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# CENTRALIZERS OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS IN DIMENSION 3

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Abstract. In this note we describe centralizers of volume preserving partially hyperbolic diffeomorphisms which are homotopic to identity on Seifert fibered and hyperbolic 3-manifolds. Our proof follows the strategy of Damjanovic, Wilkinson and Xu [10] who recently classified the centralizer for perturbations of time-1 maps of geodesic flows in negative curvature. We strongly rely on recent classification results in dimension 3 established in [5, 6].

1. **Introduction.** In [10], Damjanovic, Wilkinson and Xu investigate centralizers of certain partially hyperbolic diffeomorphisms and prove the following beautiful rigidity result: The centralizer of a perturbation of a time-1 map of an Anosov geodesic flow is either virtually $\mathbb{Z}$ or it is virtually $\mathbb{R}$. In the latter case the partially hyperbolic diffeomorphism is the time-1 map of a smooth Anosov flow.

The proof in [10] works equally well in any dimension. Here we point out that, if one considers only 3-manifolds, then some lemmas can be strengthened to obtain the rigidity result for a much broader class of partially hyperbolic diffeomorphisms.

For any diffeomorphism $f\colon M \to M$, we denote the centralizer of $f$ by

$$\mathcal{Z}(f) := \{g \in \mathrm{Diff}(M) \mid g \circ f = f \circ g\},$$

where $\mathrm{Diff}(M)$ is the space of $C^1$-diffeomorphisms of $M$.

We say that $f\colon M \to M$ is a *discretized Anosov flow* if $f$ is a partially hyperbolic diffeomorphism such that there exists a (topological) Anosov flow $\varphi^t\colon M \to M$ and a function $h\colon M \to \mathbb{R}^+$ such that $f(x) = \varphi^{h(x)}(x)$ for all $x \in M$[1].

In this note, the partially hyperbolic diffeomorphism $f$ is always assumed to be a $C^\infty$ diffeomorphism.

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[1]This is the same definition as in [5], see Appendix G of [5] for more details. Note that a discretized Anosov flow is a much broader class than what is called a discretized flow in [10], which is just a time-1 map of an Anosov flow.

**Theorem A.** *Let $f\colon M \to M$ be a volume-preserving partially hyperbolic diffeomorphism on a 3-manifold. If $f$ is a discretized Anosov flow and $\pi_1(M)$ is not virtually solvable then either $\mathcal{Z}(f)$ is virtually $\{f^n \mid n \in \mathbb{Z}\}$ or $f$ embeds into a smooth Anosov flow.*

Using the main results of [5, 6], we then deduce the following results.

**Theorem B.** *Let $f\colon M \to M$ be a volume-preserving partially hyperbolic diffeomorphism on a Seifert 3-manifold which is homotopic to the identity. Then either $\mathcal{Z}(f)$ is virtually $\{f^n \mid n \in \mathbb{Z}\}$ or $\mathcal{Z}(f)$ is virtually $\mathbb{R}$ and a power of $f$ embeds into an Anosov flow.*

**Theorem C.** *Let $f\colon M \to M$ be a volume-preserving dynamically coherent partially hyperbolic diffeomorphism on a hyperbolic 3-manifold. Then either $\mathcal{Z}(f)$ is virtually $\{f^n \mid n \in \mathbb{Z}\}$ or $\mathcal{Z}(f)$ is virtually $\mathbb{R}$ and a power of $f$ embeds into an Anosov flow.*

*Remark* 1.1. Note that Theorem B is a generalization of the 3-dimensional case of Theorem 3 of [10]. (One has to take a power of $f$ to obtain the embedding into an Anosov flow only in the case when $M$ is a $k$-cover of the unit tangent bundle of a hyperbolic surface or an orbifold, see Remark 7.4 in [5]).

*Remark* 1.2. The reason we exclude virtually solvable $\pi_1(M)$ in Theorem A is that, in this case, $f$ would be a discretized Anosov flow of a suspension of an Anosov diffeomorphism. Thus $f$ could fail to be accessible and the main motor of the proof, which is a dichotomy result by Avila, Viana and Wilkinson [2, 3], does not work. If one asks for $f$ to be accessible, then Theorem A will apply even on manifold with virtually solvable fundamental group.

In particular, any dynamically coherent, accessible, volume-preserving partially hyperbolic diffeomorphism $f$ on a manifold with virtually solvable fundamental group has centralizer virtually $\mathbb{Z}$ or virtually $\mathbb{R}$ in which case a power of $f$ embeds into an Anosov flow. (The proof follows as in section 3, but using the classification results of Hammerlindl and Potrie (see [12]) instead of [5, 6]).

As this note heavily relies on the arguments of [10] to obtain Theorem A, we did not try to make it self-contained and refer to [10] whenever an argument does not need a substantial change.

## 2. Proof of Theorem A.

Overall the proof follows the scheme of the proof of Theorem 3 of [10]. The difference is in the following lemmas which are more general (when considering the 3-dimensional case) than their counterparts in [10].

In the sequel we always assume that the fundamental group of $M$ is not virtually solvable.

For $f\colon M \to M$ a dynamically coherent partially hyperbolic diffeomorphism, we denote by $\mathcal{W}^s$, $\mathcal{W}^u$, $\mathcal{W}^{cs}$, $\mathcal{W}^{cu}$, and $\mathcal{W}^c$ the stable, unstable, center stable, center unstable and center foliations of $f$, respectively. Recall that the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ are unique, but, in general, the others are not. Thankfully, for discretized Anosov flow, they are unique.

**Lemma 2.1.** *Let $f\colon M \to M$ be a discretized Anosov flow. Then there exists a unique pair of center stable $\mathcal{W}^{cs}$ and center unstable $\mathcal{W}^{cu}$ foliations that are preserved by $f$. Hence $\mathcal{W}^c$ is also unique.*

*Proof.* Since $f$ is a discretized Anosov flow, it admits a pair of center stable and center unstable foliations such that a good lift $\widetilde{f}$ of $f$ to the universal cover $\widetilde{M}$ fixes

each leaf of the lifted foliations (see [5, Proposition G.1]). Thus, by [6, Lemma 7.6], these foliations are unique. □

As a direct consequence of Lemma 2.1, we obtain that, if $g \in \mathcal{Z}(f)$, then $g$ preserves each of the foliations $\mathcal{W}^*$, $* = c, s, u, cs, cu$.

Following [10], denote by $\mathcal{Z}^c(f)$ the subgroup of $\mathcal{Z}(f)$ consisting of elements which fix each leaf of the center foliation of $f$.

Recall that the *leaf space* of a foliation $\mathcal{W}^*$ is the space $\widetilde{M}/\widetilde{\mathcal{W}}^*$, where $\widetilde{\mathcal{W}}^*$ is the lifted foliation to the universal cover. A codimension 1 foliation is called $\mathbb{R}$-covered if its leaf space is homeomorphic to $\mathbb{R}$, and an Anosov flow is called $\mathbb{R}$-covered if its weak stable and weak unstable foliations are $\mathbb{R}$-covered (see, e.g., [4]).

Let $\mathrm{MCG}(M)$ be the mapping class group of $M$, defined as the homotopy classes of diffeomorphisms of $M$. Denote by $\mathcal{Z}_0(f)$ the kernel of the homomorphism $\mathcal{Z}(f) \to \mathrm{MCG}(M)$. We first note that, assuming that the Anosov flow associated to $f$ is transitive, then $\mathcal{Z}^c(f)$ is a subgroup of $\mathcal{Z}_0(f)$.

Indeed, consider an element $g$ of $\mathcal{Z}^c(f)$. Then the induced map $g_* \in \mathrm{Out}(\pi_1(M))$ fixes the conjugacy classes of elements represented by closed center leaves of $f$. These closed center leaves are the periodic orbits of the Anosov flow associated to $f$. Since this flow is assumed to be transitive, the periodic orbits generates the $\pi_1(M)$ ([1]). So $g_*$ is trivial on a generating set hence it is the identity in $\mathrm{Out}(\pi_1(M))$. Now, a standard obstruction theory argument shows that, when $M$ is aspherical (which is the case here, because $M$ is 3-dimensional and supports an Anosov flow), the map $\mathrm{MCG}(M) \to \mathrm{Out}(\pi_1(M))$ is injective. Thus $g \in \mathcal{Z}_0(f)$.

**Lemma 2.2.** *Let $f\colon M \to M$ be a discretized Anosov flow, and suppose that the corresponding Anosov flow $\varphi^t$ is transitive. Then, the group $\mathcal{Z}^c(f)$ has finite index in the kernel $\mathcal{Z}_0(f)$.*

*Proof.* Suppose that $g \in \mathcal{Z}_0(f)$. Since $f$ is a discretized Anosov flow, its center foliation $\mathcal{W}^c$ is the orbit foliation of a topological Anosov flow $\varphi^t$ (cf. [5, Proposition G.1]). By the preceding lemma $g$ preserves the foliation $\mathcal{W}^c$. Thus the map $g$ is a self orbit equivalence of the transitive Anosov flow $\varphi^t$ which is homotopic to the identity. Therefore Theorem 1.1 of [7] applies to $g$.

Then, either $g \in \mathcal{Z}^c(f)$ or (see case 4 of [7, Theorem 1.1]) $\varphi^t$ is $\mathbb{R}$-covered and there exists a map $\eta\colon M \to M$, homotopic to identity, and an integer $i$, $i \neq 0$, such that $g \circ \eta^i$ fixes every leaf of $\mathcal{W}^c$. More precisely, if $\widetilde{g}$ and $\widetilde{\eta}$ are lifts to $\widetilde{M}$ obtained by lifting the homotopies to identity, then $\widetilde{g} \circ \widetilde{\eta}^i$ fixes every leaf of the lifted foliation $\widetilde{\mathcal{W}}^c$. Equivalently, the map $\widetilde{g} \circ \widetilde{\eta}^i$ acts as the identity on the orbit space of the Anosov flow (this is how the integer $i$ is found, see [7, Section 2]).

Since $g$ is at least $C^1$, if $g \notin \mathcal{Z}^c(f)$, then $g$ defines a non-trivial $C^1$ action on the weak-stable leaf space of the Anosov flow $\varphi^t$, and thus, by [4, Proposition 6.6], $\varphi^t$ is orbit equivalent to a finite cover of the geodesic flow on a (orientable) hyperbolic surface or orbifold $\Sigma$. That is, we are in case 4b of Theorem 1.1 of [7]. So, there exists $k \in \mathbb{N}$ such that $\eta^k = \mathrm{Id}$ and the integer $i$ above is uniquely defined modulo $k$.

Thus, we obtained a map $\mathcal{Z}_0(f) \ni g \mapsto i \in \mathbb{Z}/k\mathbb{Z}$. This map is a homomorphism since if $g_1, g_2 \in \mathcal{Z}_0(f)$, then the action of $\widetilde{g}_1 \circ \widetilde{g}_2$ on the orbit space of the Anosov flow corresponds to $\widetilde{\eta}^{-i_1} \circ \widetilde{\eta}^{-i_2} = \widetilde{\eta}^{-i_1-i_2}$. Moreover, the kernel of the map $\mathcal{Z}_0(f) \ni g \mapsto i \in \mathbb{Z}/k\mathbb{Z}$ is, by the above, exactly $\mathcal{Z}^c(f)$.

Therefore, we found an injective homomorphism $\mathcal{Z}_0(f)/\mathcal{Z}^c(f) \ni [g] \mapsto i \in \mathbb{Z}/k\mathbb{Z}$. So $\mathcal{Z}_0(f)/\mathcal{Z}^c(f)$ is finite. □

**Lemma 2.3.** *Let $f\colon M \to M$ be a discretized Anosov flow. Then for any $g \in \mathcal{Z}(f)$ and any closed center leaf $\mathcal{W}^c(x)$, there exists $k \geq 1$ such that*

$$g^k\left(\mathcal{W}^c(x)\right) = \mathcal{W}^c(x).$$

*Proof.* This is essentially the same proof as Lemma 17 in [10], but we rewrite it since we state it in a different setting.

Let $\varphi^t\colon M \to M$ be the topological Anosov flow and $h\colon M \to \mathbb{R}^+$ be the continuous function such that $f(x) = \varphi^{h(x)}(x)$. We fix a metric on $M$ such that the orbits of $\varphi^t$ have unit speed.

Let $g \in \mathcal{Z}(f)$. Let $\tilde{\varphi}^t$, $\widetilde{f}$ and $\widetilde{g}$ be lifts of $\varphi^t$, $f$ and $g$ to the universal cover $\widetilde{M}$. We choose $\tilde{\varphi}^t$ and $\widetilde{f}$ to be lifts which fix each leaf of the lifted center foliation $\widetilde{\mathcal{W}}^c$ (= the flow foliation of $\tilde{\varphi}^t$). If $g$ reverses the orientation of the orbits of $\varphi^t$, then we replace $g$ by $g^2$. Thus we can assume that $\widetilde{g}$ preserves the ordering of points on any orbit of $\tilde{\varphi}^t$.

Recall that all orbits of $\tilde{\varphi}^t$ are lines. Hence a closed center leaf $\mathcal{W}^c(x)$ lifts to an orbit segment $[x, \widetilde{\varphi}^T(x)]$, $T > 0$ (where we write $x$ for both the point $x \in M$ and a lift of it to the universal cover $\widetilde{M}$). The orbit of $x$ under $\widetilde{f}$ is an increasing sequence of points. Hence, there exists a unique $N \geq 0$ such that $\widetilde{\varphi}^T(x)$ belongs to the orbit segment $(\widetilde{f}^N x, \widetilde{f}^{N+1}x]$. Then, for any $m \geq 1$, the points $\widetilde{g}^m\widetilde{\varphi}^T x$ belongs to the orbit segment $(\widetilde{g}^m(\widetilde{f}^N x), \widetilde{g}^m(\widetilde{f}^{N+1}x)] = (\widetilde{f}^N(\widetilde{g}^m x), \widetilde{f}^{N+1}(\widetilde{g}^m x)]$.

The center leaf $\mathcal{W}^c(g^m x)$ lifts to the orbit segment $[\widetilde{g}^m x, \widetilde{g}^m(\widetilde{\varphi}^T x)]$. By the above discussion we have $[\widetilde{g}^m x, \widetilde{g}^m(\widetilde{\varphi}^T x)] \subset [\widetilde{g}^m x, \widetilde{f}^{N+1}(\widetilde{g}^m x)]$. Hence the length of $\mathcal{W}^c(g^m x)$ is bounded by $C = (N+1)\max(h)$. Note that this bound is uniform in $m$.

Since there are only finitely many closed center leaves of length less than $C$, it follows that every closed center leaf is $g$-periodic. □

**Proposition 2.4.** *Let $f\colon M \to M$ be a discretized Anosov flow, and suppose that the Anosov flow $\varphi^t$ is transitive. Then $\mathcal{Z}(f)/\mathcal{Z}^c(f)$ is finite.*

*Proof.* By Lemma 2.2, since $\mathcal{Z}^c(f)$ has finite index in $\mathcal{Z}_0(f)$, it is sufficient to show that $\mathcal{Z}(f)/\mathcal{Z}_0(f)$ is finite, which we now proceed to do.

Let $g \in \mathcal{Z}(f)$. By Lemma 2.3, every closed center leaf in $\mathcal{W}^c$ is periodic under $g$. Now recall that each closed center leaf is a periodic orbit of the transitive Anosov flow $\varphi^t$. By [1], the (conjugacy classes of) closed orbits of the transitive Anosov flow $\varphi^t$ generate the fundamental group of $M$. Thus we can choose a generating set of closed orbits and choose $n$ large enough so that $g^n$ fixes each closed center leaf in the generating set of conjugacy classes of $\pi_1(M)$.

This implies that the element $[g^n_*] \in \mathrm{Out}(\pi_1(M))$ is the identity of the outer automorphism group of $\pi_1(M)$.

Thus $g^n$, seen as an element of $\mathrm{MCG}(M)$, is in the kernel of the homomorphism $\mathrm{MCG}(M) \to \mathrm{Out}(\pi_1(M))$.

As we recalled earlier, a standard obstruction theory argument shows that the map $\mathrm{MCG}(M) \to \mathrm{Out}(\pi_1(M))$ is injective, because $M$ is aspherical. Thus $g^n$ is the identity in $\mathrm{MCG}(M)$. Hence, we conclude that $\mathcal{Z}(f)/\mathcal{Z}_0(f)$ is a torsion subgroup of $\mathrm{MCG}(M)$.

Now, since $M$ is an irreducible 3-manifold, $\mathrm{MCG}(M)$ is virtually torsion free (see section 5 of [13])[2]. Thus, $\mathcal{Z}(f)/\mathcal{Z}_0(f)$ must be finite, since it is a torsion subgroup of $\mathrm{MCG}(M)$. □

Now that we obtained Proposition 2.4, we can copy verbatim the proof of Theorem 5 of [10] and obtain the following result that will allow us to deduce Theorem A.

**Theorem 2.5.** *Let $f$ be a discretized Anosov flow on a 3-manifold $M$ such that $\pi_1(M)$ is not virtually solvable. Suppose that $f$ preserves a volume* Vol *on $M$. Then either* Vol *has Lebesgue disintegration along $\mathcal{W}^c$ or $f$ has virtually trivial centralizer in* $\mathrm{Diff}(M)$.

*Proof.* As $\pi_1(M)$ is not virtually solvable, by [11, Theorem C], $f$ is accessible. Because $f$ is volume preserving, it is, thus, transitive ([8]). Hence there exists a center leaf which is dense in $M$, which implies that the Anosov flow $\varphi^t$ is also transitive. So Proposition 2.4 apply.

We have that $\mathcal{Z}(f)$ is virtually $\mathcal{Z}^c(f)$. Moreover, $f$ is ergodic (because it is accessible, so it is ergodic by [17, 9]) and all the elements of $\mathcal{Z}(f)$ are volume preserving (see [10, Lemma 11]).

From the the proof of Theorem F of [3] (see section 10.3 of [3]) we have the following lemma.

**Lemma 2.6.** *Under the asssumptions of Theorem 2.5, if* Vol *has singular disintegration along the leaves of $\mathcal{W}^c$, then there exists $k \geq 1$ and a full measure set $S \subset M$ that intersects every center leaf in exactly $k$ orbits of $f$.*

Using this lemma, we can now copy verbatim the proof of Theorem 5 in [10] (replacing $T^1X$ with $M$) to obtain Theorem 2.5. □

*Proof of Theorem A.* If Vol has singular disintegration along the leaves of $\mathcal{W}^c$, then the conclusion of Theorem A follows from Theorem 2.5.

Otherwise, by Theorem F of [3], $\mathcal{W}^c$ is absolutely continuous and $f = \psi^1$, where $\psi^t \colon M \to M$ is a smooth volume preserving Anosov flow. In particular, $\{\psi^t \mid t \in \mathbb{R}\} \subset \mathcal{Z}(f)$.

Now, if $g \in \mathcal{Z}^c(f)$, then, by ergodicity of $f$, the map $g$ preserves Vol, and, hence, it preserves the disintegration of Vol along $\mathcal{W}^c$. Thus $g = \psi^t$ for some $t \in \mathbb{R}$.

So $\{\psi^t \mid t \in \mathbb{R}\} = \mathcal{Z}^c(f)$ and Theorem A follows from Proposition 2.4. □

3. **Proofs of Theorems B and C.** The two main results of [5, 6] state that, if $f \colon M \to M$ is a partially hyperbolic diffeomorphism such that, either $f$ is homotopic to the identity and $M$ is Seifert, or that $f$ is dynamically coherent and $M$ is hyperbolic, then there exists $k \geq 1$ such that $f^k$ is a discretized Anosov flow.

Since $\mathcal{Z}(f) \subset \mathcal{Z}(f^k)$, we immediately deduce from Theorem A that, under the assumptions of Theorem B or Theorem C, either $\mathcal{Z}(f)$ is virtually $\{f^n \mid n \in \mathbb{Z}\}$ or $\mathcal{Z}(f^k)$ is virtually $\mathbb{R}$ and $f^k$ embeds into an Anosov flow for some $k \geq 1$.

Thus, in order to finish proving Theorems B and C, we only need to show that if $f^k$ is the time-1 map of an Anosov flow which is transitive on a Seifert or hyperbolic manifold, then the centralizer of $f$ is virtually $\mathbb{R}$.

This last step is given by the next lemma, which is in fact more general.

[2]Note that McCullough [14] proved that $\mathrm{MCG}(M)$ is virtually torsion free for Haken manifolds and it follows from Mostow Rigidity Theorem for hyperbolic manifolds, which are the only two cases we need, since, as $M$ supports an Anosov flow, it is either Haken or hyperbolic.

**Lemma 3.1.** *Suppose that $f^k$ is the time-1 map of a transitive Anosov flow that is not a constant roof suspension of an Anosov diffeomorphism. Then $\mathcal{Z}(f)$ is virtually $\mathbb{R}$.*

Recall that suspensions of Anosov diffeomorphisms are on solvmanifolds. In particular Seifert and hyperbolic manifolds do not support suspensions of Anosov diffeomorphisms, so Lemma 3.1 apply in the setting of Theorems B and C.

In order to prove Lemma 3.1, we first need a result about topologically weak-mixing Anosov flows.

**Lemma 3.2.** *Let $\varphi^t\colon M \to M$ be a topologically weak-mixing Anosov flow, then, for every $n > 0$, the set of periodic orbits of $\varphi^t$ that have period* not *a multiple of $1/n$ is dense in $M$*

*Proof.* This is a simple consequence of the spatial equidistribution of orbits of periods between $T$ and $T + \varepsilon$ for weak-mixing Anosov flow.

We let $\mathcal{P}$ be the set of periodic orbits of $\varphi^t$. For any $\gamma \in \mathcal{P}$, we let $\ell(\gamma)$ be the minimal period of $\gamma$. For any map $K\colon M \to \mathbb{R}$ that is continuous along the orbits of $\varphi^t$, and any $\varepsilon > 0$, we have, by [16, Proposition 7.3],

$$\frac{\sum_{T<\ell(\gamma)\leq T+\varepsilon} \int_\gamma K}{\sum_{T<\ell(\gamma)\leq T+\varepsilon} \ell(\gamma)} \to \int_M K d\mu_{BM}, \quad \text{as} \quad T \to +\infty,$$

where $\mu_{BM}$ is the measure of maximal entropy of $\varphi^t$.

Let $n > 0$ be fixed and let $\mathcal{P}_{\notin \frac{1}{n}\mathbb{Z}}$ be the set of periodic orbits of period not a multiple of $1/n$. If $\overline{\mathcal{P}_{\notin \frac{1}{n}\mathbb{Z}}} \neq M$ then there would exists an open set $U$ that is missed by the orbits in $\mathcal{P}_{\notin \frac{1}{n}\mathbb{Z}}$. Taking $K$ to be a smooth approximation of the characteristic function of $U$ and $\varepsilon < 1/n$, we would get that the left hand side of the above equation is zero along a subsequence, while the right hand side is strictly positive, as the measure of maximal entropy has full support. A contradiction. □

*Proof of Lemma 3.1.* Let $\varphi^t\colon M \to M$ be the Anosov flow such that $f^k = \varphi^1$, we will show that $f$ itself commutes with $\varphi^t$ for any $t \in \mathbb{R}$ which will prove the claim (since $\mathcal{Z}(f) \subset \mathcal{Z}(f^k)$ and $\mathcal{Z}(f^k)$ is virtually $\{\varphi^t \mid t \in \mathbb{R}\}$).

Since $f^k = \varphi^1$, we have that, for any $m \in \mathbb{Z}$ and any $x \in M$,

$$f(\varphi^m(x)) = \varphi^m(f(x)).$$

Now consider a periodic orbit $\gamma$ of $\varphi^t$.

**Claim 3.3.** *Let $l$ be the period of the orbit $\gamma$, then $f(\gamma)$ is a periodic orbit of period $l$.*

*Moreover, if $l$ is irrational, then, for any $x \in \gamma$ and any $t \in \mathbb{R}$, we have*

$$f(\varphi^t(x)) = \varphi^t(f(x)).$$

*On the other hand if $l$ is rational, say $l = p/n$, $\gcd(p, n) = 1$, then, for any $x \in \gamma$ and any $m \in \mathbb{Z}$, we have*

$$f(\varphi^{m/n}(x)) = \varphi^{m/n}(f(x)).$$

*Proof of Claim 3.3.* Since $f$ preserves the center foliation, $f(\gamma)$ is also a periodic orbit of $\varphi$.

Then, since $f^k = \varphi^1$, the diffeomorphism $f$ restricts to a conjugacy between the circle maps $\varphi^1|_\gamma$ and $\varphi^1|_{f(\gamma)}$ considered as diffeomorphisms of the circle. Since the rotation number is a conjugacy invariant, we deduce that $\varphi^1|_\gamma$ and $\varphi^1|_{f(\gamma)}$ have the

same rotation numbers, which implies (since $\varphi^1$ is the time-1 map of a flow) that the period of $\gamma$ and $f(\gamma)$ are both equal to $l$. So the first part of the claim is proven.

Now suppose that the period $l$ is irrational. Then for any $t \in \mathbb{R}$, and any point $y \in \gamma$ or $y \in f(\gamma)$, the sequence $\varphi^{n_k}(y)$ converges to $\varphi^t(y)$ if and only if the sequence $n_k$ converges to $t$ modulo $l$.

Pick a point $x \in \gamma$, $t \in \mathbb{R}$ and an integer sequence $(n_k)$ such that $\varphi^{n_k}(x)$ converges to $\varphi^t(x)$. Then, using continuity of $f$ and the fact that $f$ commutes with $\varphi^1$, we deduce that $\varphi^{n_k}(f(x)) = f(\varphi^{n_k}(x))$ converges to $f(\varphi^t(x))$.

Since by the above we also have that $\varphi^{n_k}(f(x))$ converges to $\varphi^t(f(x))$, we deduce that $f(\varphi^t(x)) = \varphi^t(f(x))$, as claimed.

If $l = p/n$ is rational, then if an integer $k$ is equal to $m/n$ modulo $l$, we get

$$f(\varphi^{m/n}(x)) = f(\varphi^k(x)) = \varphi^k(f(x)) = \varphi^{m/n}(f(x)).$$

This completes the proof of the claim. □

Now take any $x \in M$. We approximate $x$ by a sequence $y_i \to x$, $i \to \infty$, such that each $y_i$ is periodic. Further we choose $y_i$ in such a way that the period of $y_i$ are either irrational or rational of the form $p_i/q_i$, $\gcd(p_i, q_i) = 1$, with $q_i > i$. Such choice is possible by Lemma 3.2. By the above discussion we have

$$f(\varphi^t(y_i)) = \varphi^t(f(y_i))$$

either for every $t$ or for a $1/i$-dense set of $t$'s. In either case for any $t \in \mathbb{R}$ we can choose a sequence $t_i \to t$, $i \to \infty$, such that $f(\varphi^{t_i}(y_i)) = \varphi^{t_i}(f(y_i))$. Passing to a limit as $i \to \infty$ yields $f(\varphi^t(x)) = \varphi^t(f(x))$. Thus $\{\varphi^t \mid t \in \mathbb{R}\} \subset \mathcal{Z}(f)$, which proves the lemma. □

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