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Data-Driven Synthesis of Optimization-Based Controllers for Regulation of Unknown Linear Systems

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Abstract— **This paper proposes a data-driven framework to solve time-varying optimization problems associated with unknown linear dynamical systems. Making online control decisions to regulate a dynamical system to the solution of an optimization problem is a central goal in many modern engineering applications. Yet, the available methods critically rely on a precise knowledge of the system dynamics, thus mandating a preliminary system identification phase before a controller can be designed. In this work, we leverage results from behavioral theory to show that the steady-state transfer function of a linear system can be computed from data samples without any knowledge or estimation of the system model. We then use this data-driven representation to design a controller, inspired by a gradient-descent optimization method, that regulates the system to the solution of a convex optimization problem, without requiring any knowledge of the time-varying disturbances affecting the model equation. Results are tailored to cost functions satisfy the Polyak-Łojasiewicz inequality.**

I. Introduction

Online optimization problems have attracted significant attention in various disciplines, including machine learning [1], control systems [2], [3], and transportation management [4]. When used as a controller for dynamical systems, an online optimization method seeks to make control decisions at every time instant to minimize a loss function that is time-varying and possibly uncertain as described by the system dynamics. The vast majority of works on online optimization for dynamical systems make a strict assumption on the knowledge of the underlying system dynamics (see e.g. [2], [5]–[7]). However, besides their theoretical value, maintaining and refining full system models is often undesirable because: (i) perfect knowledge of the dynamics is rarely available in practice since it requires explicit system identification and periodic model re-updates, and (ii) identifying a full model of the system is often unnecessary since most optimization-inspired controllers rely on simpler representations of the dynamical system. To the best of our knowledge, efficient and numerically reliable online optimization methods that bypass the model identification phase are still lacking.

In this work, we take a novel approach to design online optimization controllers that relies on Willems' fundamental lemma [8] to construct a data-driven representations of the dynamical system to control, which is then used for algorithm synthesis. Precisely, we assume the availability of noise-free historical data, i.e., finite-length trajectories produced by the open-loop dynamics, and we show that the steady-state transfer function of a linear system can be computed from non steady-state input-output data. The noise-free assumption corresponds to scenarios where accurate sensors or signal-processing techniques can be utilized offline to recover noise-free sample trajectories (a similar approach was taken in [9], [10]). Then, we build upon such data-driven representation to propose a controller inspired by online optimization methods that regulates the dynamical system to an equilibrium point that minimizes a prescribed loss function despite unknown and time-varying noise terms affecting the model equation. Interestingly, our results suggest that a suitable choice of the controller stepsize is sufficient to guarantee asymptotic convergence to the desired optimizers, up to an asymptotic error that is bounded by the time-variability of the exogenous noise terms.

Related Work. The results presented here are tied to the fields of data-driven control and online optimization. The success of data-driven control methods mainly originates from the possibility of synthesizing controllers without first identifying a full system model. Among these methods, the behavioral framework has recently regained considerable attention [8], [11]. Recent extensions include distributed formulations [12], combinations with model predictive control [9], [13], trajectory tracking [10], and nonlinear systems [14], [15]. In this work, we leverage the behavioral framework to build a data-driven representation of a dynamical system that is the used for optimization purposes. Especially relevant to this work are the recent results [10], [16]–[18] that focus on the presence of noise in the data.

Online optimization approaches aim to optimize loss functions that depends on the state of an underlying and uncertain dynamical system. Linear time-invariant systems are considered in e.g., [2]–[4], [6], stable nonlinear systems in [5], [19], and switched systems in [7]. In contrast with the above line of work, which considers continuous-time dynamics, the focus of this paper is on systems and controllers that operate at discrete time. Although data-driven online optimization methods are studied in the recent work [20], results are however limited to the absence of noise and regret analysis.

Contributions. This work features two main contributions. First, we show that the steady-state transfer function of a linear time-invariant dynamical system can be obtained from historical (non steady-state) input-output trajectories generated by the open-loop dynamics, without any knowledge or estimation of the system parameters. Interestingly, our results also suggest that the steady-state transfer function can

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be computed exactly from input-output data even when the trajectories are affected by constant noise terms. Our work offers contributions with respect to the vast majority of the available literature on data-driven control, which considers disturbances affecting only the output equation (see e.g. [16], [18]), by accounting for the presence of disturbance terms affecting the model equation. Second, we propose a controller inspired from online optimization methods, which steers the dynamical system to one solution of a time-varying convex optimization problem. We prove convergence of the system with controller to the desired optimal points; precisely, we show input-to-state stability (ISS) of the controlled dynamical system with respect to exogenous disturbances affecting the dynamics. Our results build upon the theory of ISS Lyapunov functions for discrete-time dynamical systems [21], properly modified to guarantee stability with respect to compact sets of optimizers [22].

II. Preliminaries

We first outline the notation and recall few basic concepts. **Notation.** Given a symmetric matrix $M \in \mathbb{R}^{n\times n}$, $\underline{\lambda}(M)$ and $\bar{\lambda}(M)$ denote the smallest and largest eigenvalue of M, respectively; $M \succ 0$ indicates that M is positive definite. For vectors $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, $(x, u) \in \mathbb{R}^{n+m}$ denotes their concatenation. We denote by $\|u\|$ the Euclidean norm of u; $u^\top$ denotes transposition; given nonempty compact sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$, $|u|_{\mathcal{A}} = \inf_{z\in\mathcal{A}} \|z - u\|$ denotes the point-to-set distance, while $\operatorname{dist}(\mathcal{A}, \mathcal{B}) := \max\{\sup_{x\in\mathcal{A}} \inf_{y\in\mathcal{B}} \|x - y\|, \sup_{y\in\mathcal{B}} \inf_{x\in\mathcal{A}} \|x - y\|\}$ denotes the Hausdorff distance.

A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is strictly increasing in its first argument, decreasing in its second argument, $\lim_{r\to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$, and it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and, in addition, $\lim_{r\to\infty} \gamma(r) = \infty$.

Persistency of Excitation. We recall some useful facts on behavioral system theory from [8]. For a signal $k \mapsto z_k \in \mathbb{R}^\sigma$, $k \in \mathbb{Z}_{\geq 0}$, we denote by $z_{[k,k+T]}$, $k \in \mathbb{Z}$, $T \in \mathbb{Z}_{\geq 0}$, the vectorization of z restricted to the interval $[k, k+T]$, namely,

$$z_{[k,k+T]} = (z_k, \ldots, z_{k+T}).$$

Given $z_{[0,T-1]}$, $t \leq T$, and $q \leq T - t + 1$, we let $Z_{t,q}$ denote the Hankel matrix of length t associated with $z_{[0,T-1]}$:

$$Z_{t,q} = \begin{bmatrix} z_0 & z_1 & \cdots & z_{q-1} \\ z_1 & z_2 & \cdots & z_q \\ \vdots & \vdots & \ddots & \vdots \\ z_{t-1} & z_t & \cdots & z_{q+t-2} \end{bmatrix} \in \mathbb{R}^{\sigma t\times q}.$$

The signal $z_{[0,T-1]}$ is persistently exciting of order t if $Z_{t,q}$ has full row rank; that is, $\operatorname{rank}(Z_{t,q}) = \sigma t$. Notice that persistency of excitation implicitly requires $q \geq \sigma t$ and, consequently, $T \geq (\sigma + 1)t - 1$.

The linear dynamical system

$$x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k + Du_k, \qquad (1)$$

$x \in \mathbb{R}^n$, is controllable if $\mathcal{C} := [B, AB, A^2B, \ldots, A^{n-1}B]$ satisfies $\operatorname{rank}(\mathcal{C}) = n$. We recall the following two properties of (1) when its input is persistently exciting.

Lemma 2.1: ***(Fundamental Lemma)*** [8, Corollary 2] Assume (1) is controllable, let $(u_{[0,T-1]}, x_{[0,T-1]})$, $T \in \mathbb{Z}_{>0}$, be an input-state trajectory of (2). If $u_{[0,T-1]}$ is persistently exciting of order $n + L$, then:

$$\operatorname{rank} \begin{bmatrix} U_{L,q} \\ X_{1,q} \end{bmatrix} = Lm + n,$$

where $U_{L,q}$ and $X_{1,q}$ denote the Hankel matrices associated with $u_{[0,T-1]}$ and $x_{[0,T-1]}$, respectively. □

Lemma 2.2: ***(Data Characterizes Full Behavior [8, Theorem 1])*** Assume (1) is controllable and observable, let $(u_{[0,T-1]}, y_{[0,T-1]})$, $T \in \mathbb{Z}_{>0}$, be an input-output trajectory generated by (2). If $u_{[0,T-1]}$ is persistently exciting of order $n + L$, then any pair of L-long signals $(\tilde{u}_{[0,L-1]}, \tilde{y}_{[0,L-1]})$ is an input-output trajectory of (2) if and only if there exists $\alpha \in \mathbb{R}^q$ such that:

$$\begin{bmatrix} \tilde{u}_{[0,L-1]} \\ \tilde{y}_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} U_{L,q} \\ Y_{L,q} \end{bmatrix} \alpha,$$

where $U_{L,q}$ and $Y_{L,q}$ denote the Hankel matrices associated with $u_{[0,T-1]}$ and $y_{[0,T-1]}$, respectively. □

In words, persistently exciting signals generate output trajectories that can be used to express any other trajectory.

III. Problem Formulation

We consider discrete-time linear time-invariant dynamical systems subject to state noise, described by:

$$x_{k+1} = Ax_k + Bu_k + Ew_k, \qquad y_k = Cx_k, \qquad (2)$$

where $k \in \mathbb{Z}_{\geq 0}$ is the time index, $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ describes the control decision, $w_k \in \mathbb{R}^r$ denotes an unknown exogenous input or disturbance, which is assumed to be bounded at all times, $y_k \in \mathbb{R}^n$ is the measurable output, and A, B, C, E are matrices of suitable dimensions. We assume that any equilibrium point of (2) is asymptotically stable, as formalized next.

Assumption 1: ***(Stability of Plant)*** The matrix A is Schur stable, i.e., for any $Q \succ 0$ there exists $P \succ 0$ such that $A^\mathsf{T}PA - P = -Q$. Moreover, (2) is controllable and the columns of C are linearly independent. □

Remark 1: ***(Linear Independence Columns of*** C***)*** Linear-independence of the columns of C implies that the state of (2) can be fully determined at every time given the system output. This assumption can be relaxed, as shown in [23]. □

Under Assumption 1, for any fixed $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^r$, the system (2) admits a well-posed steady-state input-output relationship, given by:

$$y = \underbrace{C(I - A)^{-1}B}_{:=G}\, u + \underbrace{C(I - A)^{-1}E}_{:=H}\, w. \qquad (3)$$

We focus on cases where the matrices (A, B, C, E) are unknown, and we consider the problem of regulating (2) to an equilibrium point that is specified as the solution of the following optimization problem:

$$u_k^* \in \arg\min_{\bar{u}} \quad \phi(\bar{u}) + \psi(G\bar{u} + Hw_k), \tag{4}$$

where $\phi : \mathbb{R}^m \to \mathbb{R}$ and $\psi : \mathbb{R}^p \to \mathbb{R}$ are given cost functions describing losses associated withe the system input and output, respectively. We note that, at every time k, the value of the objective function in (4) is unknown and time-varying, since it depends on the unknown matrices (A, B, C, E) and on the unknown and time-varying disturbance w_k. The optimization problem (4) formalizes an optimal regulation problem, where the goal is to regulate (2) to an optimal equilibrium point, as described by the cost in (4).

We make the following assumption on the costs in (4).

Assumption 2: ***(Lipschitz Smoothness and PL)*** The functions $u \mapsto \phi(u)$ and $y \mapsto \psi(y)$ are differentiable and have Lipschitz-continuous gradients with constants ℓ_ϕ, ℓ_ψ, respectively. Moreover, $f(u) := \phi(u) + \psi(Gu + Hw)$ is radially-unbounded, has a nonempty set of minimizers, and satisfies the Polyak-Łojasiewicz (PL) inequality, i.e., there exists $\mu > 0$ such that $\frac{1}{2}\|\nabla f(u)\|^2 \geq \mu(f(u) - f(u^*))$, for all $u \in \mathbb{R}^m$ and all minimizers u^*. □

Lipschitz-continuity assumptions are commonly used for the study of first-order optimization methods (see e.g. [24]). The PL inequality allows us to guarantee that every critical point of (4) is a global minimizer. Also, we note that the PL condition is a weaker assumption than strong convexity (in particular, it implies invexity), and has been widely adopted to study convergence of optimization algorithms [24].

We seek to synthesize a controller that does not require any prior knowledge of the matrices (A, B, C, E) as well as of the disturbance w_k, with the following structure:

$$u_{k+1} = F_{\mathrm{c}}(u_k, y_k), \tag{5}$$

that guarantees that (2) tracks the optimizers of the optimization problem (4) (see Fig. 1 for an illustration). Two important observations are in order. First, because the exogenous input w_k is time-varying, the optimizers of (4) are also time-varying. Formally, by denoting by $\mathcal{U}_k^* := \{u_k^* : 0 = \nabla\phi(u_k^*) + \nabla\psi(Gu_k^* + Hw_k)\}$ the set of optimizers of (4) at time k, in general we have that $\mathcal{U}_{k+1}^* \neq \mathcal{U}_k^*$. Second, because we assume no prior knowledge on w_k, any controller of the form (5) can track the solutions of (4) up to an error that depends on the time-variability of $w_{k+1} - w_k$. For these reasons, we aim at guaranteeing that the output of the system (2) satisfies a tracking bound of the form:

$$\begin{aligned} |\xi_k - \xi_k^*|_{\mathcal{U}_k^*} &\leq \beta(|\xi_{k_0} - \xi_{k_0}^*|_{\mathcal{U}_{k_0}^*}, k - k_0) \\ &+ \gamma_u(\sup_{t \geq k_0} \mathrm{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)) + \gamma_w(\sup_{t \geq k_0} \|w_{t+1} - w_t\|) \end{aligned} \tag{6}$$

for all $0 \leq k_0 \leq k$, where $\xi_k := (x_k, u_k)$ denotes the joint system-controller state, $\xi_k^* := ((I - A)^{-1}(Bu_k^* + Ew_k), u_k^*)$ denotes the optimizer of (4) at time k, β is a class-$\mathcal{KL}$ function, γ_u, γ_w are class-$\mathcal{K}$ functions, and $\mathrm{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ denotes the Hausdorff distance [25] between the compact sets $\mathcal{U}_{t+1}^*$ and $\mathcal{U}_t^*$. Observe that, because the cost function is radially-unbounded (see Assumption 2), the set of optimizers of (4) is compact and thus the Hausdorff distance $\mathrm{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ is finite for all k [25].

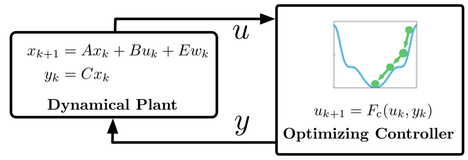

Fig. 1. Online gradient-flow optimizer used as an output feedback controller for unknown LTI systems subject to time-varying disturbances.

Remark 2: ***(Time-Varying Optimizers)*** We notice that although $\mathrm{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ may be reconducted to the time-variability of w_k, in general (6) reflects more accurately the role of individual error terms, since the precise relationship between $w_{k+1} - w_k$ and $\mathrm{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ is unknown for general cases. Furthermore, in cases where the loss $\phi(u)$ and $\psi(y)$ are time-varying, the term $\mathrm{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ can be used to account for the time-variability of the optimizers. □

IV. Data-Driven Method for Online Optimization

To track the optimizers of (4), we propose the following controller inspired from an online gradient method:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ew_k, \quad y_k = Cx_k, \\ u_{k+1} &= u_k - \eta(\nabla\phi(u_k) + G^{\mathsf{T}}\nabla\psi(y_k)), \end{aligned} \tag{7}$$

where $\eta \in \mathbb{R}_{>0}$ is a tunable controller parameter (see Fig. 1). We note that the controller in (7) does not rely on any knowledge of the system matrices (A, B, C, E), instead, it requires an exact expression for the map G. In order to implement (7), we propose a two-phase control method, where data samples are used to determine G, and then the feedback controller (7) is used to track the optimizers of (4). Fig. 2 illustrates the two phases. Similar approaches using data recorded offline were proposed in [10] for output tracking as well as in [9] for model predictive control.

A. Data-Driven Characterization of the Transfer Function

The following result shows that, when $w_k = 0$ at all times, the steady-state transfer function G can be computed from a (non-steady-state) sample trajectory of the system.

Theorem 4.1: ***(Data-Driven Characterization of Transfer Function in the Absence of Noise)*** Let Assumption 1 hold, let $u_{[0,T-1]}$ be persistently exciting of order $n+1$ and assume $W_{1,T} = 0$. Then, there exists $M \in \mathbb{R}^{q \times m}$ such that:

$$Y_{1,T}^{\mathrm{diff}} M = 0, \qquad U_{1,T} M = I, \tag{8}$$

where $Y_{1,T}^{\mathrm{diff}} = [y_1 - y_0,\ y_2 - y_1,\ \ldots\ y_T - y_{T-1}]$. Moreover, for any M that satisfies (8), the steady-state transfer function of (2) equals $G = Y_{1,T}M$. □

The proof is presented in [23]. Theorem 4.1 asserts that G can be computed from sample data originated from (2), by solving the set of linear equations (8). Two important observations are in order. First, the matrix M that satisfies (8) depends on the sample data $u_{[0,T-1]}$. Second, given a sample sequence $u_{[0,T-1]}$, in general, there exists an infinite number of choices of M that satisfy (8). Despite M not being unique,

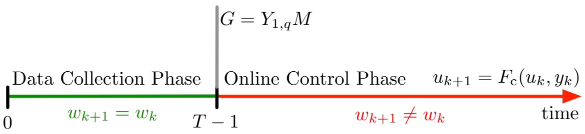

Fig. 2. Two-phase control method where noise-free historical data is used to compute the map G, and then a dynamical controller is used to track the optimizers of (4) despite unknown and time-varying disturbances.

Theorem 4.1 shows that $Y_{1,q}M$ is unique and independent of the choice of $u_{[0,T-1]}$ used to generate the data. It is worth noting that Theorem 4.1 requires one to collect T samples of the (persistently exciting) input sequence $u_{[0,T-1]}$, and $T+1$ samples of the associated output sequence $y_{[0,T]}$.

Theorem 4.1 assumes the availability of a finite-length trajectory produced by the open-loop system (2) in the absence of exogenous disturbance w_k. When w_k is non-zero but constant at all times, Theorem 4.1 can still be used to determine G, as described next. For all $k \in \mathbb{Z}_{\geq 0}$, define

$$d_k := x_{k+1} - x_k, r_k := y_{k+1} - y_k, v_k := u_{k+1} - u_k. \quad (9)$$

By substituting into (2), the new variables satisfy:

$$d_{k+1} = Ad_k + Bv_k, \qquad r_k = Cd_k. \quad (10)$$

By leveraging (10), Theorem 4.1 can be used to determine G when the sample data is affected by constant disturbance.

Corollary 4.2: ***(Data-Driven Characterization of Transfer Function With Constant Noise)*** Let Assumption 1 hold, assume $u_{[0,T]}$ is persistently exciting of order $n+1$. If $w_k = w \in \mathbb{R}^r$ is fixed for all $k \in \mathbb{Z}_{\geq 0}$, then the steady-state transfer function of (2) equals $G = R_{1,q}M$, where

$$R_{1,q}^{\text{diff}} M = 0, \qquad V_{1,q}M = I, \quad (11)$$

$R_{1,q}^{\text{diff}} = [r_1 - r_0,\ r_2 - r_1,\ \dots\ r_q - r_{q-1}]$, and $q = T+1$. □

Corollary 4.2 provides a direct way to compute the transfer function G when the sample data is affected by constant noise. Notice that, the Hankel matrices $R_{1,q}, V_{1,q}$, and $R_{1,q}^{\text{diff}}$ can be computed directly from an input-output trajectory of (2) by preprocessing the data samples as described by (9).

B. Convergence to First-Order Optimizers

We now turn our attention to the online control phase, where the map G computed according to Theorem 4.1 is used in the gradient-based controller in (7). The following result guarantees global convergence to the set of optimizers of (4) under a suitable choice of the stepsize η.

Theorem 4.3: ***(Tracking of Optimal Solutions)*** Let Assumptions 2-1 hold, let $\ell = \ell_\phi + \|G\|^2 \ell_\psi$, and assume that Q satisfies $\underline{\lambda}(Q) > \frac{a}{\epsilon(1-\epsilon)}$, where $a = \frac{1}{2}\ell_\psi^2 \|C\|^2 \|G\|^2$ and $\epsilon \in (0,1)$ is a fixed parameter. Moreover, let

$$\eta^* := \frac{1-\epsilon}{\ell/2 + b}, \qquad b := \frac{2\|A^\mathsf{T} P \bar{G}\|^2}{\epsilon \underline{\lambda}(Q)} + \|\bar{G}^\mathsf{T} P \bar{G}\|,$$

where $\bar{G} := (I-A)^{-1}B$. Then, for every $\eta < \eta^*$ there exists a class-$\mathcal{KL}$ function β and class-$\mathcal{K}$ functions γ_u, γ_w such that all solutions of (7) satisfy (6). □

The proof of Theorem 4.3 relies on the following result, which can be interpreted as an extension of the characterization of input-to-state stability for equilibrium points studied in [21] to the case of compact forward invariant sets [22].

Lemma 4.4: ***(ISS with Respect to Compact Sets)*** Consider the system $z_{k+1} = f(z_k, v_k)$ where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and v_k is a bounded input sequence. Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty and compact set that is forward invariant for the unforced system. Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous function such that

$$\alpha_1(|z|_\mathcal{A}) \leq V(z) \leq \alpha_2(|z|_\mathcal{A}), \quad (12a)$$

$$V(f(z,v)) - V(z) \leq -\alpha_3(|z|_\mathcal{A}) + \sigma(|v|_\mathcal{A}), \quad (12b)$$

hold for all $z \in \mathbb{R}^n$, and $v \in \mathbb{R}^m$, where $\alpha_1, \alpha_2, \alpha_3$ are class $\mathcal{K}_\infty$ functions and σ is of class $\mathcal{K}$. Then, there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that the system solutions satisfy

$$|z_k|_\mathcal{A} \leq \beta(|z_{k_0}|_\mathcal{A}, k - k_0) + \gamma(\sup_{t \geq k_0} \|v_t\|),$$

for all $0 \leq k_0 \leq k$, and for any $z_{k_0} \in \mathbb{R}^n$. □

Proof: The claim follows by iterating the proof of [21, Lemma 3.5]. Precisely, we note that, when $\mathcal{A}$ is nonempty and compact, the quantity $|z|_\mathcal{A}$ is well-defined and bounded for any $z \in \mathbb{R}^n$, and thus all Euclidean norms in [21, Lemma 3.5] can be replaced by the point-to-set distance $|\cdot|_\mathcal{A}$. ■

Proof of Theorem 4.3: We begin by performing a change of variables for (7). Let $\tilde{x}_k := x_k - \mathcal{M}(u_k, w_k)$, where $\mathcal{M}(u,w) = \bar{G}u - \bar{H}w$, $\bar{G} := (I-A)^{-1}B$, $\bar{H} := (I-A)^{-1}E$. In the new variables:

$$\begin{aligned} \tilde{x}_{k+1} &= A\tilde{x}_k - \mathcal{M}(u_{k+1}, w_{k+1}) + \mathcal{M}(u_k, w_k) \quad (13) \\ &= A\tilde{x}_k - \bar{G}(u_{k+1} - u_k) - \bar{H}(w_{k+1} - w_k), \\ u_{k+1} &= u_k - \eta(\nabla\phi(u_k) + G^\mathsf{T}\nabla\psi(C\tilde{x}_k + Gu_k + Hw_k)). \end{aligned}$$

Next, let $f(u) := \phi(u) + \psi(Gu + Hw_k)$, and denote by u^* any minimizer of $f(u)$. We will show that the following Lyapunov function satisfies the assumptions of Lemma 4.4:

$$U(x,u) := V(u) + W(\tilde{x}), \quad (14)$$

where $V(u) = \frac{1}{\eta}(f(u) - f(u^*))$, $W(\tilde{x}) = \tilde{x}^\mathsf{T} P \tilde{x}$.

First, (12a) follows by application of [26, Lemma 4.3] by noting that $U(x,u)$ is continuous, positive definite, and radially unbounded. Next, we show (12b). By letting $F_c(\tilde{x}, u) := -\nabla\phi(u) - G^\mathsf{T}\nabla\psi(C\tilde{x} + Gu + Hw_k)$, and by noting that $\nabla f(u) = -F_c(0,u)$, $V(\cdot)$ satisfies:

$$\begin{aligned} V(u_{k+1}) - V(u_k) &= \frac{f(u_{k+1}) - f(u_k)}{\eta} - \frac{f(u_{k+1}^*) + f(u_k^*)}{\eta} \\ &\leq \langle \nabla f(u_k), \frac{u_{k+1} - u_k}{\eta} \rangle + \frac{\ell}{2\eta} |u_{k+1} - u_k|_{\mathcal{U}_k^*}^2 \\ &\quad + \underbrace{\langle \nabla f(u_k^*), \frac{u_{k+1}^* - u_k^*}{\eta} \rangle}_{=0} + \frac{\ell}{2\eta} \|u_{k+1}^* - u_k^*\|^2 \\ &\leq -(1 - \frac{\ell\eta}{2}) |F_c(\tilde{x}_k, u_k)|_{\mathcal{U}_k^*}^2 + \frac{\ell}{2\eta} \|u_{k+1}^* - u_k^*\|^2 \\ &\quad + \ell_\psi \|C\| \|G\| |\tilde{x}|_{\mathcal{U}_k^*} |F_c(\tilde{x}_k, u_k)|_{\mathcal{U}_k^*}, \quad (15) \end{aligned}$$

where the first inequality follows from $f(u) - f(v) \leq \nabla f(v)^\mathsf{T}(u-v) + \frac{\ell}{2}\|u-v\|^2$, which holds $\forall u, v \in \mathbb{R}^m$ under Assumption 2, and the last inequality follows by noting that $|u_{k+1} - u_k|^2_{\mathcal{U}_k^*} = \eta^2 |F_\mathrm{c}(\tilde{x}_k, u_k)|^2_{\mathcal{U}_k^*}$ and by using:

$$\begin{aligned}\langle \nabla f(u_k), \frac{u_{k+1} - u_k}{\eta}\rangle &= -\langle F_\mathrm{c}(0, u_k), F_\mathrm{c}(\tilde{x}_k, u_k)\rangle \\ &= -\langle F_\mathrm{c}(0, u_k) + F_\mathrm{c}(\tilde{x}_k, u_k) - F_\mathrm{c}(\tilde{x}_k, u_k), F_\mathrm{c}(\tilde{x}_k, u_k)\rangle \\ &\leq -|F_\mathrm{c}(\tilde{x}_k, u_k)|^2_{\mathcal{U}_k^*} + \ell_\psi \|C\|\|G\|\|\tilde{x}_k\||F_\mathrm{c}(\tilde{x}_k, u_k)|_{\mathcal{U}_k^*},\end{aligned}$$

where we used Assum. 2. By completing the squares in (15):

$$\begin{aligned}V(u_{k+1}) - V(u_k) \leq -(1 - \frac{\epsilon}{2} - \frac{\ell\eta}{2})|F_\mathrm{c}(\tilde{x}_k, u)|^2_{\mathcal{U}_k^*} \qquad (16)\\ + \underbrace{\frac{\ell_\psi^2 \|C\|^2\|G\|^2}{2\epsilon}}_{:=a_1} |\tilde{x}_k|^2_{\mathcal{U}_k^*} + \underbrace{\frac{\ell}{2\eta}}_{:=a_2} \|u_{k+1}^* - u_k^*\|^2.\end{aligned}$$

By denoting in compact form $\Delta w_k := w_{k+1} - w_k$ and by recalling that $u_{k+1} - u_k = \eta F_\mathrm{c}(\tilde{x}_k, u_k)$, $W(\cdot)$ satisfies:

$$\begin{aligned}&W(\tilde{x}_{k+1}) - W(\tilde{x}_k) = \tilde{x}_k^\mathsf{T}(A^\mathsf{T} P A - P)\tilde{x}_k \\ &\quad + \eta^2 F_\mathrm{c}^\mathsf{T}(\tilde{x}_k, u_k)\bar{G}^\mathsf{T} P \bar{G} F_\mathrm{c}(\tilde{x}_k, u_k) + \Delta w_k^\mathsf{T} \bar{H}^\mathsf{T} P \bar{H} \Delta w_k \\ &\quad - 2\eta \tilde{x}_k^\mathsf{T} A^\mathsf{T} P \bar{G} F_\mathrm{c}(\tilde{x}_k, u_k) - 2\tilde{x}_k^\mathsf{T} A^\mathsf{T} P \bar{H} \Delta w_k \\ &\quad + 2\eta F_\mathrm{c}^\mathsf{T}(\tilde{x}_k, u_k)\bar{G}^\mathsf{T} P \bar{H} \Delta w_k \\ &\leq -\underline{\lambda}(Q)|x_k|^2_{\mathcal{U}_k^*} + 2\eta\|A^\mathsf{T} P \bar{G}\||\tilde{x}_k|_{\mathcal{U}_k^*}|F_\mathrm{c}(\tilde{x}_k, u_k)|_{\mathcal{U}_k^*} \\ &\quad + \eta^2\|\bar{G}^\mathsf{T} P \bar{G}\||F_\mathrm{c}(\tilde{x}_k, u_k)|^2_{\mathcal{U}_k^*} + 2\|A^\mathsf{T} P \bar{H}\||\tilde{x}_k|_{\mathcal{U}_k^*}\|\Delta w_k\| \\ &\quad + \|\bar{H}^\mathsf{T} P \bar{H}\|\|\Delta w_k\|^2 + 2\eta\|\bar{G}^\mathsf{T} P \bar{H}\||F_\mathrm{c}(\tilde{x}_k, u_k)|_{\mathcal{U}_k^*}\|\Delta w_k\|\end{aligned}$$

By completing the squares:

$$\begin{aligned}W(\tilde{x}_{k+1}) - W(\tilde{x}_k) &\leq -(1-\epsilon)\underline{\lambda}(Q)|\tilde{x}_k|^2_{\mathcal{U}_k^*} \\ &+ \eta^2 \underbrace{\left(\frac{2\|A^\mathsf{T} P \bar{G}\|^2}{\epsilon\underline{\lambda}(Q)} + \|\bar{G}^\mathsf{T} P \bar{G}\|\right)}_{:=b_1} |F_\mathrm{c}(\tilde{x}_k, u_k)|^2_{\mathcal{U}_k^*} \\ &+ \underbrace{\left(\frac{2\|A^\mathsf{T} P \bar{H}\|^2}{\epsilon\underline{\lambda}(Q)} + \|\bar{H}^\mathsf{T} P \bar{H}\|\right)}_{:=b_2} \|\Delta w_k\|^2 \\ &+ 2\eta\|\bar{G}^\mathsf{T} P \bar{H}\||F_\mathrm{c}(\tilde{x}_k, u_k)|^2_{\mathcal{U}_k^*}\|\Delta w_k\|^2. \qquad (17)\end{aligned}$$

By combining (16)-(17) and by completing the squares:

$$\begin{aligned}&U(\tilde{x}_{k+1}, u_{k+1}) - U(\tilde{x}_k, u_k) \leq -(1 - \epsilon - \frac{\ell\eta}{2})|F_\mathrm{c}(\tilde{x}_k, u)|^2_{\mathcal{U}_k^*} \\ &\quad + a_1|\tilde{x}_k|^2_{\mathcal{U}_k^*} + a_2\|u_{k+1}^* - u_k^*\|^2 - (1-\epsilon)\underline{\lambda}(Q)|\tilde{x}_k|^2_{\mathcal{U}_k^*} \\ &\quad + \eta b_1 |F_\mathrm{c}(\tilde{x}_k, u_k)|^2_{\mathcal{U}_k^*} + (b_2 + b_3)\|\Delta w_k\|^2,\end{aligned}$$

where $b_3 = 2\eta^2\|\bar{G}^\mathsf{T} P \bar{H}\|/\epsilon$. In summary, by letting $z := (|F_\mathrm{c}(\tilde{x}_k, u)|_{\mathcal{U}_k^*}, |\tilde{x}_k|_{\mathcal{U}_k^*})$, $v_1 := \|u_{k+1}^* - u_k^*\|$, and $v_2 := \|\Delta w_k\|$, if the following conditions are satisfied:

$$\eta < \frac{1-\epsilon}{\ell/2 + b_1}, \qquad \underline{\lambda}(Q) > \frac{a_1}{1-\epsilon},$$

then $U(x_k, u_k)$ satisfies (12b) with:

$$\begin{aligned}\alpha_3(z) &= \min\{(1-\epsilon) - \frac{\eta\ell}{2} - \eta b_1, (1-\epsilon)\underline{\lambda}(Q) - a_1\} z^2, \\ \sigma(v_1, v_2) &= a_2 v_1^2 + (b_2 + b_3) v_2^2.\end{aligned}$$

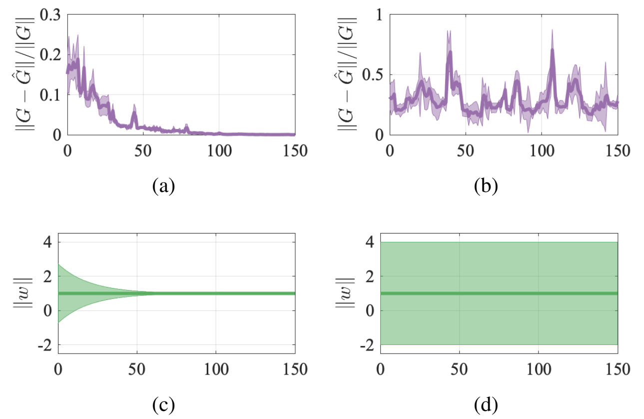

Fig. 3. Error between data-driven G and model-based G when the training data is affected by an unknown disturbance w_k. The illustration has been produced by using the output of a Montecarlo simulation where the realization of w was varied over $10,000$ samples of a IID Gaussian process. Continuous lines illustrate the mean of the trajectory, shaded areas illustrates the 3-standard deviation confidence intervals. (a)-(c) When the exogenous disturbance is constant over time (equivalently, when the variance of w_k in the Montecarlo simulations decays to zero asymptotically) then Corollary 4.2 guarantees $\|G - \hat{G}\| \to 0$. (b)-(d) When the training data is affected by time-varying disturbance (equivalently, when the variance of w_k in the Montecarlo simulations is constant over time), Theorem 4.1 allows us to approximate G up to a finite error.

Finally, the claim follows by taking the supremum among all u_k^* and by replacing the Hausdorff distance. ■

Theorem 4.3 asserts that a sufficiently-small choice of the stepsize η is guarantees convergence to the optimizers, up to an asymptotic error that depends on the time-variability of the optimal set and on the time-variability of the unknown exogenous signal w_k. Although an exact computation of η^* requires the knowledge of the system matrices (A, C), Theorem 4.3 provides an existence claim for the stepsize η. Further, we note that the requirement $\underline{\lambda}(Q) > \frac{a}{\epsilon(1-\epsilon)}$ is non-restrictive, since the choice of Q is arbitrary in Assumption 1.

Remark 3: ***(Relationship with Classical Convergence Results)*** We note that, when the plant is infinitely fast (i.e. the plant dynamics in (7) are replaced by $y_k = G u_k + H w_k$), standard results (see e.g. [24]) guarantee convergence of gradient-like dynamics for all $\eta < \eta_\text{static} := 2/\ell$. By noting that η^* in Theorem 4.3 is strictly smaller that η_static, the theorem suggests that a strictly smaller stepsize is required when the system dynamics are non-negligible. □

V. Simulation Results

To illustrate the conclusions drawn in Theorem 4.1, we consider a system with $n = 20$, $m = r = 10$, $p = n$, and matrices (A, B, C, E) with random entries that satisfy Assumption 1. Fig. 3 illustrates the error in the computed transfer function $\|G - \hat{G}\|$, where G denotes the model-based steady-state transfer function, and $\hat{G}$ denotes the transfer function computed according to Corollary 4.2. $\hat{G}$ has been computed by using a rolling-horizon window that discards old samples over time. The length of the collection window is chosen equal to the smallest number of samples needed to guarantee persistence of excitation, and all samples

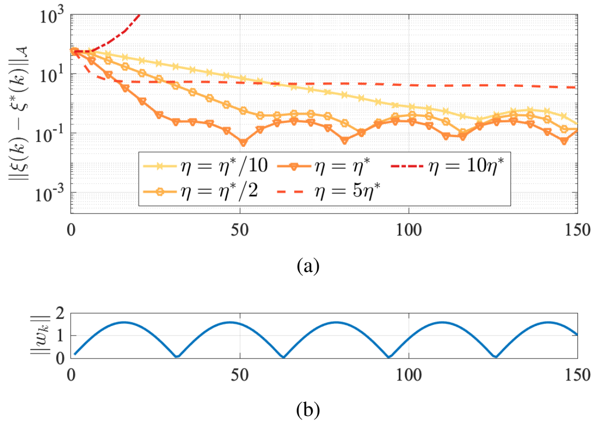

Fig. 4. (a) Tracking error of (7). (b) Time-varying disturbance. The simulation suggests that values of η larger than η^* prevent convergence.

(including the ones used for initialization) are obtain by using a random, persistently exciting, input. Fig. 3(a)-(c) validates the conclusions drawn in Theorem 4.1: it shows that when the disturbance affecting the training data w_k is constant, then the technique proposed in Corollary 4.2 allows us to compute G from a sample trajectory. Fig. 3 illustrates numerically that the error $\|G - \hat{G}\|$ is an increasing function of the time-variability of the disturbance w_k affecting the training data.

Fig. 4(a) illustrates the tracking error for a simulation of the dynamics (7), subject to the time-varying disturbance w illustrated in Fig. 4(b). In this case, the map G was computed from noiseless historical data, according to Theorem 4.3. For simplicity, we consider the case where $\phi(u) = u^\top Q_u u$, $Q_u \succ 0$ and $\psi(y) = (y - y^{\text{ref}})^\top Q_y (y - y^{\text{ref}})$, $Q_y \succ 0$. The figure validates Theorem 4.3 by showing that the tracking error is governed by two terms: (i) an error component associated with the initial conditions that decays to zero asymptotically, and (ii) an error component associated with the time-variability of w_k, which vanishes only if $w_{k+1} - w_k = 0$. The numerical simulations also suggest that values of η larger than η^* prevent the convergence of the method.

VI. Conclusions

We proposed a data-driven method to steer a dynamical system to the solution trajectory of a time-varying optimization problem. The technique does not rely on any prior knowledge or estimation of the system matrices or of the exogenous disturbances affecting the model equation. Instead, we showed that noise-free input-output data originated by the open-loop system can be used to compute the steady-state transfer function of the dynamical environment. Moreover, we showed that convergence of the proposed algorithm to the time-varying optimizers is guaranteed when the dynamics of the controller are sufficiently slower than those of the dynamical system. This work sets out several opportunities for future works, including extensions to scenarios where historical data is affected by non-constant noise terms, and the development of data-driven methods for the convergence analysis of the interconnected system.

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