentally-coupled gesture* is particularly relevant, as are his analyses of interactions that cultivate *professional vision* and offer occasions to assess it.

Methods

Our study was conducted in a third-grade classroom at an elementary school in the US Southwest serving 875 students from pre-K through 5. The student population is 40% Hispanic, 28% White, 25% Pacific Islander (mostly Marshallese), 5% Asian, 1% Black and 1% multiple races, with 82% of students qualifying for free or reduced lunch. Our partner teacher, JF, had taught at this school for 9 years and participated in professional development about measure and geometry for 3 years prior to the conduct of this study. The authors of this study (CB and RL) acted as participant observers (Jorgensen 1989), supporting the teacher both with the logistics of physical and virtual sweeping environments and with the facilitation of classroom discussions.

Curricular Context

This class's history of engaging with measurement in general, and with area measure in particular, is relevant. The teacher, JF, is a participant in a multi-year study of measurement, involving teachers across multiple grades in the elementary school. In studying length measure, students were consistently supported in seeing lengths in terms of distances that could be *traveled* – a perspective conducive to key ideas of linear measure, including connecting the starting point of measurement with the idea of having traveled zero units (Lakoff and Núñez 2000).

Within *area* measure, the students had experienced two key tasks, which they and the teacher referred to in the data we analyze here:

- *Comparing Three Rectangles* As an introduction to area measure, students were to compare three paper rectangles (A, B and C) of different, unknown dimensions, with an eye toward determining *which covered the most space*. They dissected these rectangles (by folding) to establish space-covering relations among them (Lehrer et al. 1998; Strom et al. 2001). As students dissected the rectangles, they often privileged and re-used a partition (e.g. a 1×2 rectangle) to compare the figures' areas by counting the number of them covering each one. The counting of *units of measure*, most of which were not square, emerged as a practical means for comparing areas.
- *Comparing Handprints* Students next compared the space covered by their handprints aiming to order them (Lehrer et al. 1998). As they considered this challenge, they decided to count units that covered the handprints. But, because the handprints were not rectilinear, a square or rectangular unit was not an obvious choice. Many selected units resembling the contour of the figure and that could be contained within its boundaries, including beads, dried beans or pieces of pasta versus square grid paper. In contrast to the 'natural' emergence of a unit of measure in the prior task, the children's understanding of the properties of units of area measure was hard-won here. It was only after finding inconsistent measurements with initial units that resembled the contour of the handprints that they became persuaded that square units were viable choices. Then, after selecting a square unit, strategies to include *partial* units in the unit count emerged to facilitate more accurate comparisons.

We viewed the challenges that learners encountered in both of these prior tasks and their resolutions as potential resources for introducing students to generating area dynamically and measuring it via unit dissection, which we shortly describe.

Data Analysis

We collected classroom videos, using a mobile camera and a wireless microphone worn by the teacher. Engaged as participant-observers, we also took field-notes and collected snapshots of student work products. Semi-structured, videotaped interviews of nine students from the class contributed importantly to our understanding of the students' ways of thinking, but these are not analyzed here. Videos of classroom interactions were brought into the StudioCode software and marked up with exploratory codes to identify sequences of co-operative action and to interpret students' ways of thinking. Sequences of talk and interaction that illustrated important themes in the classroom were identified and analyzed through iterative viewing and interpreting.

For both physical and virtual modalities, we decided to present classroom data exclusively from the first class session where students used that modality. We chose this sampling approach to highlight how sweeping framed and supported(a) particular patterns in students' conceptions about area measure and (b) connections to multiplicative reasoning and multiplication strategies. Moreover, this choice highlights the designed relation between the two elements of the physical/virtual duo. The collective sense-making discussion at the end of the first physical sweeping day illustrates a series of interpretive achievements that created the need for the virtual environment and grounded its core functionality in operations whose utility the class recognized.

Results

In this section, we begin by analyzing the class's first day engaging with *physical* sweeping, describing how they came to make sense of sweeping and swept figures. At individual and then collective levels, we show how students developed a version of *professional vision* (Goodwin 1994, 2018). In the whole-class discussion, we track how shared meaning-making work led to stable interpretations and also produced a shared, environmentally-coupled, representational gesture for sweeping area. Finally, analyzing the class's first day working with the *virtual* sweeping app, we show how the substrate established in enacting and interpreting physical sweeps supported students in their virtual sweeping work.

The Physical Sweeping Environment

Students began with tempera paint spread out evenly on large ceramic tiles. Using one of two types of 'sweepers' (squeegees and ice scrapers), they could then clear paint from a part of the tile, creating a figure in negative space (Kobiela et al. 2010; Kobiela and Lehrer 2019). Plastic-laminated sheets with inch rulers on their borders were provided, intended to support students in applying linear measurements to the vertical and horizontal dimensions of their tiles and to the figures created on them (see Fig. 1).

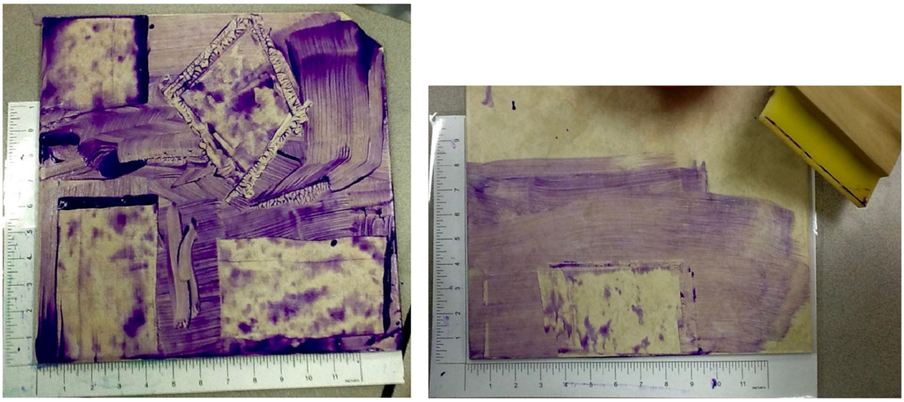

Fig. 1 Physical sweeping: two pairs' early creations and one of the available squeegees.

JF opened the class saying that whereas in prior activity students considered the areas of existing shapes that *others* had made, today they would design shapes of their own. Likening the material set-up to painting on a canvas, she demonstrated how to use a squeegee to sweep an area (see Fig. 2) and asked students to share thoughts. Their responses illustrated the salience of length measure to them. When one student pointed out the bottom ruler, the class immediately produced a family of ideas and terms connected with length ('traveling' along the edge; units, partitioned into equal sub-units; numerical values).

In describing the swept figure itself, the class also evinced a variety of thoughts on how to interpret the shape, such as, including seeing the sweep as a square (or not); as a rectangle (or not); as an imperfect figure (with one side "going like a little diagonal"); and as a shape "like Arkansas" (their home state). The idea of determining its area measure was also raised, but JF framed this as a longer-term goal ("and maybe we'll eventually get to finding the area"). JF then gave the students the task to design a figure of their own, describe their process for creating it; and make and justify any statements they could about it.

In demonstrating the sweeping action and its results via the document camera, JF created a powerful image in the class's shared experience – an image that they spent much of the class session making sense of and building upon. And in formulating their first interpretations of the shape, the class identified some of the challenges to be overcome, including the nature and status of a swept shape as a material representation, some of whose features should be attended to and others ignored.

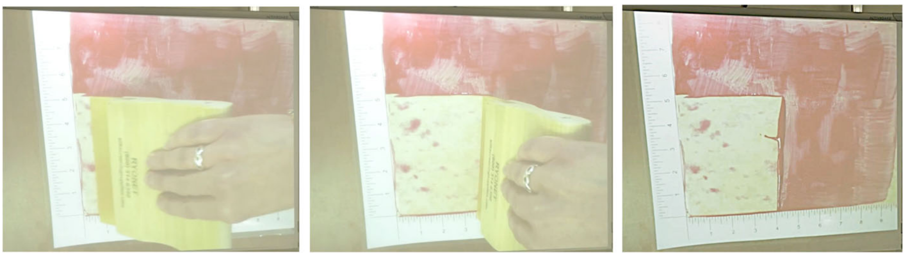

Fig. 2 JF illustrated sweeping a 5in squeegee through 4in in the paint

What Kind of Thing Is a (Physical) Sweep?

How did students understand the figures they made? In the first place, working with the tempera paint on the tiles was engaging: it was playful, a bit unusual and delightfully messy. It also was somewhat mysterious, and students initially seemed surprised by the figures that resulted from their sweeps. Further, although they were encouraged to explore many shapes, even ‘curvy’ ones, students found value in the intentional, physical focus required to keep the sweeper’s angle fixed during a sweep, which produced rectangles or, later (when the sweeper was held at an angle not perpendicular to the direction of sweep), parallelograms.

As an illuminating case, we consider Humberto’s sense-making reflections on his first figure, in interaction with JF. We do not claim that Humberto is representative of the class as a whole, but he did offer an image of one way of thinking about sweeping in connection to other work the class had done. Moreover, JF’s facilitation with Humberto was characteristic of her interactions with other students. When Humberto told JF about his figure, she asked him if he had used the rulers set alongside the tile in any way. (Unlike others in the class, Humberto was working alone on this day.)

JF: OK, you’re wiping down, you’re traveling that distance, OK. And wha... ((pointing at the two rulers and virtually tracing over both of them in an L-shaped motion)) what are these little guys? Did you use those at all?
Humberto: Yeah. I used them to measure how much I ... how long it is
JF: How long what was?
Humberto: The…the square that I made
JF: Can you show me? What do you mean you used it to measure?
Humberto: I used this ((places the squeegee at the top of the sweep, as in Fig. 3)) and I went from up to down ((re-enacts the sweep)).
JF: (OK)
Humberto: and then looked at these ((touches the 0 point and then the 7 of the vertical ruler)) and they showed me how much, how much I went
JF: How far did you go?

Fig. 3 Humberto re-enacting his sweeping motion, over the top of the swept figure

Humberto: I went seven inches long
JF: Traveled seven inches, OK…

Humberto conversed fluidly about *one* linear measure of his swept figure – the (vertical)extent of the squeegee's movement. He successfully measured this extent on his own, and JF's repeated reference to *traveling* made explicit connections to the class's prior work on length measurement. In JF's demonstration, students had also similarly been able to reason well about length measure in the direction of her (horizontal) sweep. For Humberto and others in the class, movement (or travel) made the co-ordinatized length measure of the corresponding ruler salient.

In contrast, when JF next asked Humberto "how wide" his squeegee was, turning to the other linear dimension, Humberto reacted as though her question were incomprehensible. At first, he responded by describing "why" one would use a squeegee (saying, "if there's snow … you can just use this and that'll take it off"), apparently mis-hearing "wide" for "why" in JF's question. As the discussion proceeded, it became clear that Humberto was not attending to the role of the squeegee's width–indeed, not perceiving or processing this aspect of the figure. For Humberto, the squeegee's properties were embodied, an extension of his arm and hand, a physical tool to paint a (linear) vertical magnitude on the tile.

JF continued to urge Humberto to measure the size/width of his squeegee. Eager to comply, Humberto attempted to enlist the ruler salient to him (the vertical one) to measure *some* attribute of the tool. By laying it down, he was able to bring the *height*of the squeegee into this measuring-space (see Fig. 4a). When JF pressed him instead to measure the part of the squeegee he had actually *used* to make the swept figure, Humberto objected:

Humberto: You can't because you... it would only take one number ((see Fig. 4b))
JF: It what?
Humberto: It will only take one number

Here, Humberto was still attending solely to the vertical ruler. His objection was based on the (accurate and important) observation that the squeegee tip essentially took

Fig. 4 **a** Measuring the squeegee's handle; **b** "It would only take one number"

up only one location at a time on the vertical ruler during the sweeping action – rather than occupying an *interval*, as would be necessary for a length measurement.

Humberto's struggles suggest a *perceptual* barrier: he was not actually seeing the same world as JF. The 3½ minute exchange (of which the above segment is a part) involved a series of impasses like those Goodwin (2018) described in his own experience:

> running smack into an opaque wall, a domain of phenomena that seems absolutely crucial to what the participants are doing, but that I do not understand simply by speaking the same language or living in the same country (p. 192)

Goodwin calls the perceptual knowledge that he (like Humberto) lacked *professional vision*.

JF's facilitation (above, and continuing below), allowed Humberto to struggle authentically with developing professional vision in ways that other ethnographers of scientific practice have described. For instance, like the physicists that Ochs et al. (1994) followed, Humberto (with JF), "not only direct their joint attention to static, two-dimensional graphic representations, they also animate those representations by gesturally and verbally enacting dynamic events" (p. 161), such as using the swept figure as a "stage" (p. 152) for gestures, joint re-enactments and re-animations. And, like the scientists Goodwin (2018) studied, Humberto and JF used pointing and other environmentally-coupled gestures liberally to highlight phenomena.

Moreover, Humberto verbally treated the sweeping space as a "liminal world" (Ochs et al. 1994, p. 163) – a space he could enter and an objective space to point at. For instance, his "I went seven inches long" above is similar in form to locutions made by the physicists that blend a personal pronoun ('I' or 'you') with action by an inanimate object – e.g. "I go below in temperature" or "you're fluctuating inside that barrier" (pp. 165–166).Humberto's fusion of himself and the sweeper in talk(see Nemirovsky et al. 1998) – blending the sweeping action and the swept figure – also indicated his immersion in a sense-making struggle.

This exchange ended in a sudden epiphany, a pivotal instant when Humberto's perception appeared to shift fundamentally. Setting this stage for this moment, JF offered to hold the squeegee in place for Humberto and gestured to highlight the horizontal ruler:

> JF: And what about that tool at the bottom? ((1 sec.)) Can that help you at all?
> Humberto: Yeah
> JF: How?
> Humberto: It can wipe…
> JF: Well… What about *this* tool? ((indicating the horizontal ruler with her finger and sliding along it, as shown in Fig. 5)) Can this tool help you to measure i–
> Humberto: OH!!! ((Pointing with right index finger at the ruler, shown in Fig. 6a)) … ((Switching to left finger, pointing at the zero-point and travelling the width of the swept figure, shown in Fig. 6b))It [[inaudible]]…from there ((pointing at origin)) to five ((pointing at 5))
> JF: OK, so how long do you think it is?

Fig. 5 JF held the squeegee in place and highlighted the horizontal ruler

Humberto: Five
JF: Five what
Humberto: [inaudible] Five inches long

JF's initial attempt to bring the second ruler to Humberto's attention failed: he interpreted 'tool' to mean squeegee ("it can wipe…").Her second, pivotal gesture operated at a different level; it linked the world as she was seeing it(professional vision)with the frame of action that Humberto was occupying (embodied activity). JF used her body as a scaffold for Humberto's: her right hand maintained the squeegee in the position that Humberto had held it, while she contorted slightly to configure her left hand so that it approximated the position of Humberto's own left hand. Thus, Humberto observed JF's hands operating the equipment in a physical set-up that he could (and later did) adopt. Indeed, though his 'eureka' moment was accompanied by a spontaneous *right-handed* gesture at the ruler, he then switched to his left hand to 'pick up' JF's travel/measure gesture, like an apparatus whose operation she had demonstrated.

Humberto's work with JF illustrated that, in expanding ideas of linear measure through travel to make sense of sweeping and swept figures, students encountered a barrier of professional vision: the challenge of seeing the production of area as co-

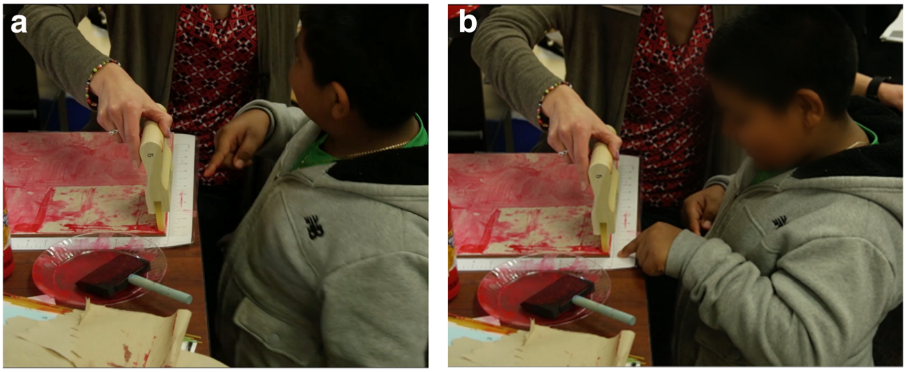

Fig. 6 **a** Epiphany about the horizontal dimension; **b** applying travel to find the squeegee's length measure

ordinating motion in two dimensions. Their exchange also identified some key features of interpersonal interactions that supported developing shared vision: jointly re-enacting sweeps; creating environmentally coupled gestures; highlighting aspects of swept figures through pointing. These interactional dynamics re-appeared at the whole-group level as the class attempted to relate and compare swept figures made by different students.

Group Discussion: Building up Resources for Comparing Two Figures as Representations of Area

We now describe how the classroom group began to establish ways of interpreting the sweeping action and the figures it produced. This episode mediated between physical and virtual sweeping, building upon physical experiences and preparing the ground for the class to make use of the virtual environment in constructing and interpreting representations of area. Moreover, the group display space – a smartboard showing a document camera – offered a connecting ground between projections of tiles with physical sweeps and of screens with virtual ones. We use co-operative action to address the question, "how are embodied knowledge and professional vision calibrated as public practice within a community?" (Goodwin 2018, p. 348).

After approximately 25 min of individual and small-group work, JF called the class back together and student groups were invited to share out. Jaime and Paloma showed their creation first using the document camera(see Fig. 7).Asked about how they created their shape, Jaime and Paloma both indicated they started at the origin ((Paloma pointing)). "Then", Paloma demonstrated, "we traveled all the way to four" ((tracing with index finger along the x-axis to the 4in mark, as in Fig. 7b)).

As a starting point for group meaning-making, Paloma's description recapitulated thinking we observed among individual groups. She and her partner viewed their sweep as a 'travel', which made the direction and extent of their sweep (and the horizontal ruler) salient. Like others, they seemed to view the sweeper as an enormously thick pen, and the group did not yet share an understanding of the sweep as a mathematical representation in two dimensions. On one hand, it was not clear which aspects of the figure were seen as to-be-measured. And on the other, there was some debate about the precision that should be taken as significant. (Although this figure was actually quite 'clean', some students disputed that the sweep came exactly to the 4in mark.)

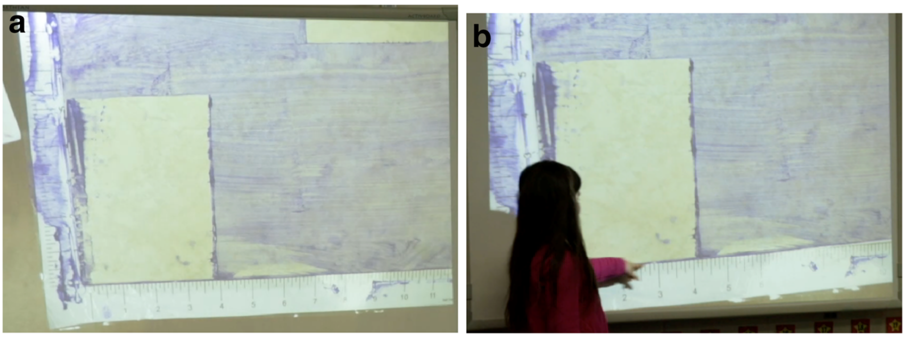

Fig. 7 **a** Jaime and Paloma's construction; **b** Paloma, "we traveled all the way to four"

After some discussion, JF requested of Paloma, “Hold your arm up like you were the squeegee…and then show me how you traveled”. Following her teacher, Paloma created the first public instance of a ‘sweeping forearm’ gesture. In executing the gesture this first time, Paloma exhibited physical intensity, bracing herself on the frame of the smartboard and holding her arm notably tight (see Fig. 8). Levy and Fowler (2000) argue that such physical intensity indicates that a gesture is introducing new information into the discourse, analogous to how prosodic cues indicate emphasis in speech.

Next, JF had Paloma repeat the gesture, with the entire class following along (see Fig. 9), helping to ensure that this ‘sweeping forearm’ was available as a group resource.

After CB documented this sweep on the whiteboard, as “Sweep 5in squeegee through 4in”,NatalieandJacinta asked if they could share their sweep, whose description turned out to be, “Sweep 4in squeegee through 5in” (Fig. 10).

Natalie told CB that her figure was “the same” as Jaime and Paloma’s. CB asked the class if they thought, “sweep five-inch squeegee through four inches [was] the same as sweep four-inch squeegee through five inches”. One student argued yes, “because the numbers are just switched”, but others were uncertain. Approximately 10 min of discussion followed, in which different ideas were forwarded, but the class remained unresolved.

As with Humberto above, the value of the class’s struggle partly derives from its fundamental nature. To a geometer’s eyes, both groups’ shapes were 5 × 4in rectangles. (Moreover, they also were ‘the same’ shape as the one JF had produced in her demonstration.) Further, these figures were both located in the lower-left corner of their tiles and oriented with their 4in sides horizontal. Thus, geometrically, the figures were as much ‘the same’ as they could possibly have been. (In particular, their ‘sameness’ did not rely even on rotational or translational congruence.)

On the other hand, as material representations in an imprecise medium, these figures required interpretation to be seen as ‘the same’. They had quite different looks, and they were constructed from different sweeping actions. The two groups had swept in different *directions*, to different *extents*, and they had used sweepers of different *sizes*

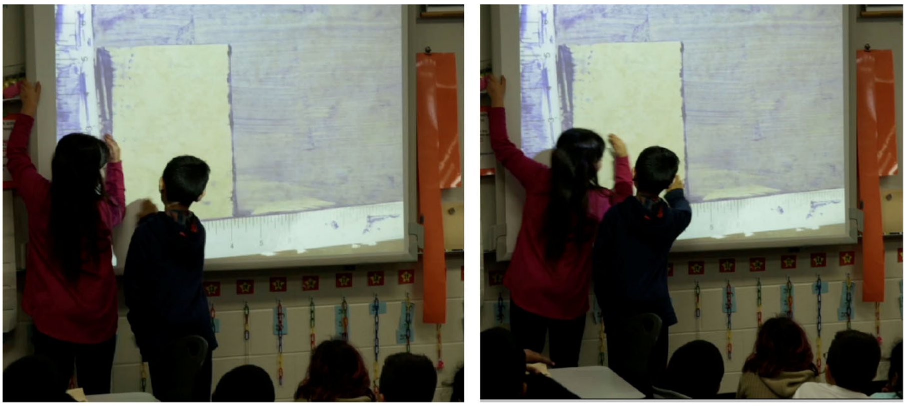

Fig. 8 **a** Paloma re-animated the left-to-right sweep that created the figure; **b** mid-gesture, Jaime joined her, to form a gestural ‘duet’

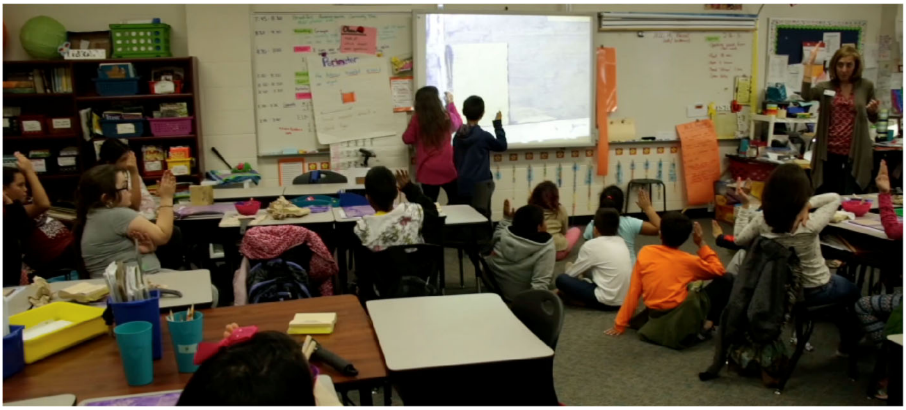

Fig. 9 In chorus, the class performed the 'sweeping forearm' as a 'choral' gesture with Paloma

and *types* (one a 5in rubber squeegee, the other a 4in ice scraper). To 'see through' these differences, the class needed to build a shared means of distinguishing and mathematizing relevant features of swept figures. Even after determining that the two shapes were both rectangles and that each had pairs of 4in and 5in sides, the class was still unsure about whether the shapes were 'the same'.

Mobilizing Resources to Achieve Spatial Structuring and Visualize Area Production

RL then asked whether the two shapes *covered the same amount of space*, and how their areas might be measured. Wanda offered to show how she and her partner thought about the space covered by a sweep. In the subsequent three-minute strip of talk and interaction, Wanda's contributions, collaboratively augmented, brought together the two groups' sweeps and built on gestural resources the class has just developed. As she presented, the projected visual space was progressively annotated with material

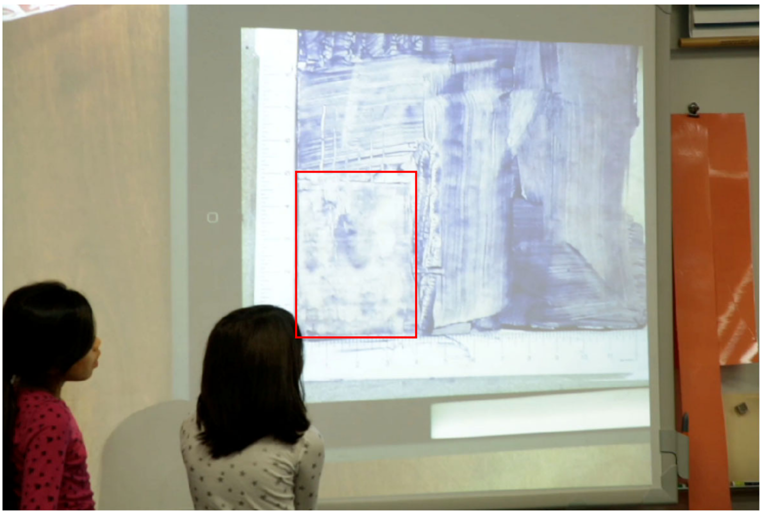

Fig. 10 Natalie & Jacinta's sweep (a rectangle is added to the figure to highlight the border)

overlays and by students' actions. Several actors entered into the action, gesturally and verbally, further using and developing the class's *substrate* of available resources.

Wanda began with her claim that the shapes were the same, using the sweeper-forearm gesture in each of the two orientations:

Wanda: If you do the five-inch squeegee brush this way (see Fig. 11a)… to the four. It's basically the same as the ***four***-inch squeegee-brush…down five (see Fig. 11b)

Focusing on the *actions*, Wanda introduced a proposition that the two sweeps were equivalent. But because her gestural re-enactment had a single swept *figure* as its background, she also implicitly made the claim that *either* sweeping action could have produced this figure.

CB then asked her about measuring the area:

CB: Do you have an idea about how…to measure the area of that shape?
Wanda: ((nodding)) If… You can have five square inches here((covering a 1-inch wide section of the ruler (see Fig. 12) and gesturing upward))

CB asked her to show where the five square inches were:

Wanda: ((indicating a region at the base of the new "column"))
This is one square inch…
((then, gesturing upward with hand and arm, from the one-inch mark on the horizontal ruler (see Fig. 13a–c)))
all the way up here, that's ... five square inches

As Wanda repeated her gesture, CB placed a piece of spaghetti along the line she cut with her hand and arm (as shown in Fig. 13d). Wanda's contribution here connected a variety of concepts and gestural ideas. Though starting from the horizontal ruler, she referred to the figure with a covering palm (see Fig. 12) where others used a pointing

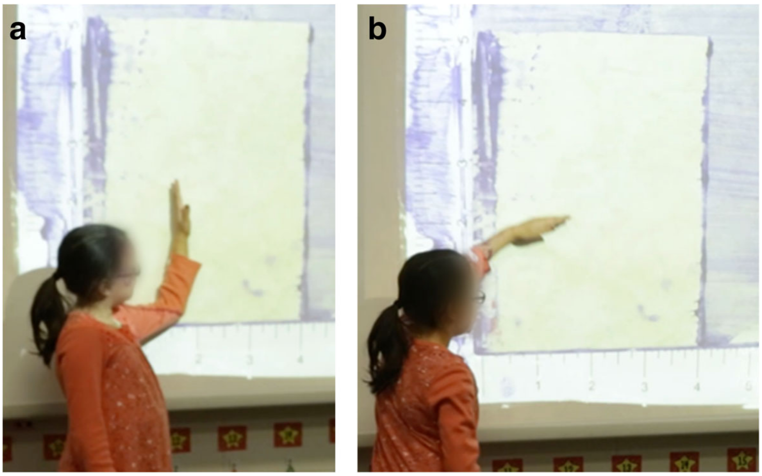

Fig. 11 **a** & **b** The two sweeps were "basically the same"

Fig. 12 "You can have five square inches here"

finger, suggesting a 2D versus a 1D referent. Then, with an upward-thrusting hand, she built on the 'sweeper-forearm' to indicate the length of the squeegee (employing a new kind of 'travel'), while also highlighting one border of the region produced by sweeping it1in.

RL then pressed Wanda to identify the square inches in this 1in-wide region:

RL: And where are the squares?
Wanda: ((Gesturing discretely at each square in a stack of imagined squares))

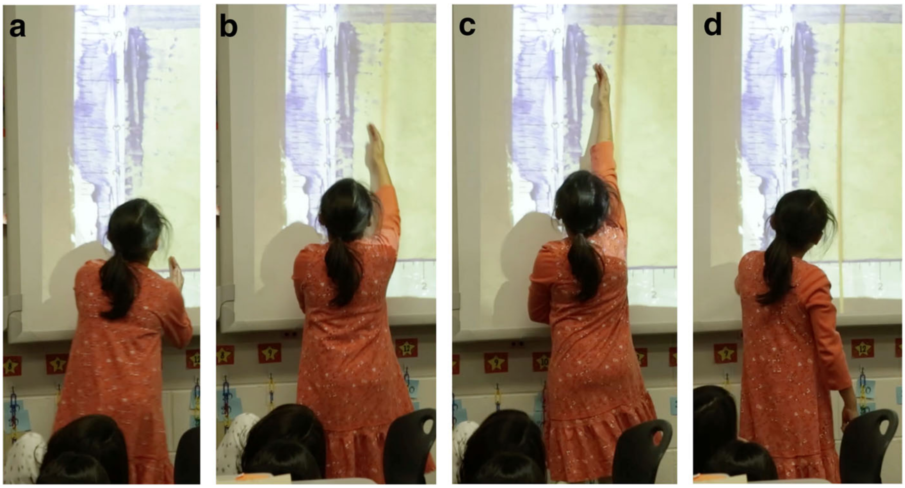

Fig. 13 a–d Wanda showed the five square inches that are produced by sweeping the five-inch squeegee through 1 in

CB asked her to repeat and, as she placed her hand at the 1in vertical mark, he offered another annotation, placing a spaghetti piece horizontally at the level Wanda indicated:

CB: So maybe I need to do something like this?((placing spaghetti as in Fig. 14c))
Wanda: Yeah. ((moving to the vertical 2in mark as in Fig. 14d)) And that's a square…
CB: ((places second spaghetti noodle)
Wanda: ((rapidly continuing)) and that's a square, and that's a square, and that's a square

Although they were ostensibly used to outline the five square inches at the left-most edge of the sweep, the spaghetti pieces extended suggestively to the right, over the rest of the swept figure. Here, the spaghetti, as a nascent notational tool, bridged between stabilizing swept area (in the column to the left) and suggesting continuation (in the horizontal rows extending rightward). When Wanda indicated subsequent columns of five square inches, her hand (consciously or not) carried forward the sweeper-forearm gesture, as in Fig. 15, moving in discrete, one-inch steps from left to right. She counted, "five, ten, fifteen, twenty" and a small chorus of voices from the class joined her.

Here, Humberto responded to an emerging figure in the shared space, arguing that more vertical lines should be added for these additional column-groups (see Fig. 16a):

Humberto: But you need f -, three more... ((pause))
You need three more
JF: And where would those go, Humberto?
Humberto: On the four, and the three and the two

After these spaghetti pieces were in place, JF asked:

JF: What have you all created? What have you now made appear up there?
S1: ((loudly whispering)) Square!
Wanda: Your squares
Ss: Square inch, square grid

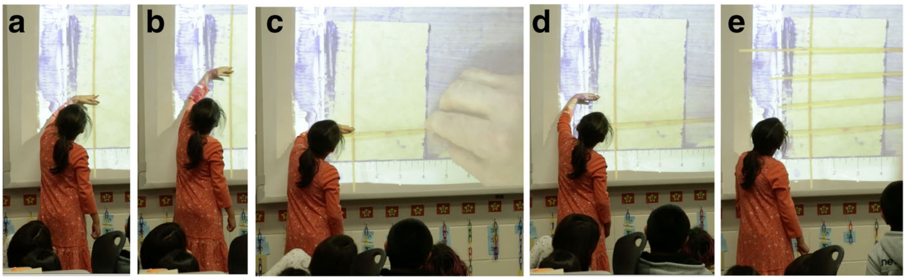

Fig. 14 **a** – **e** Wanda showed where the five square inches were; CB placed spaghetti to delineate

Fig. 15 Wanda imagined and counted 'groups of five' square inches, one group for each column

Though the final figure in Fig. 16b resembles figures created in a cover-and-(composite)-count approach to area, the process of its construction as a representation, above, stabilized shared experiences and the class's collaboratively constructed gesture. Through a process of progressive annotation, the classroom group collectively witnessed and participated in the unit structuring of the 5 × 4in rectangle. The final, structured, rectangular figure is, of course, significant, but it hides the layers of meaning developed in the discussion. Jaime and Paloma's original sweep gained in representational richness and social significance as it

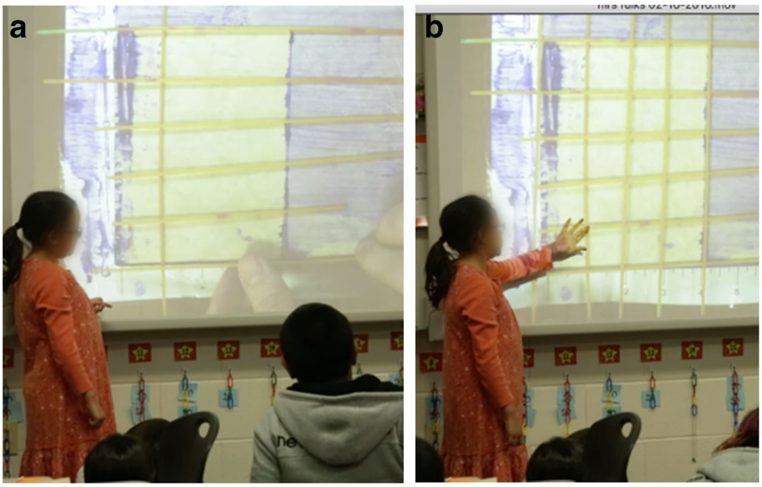

Fig. 16 **a** Humberto rose to request additional vertical spaghetti annotations; **b** the result

was re-animated, marked up, gestured upon and spoken over. The result was a new imagined action, in which, as a sweeper moved from left to right, it produced a structuring of 2D space, generating columns of area, as though at a fixed 'rate' determined by its length.

To expand on this point, JF returned to the idea that this 'same' shape could have been produced by Natalie and Jacinta's downward sweep. Animating this on top of the already-structured rectangular figure, JF used a particularly energetic and crisp set of sweeper-forearm movements, pausing after each inch of sweeping to identify the row of four square inches produced. As when Paloma first introduced the 'sweeping forearm', above, JF's physical intensity suggested that something new was being 'loaded onto' this gesture. Indeed, her action sharpened the gesture, and rendered the continuous movement of the sweeper into discrete chunks, each of which produced a composite unit of four squares. This 'chunky' sweeping motion embodied how linear units contributed to the formation of area units and to their accumulation via composite-unit iteration, both of which are typically challenging, even for older learners (e.g. Kara et al. 2011).

To solidify the discussion, JF and RL led a notational review that indicated the complexity of the equivalences that the group had been exploring. On the one hand, 'sweep 5in squeegee through 4in'describedJaime&Paloma's sweep, while'5 × 4in'described the figure produced. By the explanation initiated by Wanda, this figure's area was structured as '4 × 5sq in'– four columns of five square inches each. Analogously, Natalie & Jacinta's action was 'sweep 4in squeegee through 5in', while '4 × 5in'described the figure produced. Figure 17 show this area was structured as '5 × 4sq in'– five rows of four square inches each, a view of the entire sweep as composed of partial sweeps (Kobiela and Lehrer 2019; Panorkou 2021). Finally, counting squares showed the figure to have an area of 20 square inches (or '20 × 1sq in').

This review provoked new perspectives on area equivalence. Humberto pronounced:

Humberto: It's the same…
JF: It's the same what?
Humberto: the squares
JF: It's the same what?
Humberto: Five groups of four inch- … five groups of four and four groups of five are the same
JF: So, in other words, they cover…
S1: They just flip-flop
S2: - same area do like…
Jeremy: -They're like A, B and C
JF: Like what?
Jeremy: ((Points at the orange rectangles next to the smartboard screen))
S4: A, B and C

Fig. 17 JF enacted a 4in squeegee sweeping out the same area, but now in $4in^2$'chunks'

Jeremy's remark built on Humberto's assertion that different processes can generate the same area. His connection to the Three Rectangles task signaled the recognition of a recurring theme in which mathematical equivalences linked apparently different shapes. This student-created connection to the class's collective history encapsulated the shared sense-making effort and established relations with their prior work (see Fig. 18).

Collective meaning-making and progressive symbolization around two 4 × 5in rectangles involved intertwined processes of (a) building professional vision and (b) constructing communal gestures. In our analysis, gestures were significant because they highlighted developments in the substrate the class could draw upon in future co-operative action. LeBaron and Streeck (2000) note that such gestures, "embody experiences that have emerged in situated action"(p. 136).

Through reuse, these gestures can be a means of achieving 'discourse deixis' (Levinson 1983).(For instance, a 'sweeping forearm' could point back to the moment when JF invited Paloma to move "like you were the squeegee".)Butsuch gestures can also become symbols – "package theoretical conceptions" (Becvar et al. 2005, p. 89) – as starting points for subsequent co-operative action. In this way, JF built on the 'sweeping forearm' to amplify the image of 'chunks' of square inches being structured and produced at a 'unit-rate' by the squeegee's length. This meaning may not yet have been fully mastered by the class, but JF's modified gesture offered a direction for thought.

Virtual Sweeping with the *Sweeping Area* App

As Maschietto and Soury-Lavergne (2013) assert, the virtual component of a duo of artifacts neither replaces the physical component nor replicates all of its aspects. Instead, as a *re-mediation* (see Arzarello and Robutti 2010), it foregrounds different capacities and perhaps provides new perspectives on questions provoked in the

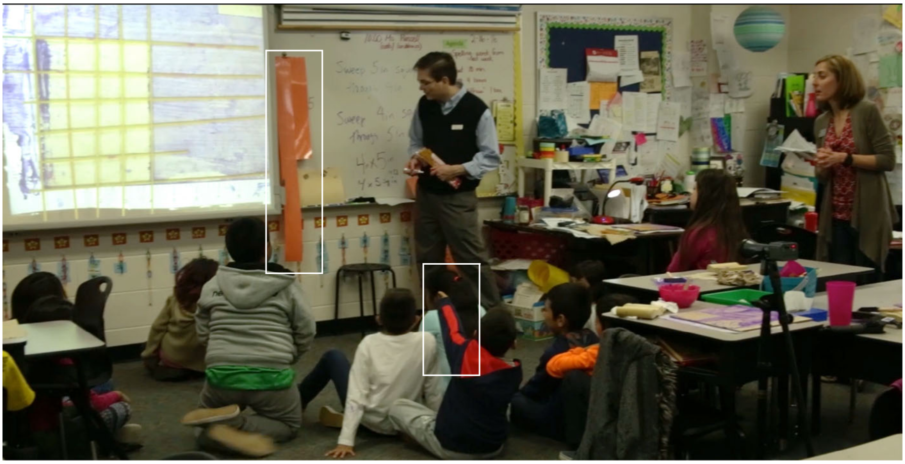

Fig. 18 Jeremy made a connection to the orange equal-area rectangles A, B and C from the Three Rectangles task

physical environment. There should be both continuities and discontinuities between the tools, defined by pedagogical design intentions.

In the above, we have described students' challenges in conceptualizing area and traced the class's development of ways of talking about different sweeping actions that could produce 'the same' figure. In building these shared ways of talking and thinking, students were learning to foreground certain features of physical sweeps and background or ignore others. They also became attentive to the appropriate level of precision to apply to swept figures, as they found that these ways of interpreting and communicating enabled them to see particular sweeps as material representations of area that could be equivalent in several ways (corresponding to congruence of figures or equality of area). Indeed, we view the student activity sequence from physical to virtual, mediated through shared meaning-making, as offering the class an opportunity to construct a shared notational system for area (see Goodman 1976), one that they could then use to frame and answer questions about the areas of families of figures.

The imprecision of the physical sweeping environment served as an asset both for the class's initial work and for their shared meaning-making. It required the students: to become explicit about representationally relevant features of swept figures; to expand their ideas about 'travel' to apply them to area-producing motion; to develop consistent, shared interpretations of figures through descriptions, gestures and arithmetic representations.

The virtual environment offered different affordances, designed to enable students to explore features of area measure as a quantity. First, in contrast with physical sweeping, the *Sweeping Area* app enabled flexible production of swept figures at scales, dimensions and levels of precision specified by the student. In addition to varying the length of their squeegee (see Fig. 19a and b), the app allowed students to alter the units of horizontal and vertical measure and/or to subdivide these units (see Fig. 19c). This permitted them to create sweepers of non-integer lengths and/or sweep through non-integer extents, with greater precision and flexibility. Together, these features enabled students to investigate *families* of related sweeps, by systematically varying the extent of sweeps or changing sweeper sizes.

Second, while in the physical medium, some students did create parallelograms by orienting the sweeper at an angle to the sweep and carefully maintaining that orientation, the app made such constructions more accessible and literally less strenuous, again allowing a proliferation of interesting cases to be explored by all members of the class and put in a broader context (see Fig. 20a and b).Finally, the app's re-mediation and extension of the 'spaghetti annotations' described above allowed students to *dissect* their swept areas (either with a full grid, see Fig. 21a, or in single custom cuts at

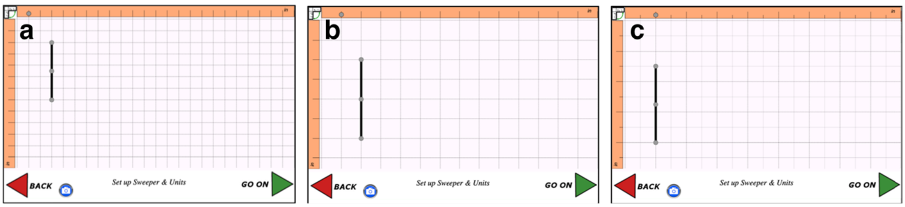

Fig. 19 **a** & **b** Varying the virtual squeegee length and the units of measure in the virtual space; **c** introducing fractional units (here half-units)

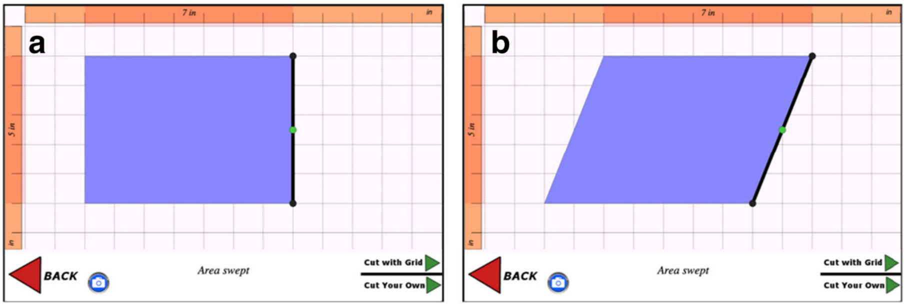

Fig. 20 **a** Sweeping a rectangle left to right; **b** sweeping a parallelogram with a tilted squeegee

horizontal or vertical locations they chose, see Fig. 21b), and then to drag and *rearrange* the resulting pieces (see Fig. 21c). Here again, the physical environment grounded or provoked the need for actions, which the virtual environment then enabled students to generalize and explore.

We introduced the app on Day 3. In that session, the features above enabled several extensions of work students had done in physical sweeping, supporting a growing network of multi-faceted ties between sweeping actions and symbolic structures of arithmetic. Through the app, students connected patterns within and across swept figures with patterns in arithmetic operations on numbers. For instance, at the start of class, Zamora used a doubling strategy to sweep a 3in sweeper through 8in and calculate its area. She explained that the initial column of 3in^2(created by sweeping 1in)would become 6in^2 when the extent doubled to 2in; then 12in^2 when it doubled again to 4in; and, finally, 24in^2 when it doubled again to 8in.As shown in Fig. 22, Zamora indicated the second copy of the 4in sweep, using an open-handed gesture reminiscent of Wanda's shown in Fig. 12.

The app's support for creating sweeps quickly also enabled students to build, "new action by decomposing and reusing with transformation resources provided by earlier actors" (Goodwin 2018, p. 429). That is, it made the class's *substrate* more accessible as a resource for conceiving new action. Journals revealed traces of these lines of inquiry, as students recorded and symbolized their work in the app. For instance, perhaps drawing on Zamora's demo of a 3in *squeegee*, Humberto's journal (see Fig. 20a) showed his experiments with a family of shapes produced by that squeegee sweeping different lengths. Other students picked up on the *doubling* theme in Zamora's example. Figure 23b shows Jeremy's work in calculating 8 × 6in by doubling from 4 × 6 = 24, an exemplar of investigations of change in one dimension on area (Panorkou 2021).

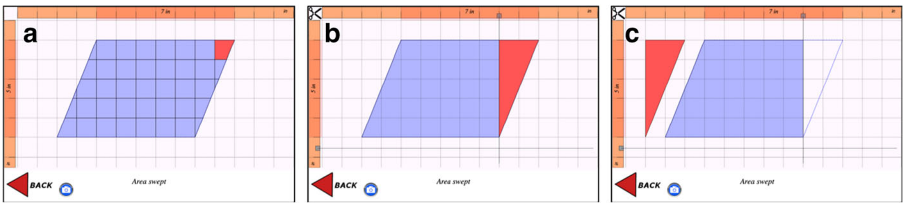

Fig. 21 **a** Dissecting a swept figure along the unit grid; **b** dissecting with specific cuts; **c** rearranging dissected pieces

Fig. 22 A repeated doubling strategy to calculate 3 × 8in

The app also facilitated creation and dissection of parallelograms. In the physical environment, Laura had swept a parallelogram, where it was a challenge (a) to maintain the sweeper's orientation to produce parallel edges and (b) to account for fractional area units in the 'spaghetti dissection'. She asked to show a virtual parallelogram to the class and then explained why she wanted to move a *part* of her figure from the right side to the left, to create a rectangle (see Fig. 24a and b). When she then *enacted* this dissection and rearrangement (see Fig. 24c), the class broke out in spontaneous applause.

The class briefly debated whether the rectangle that Laura created by moving the highlighted triangle had the same area as the original parallelogram. Some argued that, "it's the same", while others insisted that she had "changed it". A persuasive contribution then came from Wanda, who recognized a similarity to the earlier Comparing Handprints task. She noted that there, too, "you had to match the pieces to make a whole", referring to matching partial area units to make whole square units. Thus, as

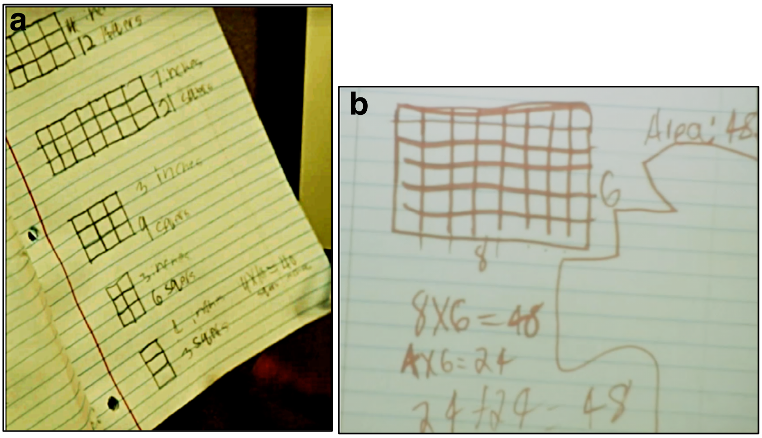

Fig. 23 **a** Humberto explored sweeps of a 3-in squeegee through 4, 7, 3, 2 and 1in; **b** Jeremy doubled24in^2 to get 48in^2

Fig. 24 **a** & **b** Laura described moving part of her parallelogram to the other side, to make a rectangle; **c** doing the rearrangement (using the trackpad on her projected Chromebook)

Laura moved the large triangle, she was seen as pairing *several* complementary partial units at once. On recognizing this, Connor, a vocal opponent of area conservation, changed his opinion, saying. "Yes, it's the s-… I thought it was different, but it's the same because if you add **them** to that piece over there, it's the same".

An interactive demonstration of dissection and rearrangement under the app's two cutting options – to cut with the full grid (see Fig. 25a) or to "Cut Your Own" along selective vertical or horizontal lines (see Fig. 25b) – deepened the connection with Comparing Handprints. And the class was ready to use dissection to extend their exploration of sweeping and calculating areas.

All students we observed were able to dissect parallelograms into rectangles, though they varied in whether they chose to match individual partial units or to use the Cut-Your-Own tool. And even among Cut-Your-Own users, there was not yet a shared understanding of how to cut optimally. Nevertheless, students generally made the connection that rearrangements helped them to produce rectangular figures whose areas could be described with number sentences.

This increasingly flexible connection between rectangular areas and arithmetic sentences suggested a final task for the day. RL introduced the idea by wondering, "whether anyone had found a quick way to find the area" of a rectangular sweep. When several responded that they had, CB wrote a scenario on the board – "Sweep **3**in through **5**in", putting boxes around the bolded numbers – and asked whether it was possible to find the area without drawing it. Students answered "15", and several offered ways of thinking about this area (knowing 3 groups of 5 = 15; knowing that 4 groups of 5 = 20 minus 5 = 15; and skip-counting 5, 10, 15). Students asserted they could visualize the area without actually constructing the sweep. A final student noted a pair of multiplication facts: "three times five equals fifteen" and "five times three is fifteen".

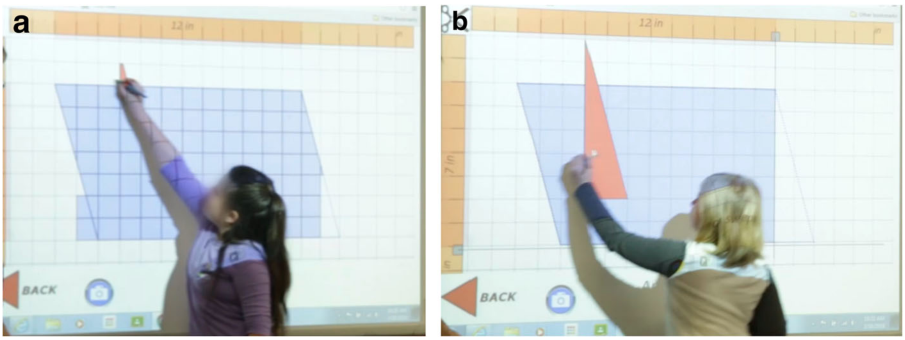

Fig. 25 **a** Rearranging a parallelogram after cutting with the grid; **b** cutting your own

CB then said he was going to change the numbers in the boxes and asked who thought they could find the area, no matter what numbers he chose. About half of the class raised their hands. JF selected Judah, who answered the challenge "Sweep **2**in through **6**in" by saying he knew the multiplication fact two times six is 12. Then, it was his turn to invent a scenario. When he said, "The sweeper will be eight inches…and it will sweep through twelve …inches", there were audible gasps in the room at the difficulty facing Monica, who had agreed to try Judah's challenge.

CB asked what Monica might type in to a calculator, if she had one, to answer the question. This provoked responses from Monica and from others in the class:

Monica: Twelve times eight
CB: Any other thing you could put in the calculator?
Jaime: Eight times twelve
S3: Twelve times eight
CB: Anything else?
S4: Twelve plus twelve plus twelve plus twelve plus twelve plus twelve …
CB: Right! I could do that how many times?
S4: Eight times.
CB: Eight times. Or if I did it the other way, it could be…
S5: eight plus eight plus eight plus eight plus eight plus…
CB: ((nodding))
Connor: Or, do it eight times, but… ten plus two
RL: Oh!

With Connor's guidance, CB annotated his proposed strategy as "8 × (10+2)". Jaime then volunteered an idea that he may or may not have connected with Connor's:

Jaime: I was gonna say, eight times… eight times twelve you can break it up. Like you can put eight times ten you just take the ten off of the, of the twelve. And then eight times, eight times one … eight times tenis eighty. And eight and eight; two eight times…equals…
JF: So, you're talking about doing…kinda two sets…two sets of parentheses? So, eight groups of ten plus eight groups of two?
CB: ((notating 8 × 10 + 8 × 2)) That's what you're saying, right?
Jaime: ((nods)).

Jaime's language of breaking up and taking off suggested a connection to the dissection and rearrangement of a rectangular figure, offering opportunities to return to the sweeping representation. In order to compare the three notational expressions:

$$8 \times 12 \; 8 \times (10 + 2) \; 8 \times 10 + 8 \times 2$$

CB created an 8in squeegee and swept it through 12 in. Monica and Jaime worked together to use dissection to represent Jaime's arithmetic strategy, 8 × 10 + 8 × 2 (see Fig. 26a), and to calculate the result (see Fig. 26b). The class session ended with the reflection that one could represent any multiplication expression with a sweep and that

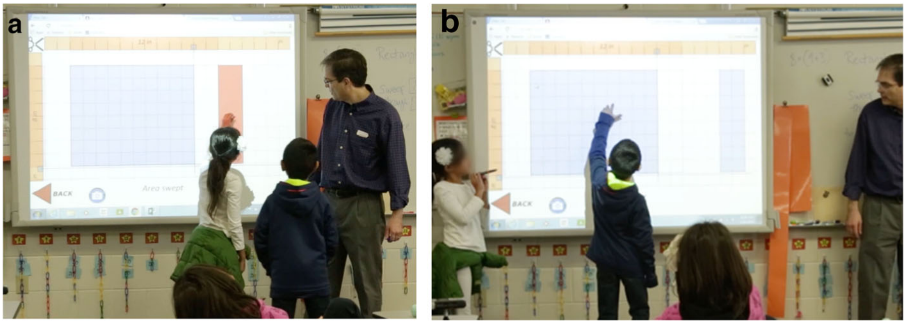

Fig. 26 **a** Monica dissecting 8 × 12in into 8 × 10in and 8 × 2in; **b** Jaime calculating the areas

related calculation strategies could be expressed with dissection of the swept figure into pieces.

Discussion

In approaching this study, we identified two mathematical issues associated with the construct of area, which also corresponded to learners' challenges with area measurement: (a) the structuring of 2D space and (b) the unit- and referent-transforming nature of area as a quantity. We showed how sweeping tasks involving individual construction and group interpretation across physical and virtual environments provided students with resources to address these challenges. And we showed how a progressively enriched representation of sweeping as dynamic area production enabled students to make connections to their prior work on area measure, to multiplicative number patterns and, through dissection, to geometric reasoning about groups of figures with equal area.

Structuring 2D Space

Sweeping introduced an *asymmetry* between dimensions in the *dynamic construction* of area. Asymmetry and dynamism created initial difficulties for students, but also provided means for overcoming these challenges. On one hand, this asymmetry encouraged students to attend to the extent-dimension of their dynamic sweeps (how far they swept), sometimes exclusively, collapsing area creation to travel. On the other, sweeping encouraged the *active seeing* of area (professional vision, 2D structuring), through *seeing the action* of sweeping in a particular way. As illustrated in Wanda's presentation, augmented by the class and by JF, the sweeping action *generated* area through a 'chunky' production of columns of square units, one for each unit of the squeegee's length.

Using the app, students then flexibly explored the effects of using (a) different sweep-extents and (b) different-sized sweepers. Moreover, in discussion, the class increasingly explored the 'commutativity' they discovered between these length-measures to assert that a given shape could be created in multiple ways (sweeping one direction or the other, or swapping the roles of squeegee length and sweep extent). Thus, under a sweeping lens, the class saw the two dimensions of area construction as *asymmetric* but *exchangeable*: squeegee length and sweep extent had different roles (*asymmetry*), but, for *any* rectangular shape, there was a pair of sweeps with roles reversed that could produce it (*exchangeability*).

Unit Transformation

Izsák and Beckmann (2019) argue that unit-transforming approaches to area (such those of Schwartz and of Thompson and colleagues), "do not address how exactly one might derive, or determine, units of the product given units of the factors" (p. 89). They assert that, to make sense of $1\ \text{cm} \times 1\ \text{cm} = 1\ \text{cm}^2$, a student, "would either have to accept a convention or already know something about tiling rectangles with square centimeters". We agree that unit transformation is a challenging concept. And yet, dynamic sweeping constructions in both physical and virtual media allowed students to interact playfully with that mystery and use it to design shapes.

Moreover, using the app in subsequent class sessions, students engaged with a variety of explorations that highlighted ways in which sweeping transformed constituent units. For instance, the class created a ½-unit squeegee and swept it through ½ unit, observing the result and rationalizing it in various ways, including connections to a 1-unit squeegee swept through ½ unit, and to a ½-unit squeegee swept through 1 unit. Students then independently explored fractional-length squeegees and/or sweeping through fractional extents more generally, visualizing the results and using dissection and rearrangement to calculate the area values. The class also used lengths measured in different metrics, including invented lengths for squeegees (e.g. "Volks") and for lengths (e.g. "Wagons") to create area measure in "Volks-Wagons". These tasks offered students further opportunities to illuminate the nature of area and its emergence through the interaction of two length quantities (Thompson 2000).

Conclusions and Directions for Future Research

The duo of physical/virtual sweeping environments illuminated and supported students' thinking about area. Moreover, the study opened several additional directions for future research, in reasoning about area and in the design of duos of physical and virtual learning environments.

Construction and Perception

Sweeping positioned students as creators of the shapes they interpreted, and whose areas they analyzed and calculated, making for 'low threshold, high ceiling' tasks (see Papert 1980). Creating small or simple shapes offered a 'way in' for children with less well-developed arithmetical skills, while deep mathematical ideas were increasingly accessible to students as their confidence grew. The continuity between low-threshold and high-ceiling constructions also fostered the growth of a shared professional vision.

Students learned to see sweeps as geometrical figures:(1) interpreting their own and their classmates' constructions as material representations and (2) attending to the role of both dimensions as a sweep unfolded. Moreover, in moving from physical to virtual environments, this professional vision could become increasingly active, as continuities between interpretive and constructive/transformational operations were established. For example, representing dissection with spaghetti pieces placed on top of physical sweeps to stabilize an aspect of interpretive vision could then be expanded and reimagined with the virtual sweeping environment's dissection tools.

The integration of perceptual and conceptual planes in activities that demand professional vision is also significant for design. Hutchins (1995) describes how navigational tools transform *conceptual* calculations into *perceptual* activities:

> These tools thus implement computation as simple manipulation of physical objects and implement conceptual judgements as perceptual inferences. (p. 171)

In the context of *learning* tasks, we propose an opposite approach. Environments that *reverse* the translation Hutchins describes can be valuable – causing conceptual activity to manifest itself crisply as *problems* in the perceptual domain. Solving such problems can become a means to build professional vision, and virtual tools can be designed to connect interpretative actions and constructive or transformational actions.

Asymmetry and Mathematizing

Part of the power of sweeping arises from the *asymmetry* it imposes on the dimensions of area, viewed as symmetric in other approaches– though see the PerContare project for another asymmetry-leveraging approach (Baccaglini-Frank 2015). Here, too, problematizing a basic feature of area may offer learning affordances. Due to asymmetry, the sweeping interpretations of arithmetic axioms seem more noteworthy to students. For example, the class wrestled with the commutative law in discussing whether Paloma and Jaime's and Natalie and Jacinta's rectangles had the same area: asymmetry gave commutativity an interpretation that children found worthy of debate in the discussion above and celebration in Humberto's realization that, "It's the same".

Similarly, the students' work shown in Fig. 26 hinted at the distributive law, interpreted as a single sweep broken up into two sub-sweeps with equal total extent (or, as sweeping with/without a pause –see Kobiela and Lehrer 2019). The noteworthiness of arithmetical properties as connected to the area measures of figures appeared in challenges of interpreting representations in the physical domain, and it was amplified and extended in the virtual, where modified sweeping actions or dissections and rearrangements could reveal new equivalences.

Dissection and Rearrangement

The discoveries that can be accessed through dissections and rearrangements suggest another direction for future research, in which the sweeping app offers an entry point into treating fundamental area concepts in an ancient Greek tradition. Dissection, rearrangement and re-composition are reversible operations for which the quantity of interest (area) is invariant – and we believe playing with figures in this way can be powerful. To support this design trajectory, we have since enhanced the app to allow students to rotate and reflect dissected parts of shapes and to work on challenges such as dissecting one figure (whether generated by sweeping or not) into another.

Strom et al. (2001) mapped classroom discourse within the Comparing Rectangles task, revealing the emergence of unit dissection to facilitate comparisons by counting. Initial dissections were less uniform, and so were not countable, but were nevertheless useful for establishing space-covering relations among rectangles. As we expand the field of figures to include parallelograms (as well as triangles, trapezoids, and other shapes), a full unit dissection approach (unit grid) gives way to identifying more strategic and economical cuts, with increased opportunities for understanding area equivalence in new ways.

Students can establish equivalence for shapes that not only do not look alike but are also produced differently and may belong to different classes of shapes (e.g. types of polygons). These are fundamental ideas in mathematics, yet the approach encourages students to see these basic findings as discussion-worthy – disrupting their prior conceptions and enabling discovery and geometric proof of area formulas and relations. This may be significant from the point of view of both conceptual development and curricular innovation.

Continuity and Discretization

A key feature to be investigated in future work with this physical/virtual sweeping duo is whether and, if so, how it supports students in viewing the sweep dimension as a *continuous* quantity, while also allowing that sweep to be *discretized*. In the fully analog physical environment, the sweep was continuous and a length unit by length unit sweep bridged between discrete and continuous perspectives on area and its measure. Students' meaning-making discussions then highlighted that their intentions in creating sweeps often focused on sweeps of integer lengths or, if fractional, involving specific fractional steps.

This feature of their investigation – seeking a precision of execution that matched their own authorial intent – connected well with the two features of the app, namely (a) 'snapping' to the grid and (b) adjusting the grid to reflect the student's choice of partitions of the length units. Integrating the two perspectives of continuity and discretization (continuous quantity and precision of area production and measurement) is important yet challenging (Kobiela and Lehrer 2019; Thompson and Carlson 2017).

Additionally, understanding a sweep as producing area as it unfolds at a 'chunked' unit rate is worthy of future research. In JF's animation of a 'chunky' sweep (see Fig. 17), we saw discretization of the *process* of area production. This theme appeared as a notable enhancement to the shared 'sweeping forearm' gesture, where it helped to show how sweeping horizontally produced column-groups of area units, one column per unit swept, while sweeping vertically produced rows, one row per unit swept. The app amplified this idea through its implementation of 'snapping': snapping occurs constantly during the sweep, rather than only when the user stops dragging. Thus, in the app as in JF's version of the gesture, the visualization of the sweeping process is discretized, highlighting the notion of chunked unit rate.

On one hand, it seems paradoxical to *lower* the threshold for the area concept by converting an extensive (amount) quantity (counting square units) into an intensive (rate) one (producing area at a rate proportional to the squeegee length). However, viewing "5 × 4in" as the figure formed when a 5in squeegee sweeps for a 4-in. extent did allow Wanda and the class to experience a squeegee as *producing* area at a rate of 5 square inches for each inch it moved. Once this 'unit rate' of area production was comprehensible, they could see the whole figure's area as (5 in^2) x 4 and apply a groups-of approach to the multiplication. Discretizing a continuous sweep into unit-sized steps may allow sweeping to become a situation like those Kaput (1985) identified, "where the intensive quantity is familiar enough to be well 'chunked' into a single familiar 'rate' entity" (p.22).

In pursuing this theme in students' thinking about area, the duo of environments jointly sustain a duality between continuous and discrete conceptions. The physical environment is inherently continuous and analog, while students' and the teacher's notations with spaghetti, animations with 'chunky' gestures and skip-count accounting of area impose a discretizing lens on the resulting figure. Dually, the virtual environment is inherently discrete and digital, while students' fluid use of it to explore families of swept figures rapidly and connect their areas impose an imagined continuity, to see a range of sweeps as modifications of one another.

Connections

Finally, sweeping offers students opportunities to make connections between area (as part of the 'mathematics of quantities') and arithmetic facts (as part of the 'mathematics of pure number'). Such connections appeared in students' recognition that columns of area produced by a physical squeegee corresponded to 'groups of' the squeegee length, whose contributions to a figure's area could be calculated with skip counting. In the virtual setting, Humberto's experiments with rectangles made by a 3in sweeper extended a similar line of thought. Both Schwartz (1996) and Kaput (1985) cite arguments by Gauss and Bolzano – and Kaput also cites Janke (1980), Lebesgue (1966) and Whitney (1968a, 1968b)– in favor of learning the mathematics of grounded, measurable quantities, as opposed to that of pure number. While the mathematics of area is elementary, it is also fundamental – and the connections we have sketched in the discussion suggest that sweeping area could be a taproot domain of activity, accessing and connecting a diverse set of important ideas.

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Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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