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Holographic hadron masses in the language of quantum mechanics

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Abstract

String theory's holographic QCD duality makes predictions for hadron physics by building models that live in five-dimensional (5D) curved space. We show that finding the hadron mass spectrum in these models amounts to finding the eigenvalues of a one-dimensional differential equation identical in form to the Schrödinger equation. Changing the structure of the 5D curved space is equivalent to altering the potential in the Schrödinger equation, which in turn alters the hadron spectrum. We illustrate this concept with three holographic QCD models possessing exact analogs in basic quantum mechanics: the free particle, the infinite square well, and the harmonic oscillator. In addition to making aspects of holographic QCD accessible to undergraduate quantum mechanics students, this formulation can provide students with intuition for the meaning of curved space. This paper is intended primarily for high-energy theoretical physicists interested in involving undergraduates in their research, but is also a suitable introduction to holographic QCD for advanced undergraduates and beginning graduate students with basic knowledge of general relativity and classical field theory.

Keywords: AdS/CFT, quantum mechanics, string theory, holographic duality, extra dimensions, field theory, general relativity

(Some figures may appear in colour only in the online journal)

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1. Introduction

We have known for over fifty years that protons are made up of quarks and gluons. We have pinned down the masses and couplings of quarks to a startling degree of accuracy. Yet we still don't know why the proton's mass is almost exactly a factor of 100 greater than the sum of the masses of its constituent quarks. This mystery persists because quarks and gluons are *strongly coupled* at low energies: they interact so forcefully and often that our usual calculational tools—based almost entirely on perturbation theory—fail.

In this article we describe an idea, holographic duality, that may help to resolve this quandary. While holographic duality has its roots in string theory, this article focuses on a concrete application of the duality—predicting the spectrum of quark-gluon bound states observed in particle physics experiments—which can be understood with a background in undergraduate quantum mechanics at the level of Griffiths [1], and elements of special and general relativity and classical field theory as in the first two chapters of Carroll [2].

Holographic duality, or 'holographic QCD' (hQCD) when it is applied to quarks and gluons [3–5], emerged in the early 2000s and quickly showed potential for tackling the problem of strong coupling. hQCD posits that strongly coupled quarks and gluons in our ordinary flat spacetime are equivalent—or 'dual'-to a weakly coupled theory living in a curved spacetime with an extra spatial dimension in addition to the usual 3.[5] This implies that one can use the higher dimensional, weakly coupled system to make predictions about strongly-coupled quarks and gluons. Indeed, the classical fields in the higher dimensional spacetime correspond to the degrees of freedom we see at low energies in our world: protons, pions, and other quark-gluon composites we collectively call 'hadrons'. The properties of the fields in the higher dimensional space produce predictions for the properties of these hadrons (as detailed further below).

This approach has proved quite successful: even rudimentary hQCD models reproduce hadron spectra and couplings to within about $\sim$15% of measured values [3–5].

Predictions by hQCD models depend crucially on the shape of the 5D spacetime. (We refer to our ordinary 3 space + 1 time dimensions as 4D, and the 4 space + 1 time of the curved space system as 5D.) The shape of the 5D spacetime is defined by the distance measure on the spacetime—its metric. The metric should, first of all, be chosen to reproduce the (myriad) patterns observed in the hadron spectrum. We focus here on the patterns highlighted in figure 1: the hadron spectrum is discrete; there are several copies of each hadron, which have all of the same quantum numbers but heavier masses; and, the masses-squared of these copies increase roughly linearly. The metric must also have a specific form ('anti-de Sitter' (AdS)) near its boundary, to guarantee the existence of a holographic map in the first place—that is, that there indeed exists a 4D system the 5D spacetime corresponds to.

In this work, we show that choosing a 5D metric satisfying these criteria amounts to choosing the right potential in an equation identical in form to the time-independent Schrödinger equation.

We first transform the equation of motion that determines the spectrum into Schrödinger form for arbitrary metric. We then examine the spectrum and Schrödinger potential in three examples of hQCD models: (1) pure AdS space, which admits holographic duality but produces a continuous hadron spectrum because its potential asymptotes to that of a free particle. (2) AdS cut off at a finite radial value (as in [3]) to produce an infinite square well potential. Here the spectrum is discrete, but mass-squared increases quadratically. (3) A metric like that of

[5] The statement of this 'holographic duality' is actually even more powerful: it is really a map between QCD at any energy and a string theory in curved space. For our purposes, however, this low energy limit is sufficient.

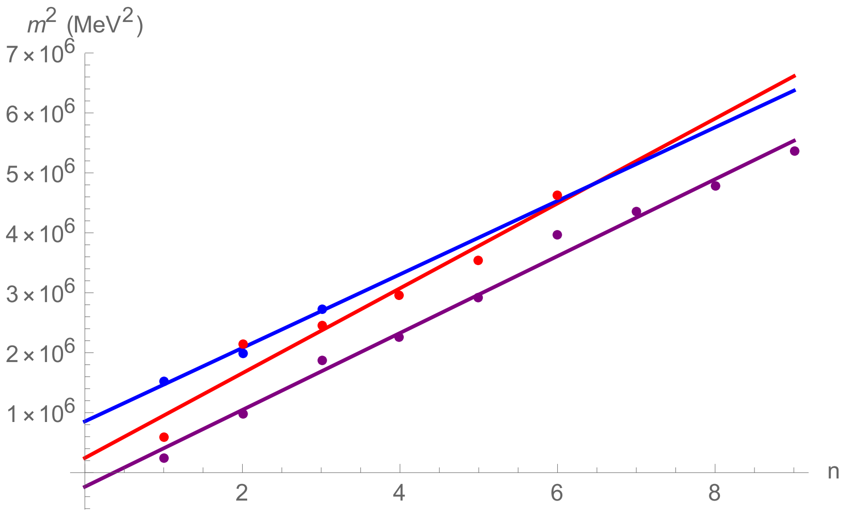


Figure 1. Measured values [6] for meson mass-squared (m^2) vs excitation number (n) for ρ, a_1, and f_0-type mesons. Other mesons and baryons display similar behavior.

[4], which yields a harmonic oscillator potential, and producing a discrete spectrum with the desired linear increase of mass-squared.

1.1. Notation and conventions

Before turning to our study of hadron masses in hQCD, we describe the metric and units conventions we will use for the rest of this article. We use (1) Einstein summation notation, in which all repeated indices are summed over; (2) a 'mostly plus' metric; and (3) 'natural' units in which $\hbar = c = 1$. With these conventions, the infinitesimal distance measure in flat 4D ('Minkowski') spacetime is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu := \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, time is parameterized by $t := x^0$ and the three space dimensions by x^1, x^2, x^3.

We will also derive an equation below that has the same form as the time-independent, 1D Schrödinger equation, whose single spatial dimension will be parametrized by z in what follows. In the 'natural' units just described, a time-independent one-dimensional Schrödinger equation for a particle of mass M and potential $\tilde{V}$ becomes

$$-\frac{\hbar^2}{2M}\Psi''(z) + \tilde{V}(z)\Psi(z) = \tilde{E}\Psi(z) \qquad \Rightarrow \qquad \Psi'' - V(z)\Psi = -m^2\Psi, \tag{2}$$

where we have redefined the potential and the eigenvalue to eliminate the constant factors of mass M as $V(z) := 2M\tilde{V}(z)$, $m^2 := 2M\tilde{E}$. In what follows, we will show that the masses of hadrons in hQCD models can be determined using an equation of this Schrödinger-like form.

2. Generic holographic models in Schrödinger form

2.1. The 5D metric and the asymptotically AdS condition

In this section, we describe a generic version of a hQCD model, and explain why the metric must be AdS near its boundary. Let's say the 5D metric is

$$ds^2 = g_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N = A(z)^2(\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathrm{d}z^2), \quad \text{where} \quad z \in (0, \infty), \tag{3}$$

for a generic function $A(z)$. Here capitalized Latin indices $M, N = 0, 1, 2, 3, 4$ run over all five spacetime coordinates, and lower case Greek indices $\mu, \nu = 0, 1, 2, 3$ run over the four Minkowski dimensions x^μ. The fifth dimension is parameterized by z, with the boundary of the space lying at $z = 0$. One can think of this metric as a series of 4D Minkowski slices along the z-direction, with distances in each 4D Minkowski space controlled by the factor $A(z)^2$.

The structure of this space gives some intuition for the holographic correspondence between 4D and 5D: one can think of the 4D Minkowski directions in (3) as housing the 4D hadrons; each hadron has a 'wavefunction' in the extra dimension (z), which encodes its characteristics and behavior. This is a bit of an oversimplification: the 4D and 5D systems exist independently of each other, and are connected by a map translating one to the other. Still, the essence of hQCD is that the 4D and 5D theories are just different descriptions of the *same* physical system. This means that physical degrees of freedom (e.g. particle states) can be mapped onto each other one-to-one. Both theories must thus contain the same *number* of physical degrees of freedom—and by matching degrees of freedom between the two theories, we can understand why the 5D metric must be AdS near its boundary.

The argument is as follows [7]: the number of degrees of freedom in the 4D quark-and-gluon theory is proportional its volume (it is an extensive quantity). Taking 3D space to be a cube of side L_4, discretized into a grid with lattice spacing ϵ, the number of degrees of freedom is then proportional to the number of lattice points,

$$N_{\mathrm{4D}} \propto (L_4/\epsilon)^3. \tag{4}$$

This counting works differently for gravitating systems, where the number of degrees of freedom is proportional to the *area of the space's boundary* [8, 9]. Because our 5D theory lives in curved spacetime, it must have gravity. Again using ϵ to define the shortest length-scale in the system, we regularize the location of the boundary at $z = 0$ in the metric (3) to $z = \epsilon$. The degrees of freedom living in the 5D spacetime is

$$N_{\mathrm{5D}} \propto \int_{z=\epsilon} \mathrm{d}^3x\sqrt{-g_\partial} = A(\epsilon)^3 \int \mathrm{d}^3x = A(\epsilon)^3 L_4^3, \tag{5}$$

where g_∂ is determinant of the metric restricted to the boundary. This factor ensures that the volume is independent of coordinate choice: $\sqrt{-g_\partial}$ cancels out the Jacobian from coordinate transformations of the volume d^3x. For N_{4D} and N_{5D} to scale identically with ϵ, we must have $A(z) \sim 1/z$. The metric of AdS space is simply (3) with $A = L/z$ for some constant L. The latter thus amounts to requiring that the metric be AdS near $z = 0$. (See [7] for more details of this derivation.)

2.2. 5D action and reduction to Schrödinger-like equation

Having established the basic structure of the 5D metric, we now introduce the classical fields that correspond to hadrons in 4D. The fields' spins and (appropriately defined) charge- and parity-conjugation properties determine which hadrons they correspond to. For example, vector fields correspond to vector mesons. As we are interested only in the heuristic scaling of the spectrum, we restrict ourselves to scalar fields, dual to spin 0 hadrons. The field's behavior is determined by its action on the 5D spacetime, the simplest form of which is the action for a free, massless, relativistic scalar field:

$$S_{\mathrm{5D\ scalar}} = \int \mathrm{d}t\, \mathrm{d}^3x\, \mathrm{d}z \sqrt{-g} \left(\frac{1}{2} \partial_M \Phi \partial_N \Phi g^{MN} \right), \tag{6}$$

where g is the determinant of the 5D metric and ∂_M is a partial derivative with respect to x^M. To understand where this comes from, recall that the action of a massless scalar field in a flat 4D spacetime is

$$S_{\mathrm{flat\ 4D}} = \int \mathrm{d}t\, \mathrm{d}^3x \left(\frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi \eta^{\mu\nu} \right) = \int \mathrm{d}t\, \mathrm{d}^3x \frac{1}{2} \left(-\partial_t \Phi \partial_t \Phi + \vec{\nabla} \Phi \vec{\nabla} \Phi \right). \tag{7}$$

The Euler–Lagrange equations for this action would give us the wave equation obeyed by the $\vec{E}$- and $\vec{B}$-fields for light waves (remember $c = 1$), or generically for any relativistic particle moving at the speed of light. Note that this action is invariant under boosts and rotations. The curved space, extra-dimensional version of this in equation (6) is identical in form except for the $\sqrt{-g}$ and the (inverse) metric g^{MN} replacing the flat Minkowski metric $\eta^{\mu\nu}$. These substitutions make the action invariant under coordinate transformations, a generalization of boosts and rotations to curved space.

Each factor of the metric g_{MN} contributes $A(z)^2$ and each inverse metric g^{MN} contributes $A(z)^{-2}$, so the action becomes

$$S_{\mathrm{scalar}} = \int \mathrm{d}^4x\, \mathrm{d}z\, A(z)^3 \left(\frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi \eta^{\mu\nu} + \frac{1}{2} \partial_z \Phi \partial_z \Phi \right), \tag{8}$$

with Euler–Lagrange equation

$$A(z)^3 \eta^{\mu\nu} \partial_\mu \partial_\nu \Phi + \partial_z (A(z)^3 \partial_z \Phi) = 1. \tag{9}$$

We now apply separation of variables using an ansatz which corresponds to a free particle in Minkowski space with constant energy E and three-momentum $\vec{p}$, and a non-trivial wavefunction in the z direction:

$$\Phi(x, z) = \mathrm{e}^{-\mathrm{i}Et}\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}} \varphi(z). \tag{10}$$

This gives

$$\begin{aligned} &(E^2 - \vec{p}^2) A(z)^3 \varphi(z) + \partial_z A(z)^3 \partial_z \varphi(z) \\ &\quad := m^2 A(z)^3 \varphi(z) + \partial_z A(z)^3 \partial_z \varphi(z) = 0\ . \end{aligned} \tag{11}$$

The 4D mass–energy relation, $(E^2 - \vec{p}^2) \equiv m^2$ appears as an eigenvalue in the final equation. m is thus the mass of the 4D hadron, which leads to the requirement that $m^2 > 0$ for physical particle states.

We can transform (11) to the form of a time-independent Schrödinger equation by defining $\varphi(z) = B(z)\psi(z)$, plugging into (11),

$$\frac{\partial_z (A^3 B')}{A^3 B} \psi + \frac{2A^3 B' + \partial_z (A^3) B}{A^3 B} \psi' + \psi'' = -m^2 \psi, \tag{12}$$

and choosing $B(z)$ such that the ψ' terms vanish:

$$2B'A^3 + \partial_z(A^3)B = 0 \qquad \Rightarrow \qquad B = A^{-3/2}. \tag{13}$$

The potential is thus

$$V(z) = \frac{1}{4}A^{-6}\left[2A^3\partial_z^2(A^3) - (\partial_z A^3)^2\right]. \tag{14}$$

Boundary conditions: solutions to this Schrödinger equation—dual to hadrons of finite mass and zero spin—must also satisfy boundary conditions. In this case, these are such that the action (8) be finite for a given solution. This is equivalent to the normalizability of the wave function in quantum mechanics, and yields the requirement

$$\int \mathrm{d}z\, A(z)^3\varphi^2 = \int \mathrm{d}z\, \psi^2 = \text{finite}, \tag{15}$$

or $\psi(z \to 0) \sim z^\alpha$ for $\alpha > -1/2$ and $\psi(z \to \infty) \sim z^\beta$ where $\beta < -1/2$. As usual, this equation can only be satisfied for special values of the eigenvalue m^2, which determines the mass spectrum of the scalar particles.

3. Holographic QCD in three examples

We now build up an hQCD model starting with a simple metric, which we modify bit by bit until we achieve the desired behavior of the experimental hadron spectrum.

3.1. Anti de Sitter space

The simplest example of a metric that is AdS near its boundary is simply AdS all the way through: $A(z) = L/z$ for some scale L. Using (14), the potential for pure AdS is

$$V_{\mathrm{AdS}} = \frac{15}{4z^2}, \tag{16}$$

shown in figure 2. The arbitrary scale L has dropped out. V_{AdS} repels particles from the boundary at $z = 0$ toward $z \to \infty$, where it dies off quickly to leave a free particle potential. (V_{AdS} is in fact identical to the centrifugal potential of the radial wave equation in 3D, which has a similar effect.) The Schrödinger-like equation

$$\psi''(z) - \frac{15}{4z^2}\psi = -m^2\psi \tag{17}$$

has solution

$$\psi(z) = N\sqrt{z}J_2(mz), \tag{18}$$

where N is a normalization constant, and J_2 is a Bessel function of the first kind. (We dropped the $\sqrt{z}Y_2(mz)$ solution, which diverges at 0.)

Recalling that $\sqrt{z}J_2(z) \sim z^{5/4}$ as $z \to 0$ and $\sqrt{z}J_2(z) \sim \cos(z + 3\pi/4)$, one can check that the wavefunction dies off near $z \to 0$, and is sinusoidal as $z \to \infty$, as expected for a scattering state. Here too there is no normalizable as solution, and the particle can have any 4-momentum (and any mass).

Indeed, AdS_5—the original playground for holographic duality—is *scale invariant*: invariant under dilations of the coordinates, or equivalently, under changing the energy scale. The

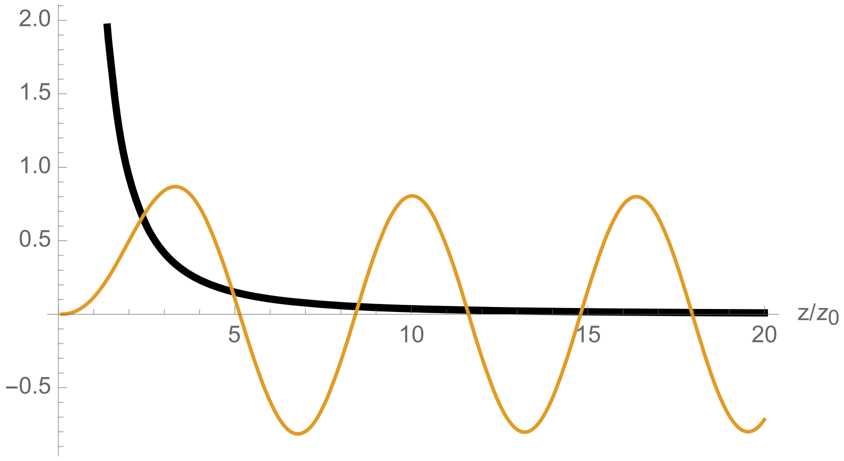


Figure 2. The potential (black) and a sample wavefunction (orange) for pure AdS space are shown as a function of a dimensionless coordinate z/z_0 (for some scale z_0). Note that the potential dies off as $z \to \infty$, leaving a free particle with the standard sinusoidal wavefunction.

only scale in the problem, L, cancelled out in the potential! The corresponding 4D theory must also look identical at different energy scales, which implies a continuous spectrum.

To find a discrete spectrum, then, we must introduce a scale into the metric.

3.2. The hard wall model

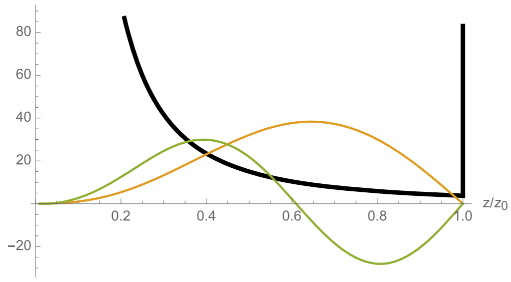
The 'hard wall' model due to Erlich *et al* [3] introduces a scale in AdS, creating an infinite square-well potential. Its metric is identical to the AdS metric, except that the radial coordinate z is cut off at a finite value, z_0:

$$\mathrm{d}s^2_{5\mathrm{D}} = \frac{L^2}{z^2}\left(\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathrm{d}z^2\right) \quad \text{with } 0 < z < z_0. \tag{19}$$

z_0 is a free parameter, which one can fix by comparing the model's predictions for hadron masses and interactions to experimental data. Since we focus only on the scaling of mass with excitation number, we leave our results in terms of z_0.

The hard wall potential is identical to that of AdS, except for the wall at $z = z_0$ (see figure 3). The allowed wavefunctions take the same form as well, except that the boundary condition imposing $\psi(z_0) = 0$ at the wall restricts the possible values of the mass, as they do for the energy in the quantum mechanical infinite square well. The masses are $m_n = \lambda_n/z_0$, where $\lambda_n = (5.1356.., 8.4172.., 11.6198.., \ldots)$ are zeroes of $J_2(mz)$. Each classical field Φ thus gives rise to a tower of hadron states. The mass spacing scales like $1/z_0$, so $z_0 \to \infty$ indeed recovers the continuous spectrum of pure AdS.

As we can see in figure 3, $m^2 \sim n^2$. This is not surprising: the scaling behavior of the spectrum is determined by the large n behavior of the eigenvalues, or, equivalently, large arguments for the Bessel function $J_2(mz)$. For large mz, the Bessel functions are identical to the wave functions of the standard 1D square well: sines and cosines. Thus the eigenvalues of the two systems also have the same large n scaling.



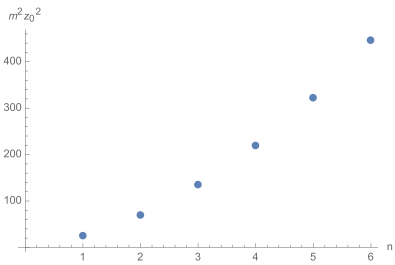


Figure 3. The hard wall model. (a) The potential (black) and the first two wavefunctions (orange and green) as a function of the dimensionless coordinate z/z_0. (b) Predictions for (dimensionless) mass-squared $m^2 z_0^2$ plotted as a function of excitation number n. Note that $m^2 \sim n^2$ as for the quantum infinite square well.

3.3. The soft wall model

Quantum mechanics provides a clue to a potential that *does* produce the correct, $m^2 \sim n$ scaling for the spectrum: a harmonic oscillator. This was indeed the basis for the next iteration on hQCD, the 'soft wall model' [4], which we derive a bit differently from the original formulation here.[6]

Consider a new metric factor $\tilde{A}(z) \equiv \frac{L}{z} C(z)^{1/3}$, where $C(z)$ is a smooth function obeying $C(z \to 0) \sim 1$ to guarantee that the near-boundary behavior is AdS. The potential becomes

$$V_{\rm sw}(z) = \frac{1}{4\tilde{A}^6}\left[2\tilde{A}^3 \partial_z^2(\tilde{A}^3) - (\partial_z \tilde{A}^3)^2\right]$$
$$= \frac{15}{4z^2} - \frac{1}{4}\left(\frac{C'}{C}\right)^2 - \frac{3}{2z}\frac{C'}{C} + \frac{C''}{2C}. \tag{20}$$

Once again, the scale L does not appear in the potential. The first term is the usual repulsive piece from the AdS factor; the remaining terms should go like z^2 as $z \to \infty$ to get a harmonic oscillator. Making the guess

$$C(z) = \mathrm{e}^{-(z/z_{\rm s})^\alpha} \tag{21}$$

for some arbitrary length scale $z_{\rm s}$ and constant α,

$$V_{\rm sw}(z) - \frac{15}{4z^2} = \frac{1}{4z^2}\left(\frac{z}{z_{\rm s}}\right)^\alpha \left(8\alpha - 2\alpha^2 + \alpha^2 \left(\frac{z}{z_{\rm s}}\right)^\alpha\right) . \tag{22}$$

The final term dominates for $\alpha > 0$, leading to the requirement $\alpha = 2$. We now have $\tilde{A}(z) = \frac{L}{z}\mathrm{e}^{-(z/z_{\rm s})^2}$ and potential

$$V_{\rm sw}(z) = \frac{1}{z_{\rm s}^2}\left[\frac{15}{4}\left(\frac{z}{z_{\rm s}}\right)^{-2} + \left(\frac{z}{z_{\rm s}}\right)^2 + 2\right] := \frac{1}{z_{\rm s}^2}\left[\frac{15}{4}\frac{1}{\tilde{z}^2} + \tilde{z}^2 + 2\right], \tag{23}$$

[6] While the soft wall model in its original form added an additional background field called the 'dilaton' on a pure AdS metric, the treatment we give here is equivalent. For practitioners: the difference amounts essentially to using Einstein frame instead of string frame.

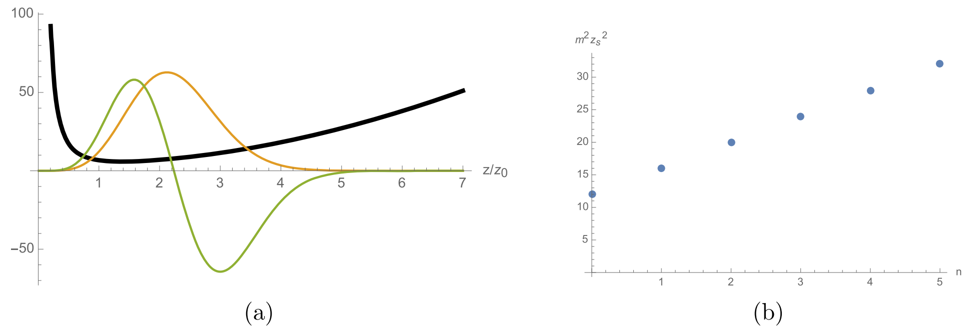


Figure 4. The soft wall model. (a) The potential (black) and the first two wavefunctions (orange and green) as a function of the dimensionless coordinate z/z_s. (b) $m^2 z_s^2$ plotted as a function of excitation number n.

where we introduce a dimensionless coordinate $\tilde{z} = \frac{z}{z_s}$ for convenience.

We can solve for the wavefunctions analytically as sketched in [4] using standard techniques from the higher-dimensional quantum harmonic oscillator problem. After absorbing the constant term of the potential (23) into the eigenvalue as $\lambda = z_s^2 m^2 - 2$, and changing variables to $\tilde{z}$, the equation takes the form

$$\psi'' - \left(\tilde{z}^2 + \frac{k^2 - \frac{1}{4}}{\tilde{z}^2}\right)\psi = -\lambda\psi, \tag{24}$$

where $k = 4$. Noting that $k^2 - \frac{1}{4} = (k + \frac{1}{2})(k - \frac{1}{2})$, we guess a solution of the form

$$\psi \propto \tilde{z}^{k+\frac{1}{2}} g(\tilde{z}) \mathrm{e}^{-\frac{\tilde{z}^2}{2}}, \tag{25}$$

where $g(\tilde{z})$ is a polynomial in $\tilde{z}$. This behaves similarly to the standard quantum harmonic oscillator wavefunctions at infinity, but also has the modification $\tilde{z}^{k+\frac{1}{2}}$ due to the $\tilde{z}^{-2}$ term in the potential. The $\tilde{z}^{k+\frac{1}{2}}$ term guarantees that the wavefunction will vanish at the origin. Furthermore, when we take the second derivative of the wavefunction, the term $(k + \frac{1}{2})(k - \frac{1}{2})\tilde{z}^{k-\frac{3}{2}} g(\tilde{z}) \mathrm{e}^{-\frac{\tilde{z}^2}{2}}$ cancels the $\tilde{z}^{-2}$ in the potential. With this ansatz, and making the substitution $u = \tilde{z}^2$, we obtain

$$ug'' + (k + 1 - u)g' - \left(\frac{k+1}{2} - \frac{\lambda}{4}\right) g = 0. \tag{26}$$

This is the second order differential equation corresponding to the associated Laguerre polynomials:

$$ug'' + (k + 1 - u)g' + (n - 1)g = 0, \tag{27}$$

where here n is a positive integer. Equating the coefficients for g we obtain the eigenvalues

$$\lambda = 4n + 2k + 2 \quad \Rightarrow \quad m^2 = \frac{1}{z_s^2}(4n + 2k + 4) \tag{28}$$

which correspond to the associated Laguerre polynomials $L^k_{n-1}(u) = L^k_{n-1}(\tilde{z}^2)$. As expected, the eigenvalues are linear in n, and the constant term in the potential gives rise to a linear shift in the mass-squared, as it would for the energies of any quantum mechanical potential. The normalized eigenfunctions for our case, $k = 4$, take the form

$$\psi_n = \tilde{z}^{\frac{9}{2}}\sqrt{\frac{2(n-1)!}{(n+3)!}}L^4_{n-1}(\tilde{z}^2)\mathrm{e}^{-\frac{\tilde{z}^2}{2}} \quad \text{where } n = 1, 2, 3, \ldots . \tag{29}$$

The first two of these, together with the potential, are shown in figure 4.

4. Conclusions and further elaborations

We have shown that the problem of choosing an appropriate 5D metric for hQCD models reduces to choosing a potential in a 1D Schrödinger equation. We then illustrated the method on examples from the literature with direct analogs to well-known quantum-mechanics problems (the free particle, infinite square well, and harmonic oscillator).

One could consider many elaborations on and generalizations of this technique. For instance:

(a) While our focus was hQCD, the method described could give students who have not studied general relativity intuition for the meaning of curved space. Minima of the potential corresponding to a given metric (or, equivalently, maxima of the ground state wavefunction) correspond to locations in spacetime where particles' energies are minimized. In the case of the asymptotically AdS spacetimes studied here, for instance, the metric pushes particles away from the boundary at $z = 0$. One can apply the same method to other spacetimes (like de Sitter, Schwarzschild, etc).

(b) We focused on scalar fields in the 5D that are dual to scalar hadrons in 4D. One can also study higher spin classical fields, dual to higher spin hadrons [4]. The essential difference will be the form of the potential (14) that appears in the Schrödinger-like equation, coming from additional factors of the metric that appear in the higher spin fields' action. For example, vector hadrons (like the ρ meson) correspond to massless vector fields in 5D. Their 5D kinetic terms go like $\partial_M A_N \partial_P A_Q g^{MP} g^{NQ}$, where the contraction of the vector field's index yields an extra factor of $A(z)^{-2}$.

(c) The hard wall model [3] differs from the infinite square well of quantum mechanics because it allows boundary conditions at $z = z_0$ other than $\psi(z_0) = 0$ (since the finite size of the z interval still allows the solution to be normalizable). In fact, maintaining certain symmetries of QCD—like isospin—requires boundary conditions like $\varphi'(z_0) = 0$ for certain fields. One can therefore explore the effect of different sets of boundary conditions (Neumann, Dirichlet, or mixed) on the spectrum.

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