Enhancement of elasto-dielectrics by homogenization of active charges

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A R T I C L E   I N F O

A B S T R A C T

We investigate the PDE system resulting from even electromechanical coupling in elastomers. Assuming a periodic microstructure and a periodic distribution of micro-charges of a prescribed order, we derive the homogenized system. The results depend crucially on periodicity (or adequate randomness) and on the type of microstructure under consideration. We also show electric enhancement if the charges are carefully tailored to the homogenized electric field and explicit that enhancement, as well as the corresponding electrostrictive enhancement in a dilute regime.

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R É S U M É

Nous étudions un système d'équations modélisant un couplage électromécanique neutre dans les élastomères. Nous obtenons un système homogénéisé en partant d'une microstructure périodique et d'une distribution périodique donnée de micro-charges. Les résultats reposent de manière cruciale sur la périodicité (ou sur une structure aléatoire adéquate) et sur le type de microstructures considéré. Nous montrons également un effet de renforcement électrique si les charges sont soigneusement choisies en fonction du champ électrique homogénéisé, de même qu'un renforcement élastostrictif, que nous explicitons dans le régime dilué.

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1. Introduction

Ever since the discovery of the piezoelectric behavior of several types of minerals — including quartz, tourmaline, and Rochelle salt — by Pierre and Jacques Curie in the 1880s [8,9], deformable dielectrics have been an object of uninterrupted interest in fields ranging from materials science to mathematics. This has been reinforced since the turn of the millennium when soft organic dielectrics were "re-discovered" as a class of materials with high technological potential.

In contrast to the odd coupling between mechanical and electric field, a characteristic of the hard deformable dielectrics investigated by the Curie brothers, soft organic dielectrics typically exhibit even electromechanical coupling. From a mathematical point of view, this means that the governing equations involved exhibit nonlinearity, even in the simplest asymptotic setting of small deformations. Furthermore, space charges varying at the length scale of the microstructure may assert their presence, as is the case, for example, in porous polymer electrets [4,20] and polymer nano-particulate composites [21,32]. This translates into equations that contain a rapidly oscillating source term and leads to anomalous behaviors [15,26].

Our goal in this study is to investigate the homogenization of elasto-dielectrics with even electromechanical coupling that contain space charges that vary at the length scale of their microstructure; a formal analysis of that problem was presented in [23]. In addition to ignoring dissipative effects, we restrict attention to materials with periodic (or with adequate random) microstructure, quasi-static electromechanical loading conditions, and further focus on the asymptotic setting of small deformations and moderate electric fields. The derivation of the relevant local governing equations goes as follows.

Consider an elastic dielectric that occupies a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega$ in its undeformed, stress-free, and polarization-free ground state. Material points are identified by their initial position vector x in Ω relative to some fixed point. Upon application of mechanical loads and electric fields, the position vector x of a material point moves to a new position $v(x) = x + u(x)$, where u denotes the displacement field. The associated deformation gradient is denoted by $F(x) = I + \nabla u(x)$. In the absence of magnetic fields, free currents, and body forces, and with no time dependence (see, e.g., [10]), Maxwell's and the momentum balance equations require that

$$\begin{cases} \operatorname{div} D = Q, \ \operatorname{curl} E = 0, \ x \in \mathbb{R}^N \\ \operatorname{div} S = 0, \ SF^T = FS^T, \ x \in \Omega, \end{cases}$$

where $D(x)$, $E(x)$, $S(x)$ stand for the Lagrangian electric displacement field, the Lagrangian electric field, and the "total" first Piola-Kirchhoff stress tensor, while $Q(x)$ stands for the density (per unit undeformed volume) of space charges. Further,

$$D(x) = -\frac{\partial W}{\partial E}(x, F(x), E(x)) \qquad \text{and} \qquad S(x) = \frac{\partial W}{\partial F}(x, F(x), E(x)),$$

where the "total" free energy $W(x, F, E)$ is an objective function of the deformation gradient tensor F and an *even* and objective function of the electric field E, namely, $W(x, F, E) = W(x, QF, E) = W(x, F, -E)$ for all $Q \in SO(N)$ and arbitrary F and E. The objectivity of W implies that the balance of angular momentum $SF^T = FS^T$ is automatically satisfied. Faraday's law $\operatorname{curl} E = 0$ can also be satisfied automatically by the introduction of an electric potential $\varphi(x)$ such that $E(x) = -\nabla\varphi(x)$. Thus, only Gauss's law $\operatorname{div} D = Q$ and the balance of linear momentum $\operatorname{div} S = 0$ remain.

Now, setting $H := F - I$, a Taylor expansion of W about the ground state $F = I$, $E = 0$ yields

$$W(x, F, E) = -\frac{1}{2}E \cdot \varepsilon(x)E + \frac{1}{2}H \cdot L(x)H + H \cdot (M(x)(E \otimes E)) - E \otimes E \cdot (\mathcal{T}(x)(E \otimes E)) + \ldots,$$
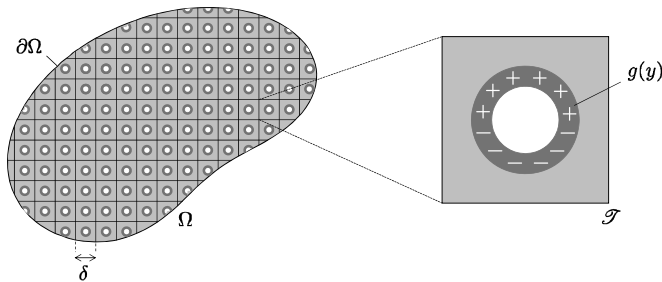
Fig. 1. (a) The elastic dielectric composite. (b) The unit cell with the space-charge density.

where $\varepsilon(x) := -\partial^2 W(x, I, 0)/\partial E^2$ is the permittivity tensor, $L(x) := \partial^2 W(x, I, 0)/\partial F^2$ is the elasticity tensor, $M(x) := 1/2\partial^3 W(x, I, 0)/\partial F \partial E^2$ is the electrostriction tensor, and $\mathcal{T}(x) := -1/24 \, \partial^4 W(x, I, 0)/\partial E^4$ is the permittivity tensor of second order. It follows that the constitutive relations that describe the electromechanical response of the elastic dielectric specialize to

$$D(x) = \varepsilon(x)E(x) + H(x) \cdot M(x)E(x) + \mathcal{T}(x)(E(x) \otimes E(x) \otimes E(x)) + \ldots$$
$$S(x) = L(x)H(x) + M(x)(E(x) \otimes E(x)) + \ldots$$

Taking the magnitude of the deformation measure H to be of order ζ, with $0 < \zeta \ll 1$, it follows in turn that the electric field E must be of order $\zeta^{1/2}$ if the elastic dielectric is to display electromechanical coupling around its ground state. To leading order, we then get

$$D(x) = \varepsilon(x)E(x) \qquad \text{and} \qquad S(x) = L(x)H(x) + M(x)(E(x) \otimes E(x)).$$

This is the so-called scaling of small deformations and moderate electric fields; within this scaling, by the same token, the space charge density Q must be of order $\zeta^{1/2}$.

Note that objectivity also implies that $L(x)$ and $M(x)$ are truly symmetric mappings from $\mathbf{M}_{\mathrm{sym}}^N$ into itself. In particular this will imply that, for any H, $L(x)H = L(x)(\frac{H+H^T}{2})$ (see for example (4.1)). Throughout, we chose to di-symmetrize the symmetrized gradients for notational convenience.

We now detail the governing equations for the problem under investigation in this work. Assuming periodicity of the microstructure, the permittivity, elasticity, and electrostriction tensors $(\varepsilon(y), L(y), M(y)$, respectively) that characterize the local elastic dielectric response of the material are defined on a unit cell (or, more precisely, on a unit torus $\mathcal{T}$) and they are periodically rescaled by a small parameter δ to reflect the size of the microstructure. The resulting tensors are respectively denoted by $\varepsilon^\delta(x), L^\delta(x), M^\delta(x)$.

Moreover, the material is assumed to contain a distribution of periodically distributed space charges with density $g(y)$ such that

$$\int_{\mathcal{T}} g(y) \, dy = 0 \tag{1.1}$$

so as to preserve local charge neutrality, rescaled in a manner similar to that of the microstructure and modulated by a slowly varying macroscopic charge $f(x)$. These space charges can be passive or active. In the case of passive charges, the slowly varying macroscopic charge $f(x)$ is fixed from the outset. Physically, this corresponds to materials wherein space charges are "glued" to material points and remain so regardless of the applied mechanical loads and electric fields. This is the case, for instance, of porous polymer electrets for which the space charges are fixed at the walls of the pores. In the case of active charges, the slowly varying macroscopic charge $f(x)$ is identified as the resulting macroscopic field for the electric potential and hence depends on the applied electric field. Physically, this corresponds to materials wherein space charge

are locally mobile. This is the case, for instance, of polymer nano-particulate composites for which the space charges are locally mobile around the interfaces between the polymer and the nano-particles. Fig. 1 illustrates a schematic of the material and of its periodic microstructure and space charge content.

The relevant governing equations are

$$\begin{cases} \operatorname{div} \varepsilon^\delta \nabla \varphi^\delta = \frac{1}{\delta} g^\delta f \\ \operatorname{div} \left[L^\delta \nabla u^\delta + M^\delta (\nabla \varphi^\delta \otimes \nabla \varphi^\delta) \right] = 0 \end{cases} \tag{1.2}$$

for the electric potential φ^δ and the displacement field u^δ. For simplicity, the boundary conditions are taken to be of Dirichlet type, that is,

$$\varphi^\delta = \Phi, \quad u^\delta = 0 \quad \text{on } \partial\Omega.$$

Note that imposing Dirichlet boundary conditions on the electric potential amounts to considering Gauss' law inside the domain Ω, and not in $\mathbb{R}^N$, a situation which corresponds to electrodes being placed along the entire boundary of Ω.

Remark 1.1. The heuristic justification of the presence of the term $1/\delta$ in front of the space charges $g^\delta f$ is as follows. Because of charge neutrality (see (1.1)), multiplication of the source term by δ^q with $q > -1$ would result in a homogenized dielectric equation without any source term, that is an equation of the form

$$\operatorname{div} \varepsilon^h \nabla \varphi = 0 \quad \text{in } \Omega,$$

where ε^h is the homogenized permittivity tensor defined later in (2.9). Thus, the lowest δ-order at which microscopically distributed charges will impact the homogenized dielectric equation is δ^{-1}. Of course, one can always add lower order source terms as emphasized in Remark 3.4 below, but, their impact will disappear in the effective behavior unless charge neutrality is forsaken for those terms. ¶

The first objective of this work is to determine the purely dielectric macroscopic behavior of the material for an arbitrary but fixed (*i.e.*, passive) distribution of space charges in the limit when the period δ of the microstructure goes to 0. This will be achieved in Section 2.

The second objective is to demonstrate that dielectric enhancement can always be achieved for the purely dielectric macroscopic behavior when adequate active space charges are introduced. To do that we need to identify $f(x)$ with the resulting macroscopic field for the electric potential. We demonstrate that for a two-phase inclusion type microstructure, it is always possible to produce enhancement. Further, in the case of dilute inclusions, we propose an argument inspired by the Clausius-Mossotti formula that yields an explicit value for that enhancement and for "manufacturable" charges. This is the object of Section 5 which we have placed at the end of this paper.

Finally, we determine the homogenized equations for the coupled elastic-dielectric behavior of the material. This is the object of Section 4. Doing so necessitates better convergence properties on the dielectric micro-macro analysis than those provided by Section 2. To do so we combine large-scale regularity due to homogenization with local regularity properties that hold for two-phase microstructures with smooth inclusions. The technical details are the object of Section 3. In the last part of Section 5, we also investigate the elastic enhancement for the dilute case already alluded to above.

Although the results and their proofs are written in the case of periodic media, they can all be extended to random media (with suitable mixing conditions), as we quickly argue in the appendix.

Notationwise, we denote by $\mathbf{M}^N$ the space of $N \times N$-matrices, by $\mathbf{M}^N_{\text{sym}}$ the subspace of symmetric $N \times N$-matrices, by Id the $N \times N$ identity matrix, and by $\cdot$ the Euclidean inner product between vectors in

$\mathbb{R}^N$ or the Fröbenius inner product between elements of $\mathbf{M}^N_{\mathrm{sym}}$, that is $e \cdot e' = \mathrm{tr}\, ee'$ with $e, e' \in \mathbf{M}^N_{\mathrm{sym}}$. We will denote by $B_r(x)$ the open ball of center x and radius r.

We will sometimes identify the torus and its subsets with the unit cube $Y = \Pi_{i=1,\dots,N}[0,1) \subset \mathbb{R}^N$ and the corresponding subsets (denoted with the corresponding roman character) through the canonical identification $\mathbf{i}$ between $\mathscr{T}$ and $\{z + Y : z \in \mathbb{Z}^N\} = \mathbb{R}^N$. Also we will adopt the following convention for a function ζ defined on $\mathscr{T}$. We will say that $\zeta \in H^1(\mathscr{T})$ if, and only $z = \zeta \circ \mathbf{i}$ is such that $z \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$; note that z is Y-periodic. Further, if $\varepsilon \in L^\infty(\mathscr{T}; \mathbf{M}^N_{\mathrm{sym}})$, we will write $\mathrm{div}\, \varepsilon \nabla \zeta$ for $\mathrm{div}\,\{(\varepsilon \circ \mathbf{i}) \nabla z\}$ and denote by ζ^δ the periodic H^1_{loc}-function $z(x/\delta)$ which we will also write as $\zeta(x/\delta)$. Similarly, we will denote by $\int_{\mathscr{T}} (\nabla) \zeta(y)\, dy$ the integral $\int_Y (\nabla) z(y)\, dy$. Also, for any set S, we will denote by χ_S the characteristic function of that set.

The rest of the notation is standard.

2. Classical homogenization of the dielectrics

In this section, we consider the dielectric part of our problem and propose to pass to the limit as the period goes to 0. As already noted, structural assumptions such as periodicity (a random distribution with good enough mixing properties would do as well), while essential in the next section, are not necessary assumptions when handling the scalar dielectric equation; see Remark 2.3 below.

So, on Ω, a bounded Lipschitz domain of $\mathbb{R}^N$, we consider the equation

$$\begin{cases} \mathrm{div}\, \varepsilon^\delta \nabla \varphi^\delta = \frac{1}{\delta} g(\frac{x}{\delta}) f(x) \\ \varphi^\delta = \phi \text{ on } \partial\Omega \end{cases} \tag{2.1}$$

with $f \in W^{1,\infty}(\Omega)$, $g \in L^2(\mathscr{T})$ and $\int_{\mathscr{T}} g(y)\, dy = 0$, $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ and $\varepsilon^\delta(x) := \varepsilon(\frac{x}{\delta})$ where $\varepsilon(y) \in L^\infty(\mathscr{T}; \mathbf{M}^N_{\mathrm{sym}})$ with $\gamma|\xi|^2 \le \varepsilon(y)\xi \cdot \xi \le \beta|\xi|^2$ for some $0 < \gamma < \beta < \infty$.

We define ψ to be the unique solution in $H^1(\mathscr{T})$ of

$$\begin{cases} \triangle \psi(y) = g(y) \\ \int_{\mathscr{T}} \psi(y)\, dy = 0 \end{cases} \tag{2.2}$$

and note that, by elliptic regularity, $\psi \in H^2(\mathscr{T})$. We set

$$\tau(y) := \nabla \psi(y), \quad \tau^\delta(x) := \nabla \psi(\frac{x}{\delta}), \tag{2.3}$$

so that (2.1) reads as

$$\begin{cases} \mathrm{div}(\varepsilon^\delta \nabla \varphi^\delta - f\tau^\delta) = -\tau^\delta \cdot \nabla f \\ \varphi^\delta = \phi \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

From (2.4) and Poincaré's inequality, we immediately obtain that

$$\varphi^\delta \text{ is bounded in } H^1(\Omega) \text{ independently of } \delta$$

and, upon setting

$$q^\delta := \varepsilon^\delta \nabla \varphi^\delta - f\tau^\delta,$$

that

$$q^\delta \text{ is bounded in } L^2(\Omega; \mathbb{R}^N) \text{ independently of } \delta.$$

Thus, up to a subsequence (not relabeled), we conclude that

$$\begin{cases} \varphi^\delta \rightharpoonup \varphi & \text{weakly in } H^1(\Omega) \\ q^\delta \rightharpoonup q & \text{weakly in } L^2(\Omega; \mathbb{R}^N). \end{cases} \tag{2.5}$$

Of course,

$$\operatorname{div} q = 0 \tag{2.6}$$

since $\tau^\delta \overset{L^2(\Omega;\mathbb{R}^n)}{\rightharpoonup} \fint_{\mathscr{T}} \nabla \psi(y) \, dy = 0$. It remains to identify q.

To that effect, consider the periodic corrector w_j defined as follows. Set ω_j to be the unique solution in $H^1(\mathscr{T})$ to

$$\begin{cases} \operatorname{div} \varepsilon \nabla(\omega_j + y_j) = 0 \\ \int_{\mathscr{T}} \omega_j \, dy = 0. \end{cases} \tag{2.7}$$

Then

$$w_j := \omega_j + y_j.$$

Set

$$w_j^\delta(x) := \delta \omega_j \left(\frac{x}{\delta}\right) + x_j \tag{2.8}$$

and note that $\nabla w_j^\delta(x) = (\nabla w)(\frac{x}{\delta})$. Then, on the one hand, the div-curl Lemma [30] (or integration by parts) implies that, for any $\zeta \in C_c^\infty(\Omega)$,

$$\int_\Omega \zeta q^\delta \cdot \nabla w_j^\delta \, dx \longrightarrow \int_\Omega \zeta q \cdot \vec{e}_j \, dx = \int_{\mathscr{T}} \zeta q_j \, dy.$$

On the other hand define, according to classical elliptic homogenization [5, Chapter 1], the symmetric constant matrix ε^h as

$$\varepsilon^h \vec{e}_j := \fint_{\mathscr{T}} \varepsilon(y) \nabla w_j \, dy. \tag{2.9}$$

Since ε is symmetric, another application of the div-curl Lemma yields

$$\int_\Omega \zeta q^\delta \cdot \nabla w_j^\delta \, dx = \int_\Omega \zeta \nabla \varphi^\delta \cdot \varepsilon^\delta \nabla w_j^\delta \, dx - \int_\Omega \zeta f \tau^\delta \cdot \nabla w_j^\delta \, dx \longrightarrow \int_\Omega \zeta \nabla \varphi \cdot \varepsilon^h \vec{e}_j \, dx - \int_\Omega \zeta f a_j \, dx,$$

with

$$a_j := \fint_{\mathscr{T}} \tau(y) \cdot \nabla w_j(y) \, dy = \fint_{\mathscr{T}} \nabla \psi(y) \cdot (\vec{e}_j + \nabla \omega_j(y)) \, dy = -\fint_{\mathscr{T}} g(y) \omega_j(y) \, dy. \tag{2.10}$$

Hence,

$$q = \varepsilon^h \nabla \varphi - af \tag{2.11}$$

with $a \in \mathbb{R}^N$ given through (2.10).

A classical result of H-convergence is that

$$\gamma |\xi|^2 \leq \varepsilon^h \xi \cdot \xi \leq \beta |\xi|^2.$$

Thus, in view of (2.6), we conclude that φ is the unique $H^1(\Omega)$-solution of

$$\begin{cases} \operatorname{div} \varepsilon^h \nabla \varphi = a \cdot \nabla f \\ \varphi = \phi \text{ on } \partial\Omega, \end{cases} \tag{2.12}$$

so that the entire sequence $(\varphi^\delta, q^\delta)$ converges to (φ, q) weakly in $H^1(\Omega) \times L^2(\Omega; \mathbb{R}^N)$.

We now strive to improve the weak convergence results with the help of correctors. To that effect we introduce $\theta \in H^1(\mathscr{T})$ to be the unique solution to

$$\begin{cases} \operatorname{div}(\varepsilon \nabla \theta - \tau) = 0 \text{ (or still } \operatorname{div} \varepsilon \nabla \theta = g) \\ \int_{\mathscr{T}} \theta(y) \, dy = 0. \end{cases} \tag{2.13}$$

We set

$$\sigma := \varepsilon \nabla \theta - \tau,$$

and

$$\theta^\delta(x) := \delta \theta(\frac{x}{\delta}) \quad \sigma^\delta(x) := \sigma(\frac{x}{\delta}), \tag{2.14}$$

so that $\sigma^\delta = \varepsilon^\delta \nabla \theta^\delta - \tau^\delta$ and note that, by symmetry of ε and (2.7), since τ has zero average over $\mathscr{T}$ and in view of (2.10),

$$\sigma_j^\delta \rightharpoonup b_j := \fint_{\mathscr{T}} \varepsilon(y) \nabla \theta(y) \cdot \nabla y_j \, dy = -\fint_{\mathscr{T}} \nabla \theta(y) \cdot \varepsilon(y) \nabla \omega_j(y) \, dy$$

$$= -\fint_{\mathscr{T}} (\varepsilon(y) \nabla \theta(y) - \tau(y)) \cdot \nabla \omega_j(y) \, dy - \fint_{\mathscr{T}} \tau(y) \cdot \nabla \omega_j(y) \, dy$$

$$= -\fint_{\mathscr{T}} \tau(y) \cdot \nabla \omega_j(y) \, dy = -\fint_{\mathscr{T}} \tau(y) \cdot \nabla w_j(y) \, dy = -a_j, \quad \text{weakly in } L^2(\Omega; \mathbb{R}^N). \tag{2.15}$$

We now follow a classical computation; see e.g. [6, section 4]. Take Φ in $C_c^\infty(\Omega; \mathbb{R}^N)$, ζ, η in $C_c^\infty(\Omega)$ and compute

$$\int_\Omega \eta^2 \varepsilon^\delta \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \, \zeta \right) \cdot \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \, \zeta \right) dx =$$

$$\int_\Omega \eta^2 \left\{ (\varepsilon^\delta \nabla \varphi^\delta - \tau^\delta f) - \sum_{j=1,..,N} \varepsilon^\delta \nabla w_j^\delta \Phi_j - \zeta \left(\varepsilon^\delta \nabla \theta^\delta - \tau^\delta \right) \right\} \cdot \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \, \zeta \right) dx$$

$$+ \int_\Omega \eta^2 (f - \zeta)\tau^\delta \cdot \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \; \zeta \right) \; dx$$

$$= \int_\Omega \eta^2 \left\{ \left(q^\delta - \sum_{j=1,..,N} \varepsilon^\delta \nabla w_j^\delta \Phi_j - \zeta \sigma^\delta \right) + (f - \zeta)\tau^\delta \right\} \cdot \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \; \zeta \right) \; dx.$$

Multiple applications of the div-curl Lemma, together with (2.11), (2.15), imply that

$$\int_\Omega \eta^2 \left(q^\delta - \sum_{j=1,..,N} \varepsilon^\delta \nabla w_j^\delta \Phi_j - \zeta \sigma^\delta \right) \cdot \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \; \zeta \right) \; dx \xrightarrow{\delta}$$

$$\int_\Omega \eta^2 (q - \varepsilon^h \Phi + \zeta a) \cdot (\nabla \varphi - \Phi) \; dx = \int_\Omega \eta^2 (\varepsilon^h (\nabla \varphi - \Phi) + a(\zeta - f)) \cdot (\nabla \varphi - \Phi) \; dx. \quad (2.16)$$

Further,

$$\tau^\delta \cdot \nabla \varphi^\delta = (\tau^\delta - \varepsilon^\delta \nabla \theta^\delta) \cdot \nabla \varphi^\delta + (\varepsilon^\delta \nabla \varphi^\delta - f\tau^\delta) \cdot \nabla \theta^\delta + f\tau^\delta \cdot \nabla \theta^\delta = -\sigma^\delta \cdot \nabla \varphi^\delta + q^\delta \cdot \nabla \theta^\delta + f\tau^\delta \cdot \nabla \theta^\delta,$$

so that, setting

$$\kappa := \fint_{\mathscr{T}} \tau(y) \cdot \nabla \theta(y) \; dy, \quad (2.17)$$

the div-curl Lemma implies, in view of (2.15), that

$$\tau^\delta \cdot \nabla \varphi^\delta \rightharpoonup (a \cdot \nabla \varphi + \kappa f), \quad \text{weakly* in } \mathcal{M}_b(\Omega). \quad (2.18)$$

Remark 2.1. Note for later use that, upon multiplication of the first equation in (2.13) by θ and integration over $\mathscr{T}$, we get $\kappa = \fint_{\mathscr{T}} \varepsilon(y)\nabla \theta(y) \cdot \nabla \theta(y) \; dy > 0$. ¶

Hence, since f is in particular continuous,

$$\int_\Omega \eta^2 (f - \zeta)\tau^\delta \cdot \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \; \zeta \right) \; dx \xrightarrow{\delta}$$

$$\int_\Omega \eta^2 (f - \zeta)(a \cdot \nabla \varphi + \kappa f - a \cdot \Phi - \kappa \zeta) \; dx = \int_\Omega \eta^2 (f - \zeta)(a \cdot (\nabla \varphi - \Phi) + \kappa (f - \zeta)) \; dx. \quad (2.19)$$

Summing the contributions (2.16) and (2.19), we finally obtain

$$\lim_\delta \int_\Omega \eta^2 \varepsilon^\delta \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \; \zeta \right) \cdot \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \; \zeta \right) \; dx$$

$$= \int_\Omega \eta^2 \left\{ (\varepsilon^h (\nabla \varphi - \Phi) + a(\zeta - f)) \cdot (\nabla \varphi - \Phi) + (f - \zeta)(a \cdot (\nabla \varphi - \Phi) + \kappa (f - \zeta)) \right\} \; dx$$

$$= \int_\Omega \eta^2 (\varepsilon^h (\nabla \varphi - \Phi) \cdot (\nabla \varphi - \Phi) + \kappa (f - \zeta)(f - \zeta)) \; dx.$$

From this and the coercivity of $\varepsilon(y)$ we conclude that, for some $C > 0$,

$$\gamma \limsup_{\delta} \int_{\Omega} \eta^2 \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \; \zeta \right) \cdot \left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \; \zeta \right) \; dx$$

$$\leq C\{\|\nabla \varphi - \Phi\|^2_{L^2(\Omega;\mathbb{R}^N)} + \|f - \zeta\|^2_{L^2(\Omega)}\}. \quad (2.20)$$

Now, assuming that

$$\begin{cases} \partial\Omega \in C^{2,\alpha}, \; 0 < \alpha < 1 \\ \phi \in C^{2,\alpha}(\partial\Omega)(\text{or, specifically, } \phi \text{ is the restriction to } \partial\Omega \text{ of } \phi \in C^{2,\alpha}(\overline{\Omega})) \\ f \in C^{1,\alpha}(\Omega) \end{cases} \quad (2.21)$$

Schauder elliptic regularity applied to (2.12) yields that $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Then, for any $\lambda > 0$ we can find Φ, ζ such that

$$\|\nabla\varphi - \Phi\|_{C^0(\overline{\Omega})} + \|f - \zeta\|_{C^0(\overline{\Omega})} \leq \lambda$$

so that, because $\nabla w_j^\delta, \nabla\theta^\delta$ are bounded in $L^2(\Omega)$ independently of δ, (2.20) together with the arbitrariness of η implies that

$$\nabla\varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} - \nabla\theta^\delta f \longrightarrow 0, \quad \text{strongly in } L^2_{\text{loc}}(\Omega; \mathbb{R}^N).$$

We have proved the following

Theorem 2.2. *Under assumptions* (2.21), φ^δ, *unique* $H^1(\Omega)$-*solution to* (2.1) *is such that*

$$\nabla\varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} - \nabla\theta^\delta f \longrightarrow 0, \quad \text{strongly in } L^2_{\text{loc}}(\Omega; \mathbb{R}^N)$$

with w_j^δ *defined in* (2.8) *and* θ^δ *defined in* (2.14).

Remark 2.3. Provided that we view the dielectric problem as in (2.4) – that is without reference to charges g^δ, but with an oscillating field τ^δ – the obtained results do not rest on periodicity and could be adapted to a general homogenization framework like that of H-convergence [31]. ¶

Remark 2.4. If we also assume that

$$g \in C^{0,\alpha}(\mathscr{T}) \quad (2.22)$$

for some $0 < \alpha < 1$, then elliptic regularity applied to (2.2) implies that $\tau \in C^{1,\alpha}(\mathscr{T};\mathbb{R}^N)$, and thus that, in particular, the convergence in (2.18) takes place weakly in $L^2(\Omega)$. Because of that, we do not need the compactly supported smooth test η in all prior computations and the result of Theorem 2.2 becomes

$$\nabla\varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} - \nabla\theta^\delta f \longrightarrow 0, \quad \text{strongly in } L^2(\Omega;\mathbb{R}^N).$$

Note that the fact that φ^δ satisfies Dirichlet boundary conditions is essential in evaluating the limit of the term $\int_\Omega \varepsilon^\delta \nabla\varphi^\delta \cdot \nabla\varphi^\delta \; dx$ in all previous computations. ¶

Remark 2.5. In the case of a two-phase particulate microstructure, that is whenever

$$\varepsilon(y) := \chi_{\mathscr{M}} \varepsilon_{\mathscr{M}} + (1 - \chi_{\mathscr{M}}) \varepsilon_{\mathscr{I}}, \text{ with } \mathscr{I} := \mathscr{T} \setminus \mathscr{M}$$

where $\mathscr{M}$ is a measurable subset of $\mathscr{T}$, we can modify the definition of ε^δ so that no inclusion intersects $\partial\Omega$. We define

$$\Omega_\delta := \bigcup_{z \in \mathbb{Z}^N \text{ s.t. } \delta(z+2Y) \subset \Omega} \delta(z + 2Y),$$

note that $|\Omega \setminus \Omega_\delta| \leq C\delta$ and set

$$\varepsilon^\delta(x) := \varepsilon(\frac{x}{\delta}) \chi_{\Omega_\delta} + \varepsilon_{\mathscr{M}} \chi_{\Omega \setminus \Omega_\delta}.$$

With this definition of ε^δ, the results of Section 2 still hold true with trivial modifications of the proofs. ¶

Unfortunately, as described in the introduction, this result is not sufficient, when plugged into the equations of elasticity, to ensure that $\nabla \varphi^\delta$ can be replaced by $\sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial \varphi}{\partial x_j} + \nabla \theta^\delta f$ in those, or that one can perform any kind of homogenization process on the resulting system. This is why the next Section is devoted to an improvement of Theorem 2.2. The framework required for the successful completion of such a task will be much more constrained than that in the current Section. In particular, periodicity, which was merely convenient so far, will become essential. Equally essential will be the assumption that the microstructure is particulate.

3. Improved estimates and correctors result for the dielectrics

This section is devoted to the proof of Theorem 3.2 below which is an improvement on Theorem 2.2, the corrector result for the dielectrics. The strong convergence in $L^2_{\text{loc}}(\Omega; \mathbb{R}^N)$ is upgraded there to a convergence in $L^p(\Omega; \mathbb{R}^N)$ for all finite $p > 1$ in the spirit of seminal results of Avellaneda and Lin [2,3] for smooth periodic coefficients, later extended to piecewise smooth coefficients in [24]. In our proof, we carefully distinguish between large-scale regularity due to homogenization, for which we refer to [1], and local regularity, for which we refer to [25].

Throughout this Section, we assume that $\partial\Omega \in C^{2,\alpha}$ for some $0 < \alpha < 1$. The unit torus $\mathscr{T}$ is of the form

$$\begin{cases} \mathscr{T} = \mathscr{M} \cup \mathscr{I}, \ \mathscr{M} \text{ closed such that } \partial\mathscr{M} \text{ is } C^{1,\beta}, \ 0 < \beta < 1 \\ \mathscr{I} = \mathscr{T} \setminus \mathscr{M} \\ \overline{\mathbf{i}(\mathscr{M})} \text{ is a connected subset of } \mathbb{R}^N \text{ (a matrix phase).} \end{cases} \quad (3.1)$$

Further,

$$\varepsilon(y) := \chi_{\mathscr{M}} \varepsilon_{\mathscr{M}} + (1 - \chi_{\mathscr{M}}) \varepsilon_{\mathscr{I}}, \text{ with } \gamma|\xi|^2 \leq \varepsilon_{\mathscr{M}, \mathscr{I}} \xi \cdot \xi \leq \gamma'|\xi|^2. \quad (3.2)$$

We define ε^δ as in Remark 2.5.

We also assume that (2.21), (2.22) hold true and also that $0 < \alpha < \frac{\beta}{(\beta+1)N}$, so as to appeal to [25].

In a first step, we prove δ-independent L^q-estimates on $\nabla \varphi^\delta$ for any $1 \leq q < \infty$. This is the object of the following

Proposition 3.1. *For all $1 \leq q < \infty$, the sequence $\nabla \varphi^\delta$ is bounded in $L^q(\Omega; \mathbb{R}^N)$, uniformly in δ.*

Proof. Step 1. First, we apply the large-scale Calderón-Zygmund estimates of [1, Theorem 7.7] to (2.4) rewritten as

$$\begin{cases} \operatorname{div} \varepsilon^\delta \nabla(\varphi^\delta - \phi) = \operatorname{div}(f\tau^\delta - \varepsilon^\delta \nabla \phi + \nabla \zeta^\delta) \\ \qquad \varphi^\delta - \phi = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where ζ^δ is the unique $H_0^1(\Omega)$-solution to

$$-\triangle \zeta^\delta = \nabla f \cdot \tau^\delta.$$

We get that, for all $\delta > 0$ and $q \geq 2$, there exists a constant C_q depending on q such that

$$\int_\Omega \left(\fint_{B_\delta(x)} \chi_\Omega(z) |\nabla(\varphi^\delta - \phi)|^2(z)\, dz \right)^{\frac{q}{2}} dx \leq C_q \Big(\|\nabla(\varphi^\delta - \phi)\|_{L^2(\Omega)}^q + \|f\tau^\delta - \varepsilon^\delta \nabla \phi + \nabla \zeta^\delta\|_{L^q(\Omega)}^q \Big).$$

Note that, since the coefficients are periodic, the random variable involved in [1, Theorem 7.7] is simply a constant (and in fact δ) in the present setting.

Now, since ψ is in $C^{1,\alpha}(\mathscr{T}; \mathbb{R}^N)$, τ^δ given by (2.3) is bounded in $L^q(\Omega; \mathbb{R}^N)$ uniformly in δ and maximal L^q-regularity for the Laplacian implies that $\nabla \zeta^\delta$ is bounded in $W^{1,q}(\Omega; \mathbb{R}^N)$ uniformly in δ.

Since, by Jensen's inequality,

$$\int_\Omega \left(\fint_{B_\delta(x)} \chi_\Omega |\nabla \phi|^2\, dz \right)^{\frac{q}{2}} dx \leq \int_{\mathbb{R}^N} \fint_{B_\delta(x)} \chi_\Omega |\nabla \phi|^q\, dz dx = \int_{\mathbb{R}^N} \chi_\Omega |\nabla \phi|^q dx = \int_\Omega |\nabla \phi|^q\, dx,$$

the first convergence in (2.5) and the assumed regularity of the functions f and ψ finally yield

$$\int_\Omega \left(\fint_{B_\delta(x)} \chi_\Omega(z) |\nabla \varphi^\delta|^2\, dz \right)^{\frac{q}{2}} dx \leq C_q \tag{3.4}$$

for some other constant C_q depending on q and on $\|f\|_{C^{1,\alpha}(\Omega)}, \|\psi\|_{C^{1,\alpha}(\mathscr{T})}, \|\phi\|_{C^{1,\alpha}(\Omega)}$. Estimate (3.4) enables us to control $\nabla \varphi^\delta$ on scales larger than δ. In a second step, we will derive an estimate for small scales, that is for scales smaller than δ.

Step 2. In this step, we crucially use the two-phase character of the microstructure, as well as the $C^{1,\beta}$-regularity of the boundary of each of those phases. Take a point $x \in \Omega$ and consider a cube $Q_{4\delta}(x)$ of sidelength 4δ centered at x. We blow up equation (2.4) so as to obtain an equation on $Q_4(0) \cap \Omega_{x,\delta}$ where $\Omega_{x,\delta} := (\Omega - \{x\})/\delta$. To that effect, we set

$$\Phi^\delta(z) := \frac{1}{\delta} \big(\varphi^\delta(x + \delta z) - \bar{\phi}_{x,\delta} \big),$$

where $\bar{\phi}_{x,\delta}$ is a constant that we will fix later. Then Φ^δ satisfies

$$\begin{cases} \operatorname{div}_z \big(\varepsilon(\frac{x}{\delta} + z) \nabla_z \Phi^\delta - f(x + \delta z)\tau^\delta(x + \delta z) \big) = -\delta \tau^\delta(x + \delta z) \cdot \nabla f(x + \delta z) \text{ in } Q_4(0) \cap \Omega_{x,\delta}, \\ \Phi^\delta(z) = \frac{1}{\delta} \big(\phi(x + \delta z) - \bar{\phi}_{x,\delta} \big) \text{ on } \partial(Q_4(0) \cap \Omega_{x,\delta}) \cap \partial\Omega_{x,\delta} := \Gamma_{x,\delta}. \end{cases} \tag{3.5}$$

Note that, if the set $\Gamma_{x,\delta}$ is empty, x is an interior point of Ω. In the opposite case, we can assume, upon changing $Q_4(0)$ to $Q_t(0)$ for t large enough that the $N-1$-dimensional Hausdorff measure of that boundary

$\Gamma_{x,\delta}$ is at least 1. Since $\partial\Omega$ is $C^{2,\alpha}$, $\partial\mathcal{M}$ is $C^{1,\beta}$, and by Remark 2.5, there exists $\Omega'_{x,\delta} \subset \Omega_{x,\delta}$ with the following properties: $\partial\Omega'_{x,\delta}$ is $C^{1,\beta}$ (uniformly in x and δ), if an inclusion intersects $\Omega'_{x,\delta}$ then it is contained in $\Omega'_{x,\delta}$, and $Q(0) \cap \Omega_{x,\delta} \subset \Omega'_{x,\delta} \subset Q_3(0) \cap \Omega_{x,\delta}$. Now, the set $\Omega'_{x,\delta}$ contains a number of inclusions bounded uniformly wrt δ: if there are p inclusions in $\mathcal{T}$, that is if $\mathcal{I}$ has p connected components, then this set contains at most $3^N p$ inclusions. Since by assumption the boundary of those inclusions is also $C^{1,\beta}$, then, in the terminology of [25], the $C^{1,\beta}$ modulus K of $\Omega'_{x,\delta}$ does not depend on δ, x and [25, Theorem 1.2] applies. It yields in particular,

$$\|\nabla_z \Phi^\delta\|_{L^\infty(Q_{1/2}(0)\cap\Omega_{x,\delta})} \leq C_K \left(\|\Phi^\delta - \frac{1}{\delta}\bar{\phi}_{x,\delta}\|_{L^\infty(\Omega'_{x,\delta})} + \|\frac{1}{\delta}(\phi(x+\delta\cdot) - \bar{\phi}_{x,\delta})\|_{C^{1,\alpha}(\Gamma_{x,\delta})} + C' \right), \quad (3.6)$$

where C_K is a constant that only depends on $K, \alpha, \beta, \gamma, \gamma'$ while C' is a constant that only depends on $\|f\|_{C^{1,\alpha}(\Omega)}$ and $\|\psi\|_{C^{1,\alpha}(\mathcal{T})}$. Choosing $\bar{\phi}_{x,\delta} := \fint_{\Gamma_{x,\delta}} \phi(x+\delta\cdot)$, we obtain

$$\|\frac{1}{\delta}(\phi(x+\delta\cdot) - \bar{\phi}_{x,\delta})\|_{C^{1,\alpha}(\Gamma_{x,\delta})} \lesssim \|\phi\|_{C^{1,\alpha}(\Omega)},$$

which we then absorb in the constant C'.

We now apply De Giorgi-Nash-Moser's theorem [16, Theorems 8.24, 8.27] to (3.5). This yields in turn the following estimate:

$$\|\Phi^\delta - \frac{1}{\delta}\bar{\phi}_{x,\delta}\|_{L^\infty(\Omega'_{x,\delta})} \leq C(\|\Phi^\delta - \frac{1}{\delta}\bar{\phi}_{x,\delta}\|_{L^2(Q_4(0)\cap\Omega_{x,\delta})} + C''), \quad (3.7)$$

where C depends only on γ, γ' and C'' depends only on $\|f\|_{C^{1,\alpha}(\Omega)}, \|\psi\|_{C^{1,\alpha}(\mathcal{T})}, \|\phi\|_{C^{1,\alpha}(\Omega)}$. Inserting (3.7) into (3.6) yields

$$\|\nabla_z \Phi^\delta\|_{L^\infty(Q_{1/2}(0)\cap\Omega_{x,\delta})} \leq C'_K \left(\|\Phi^\delta - \frac{1}{\delta}\bar{\phi}_{x,\delta}\|_{L^2(Q_4(0)\cap\Omega_{x,\delta})} + C''' \right),$$

where C'_K is a constant that only depends on $K, \alpha, \beta, \gamma, \gamma'$, while C''' is a constant that only depends on $\|f\|_{C^{1,\alpha}(\Omega)}, \|\psi\|_{C^{1,\alpha}(\mathcal{T})}, \|\phi\|_{C^{1,\alpha}(\Omega)}$.

Since $\bar{\phi}_{x,\delta} := \fint_{\Gamma_{x,\delta}} \phi(x+\delta\cdot)$, $\Phi^\delta(z) = \frac{1}{\delta}(\phi(x+\delta z) - \bar{\phi}_{x,\delta})$ on $\Gamma_{x,\delta}$, and $|\Gamma_{x,\delta}| \geq 1$, a variant of Poincaré-Wirtinger's inequality allows to upgrade the previous estimate to

$$\|\nabla_z \Phi^\delta\|_{L^\infty(Q_{1/2}(0)\cap\Omega_{x,\delta})} \leq C'_K \left(\|\nabla\Phi^\delta\|_{L^2(Q_4(0)\cap\Omega_{x,\delta})} + C''' \right). \quad (3.8)$$

Blowing (3.8) down, we conclude in particular that, for a $d > 0$ large enough, we have, for all $\delta > 0$,

$$\sup_{B_{\delta/d}(x)\cap\Omega} |\nabla\varphi^\delta| \leq C_N C'_K \left(\left(\fint_{\Omega\cap B_\delta(x)} |\nabla\varphi^\delta(y)|^2 \, dy \right)^{\frac{1}{2}} + C''' \right) \quad (3.9)$$

for a dimensional constant C_N.

Step 3. We combine the estimates obtained in the first two steps as follows.

Remark that, for all $x \in \Omega$ and all small enough δ's, the assumed regularity of $\partial\Omega$ implies the existence of $d > 0$ such that $|B_\delta(x) \cap \Omega| \geq d|B_\delta(x)|$. From (3.4) we then get

$$\int_\Omega \left(\fint_{\Omega\cap B_\delta(x)} |\nabla\varphi^\delta(y)|^2 \, dy \right)^{\frac{q}{2}} dx \leq d^{-\frac{q}{2}} \int_\Omega \left(\fint_{B_\delta(x)} \chi_\Omega(y)|\nabla\varphi^\delta(y)|^2 \, dy \right)^{\frac{q}{2}} dx \leq d^{-\frac{q}{2}} C_q.$$

In turn, from (3.9), we obtain

$$\int_\Omega |\nabla\varphi^\delta|^q \, dx \leq \int_\Omega \left(\sup_{B_{\delta/d}(x)\cap\Omega} |\nabla\varphi^\delta|^q \right) dx \leq 2^q (C'_{K,\eta})^q \left(\int_\Omega \left(\fint_{\Omega\cap B_\delta(x)} |\nabla\varphi^\delta(y)|^2 \, dy \right)^{\frac{q}{2}} dx + |\Omega|(C''')^q \right).$$

Combining the inequalities above, we finally conclude that

$$\int_\Omega |\nabla\varphi^\delta|^q \, dx \leq 2^q (C'_{K,\eta})^q \left(d^{-\frac{q}{2}} C_q + |\Omega|(C''')^q \right).$$

This completes the proof of the proposition. □

We are now in a position to improve on the convergence result of Theorem 2.2. We obtain the following.

Theorem 3.2. *Under assumptions* (2.21), (2.22), (3.1), (3.2), φ^δ, *unique* $H^1(\Omega)$-*solution to* (2.1) *is such that, for any* $1 \leq q < \infty$,

$$\nabla\varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} - \nabla\theta^\delta f \longrightarrow 0, \quad \textit{strongly in } L^q(\Omega, \mathbb{R}^N)$$

with w_j^δ *defined in* (2.8) *and* θ^δ *defined in* (2.14).

Proof. First note that the regularity assumptions on the domain, f and ϕ and classical Schauder regularity imply that φ, the solution to (2.12), is in $C^{2,\alpha}(\Omega)$. Further, the regularity assumption on g and classical Schauder regularity imply that $\tau \in C^{1,\alpha}(\mathscr{T})$. Then another application of [25, Theorem 1.1], this time on $\mathscr{T}$ which has no boundary, implies in particular that

$$\nabla\theta, \nabla\omega_j \in L^\infty(\mathscr{T}), \tag{3.10}$$

hence ∇w_j as well. We can thus assume that, for any $r \geq 1$, the term

$$\sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} + \nabla\theta^\delta f \text{ is bounded in } L^r(\Omega) \text{ independently of } \delta.$$

Set $\theta = 1/(q-1)$. In view of Remark 2.4 and Proposition 3.1, the uniform bound derived above yields that, for some constant C depending on q and all the data,

$$\|(\nabla\varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} - \nabla\theta^\delta f)\|_{L^q(\Omega)} \leq \|(\nabla\varphi^\delta - \sum_{j=1,...,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} - \nabla\theta^\delta f)\|_{L^2(\Omega)}^\theta \times$$

$$\|(\nabla\varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} - \nabla\theta^\delta f)\|_{L^{2q}(\Omega)}^{1-\theta} \leq C\|(\nabla\varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} - \nabla\theta^\delta f)\|_{L^2(\Omega)}^\theta \xrightarrow{\delta} 0.$$

Hence the result. □

Remark 3.3. We have assumed throughout this Section that the composite is made of two phases. Nothing would change if we considered n phases instead of 2, provided that we keep the same regularity assumptions. Theorem 3.2 would still hold true, and Theorem 4.1 below as well. ¶

Remark 3.4. All results of Sections 2, 3 remain valid if a source term of the form $h^\delta v$ is added to the right hand-side of (1.2) with $h \in L^\infty(\mathscr{T})$ and $v \in L^\infty(\Omega)$. Then, one has to add the term $\left(\int_{\mathscr{T}} h(y)\, dy \right) v$ to the right hand-side of the homogenized equation (2.12). All other results remain unchanged.   ¶

4. Homogenization of the elasto-dielectrics

We now address the elasticity part of the problem. Recall that $L^\delta(x) := L(\frac{x}{\delta})$ where $L(y)$ is a measurable, symmetric linear mapping from $\mathbf{M}^N_{\text{sym}}$ into itself with the properties that for all $e \in \mathbf{M}^N_{\text{sym}}$, $\gamma|e|^2 \leq L(y)e \cdot e \leq \gamma'|e|^2$ for a.e. $y \in \mathscr{T}$ and some $0 < \gamma < \gamma' < \infty$. Also $M^\delta(x) := M(\frac{x}{\delta})$ with $M(y)$ a bounded, measurable, linear mapping from $\mathbf{M}^N_{\text{sym}}$ into itself.

The equations are

$$\begin{cases} \operatorname{div}(L^\delta \nabla u^\delta + M^\delta(\nabla \varphi^\delta \otimes \nabla \varphi^\delta)) = 0 \\ u^\delta = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

We assume that assumptions (2.21), (2.22), (3.1), (3.2) hold true throughout this Section.

In particular, we can apply Proposition 3.1 and we conclude, with the help of Korn's and Poincaré's inequalities, that

$$u^\delta \text{ exists and is bounded in } H^1_0(\Omega; \mathbb{R}^N) \text{ independently of } \delta.$$

We can also apply Theorem 3.2 and we immediately obtain that, for any $0 < r < \infty$,

$$M^\delta(\nabla\varphi^\delta \otimes \nabla\varphi^\delta) - M^\delta \left[\left(\sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} + \nabla\theta^\delta f \right) \otimes \left(\sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} + \nabla\theta^\delta f \right) \right]$$

$$\xrightarrow{\delta} 0, \text{ strongly in } L^r(\Omega). \tag{4.2}$$

We will only use the value $r = 2$ hereafter. Set

$$Z^\delta(x) := M^\delta \left[\left(\sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} + \nabla\theta^\delta f \right) \otimes \left(\sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial\varphi}{\partial x_j} + \nabla\theta^\delta f \right) \right]. \tag{4.3}$$

Because of convergence (4.2), if $\tilde{u}^\delta$ is the unique H^1_0-solution to

$$\begin{cases} \operatorname{div}(L^\delta \nabla \tilde{u}^\delta + Z^\delta) = 0 \\ u^\delta = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.4}$$

then

$$u^\delta - \tilde{u}^\delta \xrightarrow{\delta} 0, \text{ strongly in } H^1_0(\Omega; \mathbb{R}^N). \tag{4.5}$$

We undertake a homogenization process for the system (4.4). To that effect, we introduce the periodic corrector W_{ij} defined as follows. Set X_{ij} to be the unique solution in $H^1(\mathscr{T}; \mathbb{R}^N)$ to

$$\begin{cases} \operatorname{div} L\nabla(X_{ij} + x_i \vec{e}_j) = 0 \\ \int_{\mathscr{T}} X_{ij}\, dy = 0. \end{cases}$$

Then set

$$W_{ij}(y) := X_{ij} + y_i \vec{e}_j,$$

and

$$W_j^\delta(x) := \delta X_{ij}(\frac{x}{\delta}) + x_i \vec{e}_j$$

and note that $\nabla W_j^\delta(x) = (\nabla W)(\frac{x}{\delta})$.

Then an argument near identical to that which led to (2.12) would yield that

$$\tilde{u}^\delta \rightharpoonup u \ \text{ in } H_0^1(\Omega; \mathbb{R}^N), \tag{4.6}$$

with u, unique $H_0^1(\Omega; \mathbb{R}^N)$-solution to

$$\begin{cases} \operatorname{div}(L^h \nabla u + Z) = 0 \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.7}$$

with L^h defined as

$$L_{ijkh}^h := \fint_{\mathscr{T}} L(y) \nabla W_{ij} \cdot \nabla W_{kh} \ dy \tag{4.8}$$

while Z is defined through

$$Z := M^h(\nabla\varphi \otimes \nabla\varphi) + 2fN^h \nabla\varphi + P^h f^2 \tag{4.9}$$

where

$$\begin{cases} M_{ijkh}^h := \fint_{\mathscr{T}} M(y)(\nabla w_k(y) \otimes \nabla w_h(y)) \cdot \nabla W_{ij}(y) \ dy \\ N_{ijk}^h := \fint_{\mathscr{T}} M(y)(\nabla w_k(y) \otimes \nabla\theta(y)) \cdot \nabla W_{ij}(y) \ dy \\ P_{ij}^h := \fint_{\mathscr{T}} M(y)(\nabla\theta(y) \otimes \nabla\theta(y)) \cdot \nabla W_{ij}(y) \ dy. \end{cases} \tag{4.10}$$

Convergences (4.5) and (4.6) imply the following homogenization result.

Theorem 4.1. *Under assumptions* (2.21), (2.22), (3.1), (3.2), u^δ, *unique* $H_0^1(\Omega; \mathbb{R}^N)$-*solution to* (4.1), *converges weakly in* $H^1(\Omega; \mathbb{R}^N)$ *to the unique* $H_0^1(\Omega; \mathbb{R}^N)$-*solution* u *to* (4.7) *with* L^h *defined in* (4.8) *and* Z *defined through* (4.9), (4.10).

Remark 4.2. Note that, since $X_{ij} = X_{ji}$, M^h enjoys the same symmetry properties as $M(y)$, that is $M_{ijkh}^h = M_{jikh}^h = M_{ijhk}^h$ while $N_{ijk}^h = N_{jik}^h$ and $P_{ij}^h = P_{ji}^h$. Of course L^h has the usual symmetries of elasticity, that is $L_{ijkh}^h = L_{jikh}^h = L_{khij}^h$. ¶

Remark 4.3. We could, in the spirit of the previous Sections, provide a corrector result for u^δ but will refrain from doing so because of the notational complexity. ¶

Remark 4.4. Our result should be compared to that in [35], which investigates the case $g = 0$. In such a case $N^h = P^h = 0$ and the corrector results of the previous sections are markedly simpler. Also note that the results in [35] are derived under the *a priori* assumption that $\nabla\varphi^\delta$ is bounded in $L^4(\Omega; \mathbb{R}^N)$ independently of δ; no justification for such an estimate is offered in that work. ¶

5. The case for active charges

5.1. Setting of the problem

The homogenized dielectric equation obtained in (2.12) can be equivalently rewritten as

$$\operatorname{div}(\varepsilon^h \nabla\varphi - af) = 0.$$

Since a is a vector of size N, one can act on N different directions. This is why, in practice, there should be several collections of charges, that is that, in lieu of a charge of the form $g^\delta f$, one should envision a charge of the form $\sum_{p=1,\dots,N} g_p^\delta f_p$ where each pair (g_p, f_p) is endowed with the same properties, namely $\int_{\mathscr{T}} g_p(y)\, dy = 0$, and the necessary regularities of g_p and f that were introduced in the previous Subsections.

Provided that those are met, the homogenization results remain unchanged by linearity. In particular the homogenized dielectric equation becomes

$$\begin{cases} \operatorname{div}(\varepsilon^h \nabla\varphi - \sum_{p=1,\dots,N} a_p f_p) = 0 \\ \varphi = \phi \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

with $a_p \in \mathbb{R}^N$ defined as (see (2.10))

$$(a_p)_j := a_{jp} := \fint_{\mathscr{T}} \tau_p(y) \cdot \nabla w_j(y)\, dy = \fint_{\mathscr{T}} \nabla\psi_p(y) \cdot (\vec{e}_j + \nabla\omega_j(y))\, dy = -\fint_{\mathscr{T}} g_p(y)\omega_j(y)\, dy. \tag{5.2}$$

In (5.2), $\tau_p = \nabla\psi_p$ with ψ_p defined as ψ was in (2.2) upon replacing g by g_p.

We can thus view a as a $N \times N$-matrix with j, p coefficient a_{jp}. Note that that matrix is not necessarily symmetric.

Active charges, if they exist, consist in an appropriate choice of f_p so that the homogenized dielectric displays an enhancement (or degradation) of its permittivity. In other words, one would like to choose $f_p = \partial\varphi/\partial x_p$ so that φ is the solution to (5.1). Then (5.1) reads as

$$\begin{cases} \operatorname{div} \tilde{\varepsilon}^h \nabla\varphi = 0 \\ \varphi = \phi \text{ on } \partial\Omega \\ \tilde{\varepsilon}^h := \varepsilon^h - a. \end{cases} \tag{5.3}$$

Furthermore, if desiring electric enhancement, and not electric degradation, one should ensure that $\tilde{\varepsilon}^h$ admits at least one positive eigenvalue with a value greater than those of ε^h.

In this two-step process, one should first ensure existence of active charges, that is existence of a field φ that satisfies (5.3). This amounts to choosing g so that $\tilde{\varepsilon}^h$ is strongly elliptic. To this aim, take

$$g_p \equiv \omega_p$$

and note that, provided that (2.21), (3.1), (3.2) hold true, we are indeed in the setting of Section 3 thanks to (3.10). Then from (5.2)

$$a_{jp} = a_{pj} = -\fint_{\mathscr{T}} \omega_j(y)\omega_p(y) \ dy,$$

so that, for any $\xi \in \mathbb{R}^N \neq 0$, $-\sum_{j,p=1,\dots,N} a_{jp}\xi_j\xi_p = \fint_{\mathscr{T}} (\sum_{k=1,\dots,N} \omega_k(y)\xi_k)^2 \ dy > 0$ unless $\omega(y) \perp \xi$, a.e. on $\mathscr{T}$, where $\omega(y)$ is the vector with components $\omega_k(y)$, $k = 1, \dots, N$. In that case, taking $\xi/|\xi|$ as first unit vector we find that $\omega_1(y) \equiv 0$ on $\mathscr{T}$ and thus, recalling (2.7) and the canonical identification between $\mathscr{T}$ and the Y-periodic paving of $\mathbb{R}^N$, that,

$$(\varepsilon_{\mathscr{M}} - \varepsilon_I)\nu_1(y) = 0 \text{ on } \partial M \cap \partial I$$

where ν is the exterior normal to ∂M. So M is a cylinder with axis parallel to ξ. But, in such a case, $\mathscr{M}$ will not satisfy the last assumption in (3.1). Consequently it will always be so that

$$a \text{ is symmetric negative definite.} \tag{5.4}$$

Upon multiplication of $g_p = \omega_p$ by a large enough factor λ independent of p we conclude to the existence of large enough charges such that $\tilde{\varepsilon}^h$ is a symmetric positive definite matrix whose eigenvalues can be arbitrarily large upon choosing λ large enough. Thus, we can always solve (5.3) and obtain a large enhancement.

We have proved the following

Proposition 5.1. *Under assumptions* (2.21), (3.1), (3.2), (5.4) *always holds true if choosing g_p to be $\lambda\omega_p$ for $p = 1, \dots, N$ and $\lambda > 0$. Then, $\tilde{\varepsilon}^h$ can have arbitrarily large positive eigenvalues with an appropriate choice of λ.*

Of course, Proposition 5.1 provides no answer to the more useful question of finding a manufacturable set of micro-charges such that enhancement can occur. Indeed, Proposition 5.1 relies on the knowledge of the correctors ω_p. There are two possible routes if one wishes to take advantage of this observation in practice. On the one hand, one can resort to numerical simulations of ω_p, and use the output to devise a distribution of charges that will enhance the permittivity. On the other hand, one can resort to asymptotic analysis for some specific geometries. This is what we will do in the next subsection devoted to the case of periodically distributed small charged inclusions.

5.2. Enhancement for dilute inclusions

The following is inspired by the Clausius-Mossotti formula for dilute spherical inclusions. In this paragraph, C denotes a finite constant that may change from line to line but which is independent of λ and ℓ (see below). Let B denote the unit ball, and $B_{1+\eta}$ the ball of radius $1 + \eta$ for some fixed $\eta > 0$. We shall confine charges to the intermediate phase $B_{1+\eta} \setminus B$ (the coating). We need an additional scale $\ell \geq 1$ to quantify dilution, and define ε_ℓ to be the permittivity tensor associated with the ℓ-periodic extension of the map defined on $\ell[-\frac{1}{2}, \frac{1}{2})^N$ by

$$\varepsilon_\ell(y) := (1 + (\bar{\varepsilon} - 1)\chi_B(y))\mathsf{Id},$$

which models a background medium of permittivity 1 perturbed by spherical inclusions of a medium of permittivity $\bar{\varepsilon}$ centered on the grid $(\ell\mathbb{Z})^d$. In particular, the density of inclusions is $\ell^{-d}|B|$. We correspondingly set in $\mathbb{R}^N$

$$\varepsilon_\infty(x) := (1 + (\bar{\varepsilon} - 1)\chi_B(x))\mathsf{Id}.$$

When ℓ is very large, Clausius and Mossotti [7,28,29] argued that at first order the spherical inclusions do not interact with each other. Denote by $\omega_{\ell p}$ the ℓ-periodic corrector associated with ε_ℓ and direction $\vec{e_p}$ as in (2.7), and denote by $\omega_{\infty p}$ the solution of the "Maxwell" [27, Chapter IX] or "Eshelby" problem [14]

$$\begin{cases} \operatorname{div} \varepsilon_\infty \nabla(\omega_{\infty p} + x_p) = 0 \\ \omega_{\infty p}(x) \overset{|x|\to\infty}{\longrightarrow} 0. \end{cases} \tag{5.5}$$

Equation (5.5) can be solved explicitly, yielding, for $|x| \geq 1$

$$\omega_{\infty p}(x) = \left(\frac{1 - \bar{\varepsilon}}{\bar{\varepsilon} + N - 1}\right)\frac{x_p}{|x|^N}. \tag{5.6}$$

On the other hand, as proved in [34], $\omega_{\ell,p}$ is close to $\omega_{\infty p}$ on $B_{1+\eta}$ in the sense that for $\ell \geq 1$ we have

$$\|\nabla(\omega_{\infty p} - \omega_{\ell p})\|_{L^2(B_{1+\eta})} \leq C\ell^{-N}. \tag{5.7}$$

We then take $g_{\ell p}$ as the ℓ-periodic extension of the map defined on $\ell[-\frac{1}{2}, \frac{1}{2})^N$ via

$$g_{\ell p}(y) := \omega_{\infty p}(y)\chi_{B_{1+\eta}\setminus B}(y),$$

which has vanishing average in view of (5.6). Define the matrix $\bar{a}$ by

$$\bar{a}_{pj} := \int_{B_{1+\eta}\setminus B} \omega_{\infty p}(y)\omega_{\infty j}(y)\, dy, \ 1 \leq p, j \leq N,$$

so that

$$\bar{a} = \left(\frac{1 - \bar{\varepsilon}}{\bar{\varepsilon} + N - 1}\right)^2 1/N \int_{B_{1+\eta}\setminus B} |x|^{2(1-N)}\, dx\ \mathsf{Id}$$

$$= \left(\frac{1 - \bar{\varepsilon}}{\bar{\varepsilon} + N - 1}\right)^2 \begin{Bmatrix} \eta : N = 1 \\ \pi\log(1 + \eta) : N = 2 \\ \frac{|\mathcal{S}^{N-1}|}{N}(1 - (1 + \eta)^{2-N}) : N > 2 \end{Bmatrix} \mathsf{Id}, \tag{5.8}$$

where $|\mathcal{S}^{N-1}|$ denotes the surface of the unit sphere in dimension N. The combination of (5.6) and (5.7) together with the Poincaré-Wirtinger inequality allows us to conclude that a_ℓ defined as in (5.2) is quantitatively close to $-\ell^{-N}\bar{a}$, namely,

$$|a_{\ell pj} + \ell^{-N}\bar{a}_{pj}| = \ell^{-N}\left|\int_{B_{1+\eta}\setminus B} \omega_{\infty p}(y)(\omega_{\infty j} - \omega_{\ell j})\, dy\right| + C\ell^{-2N} \leq C\ell^{-2N},$$

from which we deduce that for all $\ell \geq 1$ large enough, a_ℓ is diagonalizable with negative eigenvalues of order ℓ^{-N}. Upon multiplying $g_{\ell p}$ by the factor $\ell^N\lambda$ for some $\lambda \geq 1$, one obtains for $\tilde{\varepsilon}_\ell^h$ defined as in (5.3) (and ε_ℓ^h defined as in (2.9))

$$\left|\tilde{\varepsilon}_\ell^h - (\varepsilon_\ell^h + \lambda\bar{a})\right| \leq C\ell^{-N}\lambda,$$

with $\bar{a}$ defined in (5.8). Note that here ε_ℓ^h is the homogenized permittivity associated with a spherical inclusion of radius 1 in a cell of side-length ℓ, or, equivalently, a spherical inclusion of radius $1/\ell$ in the unit

cell Y. Since ε_ℓ^h is of order 1 (uniformly in ℓ), for ℓ large enough (in function of C), all the eigenvalues of $\tilde{\varepsilon}_\ell^h$ are larger than $\frac{1}{C}\lambda$, which yields the claimed enhancement.

Let us give the leading order part of f in the regime $\lambda, \ell^N \geq 1$. Since $\bar{a}$ is a multiple of the identity, the equation for φ takes the form

$$
\begin{cases}
\operatorname{div}(\operatorname{Id} + \kappa_{\lambda\ell})\nabla\varphi = 0 \\
\varphi = \phi \text{ on } \partial\Omega
\end{cases}
\tag{5.9}
$$

with $\|\kappa_{\lambda\ell}\|_\infty \leq C(\ell^{-N} + \lambda^{-1})$ so that $\|f_p - \partial\tilde{\varphi}/\partial x_p\|_{L^2(\Omega)} \leq C(\ell^{-N} + \lambda^{-1})$ with $\tilde{\varphi}$ solving

$$
\begin{cases}
-\triangle\tilde{\varphi} = 0 \\
\tilde{\varphi} = \phi \text{ on } \partial\Omega.
\end{cases}
\tag{5.10}
$$

We conclude this subsection with an investigation of the impact of the dielectric enhancement on the elastic response of dilute inclusions.

In the statement of Theorem 4.1, which yields the homogenized equation for the displacement field, the only contribution of the active charges $\sum_{p=1}^N \lambda\ell^N g_{\ell p}$ are the terms $\sum_{p=1}^N (2f_p N_{\lambda\ell}^{ph}\nabla\varphi + P_{\lambda\ell}^{ph} f_p^2)$ involving the tensors $N_{\lambda\ell}^{ph}$ and $P_{\lambda\ell}^{ph}$ in the forcing Z; cf. (4.9) and (4.10) (the sum over p follows by linearity). In the setting of the example of dilute inclusions, these tensors take the form

$$
\begin{cases}
(N_{\lambda\ell}^{hp})_{ijk} := \lambda\ell^N \fint_{\ell\mathcal{T}} M_\ell(y)(\nabla w_{\ell k}(y) \otimes \nabla\theta_{\ell p}(y)) \cdot \nabla W_{\ell ij}(y) \, dy \\
(P_{\lambda\ell}^{hp})_{ij} := (\lambda\ell^N)^2 \fint_{\ell\mathcal{T}} M_\ell(y)(\nabla\theta_\ell^p(y) \otimes \nabla\theta_{\ell p}(y)) \cdot \nabla W_{\ell ij}(y) \, dy,
\end{cases}
\tag{5.11}
$$

where $\theta_{\ell p}$ solves (see (2.13))

$$
\begin{cases}
\operatorname{div}(\varepsilon_\ell \nabla\theta_{\ell p}) = g_{\ell p} \\
\int_{\ell\mathcal{T}} \theta_{\ell p}(y) \, dy = 0.
\end{cases}
\tag{5.12}
$$

Let us determine the scalings of these contributions. We claim that $N_{\lambda\ell}^{hp}$ is at most of order λ whereas $P_{\lambda\ell}^{hp}$ is at most of order $\lambda^2\ell^N$.

We start with $N_{\lambda\ell}^{hp}$ and recall the following properties on w_ℓ and W_ℓ

$$
\nabla w_{\ell k}(y) = e_k + \nabla\omega_{\ell k}, \int_{\ell\mathcal{T}} |\nabla\omega_{\ell k}|^2 \leq C, \nabla W_{\ell ij}(y) = e_i \otimes e_j + \nabla X_{\ell ij}, \int_{\ell\mathcal{T}} |\nabla X_{\ell ij}|^2 \leq C,
$$

which follow from energy estimates (note that the bound is uniform with respect to ℓ). Likewise, we have $\int_{\ell\mathcal{T}} |\nabla\theta_{\ell p}|^2 \leq C$. This allows us to split the contribution in $(N_{\lambda\ell}^{hp})_{ijk}$ into four parts:

$$
(N_{\lambda\ell}^{hp})_{ijk} = \lambda\ell^N \fint_{\ell\mathcal{T}} M_\ell(y)(e_k \otimes \nabla\theta_{\ell p}(y)) \cdot e_i \otimes e_j \, dy
$$

$$
+ \lambda\ell^N \fint_{\ell\mathcal{T}} M_\ell(y)(\nabla\omega_{\ell k}(y) \otimes \nabla\theta_{\ell p}(y)) \cdot e_i \otimes e_j \, dy
$$

$$
+ \lambda\ell^N \fint_{\ell\mathcal{T}} M_\ell(y)(e_k \otimes \nabla\theta_{\ell p}(y)) \cdot \nabla X_{\ell ij}(y) \, dy
$$

$$+\lambda \ell^N \fint_{\ell \mathscr{T}} M_\ell(y)(\nabla \omega_{\ell k}(y) \otimes \nabla \theta_{\ell p}(y)) \cdot \nabla X_{\ell i j}(y) \ dy.$$

We treat the second and third terms alike using Cauchy-Schwarz' inequality to the effect that

$$\left| \int_{\ell \mathscr{T}} M_\ell(y)\Big((\nabla \omega_{\ell k}(y) \otimes \nabla \theta_{\ell p}(y)) \cdot e_i \otimes e_j + (e_k \otimes \nabla \theta_{\ell p}(y)) \cdot \nabla X_{\ell i j}(y)\Big) \ dy \right|$$

$$\leq C \Big(\int_{\ell \mathscr{T}} |\nabla \theta_{\ell p}|^2 \Big)^{\frac{1}{2}} \Big(\int_{\ell \mathscr{T}} |\nabla X_{\ell i j}|^2 + |\nabla \omega_{\ell k}|^2 \Big)^{\frac{1}{2}} \leq C.$$

For the first term we use that $\nabla \theta_{\ell p}$ integrate to zero on $\ell \mathscr{T}$ by periodicity and the specific form $M_\ell = M_1 + (M_2 - M_1)\chi_B$, followed by Cauchy-Schwarz' inequality, so that

$$\left| \int_{\ell \mathscr{T}} M_\ell(y)(e_k \otimes \nabla \theta_{\ell p}(y)) \cdot e_i \otimes e_j \ dy \right| = \left| \int_B M_2(e_k \otimes \nabla \theta_{\ell p}(y)) \cdot e_i \otimes e_j \ dy \right|$$

$$\leq C \Big(\int_B |\nabla \theta_{\ell p}|^2 \Big) \leq C.$$

For the fourth term we need to use more information on $\nabla \theta_{\ell p}$. By (3.10) and the analogue of (3.6), (3.9), we have for all $y \in \ell \mathscr{T}$

$$|\nabla \theta_{\ell p}(y)| \leq C \Big(1 + \int_{\ell \mathscr{T}} |\nabla \theta_{\ell p}|^2 \Big)^{\frac{1}{2}} \leq C.$$

Hence, using $|ab| \leq \frac{1}{2}(a^2 + b^2)$,

$$\left| \int_{\ell \mathscr{T}} M_\ell(y)(\nabla \omega_{\ell k}(y) \otimes \nabla \theta_{\ell p}(y)) \cdot \nabla X_{\ell i j}(y) \ dy \right| \leq C \int_{\ell \mathscr{T}} |\nabla \omega_{\ell k}|^2 + |\nabla X_{\ell i j}|^2 \leq C.$$

We have thus proved that $|N_{\lambda \ell}^{hp}| \leq C\lambda$. The argument to control $P_{\lambda \ell}^{hp}$ is similar and we obtain $|P_{\lambda \ell}^{hp}| \leq C\lambda^2 \ell^N$.

It remains to check that $P_{\lambda \ell}^{hp}$ is indeed of order $\lambda^2 \ell^N$. To this aim, it is enough to replace correctors by their explicit approximation using the single-inclusion problem on the whole space which we denote by $X_{\infty i j}$ and $\theta_{\infty p}$. We then define

$$P_{\infty i j}^{hp} := \int_{\mathbb{R}^N} M_\infty(y)(\nabla \theta_{\infty p}(y) \otimes \nabla \theta_{\infty p}(y)) \cdot \nabla W_{\infty i j}(y) \ dy$$

On the one hand, a direct calculation using explicit formulas (see [36, Section 17.2.1]) shows that P_∞^{hp} is of order 1 and that $|(\nabla \theta_{\infty p}(y) \otimes \nabla \theta_{\infty p}(y)) \cdot \nabla W_{\infty i j}(y)| \leq C(1 + |y|)^{-2N}$. On the other hand, by [34], one has

$$\|\nabla(X_{\ell i j} - X_{\infty i j})\|_{L^2(\ell Y)} + \|\nabla(\theta_{\ell p} - \theta_{\infty p})\|_{L^2(\ell Y)} \leq C\ell^{-\frac{N}{2}}$$

so that $|P_{\lambda \ell}^{hp} - \lambda^2 \ell^N P_\infty^{hp}| \leq C\lambda^2 \ell^{\frac{N}{2}}$, and therefore $\frac{1}{C}\lambda^2 \ell^N \leq \|P_{\lambda \ell}^{hp}\| \leq C\lambda^2 \ell^N$.

As we did above for φ when $\lambda, \ell^N \geq 1$, one can identify the leading order contribution $\tilde{u}$ to the solution u of (4.7) in the sense that $\|u - \lambda^2 \ell^N \tilde{u}\|_{H^1(\Omega)} \leq C(\lambda^2 + \lambda \ell^N)$, where $\tilde{u}$ solves

$$\begin{cases} \operatorname{div}(L_\ell^h \nabla \tilde{u}) = -\operatorname{div} \bar{P}_\ell^p f_p^2 \\ \tilde{u} = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.13}$$

where $(\bar{P}_\ell^p)_{ij} =: (\lambda^2 \ell^N)^{-1} (P_{\lambda\ell}^{hp})_{ij}$ is of order 1, as well as $\|f_p\|_{L^2(\Omega)}$. In particular, $\|\tilde{u}\|_{H^1(\Omega)}$ is of order 1, and the above yields enhancement of elastostriction by a factor $\lambda^2 \ell^N$ within an error of order $\lambda^2 + \lambda\ell^N$ (which is relatively small with respect to $\lambda^2 \ell^N$ by a factor $\ell^{-N} + \lambda^{-1}$).

Acknowledgments

Appendix A. Extension to the random setting

For simplicity, we only consider the dielectric problem (the coupling to elasticity is straightforward once the needed regularity results are proved) and we place ourselves in a setting similar to the example of Section 5 with spherical inclusions. Let $\mathcal{P} = \{x_n, n \in \mathbb{N}\}$ be a stationary ergodic point process on $\mathbb{R}^N$ such that for all $n \neq m$, $|x_n - x_m| \geq 2 + \delta$ for some deterministic $\delta > 0$ almost surely (we denote by $\mathbb{E}[\cdot]$ the associated expectation), and $\mathscr{I} = \cup_{x \in \mathcal{P}} B(x)$ denote the (random) set of inclusions in $\mathbb{R}^N$. We then define

$$\varepsilon : x \mapsto \mathsf{Id} + (\bar{\varepsilon} - 1)\mathsf{Id}\chi_{\mathscr{I}}.$$

In this appendix, C denotes a constant that may change from line to line, depends on N, Ω, $\bar{\varepsilon}$, controlled norms of ϕ and f, and the law of $\mathcal{P}$, but which is independent of g and ψ (see below), unless otherwise explicitly stated (using subscripts).

A.1. Definition of g and ψ

We start with the definition of g and ψ, cf. (2.2) in the periodic setting. For conciseness, for all $x \in \mathbb{R}^N$ we denote by $\mathcal{B}(x)$ the ball $B_{1+\delta/2}(x)$. In particular, by definition, the balls $\{\mathcal{B}(x)\}_{x \in \mathcal{P}}$ are at distance at least $\delta/2$ from one another. We assume that $g : \mathbb{R}^N \to \mathbb{R}^N$ is a stationary random field supported on $\cup_{x \in \mathcal{P}} \mathcal{B}(x)$ and satisfying $\int_{\mathcal{B}(x)} g = 0$ for all $x \in \mathcal{P}$ and $\|g\|_{L^\infty(\mathbb{R}^N)} < \infty$. Under this specific assumption, there exists a stationary field $\nabla\psi \in L^2_{\text{loc}}(\mathbb{R}^N)$ with vanishing expectation $\mathbb{E}[\nabla\psi] = 0$ and finite second moment $\mathbb{E}[|\nabla\psi|^2] < \infty$ satisfying almost surely

$$\triangle\psi(y) = g(y). \tag{A.1}$$

Let us give the short argument in favor of the well-posedness of (A.1) for completeness. As customary in the field, we first add a massive regularization of order $T \gg 1$ and consider the equation on $\mathbb{R}^N$

$$\frac{1}{T}\psi_T(y) - \triangle\psi_T(y) = -g(y), \tag{A.2}$$

which is well-posed in the space $H^1_{\text{uloc}}(\mathbb{R}^N) = \{\zeta \in H^1_{\text{loc}}(\mathbb{R}^N) \mid \sup_{x \in \mathbb{R}^N} \int_{B(x)} (\zeta^2 + |\nabla\zeta|^2) < \infty\}$. The argument is standard (see e.g. [19, Lemma 2.7]): we first solve the equation on balls B_R with homogeneous Dirichlet boundary conditions, and pass to the limit $R \nearrow +\infty$ using a uniform a priori bound in $H^1_{\text{uloc}}(\mathbb{R}^N)$. This bound relies on the Caccioppoli inequality, which we presently work out in our setting. We display the argument in the whole space, assuming that all the quantities that appear are finite. The argument is

the same for the approximations on the balls B_R (for which all quantities are indeed finite). Let $\eta_T : x \mapsto \exp(-c|x|/\sqrt{T})$, and test the equation with $\eta_T^2 \psi_T$. This yields after integration by parts and rearranging the terms

$$\int_{\mathbb{R}^N} \frac{1}{T} \psi_T^2 \eta_T^2 + \int_{\mathbb{R}^N} \eta_T^2 |\nabla \psi_T|^2 = -\int_{\mathbb{R}^N} g \eta_T^2 \psi_T - 2 \int_{\mathbb{R}^N} \eta_T \psi_T \nabla \eta_T \cdot \nabla \psi_T.$$

The second term is standard: since $|\nabla \eta_T| \leq \frac{c}{\sqrt{T}} \eta_T$, we have for c small enough,

$$\left| 2 \int_{\mathbb{R}^N} \eta_T \psi_T \nabla \eta_T \cdot \nabla \psi_T \right| \leq \frac{1}{4} \int_{\mathbb{R}^N} \eta_T^2 |\nabla \psi_T|^2 + 4 \int_{\mathbb{R}^N} |\nabla \eta_T|^2 \psi_T^2 \leq \frac{1}{4} \int_{\mathbb{R}^N} \eta_T^2 |\nabla \psi_T|^2 + \frac{1}{4} \int_{\mathbb{R}^N} \frac{1}{T} \psi_T^2 \eta_T^2.$$

We then use the specific properties of g to reformulate the first term as

$$\int_{\mathbb{R}^N} g \eta_T^2 \psi_T = \sum_{n \in \mathbb{N}} \int_{\mathcal{B}(x_n)} g(\eta_T^2 \psi_T - \gamma_n)$$

where the γ_n's are arbitrary constants (since g has vanishing average on the $\mathcal{B}(x_n)$'s). By Poincaré-Wirtinger's inequality on the $\mathcal{B}(x_n)$'s, we thus have

$$\left| \int_{\mathcal{B}(x_n)} g(y)(\eta_T^2 \psi_T - \gamma_n) \right| \leq C \left(\int_{\mathcal{B}(x_n)} g^2 \right)^{\frac{1}{2}} \left(\int_{\mathcal{B}(x_n)} |\nabla(\eta_T^2 \psi_T)|^2 \right)^{\frac{1}{2}}$$

and we expand the second factor as, using again that $|\nabla \eta_T| \leq \frac{c}{\sqrt{T}} \eta_T$,

$$\int_{\mathcal{B}(x_n)} |\nabla(\eta_T^2 \psi_T)|^2 \leq C \int_{\mathcal{B}(x_n)} \eta_T^4 |\nabla \psi_T|^2 + \eta_T^2 |\nabla \eta_T|^2 \psi_T^2 \leq \left(\sup_{\mathcal{B}(x_n)} \eta_T^2 \right) \int_{\mathcal{B}(x_n)} \eta_T^2 |\nabla \psi_T|^2 + \eta_T^2 \frac{c^2}{T} \psi_T^2.$$

By the inequality $ab \leq \frac{1}{2}(\frac{1}{C^2} a^2 + C^2 b^2)$ for an appropriate constant C, this yields

$$\left| \int_{\mathcal{B}(x_n)} g(y)(\eta_T^2 \psi_T - \gamma_n) \right| \leq C \sup_{\mathcal{B}(x_n)} \eta_T^2 \int_{\mathcal{B}(x_n)} g^2 + \frac{1}{4} \int_{\mathcal{B}(x_n)} \frac{1}{T} \eta_T^2 \psi_T^2 + \eta_T^2 |\nabla \psi_T|^2.$$

Altogether, these estimates combine to the a priori estimate

$$\int_{\mathbb{R}^N} \frac{1}{T} \psi_T^2 \eta_T^2 + \int_{\mathbb{R}^N} \eta_T^2 |\nabla \psi_T|^2 \leq C \|g\|_{L^\infty(\mathbb{R}^N)}^2 \sum_{n \in \mathbb{N}} \sup_{\mathcal{B}(x_n)} \eta_T^2. \tag{A.3}$$

From this, it is now standard to deduce that there exists a unique random field ψ such that $\nabla \psi$ is stationary and has finite second moment

$$\mathbb{E}[|\nabla \psi|^2]^{\frac{1}{2}} \leq C \mathbb{E}[\chi_{\mathscr{I}}(0)] \|g\|_{L^\infty(\mathbb{R}^N)}, \tag{A.4}$$

and ψ solves (A.1) almost surely in the distributional sense, cf. [33].

We conclude with a quick argument in favor of the additional a priori bound

$$\mathbb{E}[|\nabla^2 \psi|^2] \leq \mathbb{E}[g^2]. \tag{A.5}$$

Our starting point is (A.2) which is satisfied almost surely on $\mathbb{R}^N$ and implies that $\nabla^2 \psi_T \in L^2_{\text{loc}}(\mathbb{R}^N)$. Again, it is enough to prove an a priori estimate on $\nabla^2 \psi_T$ and pass to the limit $T \nearrow +\infty$. Up to proceeding at the level of approximations on balls B_R, we may assume that all the quantities involved below are finite. We then test (A.2) with $-\eta_T^2 \triangle \psi_T$. Treating the massive term $\frac{1}{T}\eta_T^2 \psi_T \triangle \psi_T$ as above, and using (A.3), this yields

$$\int\limits_{\mathbb{R}^N} \eta_T^2 (\triangle \psi_T)^2 \leq \int\limits_{\mathbb{R}^N} \eta_T^2 |g||\triangle \psi_T| + \frac{C}{T}\|g\|^2_{L^\infty(\mathbb{R}^N)} \sum_{n \in \mathbb{N}} \sup_{\mathcal{B}(x_n)} \eta_T^2,$$

which, using $\int_{\mathbb{R}^N} \eta_T^2 |g||\triangle \psi_T| \leq \frac{1}{2}\int_{\mathbb{R}^N} \eta_T^2 (g^2 + (\triangle \psi)^2)$ and absorbing the second term in the left-hand side, implies

$$\int\limits_{\mathbb{R}^N} \eta_T^2 (\triangle \psi_T)^2 \leq \int\limits_{\mathbb{R}^N} \eta_T^2 g^2 + \frac{2C}{T}\|g\|^2_{L^\infty(\mathbb{R}^N)} \sum_{n \in \mathbb{N}} \sup_{\mathcal{B}(x_n)} \eta_T^2.$$

It remains to reformulate the left-hand side to recognize the Hessian. After two integrations by parts, we have

$$\int\limits_{\mathbb{R}^N} \eta_T^2 (\triangle \psi_T)^2 = \int\limits_{\mathbb{R}^N} \eta_T^2 |\nabla^2 \psi_T|^2 + 2 \sum_{ij} \int\limits_{\mathbb{R}^N} \eta_T \partial_j \eta_T \partial_j \psi_T \partial_{ij}^2 \psi_T - 2 \sum_i \eta_T \partial_i \eta_T \partial_i \psi_T \triangle \psi_T,$$

which we rewrite, using that $|\nabla \eta_T| \leq \frac{c}{\sqrt{T}}\eta_T$ for c small enough, as

$$\int\limits_{\mathbb{R}^N} \eta_T^2 |\nabla^2 \psi_T|^2 \leq (1 + \frac{C}{T}) \int\limits_{\mathbb{R}^N} \eta_T^2 (\triangle \psi_T)^2 + \frac{C}{T} \int\limits_{\mathbb{R}^N} \eta_T^2 |\nabla \psi_T|^2$$

for some C depending only on N and c. We have thus proved

$$\int\limits_{\mathbb{R}^N} \eta_T^2 |\nabla^2 \psi_T|^2 \leq (1 + \frac{C}{T}) \int\limits_{\mathbb{R}^N} \eta_T^2 g^2 + \frac{C}{T}\|g\|^2_{L^\infty(\mathbb{R}^N)} \sum_{n \in \mathbb{N}} \sup_{\mathcal{B}(x_n)} \eta_T^2.$$

Taking the expectation and letting $T \nearrow +\infty$ yields the claim (A.5).

A.2. Qualitative homogenization of the dielectrics

Once g and ψ are defined as above, the proof of the qualitative homogenization of the dielectrics follows the proof of Section 2, replacing periodicity by stationarity – the adaptation is standard and left to the reader (see e.g. [22, Chapter 7] or [33]).

In particular, (2.9) is replaced by $\varepsilon^h \vec{e}_j := \mathbb{E}[\varepsilon \nabla w_j]$, and the formula (2.10) for a takes the form $a_j := \mathbb{E}[\nabla \psi \cdot \nabla w_j]$.

A.3. Improved integrability in homogenization of the dielectrics

In this paragraph we extend the results of Section 3 to the random setting. Since the results are based on large-scale regularity for random elliptic operators, we need to make some quantitative mixing assumptions on the point process $\mathcal{P}$. In particular, a hardcore Poisson point process or the random parking measure will do, cf. [12,18].

We start with the extension of Proposition 3.1.

Proposition A.1. *For all $2 \leq q, q' < \infty$, the sequence $\nabla \varphi^\delta$ satisfies $\mathbb{E}[\|\nabla \varphi^\delta\|_{L^q(\Omega;\mathbb{R}^N)}^{qq'}] \leq C_{q,q',g}$, uniformly in δ. In particular, $\nabla \varphi^\delta$ is bounded in $L^q(\Omega;\mathbb{R}^N)$ along any subsequences of δ almost surely.*

Proof. Step 1. First, we apply the large-scale Calderón-Zygmund estimates of [1, Theorem 7.7] to (3.3). In this random setting, the estimate involves a stationary random field $r_* \geq 1$ on $\mathbb{R}^N$, which, in the examples considered above, satisfies $\mathbb{E}[\exp(\frac{1}{C} r_*(0))] \leq 2$ for some finite constant C, cf. [18, Theorem 4] (by stationarity, this holds for $r_*(0)$ replaced by $r_*(x)$ for all $x \in \mathbb{R}^N$). We then obtain for all $q \geq 2$, with the short-hand notation $B_{*,\delta}(x) := B_{\delta r_*(\frac{x}{\delta})}(x)$

$$\int_\Omega \left(\fint_{B_{*,\delta}(x)} \chi_\Omega(z) |\nabla(\varphi^\delta - \phi)|^2(z) \, dz \right)^{\frac{q}{2}} dx \leq C_q \left(\|\nabla(\varphi^\delta - \phi)\|_{L^2(\Omega)}^q + \|f\tau^\delta - \varepsilon^\delta \nabla \phi + \nabla \zeta^\delta\|_{L^q(\Omega)}^q \right).$$

As in the periodic setting, since ζ^δ is the unique $H_0^1(\Omega)$-solution to $\triangle \zeta^\delta = \nabla f \cdot \tau^\delta$, we have by maximal regularity for the Laplacian

$$\|f\tau^\delta - \varepsilon^\delta \nabla \phi + \nabla \zeta^\delta\|_{L^q(\Omega)}^q \leq C \left((\|f\|_{C^{1,\alpha}(\Omega)} + 1) \|\tau^\delta\|_{L^q(\Omega)} + \|\phi\|_{C^{1,\alpha}(\Omega)} \right)^q$$

and we have to control $\|\tau^\delta\|_{L^q(\Omega)}$. As opposed to the periodic setting, this is a random quantity. We proceed in two steps. First, taking the derivative of (A.1) and using deterministic Calderón-Zygmund estimates for the Laplacian, we have for all $R \geq 1$

$$\fint_{B_R} |\nabla^2 \psi|^{qq'} \leq C_{q,q'} \left(\left(\fint_{B_{2R}} |\nabla^2 \psi|^2 \right)^{qq'/2} + \fint_{B_R} |g|^{qq'} \right)$$

so that by taking the limit $R \nearrow +\infty$ we obtain by stationarity of these random fields and the ergodic theorem

$$\mathbb{E}[|\nabla^2 \psi|^{qq'}] \leq C_{q,q'} \left(\mathbb{E}[|\nabla^2 \psi|^2]^{qq'/2} + \mathbb{E}[|g|^{qq'}] \right).$$

Using (A.5) and Hölder's inequality in probability, this turns into $\mathbb{E}[|\nabla^2 \psi|^{qq'}] \leq C_{q,q'} \mathbb{E}[|g|^{qq'}]$. Combined with (A.4) and Poincaré's inequality in form of $\|\nabla \psi\|_{L^{qq'}(B)} \leq C \left(\|\nabla^2 \psi\|_{L^{qq'}(B_2)} + \|\nabla \psi\|_{L^2(B)} \right)$, and using the ergodic theorem as above, this yields

$$\mathbb{E}[|\nabla \psi|^{qq'}] \leq C_{q,q'} \|g\|_{L^\infty(\mathbb{R}^N)}^{qq'}, \tag{A.6}$$

that is, the desired control $\mathbb{E}[\|\tau^\delta\|_{L^q(\Omega)}^{qq'}] \leq C_{q,q'} \|g\|_{L^\infty(\mathbb{R}^N)}^{qq'}$. As in the periodic setting, we also have

$$\int_\Omega \left(\fint_{B_{*,\delta}(x)} \chi_\Omega |\nabla \phi|^2 \, dz \right)^{\frac{q}{2}} dx \leq \int_{\mathbb{R}^N} \fint_{B_{*,\delta}(x)} \chi_\Omega |\nabla \phi|^q \, dz dx \leq C \int_{\mathbb{R}^N} \chi_\Omega |\nabla \phi|^q dx = \int_\Omega |\nabla \phi|^q \, dx,$$

where we used [18, (140)] in form of $\int_{\mathbb{R}^N} \fint_{B_{*,\delta}} \sim \int_{\mathbb{R}^N}$. Hence, we have proved that

$$\mathbb{E}\left[\left(\int_\Omega \left(\fint_{B_{*,\delta}(x)} \chi_\Omega |\nabla \varphi^\delta|^2 \, dz \right)^{\frac{q}{2}} dx \right)^{q'} \right] \leq C_{q,q',g} \tag{A.7}$$

for some finite constant $C_{q,q'}$ depending on q, q' and on $\|f\|_{C^{1,\alpha}(\Omega)}$, $\|\phi\|_{C^{1,\alpha}(\Omega)}$, and $\|g\|_{L^\infty}$. As opposed to the periodic setting, we have to reformulate this estimate in order to remove the stochastic dependence of the local averages upon the random field r_*. The rest of this step is dedicated to the proof of

$$\mathbb{E}\left[\left(\int_\Omega \left(\fint_{B_\delta(x)} \chi_\Omega |\nabla\varphi^\delta|^2 \, dz\right)^{\frac{q}{2}} dx\right)^{q'}\right] \leq C_{q,q',\Omega,g}. \tag{A.8}$$

As customary in the field, this can be done at the price of some (arbitrarily small) loss of stochastic integrability (the dependence of the constants in (A.7) and (A.8) with respect to q and q' are different). Since we are not interested in the precise stochastic integrability in this contribution, we display an elementary (and suboptimal) proof of this improvement of (A.7). The argument relies on the estimate

$$\int_\Omega \left(\fint_{B_\delta(x)} \chi_\Omega |\nabla\varphi^\delta|^2 \, dz\right)^{\frac{q}{2}} dx \leq C\left(\frac{\delta \inf_\Omega r_*(\frac{\cdot}{\delta})^N}{|\Omega|} + 1\right) \int_\Omega r_*(\tfrac{x}{\delta})^{N(q-2)} \left(\fint_{B_{*,\delta}(x)} \chi_\Omega |\nabla\varphi^\delta|^2 \, dz\right)^{\frac{q}{2}} dx \tag{A.9}$$

in favor of which we presently argue (note that the averages on the left-hand side are made on balls of fixed radius δ). Following [18], we replace the integral of local averages by a sum on a partition. In particular, by [18, (139)], there exists a partition of $\mathbb{R}^N$ into a family of cubes $\mathcal{Q} := \{Q\}_Q$ such that $\sup_Q r_* \leq C \inf_Q r_*$, $\operatorname{diam}(Q) \sim \inf_Q r_*$ and for all functions $h \geq 0$ and exponents $\gamma \geq 1$ we have $\int_{\mathbb{R}^N} \left(\fint_{B_*(x)} h\right)^\gamma \sim \sum_Q |Q| (\fint_Q h)^\gamma$. Denote by $\mathcal{Q}_\delta(\Omega)$ the smallest subset of $\mathcal{Q}$ which contains $\frac{1}{\delta}\Omega$ in the sense that $\frac{1}{\delta}\Omega \subset \cup_{Q \in \mathcal{Q}_\delta(\Omega)} Q$. For convenience, we call $Q_\delta(x)$ the cube of radius δ centered at $x \in \mathbb{R}^N$. Then we have by the discrete $\ell^1 - \ell^\gamma$ estimate

$$\int_\Omega \left(\fint_{B_\delta(x)} h\chi_\Omega\right)^\gamma dx \leq C \sum_{Q \in \mathcal{Q}_\delta(\Omega)} \int_{\delta Q} \left(\fint_{Q_\delta(x)} h\chi_\Omega\right)^\gamma dx$$

$$\leq 3^N C \sum_{Q \in \mathcal{Q}_\delta(\Omega)} \operatorname{diam}(Q)^{(\gamma-1)N} \int_{\delta Q} \left(\fint_{\delta Q} h\chi_\Omega\right)^\gamma dx.$$

We now distinguish two cases: If $\sup_{Q \subset \mathcal{Q}_\delta(\Omega)} \operatorname{diam}(Q) \leq \frac{1}{\delta}\operatorname{diam}(\Omega)$, then [18, (146)] combined with the property $\sup_Q r_* \leq C \inf_Q r_*$ yields

$$\int_\Omega \left(\fint_{B_\delta(x)} h\chi_\Omega\right)^\gamma dx \leq C \int_\Omega r_*(\tfrac{x}{\delta})^{(\gamma-1)N} \left(\fint_{B_{*,\delta}(x)} h(z)\chi_\Omega(z) \, dz\right)^\gamma dx,$$

whereas if $\sup_{Q \subset \mathcal{Q}_\delta(\Omega)} \operatorname{diam}(Q) > \frac{1}{\delta}\operatorname{diam}(\Omega)$, then

$$\int_\Omega \left(\fint_{B_\delta(x)} h\chi_\Omega\right)^\gamma dx \leq C\left(\frac{\delta \inf_\Omega r_*(\frac{\cdot}{\delta})^N}{|\Omega|} + 1\right) \int_\Omega r_*(\tfrac{x}{\delta})^{(\gamma-1)N} \left(\fint_{B_{*,\delta}(x)} h(z)\chi_\Omega(z) \, dz\right)^\gamma dx.$$

Applied to $h = |\nabla\varphi^\delta|^2$ and $\gamma = \frac{q}{2}$ this proves (A.9). We conclude by deriving (A.8) from (A.9). To that end, we control the infimum of r_* on $\frac{1}{\delta}\Omega$ by its average, use several times Hölder's inequality in probability, the moment bound $\mathbb{E}[\exp(\frac{1}{C}r_*)] \leq 2$, and the triangle inequality in probability in the form $\mathbb{E}[(\int_\Omega |h|)^{q'}]^{\frac{1}{q'}} \leq \int_\Omega \mathbb{E}[|h|^{q'}]^{\frac{1}{q'}}$. This yields

$$\mathbb{E}\Big[\Big(\int_\Omega \Big(\fint_{B_\delta(x)} \chi_\Omega |\nabla\varphi^\delta|^2\Big)^{\frac{q}{2}} dx\Big)^{q'}\Big]^{\frac{1}{q'}}$$

$$\leq C\mathbb{E}\Big[\Big(1+\delta\fint_\Omega r_*(\tfrac{\cdot}{\delta})\Big)^{2q'}\Big]^{\frac{1}{2q'}} \mathbb{E}\Big[\Big(\int_\Omega r_*(\tfrac{x}{\delta})^{\frac{q-2}{2}N}\Big(\fint_{B_{*,\delta}(x)} \chi_\Omega|\nabla\varphi^\delta|^2\Big)^{\frac{q}{2}} dx\Big)^{2q'}\Big]^{\frac{1}{2q'}}$$

$$\leq C_{q,q'} \int_\Omega \mathbb{E}\Big[r_*(\tfrac{x}{\delta})^{Nq'(q-2)}\Big(\fint_{B_{*,\delta}(x)} \chi_\Omega|\nabla\varphi^\delta|^2\, dz\Big)^{qq'}\Big]^{\frac{1}{2q'}} dx$$

$$\leq C_{q,q'} \int_\Omega \mathbb{E}\Big[r_*(\tfrac{x}{\delta})^{2Nq'(q-2)}\Big]^{\frac{1}{4q'}} \mathbb{E}\Big[\Big(\fint_{B_{*,\delta}(x)} \chi_\Omega|\nabla\varphi^\delta|^2\, dz\Big)^{2qq'}\Big]^{\frac{1}{4q'}} dx$$

$$\leq C_{q,q',\Omega}\mathbb{E}\Big[\int_\Omega \Big(\fint_{B_{*,\delta}(x)} \chi_\Omega|\nabla\varphi^\delta|^2\, dz\Big)^{2qq'} dx\Big]^{\frac{1}{4q'}}.$$

Combined with (A.7), this entails (A.8), which enables us to control $\nabla\varphi^\delta$ on scales larger than δ. In a second step, we will derive an estimate for small scales, that is for scales smaller than δ.

Step 2. Take a point $x \in \Omega$ and consider the cube $Q_{2\delta}(x)$ of side-length 2δ centered at x. We blow up equation (2.4) so as to obtain an equation on $Q_2(0)$.

To that effect, we set $\Phi^\delta(z) := \frac{1}{\delta}\varphi^\delta(x+\delta z)$, which satisfies

$$\operatorname{div}_z(\varepsilon(\tfrac{x}{\delta}+z)\nabla_z\Phi^\delta - f(x+\delta z)\tau^\delta(x+\delta z)) = -\delta\tau^\delta(x+\delta z)\cdot\nabla f(x+\delta z) \text{ in } (\Omega-\{x\})/\delta \cap Q_2(0). \quad \text{(A.10)}$$

The only difference with the periodic setting is that τ^δ is now random. By elliptic regularity for the Laplacian in (A.1), we have

$$\|\nabla\psi\|_{C^{1,\alpha}(B(x))} \leq C\Big(\|\nabla\psi\|_{H^1(B_2(x))} + \|g\|_{C^{0,\alpha}(B_2(x))}\Big).$$

Using this bound, the same argument as in the periodic setting allows to conclude that

$$\sup_{B_{\delta/d}(x)\cap\Omega} |\nabla\varphi^\delta| \leq C\Big(\Big(\fint_{\Omega\cap B_\delta(x)} |\nabla\varphi^\delta(y)|^2\, dy\Big)^{\frac{1}{2}} + C' + \|\nabla\psi\|_{H^1(B_2(x/\delta))} + \|g\|_{C^{0,\alpha}(B_2(x/\delta))}\Big), \quad \text{(A.11)}$$

where C' is a constant that only depends on $\|f\|_{C^{1,\alpha}(\Omega)}$ and $\|\phi\|_{C^{1,\alpha}(\Omega)}$.

Step 3. We combine the estimates obtained in the first two steps as in the periodic setting. This yields

$$\mathbb{E}\Big[\Big(\int_\Omega |\nabla\varphi^\delta|^q\Big)^{q'}\Big] \leq C\mathbb{E}\Big[\Big(\int_\Omega \Big(\fint_{B_\delta(x)} \chi_\Omega|\nabla\varphi^\delta|^2\, dz\Big)^{\frac{q}{2}} dx\Big)^{q'}\Big]$$

$$+ \mathbb{E}\Big[\Big(\int_\Omega \|\nabla\psi\|^q_{H^1(B_2(x/\delta))} + \|g\|^q_{C^{0,\alpha}(B_2(x/\delta))}\Big)^{q'}\Big],$$

which, by (A.8), stationarity of $\nabla\psi$ and g and by (A.6), entails $\mathbb{E}\Big[\Big(\int_\Omega |\nabla\varphi^\delta|^q\Big)^{q'}\Big] \leq C_{q,q'}$, as claimed. $\quad\square$

A.4. Enhancement of the dielectric coefficient

The successful strategy we used in the periodic setting to prove the enhancement of the dielectric coefficient in the dilute case can be implemented in the random setting considered here, using the same g as in the periodic setting around the spherical inclusions. The proof raises additional technicalities, which can all be dealt with as we did above for the other results. The analysis is again inspired by recent results on the Clausius-Mossotti formula. As opposed to the periodic setting, there are many ways to thin a random point process and reach the dilute regime. As in the periodic setting, one may use geometric dilation and consider $\mathcal{P}_\ell = \ell\mathcal{P}$, but one may also attach a Bernoulli variable to each point and discard it if the variable is 0 – this leads to thinning by random deletion. For general thinning, we refer to the recent work [13] on the Einstein formula. In the case of geometric dilation, [34] provides tools which extend the results we used above in the periodic setting (this is however more involved since massive regularization is needed and this generates an additional error term that can be controlled using results of [17]). Likewise, for random deletion, one can combine tools introduced in [11] with [17].

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