Chern-Simons Functional and the Homology Cobordism Group

ALIAKBAR DAEMI*

February 3, 2020

Abstract

For each integral homology sphere Y, a function Γ_Y on the set of integers is constructed. It is established that Γ_Y depends only on the homology cobordism class of Y and it recovers the Frøyshov invariant. A relation between Γ_Y and Fintushel-Stern's R-invariant is stated. It is shown that the value of Γ_Y at each integer is related to the critical values of the Chern-Simons functional. Some topological applications of Γ_Y are given. In particular, it is shown that if Γ_Y is trivial, then there is no simply connected homology cobordism from Y to itself.

1 Introduction

Various Floer homology theories provide powerful tools in 3-manifold topology [Flo88, OS04, KM07]. The definitions of these invariants follow a similar pattern. To a given 3-manifold Y, one associates a pair of an infinite dimensional space $\mathcal{B}(Y)$ and a functional CS, defined on $\mathcal{B}(Y)$.[1] Then the relevant Floer homology of Y is obtained by applying the Morse homological methods to the functional CS. A unique feature of *instanton homology* [Flo88] among other Floer homology theories is that both $\mathcal{B}(Y)$ and the functional CS are topological. On the other hand, one needs to fix additional auxiliary structures for *monopole Floer homology* (in the form of a Riemannian metric for Y) and *Heegaard Floer homology* (including a Heegaard diagram for Y) to define $\mathcal{B}(Y)$ and CS. In the present article, we exploit this property of instanton Floer homology to introduce an invariant of integral homology 3-spheres which is preserved by *homology cobordisms*. The definition of this invariant is partly inspired by ideas from Min-Max theory.[2] This homology cobordism invariant provides a platform to unify works of various authors including Donaldson [Don83], Fintushel-Stern [FS85, FS90], Frøyshov [Frø02], Furuta [Fur90], Hedden-Kirk [HK12].

*The work of the author was supported by NSF Grant DMS-1812033.

[1] Here CS stands for the Chern-Simons functional, which is the relevant functional in the case of instanton Floer homology. This is not a standard notation for other 3-manifold Floer homologies. We use this notation to emphasize on formal similarities among the definitions of these 3-manifold Floer homology theories.

[2] See Appendix A for an elaboration on this point.

1.1 Statement of Results

Let Y and Y' be two integral homology spheres. A cobordism W from Y to Y' is a smooth 4-manifold with boundary $-Y \sqcup Y'$. The 3-manifolds Y and Y' are *homology cobordant*, if there is a cobordism W from Y to Y' such that $H_*(W, Y; \mathbf{Z}) = H_*(W, Y'; \mathbf{Z}) = 0$. The collection of all integral homology spheres modulo homology cobordism relation is called the *homology cobordism group* and is denoted by $\Theta^3_{\mathbf{Z}}$.

Suppose Y is an integral homology sphere. As the main construction of the present article, we introduce a function $\Gamma_Y : \mathbf{Z} \to \overline{\mathbf{R}}^{\geqslant 0}$, where $\overline{\mathbf{R}}^{\geqslant 0}$ is the extended positive real line $\mathbf{R}^{\geqslant 0} \cup \{\infty\}$, equipped with the obvious ordering. The following theorem states some of the basic properties of this function:

Theorem 1. The function Γ_Y satisfies the following properties:

(i) Γ_Y is non-decreasing.

(ii) If there is a homology cobordism from the integral homology sphere Y to another integral homology sphere Y', then $\Gamma_Y = \Gamma_{Y'}$.

Both parts of the above theorem can be strengthened. Before stating an improvement of Theorem 1 (i), we need to give a definition:

Definition 1.1. An integral homology sphere Y is $\mathrm{SU}(2)$-*non-degenerate*, if all $\mathrm{SU}(2)$ flat connections on Y are non-degenerate. That is to say, for any flat connection α on Y, we have $H^1(Y; ad_\alpha) = 0$ where ad_α is the flat vector bundle of rank 3 associated to the adjoint representation of α.

Theorem 2. Suppose Y is an integral homology sphere. There is a positive constant $\tau(Y)$ such that for any positive integer i, we have:

$$\Gamma_Y(i) \geqslant \tau(Y).$$

Moreover, if Y is $\mathrm{SU}(2)$-non-degenerate, then there is a positive constant $\tau'(Y)$ such that for any positive integer i, we have:

$$\Gamma_Y(i+1) \geqslant \Gamma_Y(i) + \tau'(Y).$$

The constants $\tau(Y)$ and $\tau'(Y)$ in the above theorem can be explicitly defined in terms of the moduli spaces of anti-self-dual $\mathrm{SU}(2)$-connections on $\mathbf{R} \times Y$. (See Definitions 3.28 and 3.40.) The following theorem gives a generalization of part (ii) of Theorem 1:

Theorem 3. Suppose W is a cobordism from an integral homology sphere Y to another homology sphere Y' such that $b_1(W) = 0$ and the intersection form of W is negative definite, i.e., $b^+(W) = 0$. Then there is a non-negative constant $\eta(W)$ such that for any positive integer i and non-positive integer j, we have[3]:

$$\Gamma_{Y'}(i) \leqslant \Gamma_Y(i) - \eta(W) \qquad \Gamma_{Y'}(j) \leqslant \max(\Gamma_Y(j) - \eta(W), 0).$$

Moreover, the constant $\eta(W)$ is positive unless there is an $\mathrm{SU}(2)$-representation of $\pi_1(W)$ which extends non-trivial representations of $\pi_1(Y)$ and $\pi_1(Y')$.

[3]Here we define $\infty - k$ to be ∞ for $k \in \overline{\mathbf{R}}^{\geqslant 0}$.

Gauge theoretical methods provide an important source of tools to study the group $\Theta^3_{\mathbf{Z}}$. In his groundbreaking work [Frø02], Frøyshov introduced a homomorphism $h : \Theta^3_{\mathbf{Z}} \to \mathbf{Z}$ which is defined using Floer's instanton homology of integral homology spheres. Frøyshov's construction motivated the definition of numerical invariants of homology cobordism group in the other Floer homologies [Frø96, OS03, KM07, Frø10, Man16, Lin18, HM1705], which have many interesting applications in low dimensional topology. An invariant with similar flavor is introduced in [HHL] by building on the constructions of [Man16, HM1705]. The invariant Γ_Y can be regarded as a refinement of Frøyshov's h-invariant:

Theorem 4. The function Γ_Y takes a finite value at an integer k if and only if $k \leqslant 2h(Y)$.

The invariant $h(Y)$ gives constraints on the intersection form of 4-manifolds X which fill the 3-manifold Y. For example, if $h(Y) \leqslant 0$, then there is no negative definite 4-manifold X with boundary Y such that the intersection form of W is not diagonal [Frø02, Theorem 3]. If we let $Y = S^3$, then this result specializes to Donaldson's groundbreaking *diagonalizability theorem* [Don83]. Negative definite 4-manifolds can be also used to obtain constraints on Γ_Y:

Theorem 5. Suppose X is a 4-manifold whose boundary is an integral homology sphere Y. Suppose the intersection form Q of X on $H^2(X; \mathbf{Z})/\mathrm{Tor}$ has the following form

$$Q = (-1) \oplus \cdots \oplus (-1) \oplus \mathcal{L}.$$

Here $\mathcal{L}$ is a non-trivial negative definite lattice such that:

$$m(\mathcal{L}) := \min_{\alpha \in \mathcal{L},\, \alpha \neq 0} \{|Q(\alpha)|\} \tag{1.2}$$

is greater than 1. Then we have:

$$\Gamma_Y(i) \leqslant \frac{m(\mathcal{L})}{4} \qquad \text{for } \ i \leqslant \lfloor \frac{m(\mathcal{L})}{2} \rfloor \tag{1.3}$$

In particular, $\Gamma_Y(1)$ is a finite number.

For a slightly more general version of Theorem 5 see Propositions 5.1 and 5.3.

In the case that $Y = S^3$, we have:

$$\Gamma_{S^3}(k) = \begin{cases} \infty & k > 0 \\ 0 & k \leqslant 0 \end{cases} \tag{1.4}$$

This is an immediate consequence of the definition of Γ_Y. It also follows from Theorem 3 applied to the product cobordism between two copies of S^3 and Theorem 4. For an integral homology sphere, we say Γ_Y is trivial, if $\Gamma_Y = \Gamma_{S^3}$. Theorems 3 and 5 imply that:

Corollary 1. Let W be a homology cobordism from Y to Y' such that Γ_Y is non-trivial (and hence $\Gamma_{Y'}$ is non-trivial). Then the inclusion of Y and Y' in W induce non-trivial maps of fundamental groups. In particular, there does not exist a simply-connected homology cobordism from Y to itself if $h(Y) \neq 0$.

This corollary is an extension of [Tau87, Proposition 1.7], which was originally proved using gauge theory on manifolds with periodic ends. (See Remark 3.52 about the relation between Corollary 1 and [Tan17].) It also answers a variant of the following question asked by Akbulut [Kir95, Problem 4.95]. In this question, μ denotes the Rokhlin homomorphism:

Question 1.5. *Does there exist a simply connected homology cobordism W from Y to Y' such that $\mu(Y) \neq 0$ (and hence $\mu(Y') \neq 0$)?*

Remark 1.6 (Levin and Lidman). The following question was raised in [HLL18, Remark 1.13]: given an integral homology sphere Z which bounds a homology 4-ball and $\gamma \in \pi_1(Z)$, does there exist a homology 4-ball X such that $\partial X = Z$ and γ is null-homotopic in X? Corollary 1 may be used to give a negative answer to this question. Suppose Y is an integral homology sphere with a weight one fundamental group. Suppose γ is a closed curve in Y which normally generates $\pi_1(Y)$. We also assume that $\Gamma_Y \neq \Gamma_{S^3}$. For example, we may take Y to be the poincaré homology sphere $\Sigma(2,3,5)$. (See Example 3.21.) The 3-manifold $Z = Y\# - Y$ bounds a homology 4-ball. However, there is no homology 4-ball X with $\partial X = Z$ such that γ vanishes in $\pi_1(X)$ because Corollary 1.5 asserts that the inclusion map induces a non-trivial map from $\pi_1(Y)$ (and also $\pi_1(-Y)$) to $\pi_1(X)$.

Fintushel and Stern introduced an invariant for any Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ in [FS85], which is denoted by $R(a_1, \dots, a_n)$ and is defined as follows:

$$R(a_1, \dots, a_n) = \frac{2}{a} - 3 + n + \sum_{i=1}^{n} \frac{2}{a_i} \sum_{k=1}^{a_i - 1} \cot\left(\frac{\pi k a}{a_i^2}\right) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi k}{a_i}\right). \tag{1.7}$$

Here $a = a_1 \cdot a_2 \dots a_n$. It turns out that the above number is an odd integer, not smaller than -1. The following theorem states that Γ_Y is related to Fintushel and Stern's R-invariant:

Theorem 6. Let Y be an integral homology sphere which has the form:

$$Y = Y_1 \# Y_2 \dots \# Y_k$$

where Y_i are (not necessarily distinct) Seifert fibered homology spheres with positive values of the R-invariant. Let Y_1 be the Seifert fibered space $\Sigma(a_1, a_2, \dots a_n)$ such that the value of the product $a_1 a_2 \dots a_n$ is maximum among all Seifert spaces Y_i. Then for $1 \leqslant i \leqslant \lfloor \frac{R(a_1,\dots,a_n)+3}{4} \rfloor$, we have:

$$\Gamma_Y(i) = \frac{1}{4 a_1 a_2 \dots a_n}.$$

In particular, $h(\Sigma(a_1, \dots, a_n)) \geqslant \frac{1}{2} \lfloor \frac{R(a_1,\dots,a_n)+3}{4} \rfloor$.

The main theorem of [FS85] asserts that if $R(a_1, \dots, a_n) > 0$, then there is no negative definite 4-manifold X whose boundary is $-\Sigma(a_1, \dots, a_n)$. This result follows immediately from Theorems 3 and 6. The following well-known theorem of Furuta is another corollary of Theorem 6.

Corollary 2 ([Fur90])**.** Suppose $\{Y_i = \Sigma(a_{i,1}, \dots, a_{i,n_i})\}_{i \in I}$ is a collection of Seifert fibered homology spheres with positive R-invariants such that the positive integers $a_i := a_{i,1} a_{i,2} \dots a_{i,n_i}$ are distinct. Then the integral homology spheres Y_i determine linearly independent elements of $\Theta^3_{\mathbf{Z}}$.

As a special case, Furuta's result implies that the integral homology spheres $\{\Sigma(p,q,pqk-1)\}_{k\in\mathbf{Z}^{>0}}$, for coprime positive integers p, q, span a subgroup of $\Theta^3_{\mathbf{Z}}$ isomorphic to $\mathbf{Z}^\infty$ because $R(p,q,pqk-1)=1$.

Corollary 3. Suppose $D(T_{p,q})$ denotes the Whitehead double of the (p,q)-torus knot and $Y_{p,q}$ is the 3-manifold $-\Sigma(D(T_{p,q}))$, the branched double cover of $D(T_{p,q})$ with the reverse orientation. Then we have:

$$\frac{1}{4pq(4pq-1)} \leqslant \Gamma_{Y_{p,q}}(1) \leqslant \frac{1}{4pq(2pq-1)} \tag{1.8}$$

More generally, if $Y = n_1 \cdot Y_{p_1,q_1} \# \ldots \# n_k \cdot Y_{p_k,q_k}$ for integers n_i, p_i, q_i such that p_i, q_i are coprime integers greater than 1 and n_i is a positive integer, then:

$$\min_{1\leqslant i\leqslant k} \frac{1}{4p_iq_i(4p_iq_i-1)} \leqslant \Gamma_Y(1) \leqslant \min_{1\leqslant i\leqslant k} \frac{1}{4p_iq_i(2p_iq_i-1)} \tag{1.9}$$

This corollary is a consequence of Theorems 2 and 6 using the ingredients provided by [HK12]. In [HK12], a lower bound for the positive values of the Chern-Simons functional on the set of flat $\mathrm{SU}(2)$-connections of $Y_{p,q}$ is given. This lower bound allows us to obtain the lower bound for Γ_Y in (1.9). The upper bound in (1.9) is verified with the aid of a negative definite cobordism from $\Sigma(p,q,2pq-1)$ to $Y_{p,q}$. In fact, Proposition 1 below implies that $\Gamma_{Y_{p,q}}(1)$ is one of the following values and it is natural to ask which of these values are equal to $\Gamma_{Y_{p,q}}(1)$:

$$\frac{1}{4pq(4pq-1)} \qquad \frac{1}{2pq(4pq-1)} \qquad \frac{1}{4pq(2pq-1)}.$$

Corollary 3 can be used to conclude the following theorem proved in [HK12].

Corollary 4 ([HK12])**.** The knots $\{D(T_{2,2^n-1})\}_{n\geqslant 2}$ are linearly independent in the smooth concordance group.

A stronger version of Corollary 4 for a more general family of knots is proved in [PC17]. Analogous to Corollary 4, it is possible to reformulate the results of [PC17] in terms of Γ_Y. In particular, the arguments of [PC17] can be used to obtain information about Γ_Y where Y is the branched double cover of S^3, branched along one of the knots studied in [PC17].

There are properties and invariants of integral homology spheres which do not respect the homology cobordism relation. But we can use Γ_Y to show that the homology cobordism group is not completely blind to them. For instance, being an $\mathrm{SU}(2)$-non-degenerate integral homology sphere is not preserved by homology cobordisms. If Y has an irreducible $\mathrm{SU}(2)$ flat connection, then $Y\#-Y$ is $\mathrm{SU}(2)$-degenerate and it is homology cobordant to S^3, an $\mathrm{SU}(2)$-non-degenerate integral homology sphere. Nevertheless, we have the following corollary of Theorems of 2 and 6:

Corollary 5. Suppose $\Sigma(a_1,a_2,\ldots a_n)$ is a Seifert fibered space with $R(a_1,a_2,\ldots a_n) \geqslant 5$. Then $\Sigma(a_1,a_2,\ldots a_n)$ is not homology cobordant to an $\mathrm{SU}(2)$-non-degenerate integral homology sphere. In particular, $\Sigma(a_1,a_2,\ldots a_n)$ is not homology cobordant to a Brieskorn homology sphere $\Sigma(p,q,r)$.

Note that there are Seifert fibered homology spheres $\Sigma(a_1,a_2,\ldots a_n)$ with arbitrarily large values of $R(a_1,a_2,\ldots a_n)$. (See Example 4.16.)

The Chern-Simons functional of an integral homology sphere Y takes finitely many values on the space of flat $\mathrm{SU}(2)$-connections. The homology cobordism relation might change this set. For example, the connected sum $Y\#-Y$, which is homology cobordant to S^3, can take non-trivial values on the space of flat $\mathrm{SU}(2)$-connections. On the other hand, the following proposition implies that if Y is homology cobordant to Y', then the values of Chern-Simons functionals of Y and Y' on the space of flat connections share the set of finite values in the image of Γ_Y and $\Gamma_{Y'}$:

Proposition 1. For any integral homology sphere Y and any integer k, either $\Gamma_Y(k) = \infty$, $\Gamma_Y(k) = 0$, or there is an irreducible flat connection α such that $\Gamma_Y(k)$ is equal to $\mathrm{CS}(\alpha)$ mod $\mathbf{Z}$.

It is natural to ask whether Γ_Y takes any irrational value. Proposition 1 implies that if Y is a linear combination of Seifert fibered homology spheres (or more generally plumbed 3-manifolds), then Γ_Y is always rational valued. In order to find Y with an irrational value in the image of Γ_Y, we firstly need to find an $\mathrm{SU}(2)$-flat connection α on an integral homology sphere such that $\mathrm{CS}(\alpha)$ is an irrational number. Even the existence of such flat connections is an open question.

We end this part of the introduction by discussing a filtration on $\Theta^3_{\mathbf{Z}}$. Given two integral homology spheres Y, Y', we define $Y \succeq_\Gamma Y'$ if $\Gamma_Y(1) \leqslant \Gamma_{Y'}(1)$. This defines a total quasi-order[4] on $\Theta^3_{\mathbf{Z}}$. There is also a well-known quasi-order $\succeq_N$ on $\Theta^3_{\mathbf{Z}}$ where $Y \succeq_N Y'$ if there is a negative definite manifold cobordism W from Y' to Y. Theorem 3 implies that if $Y \succeq_N Y'$, then $Y \succeq_\Gamma Y'$. (If $W : Y \to Y'$ is a negative definite cobordism, then performing surgery on a set of loops representing a basis for $H_1(W, \mathbf{R})$ gives rise to a negative definite cobordism from Y to Y' with vanishing b_1.) Theorem 6 implies that if we pick the sequence $\{Y_k := \Sigma(p, q, pqk - 1)\}_k$, then for any integer N we have $Y_k \succeq_\Gamma N \cdot Y_{k-1}$. It would be interesting to study the behavior of this filtration (or some refined version of it) with respect to connected sums of 3-manifolds. Of course, this requires a better understanding of the invariant Γ_Y with respect to connected sums which will be investigated in [DST].

1.2 Outline of Contents

For the purpose of this paper, we need to work with a version of instanton Floer homology which is defined with coefficients in a Novikov field. This version of instanton Floer homology is discussed in Section 2. Subsection 2.2 is devoted to a review of the functoriality of instanton Floer homology with respect to negative definite cobordisms with vanishing b_1. In [Don02], various *equivariant instanton Floer homology theories* for integral homology spheres are introduced. In Subsection 2.3, we introduce three such equivariant theories and show that they fit into an exact triangle. The construction of this exact triangle is inspired by similar objects in the context of monopole Floer homology [KM07] and Heegaard Floer homology [OS04].

The definition of Γ_Y is given in Subsection 3.1 using the exact triangle of equivariant instanton Floer homologies. As it is explained there, one can avoid equivariant theories in the definition of Γ_Y. However, I find it more instructive to use the language of equivariant instanton Floer theories. I also believe that this approach would be more efficient in studying the behavior of Γ_Y with respect to topological constructions

[4] A quasi-order $\succeq$ is a reflexive and transitive binary relation. A total quasi-order on a set S is a quasi-order $\succeq$ on S such that for any two elements a, b of S at least one of the relations $a \succeq b$ or $b \succeq a$ holds.

such as surgery along a knot and taking connected sum. In Subsection 3.2, we verify the basic properties of Γ_Y claimed above. In Section 4, we study the relation between Γ_Y and Fintushel and Stern's R invariant. Section 5 is devoted to the proof of Theorem 5 and its generalizations. In Appendix A, we discuss a finite dimensional toy model to motivate the main construction of the paper.

Acknowledgements. This work is partly inspired by the ongoing collaboration of the author with Kenji Fukaya. I am grateful to him for many enlightening discussions. The definition of the exact triangle of Subsection 2.3 is motivated by the author's discussions with Michael Miller. I am thankful to him for explaining to me his work on instanton Floer homology of rational homology spheres. I thank Masaki Taniguchi for brining his work to the author's attention. I am also grateful to Simon Donaldson, Peter Kronheimer and Christopher Scaduto for many interesting conversations about the present work.

2 A Review of Instanton Floer Homology

2.1 Instanton Floer Chain Complexes

Suppose Y is an integral homology sphere and P is an $\mathrm{SU}(2)$-bundle on Y, which is necessarily a trivial bundle. Let $\mathcal{A}(Y)$ be the space of $\mathrm{SU}(2)$-connections on P. We fix a trivialization of P and denote the associated trivial connection by Θ. Other connections on P are given by adding elements of $\Omega^1(Y, \mathfrak{su}(2))$ to the trivial connection. Here $\Omega^1(Y, \mathfrak{su}(2))$ is the space of smooth 1-forms on Y with coefficients in $\mathfrak{su}(2)$, the Lie algebra of $\mathrm{SU}(2)$. For a connection $A = \Theta + a$, with a being an element of $\Omega^1(Y, \mathfrak{su}(2))$, the Chern-Simons functional of A is defined to be:

$$\widetilde{\mathrm{CS}}(A) := -\frac{1}{8\pi^2}\int_Y \mathrm{tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a)$$

Suppose $\mathcal{G}(Y)$ is the space of smooth automorphisms of P. Given an element of $g \in \mathcal{G}(Y)$, we can pull-back a connection using the automorphism g^{-1}. This determines an action of $\mathcal{G}(Y)$ on $\mathcal{A}(Y)$. A connection in $\mathcal{A}(Y)$ is called irreducible if its stabilizer with respect to the action of $\mathcal{G}(Y)$ is $\pm\mathrm{id}$. The action of each element of $\mathcal{G}(Y)$ changes the value of the Chern-Simons functional by an integer. Therefore, we have an induced map $\mathrm{CS} : \mathcal{B}(Y) \to \mathbf{R}/\mathbf{Z}$ where $\mathcal{B}(Y)$ is the quotient space $\mathcal{A}(Y)/\mathcal{G}(Y)$.

Instanton Floer homology of Y can be regarded as the "Morse homology group" associated to the Chern-Simons functional CS. The critical points of CS are represented by flat $\mathrm{SU}(2)$-connections on Y and form a compact subspace of $\mathcal{B}(Y)$. The trivial connection (more precisely, the class represented by Θ) is a singular point of $\mathcal{B}(Y)$, because the stabilizer of Θ consists of constant automorphisms of P. On the other hand, all non-trivial flat connections on P are irreducible. The trivial connection is a non-degenerate critical point of CS, namely, the Hessian of CS at Θ is non-degenerate modulo the action of the gauge group. For now, we assume that the other critical points are also non-degenerate and hence isolated. Consequently, there are only finitely many critical points of CS.

We can form the analogues of downward gradient flow lines for the Chern-Simons functional. Fix two critical points α and β of CS which are represented by flat connections $\widetilde{\alpha}, \widetilde{\beta} \in \mathcal{A}(Y)$. Suppose A_0 is a smooth connection on $\mathbf{R} \times Y$, which is equal to the pull-backs of $\widetilde{\alpha}$ and $\widetilde{\beta}$ on the ends $(-\infty, -1] \times Y$ and $[1, \infty) \times Y$, respectively. The restrictions of A_0 to $\{t\} \times Y$ determines a path z in $\mathcal{B}(Y)$ from α

to β. Fix an integer $l \geqslant 3$ and define $\mathcal{A}_z(\alpha, \beta)$ to be the set of Sobolev connections A on $\mathbf{R} \times Y$ such that $\|A - A_0\|_{L^2_l} < \infty$. As the notation suggests, this space depends only on the path z. We define the topological energy of an element $A \in \mathcal{A}_z(\alpha, \beta)$ to be:

$$\frac{1}{8\pi^2} \int_{\mathbf{R}\times Y} \operatorname{tr}(F(A) \wedge F(A))$$

The topological energy is independent of A, and we will denote it by $\mathcal{E}(z)$. It is also related to the Chern-Simons functional as follows:

$$\mathcal{E}(z) \equiv \mathrm{CS}(\alpha) - \mathrm{CS}(\beta) \mod \mathbf{Z} \tag{2.1}$$

Suppose $\mathcal{G}_z(\alpha, \beta)$ denotes the group of automorphisms g of the trivial $\mathrm{SU}(2)$-bundle on $\mathbf{R} \times Y$ such that $\|\nabla_{A_0} g\|_{L^2_l} < \infty$. This group acts on $\mathcal{A}_z(\alpha, \beta)$ and the quotient is denoted by $\mathcal{B}_z(\alpha, \beta)$.

Fix a metric on Y and equip $\mathbf{R} \times Y$ with the product metric. A connection $A \in \mathcal{A}_z(\alpha, \beta)$ is *Anti-Self-Dual*, if the self-dual part of its curvature, $F^+(A)$, vanishes. The path $\{A|_{\{t\}\times Y}\}_{t\in\mathbf{R}}$, for an ASD connection A, can be regarded as the downward gradient flow line of the Chern-Simons functional where we use the following metric on $\mathcal{A}(Y)$:

$$\langle a, b\rangle := \frac{1}{8\pi^2} \int_Y -\operatorname{tr}(a \wedge *b) \qquad a,\ b \in \Omega^1(Y, \mathfrak{su}(2))$$

The ASD equation is invariant with respect to the action of $\mathcal{G}_z(\alpha, \beta)$ and the quotient space of the space of solutions is denoted by $\mathcal{M}_z(\alpha, \beta)$. The space $\mathcal{M}_z(\alpha, \beta)$ is non-empty only if $\mathcal{E}(z) > 0$ or z is homotopic to the constant path. In the latter case, $\mathcal{E}(z) = 0$ and $\alpha = \beta$. We say a point of $\mathcal{M}_z(\alpha, \beta)$ is *regular* if a representative connection (and hence any representative connection) is a regular point of the map $A \to F^+(A)$. The space $\mathcal{M}_z(\alpha, \beta)$ is an orientable smooth manifold in a neighborhood of any regular point. For now, we assume that all elements of the moduli spaces $\mathcal{M}_z(\alpha, \beta)$ are regular, and address the general case below.

To any α in $\mathcal{B}(Y)$, we can associate $\deg^+(\alpha), \deg^-(\alpha) \in \mathbf{Z}/8\mathbf{Z}$ such that [Flo88, Don02]:

$$\dim(\mathcal{M}_z(\alpha, \beta)) \equiv \deg^+(\alpha) - \deg^-(\beta) \mod 8 \tag{2.2}$$

Moreover, $\deg^+(\Theta) = -3$, $\deg^-(\Theta) = 0$, and $\deg^+(\alpha) = \deg^-(\alpha)$ for an irreducible flat connection α (under the standing assumption that the irreducible flat connections are non-degenerate). The grading on the elements of $\mathcal{B}(Y)$ defined by $\deg^-$ is called the *Floer grading*. Translation along the $\mathbf{R}$ direction gives an action of $\mathbf{R}$ on $\mathcal{M}_z(\alpha, \beta)$. Unless $\alpha = \beta$ and z is homotopic to the constant path, this action is free and $\breve{\mathcal{M}}(\alpha, \beta)$ denotes the quotient space with respect to the $\mathbf{R}$-action.

Suppose Λ is the following field:

$$\Lambda := \{\sum_{i=0}^{\infty} a_i \lambda^{r_i} \mid a_i \in \mathbf{Q},\ r_i \in \mathbf{R},\ \lim_{i\to\infty} r_i = \infty\}.$$

Define $C_*(Y)$ to be the Λ-module generated by non-trivial flat connections on Y. Define the endomorphism $d : C_*(Y) \to C_*(Y)$ by:

$$d(\alpha) := \sum_{z:\alpha\to\beta} \#\breve{\mathcal{M}}_z(\alpha, \beta) \cdot \lambda^{\mathcal{E}(z)}\beta \tag{2.3}$$

where α and β are generators of $C_*(Y)$ and z is a path from α to β. In the above expression, $\#\breve{\mathcal{M}}_z(\alpha,\beta)$ denotes the signed number of the points in $\breve{\mathcal{M}}_z(\alpha,\beta)$, with the understanding that this sum is non-zero only if the moduli space is 0-dimensional. In the case that $\breve{\mathcal{M}}_z(\alpha,\beta)$ is 0-dimensional, this moduli space is compact and hence it consists of finitely many points. Moreover, for a general path z, the moduli space $\mathcal{M}_z(\alpha,\beta)$ is orientable. However, to make (2.3) rigorous, we need to fix an orientation of $\mathcal{M}_z(\alpha,\beta)$ for each α, β and z. We refer to [Don02] for a consistent way of fixing orientations for these moduli spaces. The flat connection β contributes to the sum (2.3) only if $\deg^-(\alpha)-\deg^-(\beta)=1$. Therefore, the degree of d with respect to the Floer grading is equal to -1. The subspace of the elements of degree i in $C_*(Y)$ is denoted by $C_i(Y)$.

The moduli spaces of ASD connections asymptotic to the trivial connection on one end can be utilized to construct two other maps. Following [Don02], let $D_1 : C_*(Y) \to \Lambda$ and $D_2 : \Lambda \to C_*(Y)$ be Λ-homomorphisms defined as follows:

$$D_1(\alpha) := \sum_{z:\alpha\to\Theta} \#\breve{\mathcal{M}}_z(\alpha,\Theta)\cdot\lambda^{\mathcal{E}(z)} \qquad D_2(1) := \sum_{z:\Theta\to\beta} \#\breve{\mathcal{M}}_z(\Theta,\beta)\cdot\lambda^{\mathcal{E}(z)}\beta$$

From the definition, it is clear that D_1 is non-zero only on $C_1(Y)$, and D_2 takes values in the summand $C_4(Y)$ of $C_*(Y)$.

There is another interesting operator U which acts on $C_*(Y)$. The map U is defined with the same formula as (2.3) except that $\breve{\mathcal{M}}_z(\alpha,\beta)$ is replaced with $\mathcal{N}_z(\alpha,\beta)$, a co-dimension 4 submanifold of $\mathcal{M}_z(\alpha,\beta)$,

$$U(\alpha) := \sum_{z:\alpha\to\beta} -\frac{1}{2}\#\mathcal{N}_z(\alpha,\beta)\cdot\lambda^{\mathcal{E}(z)}\beta. \tag{2.4}$$

The factor $-\frac{1}{2}$ does not have a great significance and is included to simplify subsequent formulas. Each homology class $\sigma \in H_i(Y)$ gives rise to a cohomology class $\mu(\sigma) \in H^{4-i}(\mathcal{B}_z(\alpha,\beta))$ [DK90, Don02]. The submanifold $\mathcal{N}_z(\alpha,\beta)$ of $\mathcal{M}_z(\alpha,\beta)$ can be regarded as a subspace of $\mathcal{M}_z(\alpha,\beta)$ representing the cohomology class $\mu(4\cdot x)$ where x is the generator of $H_0(Y)$. To be more specific, let $\mathcal{B}^*((-1,1)\times Y)$ be the configuration space of irreducible $\mathrm{SU}(2)$-connections on $(-1,1)\times Y$ with finite L^2_l norms. Associated to the base point $(0,y_0)\in(-1,1)\times Y$, there is a base point fibration $\mathbb{E}$ on $\mathcal{B}^*((-1,1)\times Y)$, which is a rank 3 real vector bundle with structure group $SO(3)$. We fix two sections s_1, s_2 of the complexified bundle $\mathbb{E}\otimes\mathbf{C}$ and define $\mathcal{N}_z(\alpha,\beta)$ as follows:

$$\mathcal{N}_z(\alpha,\beta) := \{[A]\in\mathcal{M}_z(\alpha,\beta) \mid s_1(r([A])) \text{ and } s_2(r([A])) \text{ are linearly dependent.}\} \tag{2.5}$$

Here $r : \mathcal{M}_z(\alpha,\beta)\to\mathcal{B}^*((-1,1)\times Y)$ is given by the restriction of a connection to $(-1,1)\times Y$ and is well-defined because of unique continuation. We also assume that s_1 and s_2 are chosen generically such that $\mathcal{N}_z(\alpha,\beta)$ is cut down transversely. We refer the reader to [Don02] and [Frø02] for more details[5]. By definition, U has degree -4 because $\mathcal{N}_z(\alpha,\beta)$ is 0-dimensional only if $\deg^-(\alpha)-\deg^-(\beta)=4$.

Let $\widetilde{C}_*(Y) := C_*(Y)\oplus\Lambda\oplus C_{*-3}(Y)$. Here $C_{*-3}(Y)$ is the same Λ-module as $C_*(Y)$ whose grading is shifted up by 3. The summand Λ is in correspondence with the trivial connection and we define

[5]The corresponding operator in [Don02] is equal to $-\frac{1}{4}U$ and the corresponding operator in [Frø02] is $-2U$.

its Floer grading to be 0. The above maps can be combined to form an operator $\widetilde{d} : \widetilde{C}_*(Y) \to \widetilde{C}_*(Y)$:

$$\begin{bmatrix} d & 0 & 0 \\ D_1 & 0 & 0 \\ U & D_2 & -d \end{bmatrix} \tag{2.6}$$

Proposition 2.7. *The map $\widetilde{d}$ defines a differential of degree -1 on $\widetilde{C}$.*

Proof. The same proposition with the coefficient ring $\mathbf{Q}$ is proved in [Don02]. (See also [Frø02].) The argument there can be easily adapted to our case where the coefficient ring is Λ and the differential is weighted by powers of λ. We only need to observe that the topological energy of the concatenation of two paths z_1 and z_2 in $\mathcal{B}(Y)$ is equal to the sum of the topological energies. □

Define the *minimal-degree* of a non-zero element $\eta = \sum_{i=0}^{\infty} a_i \lambda^{r_i}$ of Λ by

$$\mathrm{val}(\eta) = \min_i \{r_i \mid a_i \neq 0\}$$

We can also extend the minimal-degree to the case that $\eta = 0$ by defining that $\mathrm{val}(0) = \infty$. The minimal-degree of $(\eta_1, \ldots, \eta_k) \in \Lambda^k$ is defined as:

$$\mathrm{val}(\eta_1, \ldots, \eta_k) := \min_i \{\mathrm{val}(\eta_i)\}$$

In particular, val can be defined on $\widetilde{C}_*(Y)$. The differential $\widetilde{d}$ increases the minimal degree, because the moduli spaces involved in the definition of this differential are non-empty only if the energy of the corresponding path is positive.

One is often faced with homology spheres Y such that the Chern-Simons functional has degenerate critical points or the moduli spaces $\mathcal{M}_z(\alpha, \beta)$ are not regular. In these cases, we need to perturb the Chern-Simons functional. The classical choice of such perturbations are given by *holonomy perturbations* [Don87, Tau90, Flo88, Don02, Kro05, KM11b]. Without going into details of holonomy perturbations, we explain what such perturbations provide for us. Here we follow the approach in [Don02, KM11b]. For each 3-manifold Y, one can define a family of holonomy perturbations parametrized by a Euclidean space $\mathcal{P}$: to each $\pi \in \mathcal{P}$, one can associate a bounded continuous function $f_\pi : \mathcal{B}(Y) \to \mathbf{R}$ and this association is linear. If α and β are two critical points of $\mathrm{CS} + f_\pi$ and z is a path from α to β, then the moduli space of downward gradient flow lines from α to β is denoted by $\mathcal{M}_z^\pi(\alpha, \beta)$. More precisely, $\mathcal{M}_z^\pi(\alpha, \beta)$ consists of the gauge equivalence classes of connections $A \in \mathcal{A}_z(\alpha, \beta)$ such that:

$$F^+(A) + (dt \wedge \nabla_{A_t} f_\pi)^+ = 0$$

where $A_t := A|_{\{t\} \times Y}$ and $\nabla_{A_t} f_\pi$ is the formal gradient of f_π at the connection A_t.

Proposition 2.8 ([Don02, KM11b])**.** *For any positive real number ϵ, there is an element $\pi \in \mathcal{P}$ such that the following properties hold:*

(i) $|\pi|_{\mathcal{P}} < \epsilon$;

(ii) the non-trivial critical points of the functional $\mathrm{CS} + f_\pi$ *are non-degenerate and irreduicble;*

(iii) for any two critical points α *and* β *of* $\mathrm{CS} + f_\pi$ *and any path* z *with* $\mathrm{index}(z) < 8$*, the moduli space* $\mathcal{M}_z^\pi(\alpha, \beta)$ *consists of regular solutions, i.e., any element of this space is cut down transversely;*

(iv) if a flat connection is already non-degenerate, then we can assume that f_π *vanishes in a neighborhood of this flat connection.*

Remark 2.9. Strictly speaking, the proof of part (iv) of Proposition 2.8 does not appear in [Don02, KM11b]. However, the proofs there can be easily adapted to treat this part. The functions f_π are defined in terms of the holonomies of connections in $\mathcal{B}(Y)$ along a set of loops embedded in Y. If a flat connection α is non-degenerate, then there is an open neighborhood of the equivalence class of this connection in $\mathcal{B}(Y)$, which does not contain any other flat connection. In fact, we can pick this neighborhood such that for any flat connection in the complement of the class of α there is a finite set of loops in Y such that holonomies along these loops distinguish this connection from the connections in the chosen neighborhood. Using this observation one can easily adapt the arguments in [Don02, Chapter 5] to obtain the desired result.

Fix a positive real number ϵ, and let f_π be a perturbation given by Proposition 2.8. The trivial connection is always a critical point of $\mathrm{CS} + f_\pi$, and if ϵ is small enough, all the other critical points are irreducible. Any such π is called an ϵ-*admissible* perturbation. We also say a perturbation is admissible if it is an ϵ-admissible perturbation for some ϵ. By replacing CS with $\mathrm{CS} + f_\pi$, we can construct a chain complex $(\widetilde{C}_*^\pi(Y), \widetilde{d})$ in an analogous way. The chain group can be also equipped with the Floer grading and val. The differential decreases the Floer grading by -1. However, the differential $\widetilde{d}$ does not necessarily increase the minimal-degree anymore. Nevertheless, there is an upper bound in terms of ϵ on how much $\widetilde{d}$ decreases val.

Lemma 2.10. *For any positive real number* δ*, there exists a positive* ϵ *such that for any* ϵ*-admissible perturbation* π *and any* $\zeta \in \widetilde{C}_*^\pi(Y)$*, we have:*

$$\mathrm{val}(\widetilde{d}\zeta) \leqslant \mathrm{val}(\zeta) + \delta.$$

Proof. For critical points α, β of $\mathrm{CS} + f_\pi$, an element of the moduli space $\mathcal{M}_z^\pi(\alpha, \beta)$ can be regarded as a downward gradient flow line of $\mathrm{CS} + f_\pi$. In particular, the non-emptiness of this moduli space implies that:

$$\mathcal{E}(z) + f_\pi(\alpha) - f_\pi(\beta) \geqslant 0.$$

If ϵ is small enough, then we can assume that $|f_\pi|$ at any point of $\mathcal{B}(Y)$ is less than $\frac{\delta}{2}$. Therefore, $\mathcal{E}(z) \geqslant -\delta$, which implies that the differential $\widetilde{d}$ decreases the minimal-degree on $\widetilde{C}_*^\pi(Y)$ by at most δ. □

2.2 Cobordism Maps

In this subsection, we discuss the functorial properties of instanton Floer homology with respect to cobordisms. Suppose Y , Y' are two integral homology spheres with base points, and π, π' are ϵ-admissible perturbations of the Chern-Simons functional on Y, Y', respectively. The associated chain

complexes to these perturbations are denoted by $(\widetilde{C}_*^{\pi}(Y), \widetilde{d})$ and $(\widetilde{C}_*^{\pi'}(Y'), \widetilde{d})$. Let $W : Y \to Y'$ be a cobordism with a choice of a path between the base points of Y, Y' such that $b^+(W) = 0$ and $H_1(W, \mathbf{Z}) = 0$. We define below a chain map $\widetilde{C}_W : \widetilde{C}_*^{\pi}(Y) \to \widetilde{C}_*^{\pi'}(Y')$, which has the following matrix form with respect to the standard direct sum decompositions of $\widetilde{C}_*^{\pi}(Y)$ and $\widetilde{C}_*^{\pi'}(Y')$:

$$\begin{bmatrix} \varphi(W) & 0 & 0 \\ \Delta_1(W) & 1 & 0 \\ \mu(W) & \Delta_2(W) & \varphi(W) \end{bmatrix} \tag{2.11}$$

As in the case of of the differentials of the Floer complexes, these maps are defined by the moduli space of ASD connections. Firstly, we need to fix a Riemannian metric on W. We assume that this metric is the product metric in a collar neighborhood of its boundary corresponding to the chosen metrics on Y and Y'. Let also W^+ be the non-compact manifold that is given by gluing the cylinders $(-\infty, 0] \times Y$ and $[0, \infty) \times Y'$ with the product metric to W.

Let α, α' be respectively generators of $C_*^{\pi}(Y)$, $C_*^{\pi'}(Y')$ and fix connections representing these elements of $\mathcal{B}(Y)$, $\mathcal{B}(Y')$. Let A be a connection on the trivial $\mathrm{SU}(2)$-bundle over W^+ which is equal to the pull back of the chosen representatives on the cylindrical ends. If A' is another connection with the similar properties, then we say A and A' are equivalent to each other if there is an automorphism g of the trivial $\mathrm{SU}(2)$-bundle such that $g^*(A) - A'$ is compactly supported. An equivalence class of this relation is called a *path from α to α' along W*. The fundamental group of $\mathcal{B}(Y)$ (resp. $\mathcal{B}(Y')$), based at the connection α (resp. α'), acts faithfully and transitively on the space of paths from α to α' by concatenation. The topological energy of a path z, represented by a connection A, is defined to be:

$$\mathcal{E}(z) := \frac{1}{8\pi^2} \int_{W^+} \mathrm{tr}(F(A) \wedge F(A)).$$

This energy is well-defined and only depends on z. Moreover, we have the following generalization of (2.1):

$$\mathcal{E}(z) \equiv \mathrm{CS}(\alpha) - \mathrm{CS}(\alpha') \mod \mathbf{Z} \tag{2.12}$$

For a path z from α to α' represented by a connection A_0, define $\mathcal{A}_z(W; \alpha, \alpha')$ to be the space of $\mathrm{SU}(2)$-connections on W^+ such that $\|A - A_0\|_{L^2_l} < \infty$. Let $\mathcal{G}_z(W; \alpha, \alpha')$ be the group of the automorphisms g of the trivial $\mathrm{SU}(2)$-bundle such that $\|\nabla_{A_0} g\|_{L^2_l} < \infty$. The group $\mathcal{G}_z(W; \alpha, \alpha')$ acts on $\mathcal{A}_z(W; \alpha, \alpha')$ and the quotient space is denoted by $\mathcal{B}_z(W; \alpha, \alpha')$. We consider perturbations of the ASD equation on the space $\mathcal{B}_z(W; \alpha, \alpha')$ compatible with the perturbations π and π'. This equation has the following form:

$$F^+(A) + G_0(A) + G_1(A) = 0. \tag{2.13}$$

The term $G_0(A)$ is defined using π and π' in the following way:

$$G_0(A) := \phi(t)(dt \wedge \nabla_{A_t} f_{\pi})^+ + \phi'(t)(dt \wedge \nabla_{A_t} f_{\pi'})^+$$

where ϕ (respectively, ϕ') is a smooth function on $\mathbf{R}$ that is equal to 1 on $(-\infty, -2]$ (respectively, $[2, \infty)$) and equal to 0 on $[-1, 0]$ (respectively, $[0, 1]$). The functions ϕ and ϕ' clearly determine functions on W^+ which are respectively supported on the incoming end and the outgoing end of W^+. The term $G_1(A)$

in (2.13) is a secondary holonomy perturbation which is supported on a compact subspace of W^+. The parametrizing space for this secondary holonomy perturbation is a Euclidean space $\overline{\mathcal{P}}$. Let $\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$ be the moduli space of solutions to the equation in (2.13).

As in the case of cylinders, we can associate an integer $\mathrm{index}(z)$ to the path z, which is the expected dimension of the moduli space $\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$. This integer is defined to be the index of the linearization of the equation in (2.13) modulo the action of the gauge group and it satisfies the following identity:

$$\mathrm{index}(z) \equiv \deg^+(\alpha) - \deg^-(\alpha') \mod 8 \tag{2.14}$$

The maps involved in the cobordism map $\widetilde{C}_W$ are defined using low dimensional moduli spaces of the form $\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$ where $\overline{\pi}$ is chosen such that all these moduli spaces are cut down transversely:

Proposition 2.15 ([Don02])**.** *Given a positive real number ϵ, there is a secondary perturbation term $\overline{\pi} \in \overline{\mathcal{P}}$ such that $|\overline{\pi}| < \epsilon$ and all the moduli spaces $\mathcal{M}_z^{\overline{\pi}}(W;\alpha_0,\alpha_1)$ with $\mathrm{index}(z) < 8$ consist of regular solutions.*

Any secondary perturbation $\overline{\pi}$ that satisfies the properties of Proposition 2.15 is called an *ϵ-admissible secondary perturbation*. We say $\overline{\pi}$ is admissible if it is ϵ-admissible for some choice of ϵ. For an ϵ-admissible perturbation, $\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$ is a smooth manifold whose dimension is equal to $\mathrm{index}(z)$.

We start with the definition of the map $\varphi : C_*^{\pi}(Y) \to C_*^{\pi'}(Y')$. We firstly fix an an ϵ-admissible secondary perturbation $\overline{\pi}$. Let α be a generator of $C_*^{\pi}(Y)$ and define:

$$\varphi(\alpha) := \sum_{z:\alpha\to\alpha'} \#\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha') \cdot \lambda^{\mathcal{E}(z)}\alpha' \tag{2.16}$$

where the sum is over all paths z that the moduli space $\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$ is 0-dimensional. In fact, for each α' there is at most one path z from α to α' such that the moduli space $\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$ is 0-dimensional and for this choice of z, the moduli space is compact. Therefore, it consists of finitely many points. As in the case of differentials, there is a canonical choice of orientation for this moduli space [Don02] and for this choice φ is a chain map.

The other terms in (2.11) are defined in an analogous way. For example, $\mu : C_*^{\pi}(Y) \to C_{*-3}^{\pi'}(Y')$ is defined similar to the map φ in (2.16) by replacing $\#\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$ with $-\frac{1}{2}\#\mathcal{N}_z^{\overline{\pi}}(W;\alpha,\alpha')$ where $\mathcal{N}_z^{\overline{\pi}}(W;\alpha,\alpha')$ is a codimension 3 submanifold of $\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$. The definition of $\mathcal{N}_z^{\overline{\pi}}(W;\alpha,\alpha')$ is similar to $\mathcal{N}_z(\alpha,\beta)$ and uses the path between base points. We refer the reader to [Frø02, Theorem 6] for the definition of μ. The maps $\Delta_1(W) : C_*^{\pi}(Y) \to \Lambda$ and $\Delta_2(W) : \Lambda \to C_*^{\pi'}(Y')$ are also defined by considering the moduli spaces $\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$ where either α or α' is equal to the trivial connection. A standard argument using 1-dimensional moduli spaces over W^+ verifies the following proposition:

Proposition 2.17. *$\widetilde{C}_W : \widetilde{C}_*^{\pi}(Y) \to \widetilde{C}_*^{\pi'}(Y')$ is a chain map.*

In the following, when the choice of the cobordism W is clear from the context, we drop W from our notation and denote the above maps with φ, Δ_1, Δ_2 and μ.

The chain map $\widetilde{C}_W$ behaves well with respect to the Floer grading and val. Identity (2.14) implies that the map $\widetilde{C}_W$ preserves the Floer grading. In order to study the behavior with respect to val, let

$\mathcal{M}_z^{\overline{\pi}}(W;\alpha,\alpha')$ be a non-empty moduli space that contains the class represented by a connection A. Then $\mathcal{E}(z)$ can be written as the sum of three terms $\mathcal{E}_0(z)$, $\mathcal{E}_1(z)$ and $\mathcal{E}_2(z)$, which are defined in the following way:

$$\mathcal{E}_0(z) := \frac{1}{8\pi^2}\int_{(-\infty,-T]\times Y_0} \mathrm{tr}(F(A)\wedge F(A)) \qquad \mathcal{E}_1(z) := \frac{1}{8\pi^2}\int_{[T,\infty)\times Y_1} \mathrm{tr}(F(A)\wedge F(A))$$

$$\mathcal{E}_3(z) := \frac{1}{8\pi^2}\int_{W^c} \mathrm{tr}(F(A)\wedge F(A))$$

where W^c is the complement of $(-\infty,-T]\times Y_0$ and $[T,\infty)\times Y_1$ in W^+, and T is chosen such that the secondary perturbation term is supported on W^c. We can argue as in the previous subsection that for any positive constant δ there is ϵ such that $\mathcal{E}_0(z),\mathcal{E}_1(z)\geqslant -2\delta$. We also have:

$$\mathcal{E}_3(z) = \frac{1}{8\pi^2}\int_{W^c} |F^-(A)|^2 - |F^+(A)|^2 \geqslant -\frac{1}{8\pi^2}\int_{W^c} |G_0(A)+G_1(A)|^2$$

Therefore, if $\overline{\pi}$ is given by Proposition 2.15 and ϵ is small enough, then we can conclude that $\mathcal{E}_3(z)\geqslant -\delta$. Consequently, we can ensure that $\widetilde{C}_W$ does not decrease val by more than a given positive number once ϵ is small enough.

We can relax the condition $H_1(W,\mathbf{Z})=0$ to $b_1(W)=0$ using the ideas of [Don87]. In the case that $b_1(W)=0$, we have two additional types of reducible flat connections on W^+:

(i) flat connections induced by representations of $\pi_1(W)$ into $\mathbf{Z}/2\mathbf{Z}$. The stabilizer of any such reducible connection is a copy of $\mathrm{SU}(2)$;

(ii) flat connections induced by representations ρ of $\pi_1(W)$ into S^1 such that there are elements in the image of ρ which do not have order 2. The stabilizer of any such reducible connection is a copy of S^1. The flat connections obtained from ρ and ρ^{-1} are gauge equivalent to each other.

Using holonomy perturbations we may assume that any reducible flat connection of type (i) is regular. Moreover, we may assume that the linearization of the perturbed ASD equation at any reducible connection of type (ii) has 1-dimensional co-kernel. See [Don87] and [CDX17, Subsection 7.3] for more details on the proofs of these facts.

The presence of flat connections of types (i) and (ii) requires us to modify the proof of Proposition 2.17. More specifically, we have the following new equations:

$$D_1'\circ\lambda = c(W)\cdot D_1 + \Delta_1\circ d, \qquad \lambda\circ D_2 = c(W)\cdot D_2' - d'\circ\Delta_2, \tag{2.18}$$

where $c(W)$ is the number of the elements of $H_1(W,\mathbf{Z})$, which is a finite positive integer. The first equation in (2.18) is obtained by looking at the 1-dimensional moduli space associated to a path z along W from a generator of $C_*^{\pi}(Y)$ to the trivial connection Θ' on Y'. The first term on the right hand side of the first equation in (2.18) is obtained by counting the ends associated to gluing an element of the 0-dimensional moduli space $\breve{\mathcal{M}}_z^{\pi}(Y\times\mathbf{R},\alpha,\Theta)$ to flat connections of type (i) and (ii). Using the arguments of [Don87], any flat connection of type (i) contributes one end to the glued up moduli space and any flat connection of type (ii) contributes two ends. Therefore, in total we have the coefficient $c(W)$ in our

formula. A similar explanation applies to the second equation in (2.18). In summary, Proposition 2.17 holds if we replace (2.11) with (2.19):

$$\begin{bmatrix} \varphi(W) & 0 & 0 \\ \Delta_1(W) & c(W) & 0 \\ \mu(W) & \Delta_2(W) & \varphi(W) \end{bmatrix} \tag{2.19}$$

2.3 Equivariant Instanton Floer Homology Groups

In this section, we review the definition of three different *equivariant* Floer homology groups $\widehat{\mathrm{I}}_*(Y)$, $\breve{\mathrm{I}}_*(Y)$ and $\bar{\mathrm{I}}_*(Y)$ for any integral homology sphere Y. These Floer homology groups are $\Lambda[x]$-modules and are constructed algebraically from the Floer chain complexes of Subsection 2.1 without any other geometrical input. The Floer homology groups $\widehat{\mathrm{I}}_*(Y)$ and $\breve{\mathrm{I}}_*(Y)$ are essentially the same as the Floer homology groups $\overline{\overline{\mathrm{HF}}}(Y)$, $\underline{\underline{\mathrm{HF}}}(Y)$ introduced in [Don02]. The notation here is motivated by the notations used in the context of monopole Floer homology [KM07]. Similar to the three flavors of monopole Floer homology, the Floer homology groups $\widehat{\mathrm{I}}(Y)$, $\breve{\mathrm{I}}(Y)$ and $\bar{\mathrm{I}}(Y)$ are modules over $\Lambda[x]$ and they fit into an exact sequence of the form:

$$\begin{array}{ccc} \breve{\mathrm{I}}_*(Y) & \xrightarrow{\quad j_* \quad} & \widehat{\mathrm{I}}_*(Y) \\ & {\scriptstyle i_*} \nwarrow \quad \swarrow {\scriptstyle p_*} & \\ & \bar{\mathrm{I}}_*(Y) & \end{array} \tag{2.20}$$

Following Subsection 2.1, we fix an ϵ-admissible perturbation π for the Chern-Simons functional of Y and form the chain complex $(C^\pi_*(Y), d)$ and the maps U, D_1 and D_2. The Floer homology group $\widehat{\mathrm{I}}_*(Y)$ (pronounced as "I-from") is the homology of the following chain complex:

$$\widehat{C}^\pi_*(Y) := C^\pi_*(Y) \oplus \Lambda[x] \qquad \widehat{d}(\alpha, \sum_{i=0}^{N} a_i x^i) = (d\alpha - \sum_{i=0}^{N} U^i D_2(a_i), 0)$$

The $\Lambda[x]$-module structure is also induced by the map:

$$x \cdot (\alpha, \sum_{i=0}^{N} a_i x^i) := (U\alpha, D_1(\alpha) + \sum_{i=0}^{N} a_i x^{i+1})$$

It is helpful to think about the summand $\Lambda[x]$ in $\widehat{C}^\pi_*(Y)$ as a free $\Lambda[x]$-module generated by the trivial connection. Similar comments apply to $\breve{\mathrm{I}}_*(Y)$ and $\bar{\mathrm{I}}_*(Y)$ described below.

The Floer homology group $\breve{\mathrm{I}}_*(Y)$ (pronounced as "I-to") is the homology of the following chain complex:

$$\breve{C}^\pi_*(Y) := C^\pi_*(Y) \oplus \Lambda[\![x^{-1}, x]\!]/\Lambda[x] \qquad \breve{d}(\alpha, \sum_{i=-\infty}^{-1} a_i x^i) = (d\alpha, \sum_{i=-\infty}^{-1} D_1 U^{-i-1}(\alpha) x^i)$$

Here $\Lambda[\![x^{-1}, x]$ is the ring of Laurent power series in x^{-1}. The action of $x \in \Lambda[x]$ on this vector space over Λ is given by multiplication by x. The quotient module $\Lambda[\![x^{-1}, x]/\Lambda[x]$ can be identified in an obvious way with the set of power series in x^{-1} with vanishing constant terms. The $\Lambda[x]$-module structure on $\check{C}^{\pi}_*(Y)$ is given by the map:

$$x \cdot (\alpha, \sum_{i=-\infty}^{-1} a_i x^i) := (U\alpha + D_2(a_{-1}), \sum_{i=-\infty}^{-2} a_i x^{i+1})$$

The Floer homology group $\bar{\mathrm{I}}_*(Y)$ (pronounced as "I-bar") has a very simple form, analogous to the monopole Floer homology group $\overline{HM}(Y)$. Let $\overline{C}^{\pi}_*(Y) = \Lambda[\![x^{-1}, x]$ with the obvious $\Lambda[x]$-module structure and $\overline{d} = 0$. In particular, $\overline{C}^{\pi}_*(Y)$ is independent of the choice of the perturbation term π. Then $\bar{\mathrm{I}}_*(Y)$, defined as the homology of the complex $(\overline{C}^{\pi}_*(Y), \overline{d})$, can be simply identified with $\Lambda[\![x^{-1}, x]$.

Next, we define chain maps:

$$\begin{array}{ccc} \check{C}^{\pi}_*(Y) & \xrightarrow{\quad j \quad} & \widehat{C}^{\pi}_*(Y) \\ & {\scriptstyle i} \nwarrow \quad \swarrow {\scriptstyle p} & \\ & \overline{C}_*(Y) & \end{array} \tag{2.21}$$

by the formulas:

$$i(\sum_{i=-\infty}^{N} a_i x^i) = (\sum_{i=0}^{N} U^i D_2(a_i), \sum_{i=-\infty}^{-1} a_i x^i), \tag{2.22}$$

$$j(\alpha, \sum_{i=-\infty}^{-1} a_i x^i) = (\alpha, 0), \tag{2.23}$$

$$p(\alpha, \sum_{i=0}^{N} a_i x^i) = \sum_{i=-\infty}^{-1} D_1 U^{-i-1}(\alpha) x^i + \sum_{i=0}^{N} a_i x^i. \tag{2.24}$$

Lemma 2.25. *The maps i and p are $\Lambda[x]$-module homomorphisms which are also chain maps, and j is a chain map which commutes with the action of x up to a chain homotopy. That is to say, there is a map $h : \check{C}^{\pi}_*(Y) \to \widehat{C}^{\pi}_*(Y)$ such that:*

$$i \circ x = x \circ i, \qquad j \circ x - x \circ j = \widehat{d} \circ h + h \circ \check{d}, \qquad p \circ x = x \circ p. \tag{2.26}$$

In particular, if i_*, j_* and p_* are the maps induced by i, j and p at the level of homology, then they are $\Lambda[x]$-module homomorphisms. Moreover, i_*, j_* and p_* form an exact triangle as in (2.20)*.*

Proof. It is straightforward to check that the first and the third identifies in (2.26) hold. If we define:

$$h(\alpha, \sum_{i=-\infty}^{-1} a_i x^i) := (0, -a_{-1}).$$

then we obtain the second identity in (2.26). We define maps $\mathfrak{k}$, $\mathfrak{l}$ and $\mathfrak{r}$ as in the diagram:

$$\begin{array}{ccc} \check{C}^{\pi}_*(Y) & \xleftarrow{\quad\mathfrak{l}\quad} & \widehat{C}^{\pi}_*(Y) \\ & {\scriptstyle\mathfrak{k}}\searrow \quad \nearrow{\scriptstyle\mathfrak{r}} & \\ & \overline{C}_*(Y) & \end{array} \tag{2.27}$$

using the formulas:

$$\mathfrak{k}(\alpha, \sum_{i=-\infty}^{-1} a_i x^i) = -\sum_{i=-\infty}^{-1} a_i x^i, \qquad \mathfrak{l}(\alpha, \sum_{i=0}^{N} a_i x^i) = ((-1)^{|\alpha|}\alpha, 0), \tag{2.28}$$

and:

$$\mathfrak{r}(\sum_{i=-\infty}^{N} a_i x^i) = (0, \sum_{i=0}^{N} a_i x^i). \tag{2.29}$$

In the definition of $\mathfrak{l}$, the term $|\alpha|$ denotes the $\mathbf{Z}/8$-grading of α. In particular, we assume that α is a homogenous element. These maps satisfy the identities:

$$p\circ j + \mathfrak{k}\circ\check{d} = 0 \qquad i\circ p + \mathfrak{l}\circ\widehat{d} + \check{d}\circ\mathfrak{l} = 0 \qquad j\circ i + \widehat{d}\circ\mathfrak{r} = 0 \tag{2.30}$$

It is also straightforward to check that the following maps are respectively isomorphisms of the chain complexes $\check{C}^{\pi}_*(Y)$, $\widehat{C}^{\pi}_*(Y)$ and $\overline{C}_*(Y)$:

$$\mathfrak{l}\circ j + i\circ\mathfrak{k} \qquad \mathfrak{r}\circ p + j\circ\mathfrak{l} \qquad \mathfrak{k}\circ i + p\circ\mathfrak{r}$$

This implies that (2.21) determines an exact triangle at the level of homology groups. (See [OS05, Lemma 4.2] and [KM11a, Lemma 7.1].) □

Equivariant instanton Floer homology groups and the exact triangle in (2.20) are functorial with respect to cobordisms. Suppose Y and Y' are two integral homology spheres and $W : Y \to Y'$ is a cobordism with $b_1(W) = b^+(W) = 0$. Suppose ϵ-admissible perturbations π and π' of the Chern-Simons functionals of Y and Y' are fixed, and these perturbations are extended to an ϵ-admissible perturbation of the ASD equation on W^+. As in the previous subsection, we can associate the maps φ, μ, Δ_1 and Δ_2 to W by the chosen perturbations. We use these maps as the only geometrical input to obtain the functoriality of the exact triangle in (2.20). We firstly define a homomorphism $\widehat{C}_W : \widehat{C}^{\pi}_*(Y) \to \widehat{C}^{\pi'}_*(Y')$ as follows:

$$\begin{aligned} \widehat{C}_W(\alpha, \sum_{i=0}^{N} a_i x^i) = \Bigg(& \varphi(\alpha) + \sum_{i=0}^{N} (U')^i \Delta_2(a_i) + \sum_{i=0}^{N}\sum_{k=0}^{i-1} (U')^k \mu U^{i-1-k} D_2(a_i), c(W)\cdot \sum_{i=0}^{N} a_i x^i \\ & + \sum_{i=0}^{N-1} (\sum_{k=i+1}^{N} D_1'(U')^{k-i-1}\Delta_2(a_k))x^i + \sum_{i=0}^{N-1} (\sum_{k=i+1}^{N} \Delta_1 U^{k-i-1} D_2(a_k))x^i \\ & + \sum_{i=0}^{N-2} (\sum_{j=i+1}^{N-1}\sum_{k=j+1}^{N} D_1'(U')^{j-i-1}\mu U^{k-j-1} D_2(a_k))x^i \Bigg) \end{aligned}$$

Here $c(W)$ denotes the number of the elements of $H_1(W, \mathbf{Z})$. Similarly, $\check{C}_W : \check{C}^{\pi}_*(Y) \to \check{C}^{\pi'}_*(Y')$ is defined to be:

$$\begin{aligned}\check{C}_W(\alpha, \sum_{i=-\infty}^{-1} a_i x^i) = \Bigg(&\varphi(\alpha), \sum_{i=-\infty}^{-1} \Delta_1 U^{-i-1}(\alpha) x^i + \sum_{i=-\infty}^{-2} \sum_{j=i+1}^{-1} D_1'(U')^{-j-1} \mu U^{j-i-1}(\alpha) x^i \\ &+ (\sum_{i=-\infty}^{-1} a_i x^i) \cdot (c(W) + \sum_{j=-\infty}^{-1} \Delta_1 U^{-j-1} D_2(1) x^j + \sum_{j=-\infty}^{-1} D_1'(U')^{-j-1} \Delta_2(1) x^j \\ &+ \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{-1} D_1'(U')^{-j-1} \mu U^{-k-1} D_2(1) x^{j+k}) \Bigg)\end{aligned}$$

Finally, we define $\overline{C}_W : \overline{C}^{\pi}_*(Y) \to \overline{C}^{\pi'}_*(Y')$:

$$\begin{aligned}\overline{C}_W(\sum_{i=-\infty}^{N} a_i x^i) = (\sum_{i=-\infty}^{N} a_i x^i) \Bigg(&c(W) + \sum_{i=-\infty}^{-1} \Delta_1 U^{-i-1} D_2(1) x^i + \sum_{i=-\infty}^{-1} D_1'(U')^{-i-1} \Delta_2(1) x^i \\ &+ \sum_{k=-\infty}^{-1} \sum_{j=-\infty}^{-1} D_1'(U')^{-j-1} \mu U^{-k-1} D_2(1) x^{k+j} \Bigg)\end{aligned}$$

Proposition 2.31. *The maps $\widehat{C}_W$, $\check{C}_W$ and $\overline{C}_W$ are chain maps. The map $\overline{C}_W$ is a $\Lambda[x]$-module homomorphism, and $\widehat{C}_W$ and $\check{C}_W$ commute with the action of x up to chain homotopies. That is to say, there are maps $\mathfrak{K}$: and $\mathfrak{L}$ such that:*

$$x \circ \widehat{C}_W - \widehat{C}_W \circ x = \mathfrak{K} \circ \widehat{d} + \widehat{d}' \circ \mathfrak{K} \qquad x \circ \check{C}_W - \check{C}_W \circ x = \mathfrak{L} \circ \check{d} + \check{d}' \circ \mathfrak{L}$$

Moreover, there are maps $K : \widehat{C}^{\pi}_*(Y) \to \overline{C}^{\pi}_*(Y')$ and $L : \overline{C}^{\pi}_*(Y) \to \check{C}^{\pi}_*(Y')$ such that:

$$p' \circ \widehat{C}_W - \overline{C}_W \circ p = K \circ \widehat{d} \qquad j' \circ \check{C}_W = \widehat{C}_W \circ j \qquad i' \circ \overline{C}_W - \check{C}_W \circ i = \check{d}' \circ L$$

In particular, $\widehat{C}_W$, $\check{C}_W$ and $\overline{C}_W$ induce a $\Lambda[x]$-module homomorphism of exact triangles at the level of homology.

Proof. The first part is easy to verify. For the remaining parts, we can define $\mathfrak{K}$, $\mathfrak{L}$, K and L as follows:

$$\mathfrak{K}(\alpha, \sum_{i=0}^{N} a_i x^i) = (\mu(\alpha), \Delta_1(\alpha)), \qquad \mathfrak{L}(\alpha, \sum_{i=-\infty}^{-1} a_i x^i) = (\mu(\alpha) + \Delta_2(a_{-1}), 0), \tag{2.32}$$

$$K(\alpha, \sum_{i=0}^{N} a_i x^i) = \sum_{i=-\infty}^{-1} \Delta_1 U^{-i-1}(\alpha) x^i + \sum_{k=-\infty}^{-1} \sum_{j=-\infty}^{-1} D_1'(U')^{-j-1} \mu U^{-k-1}(\alpha) x^{k+j}, \tag{2.33}$$

$$L(\sum_{i=-\infty}^{N} a_i x^i) = (\sum_{i=0}^{N} (U')^i \Delta_2(a_i) + \sum_{i=0}^{N} \sum_{j=0}^{i-1} (U')^j \mu U^{i-1-j} D_2(a_i), 0). \tag{2.34}$$

□

Remark 2.35. Suppose $W : Y \to Y'$ is a negative definite cobordism with trivial b_1. Standard continuation maps can be used to show that if one changes the perturbation term $\overline{\pi}$ and the Riemannian metric on Y, then $\widetilde{C}_W$ changes by a chain homotopy of the following form:

$$\begin{bmatrix} \psi & 0 & 0 \\ K_1 & 0 & 0 \\ \nu & K_2 & -\psi \end{bmatrix}. \tag{2.36}$$

This input can be used to show that the chain homotopy type of the maps $\widehat{C}_W$, $\check{C}_W$ and $\overline{C}_W$ are also independent of the perturbation term and the metric.

Suppose $W' : Y' \to Y''$ is another negative definite cobordism with trivial b_1. Then we can form the composite cobordism $W \# W' : Y \to Y''$. Standard neck stretching argument shows that the map $\widetilde{C}_{W\#W'}$ is chain homotopic to $\widetilde{C}_{W'} \circ \widetilde{C}_W$. Using this input, it is straightforward (but slightly cumbersome) to show that $\widehat{C}_{W\#W'}$ is chain homotopic to $\widehat{C}_{W'} \circ \widehat{C}_W$. Similar results hold for $\check{C}_{W\#W'}$ and $\bar{C}_{W\#W'}$. We do not attempt to write down these chain homotopies here partly because we do not need them. In fact, there are larger *models* for the homology groups $\widehat{\mathrm{I}}_*(Y)$, $\check{\mathrm{I}}_*(Y)$ and $\bar{\mathrm{I}}_*(Y)$ where these results are easier to verify. In particular, these alternative models will behave better with respect to connected sum of integral homology spheres [DST].

We can extend the definition of the grading val to the non-zero elements of the complexes $\widehat{C}^{\pi}_*(Y)$, $\check{C}^{\pi}_*(Y)$ and $\overline{C}^{\pi}_*(Y)$:

$$\mathrm{val}(\alpha, \sum_{i=0}^{N} a_i x^i) = \begin{cases} \mathrm{val}(a_0, a_1, \dots, a_N) & \text{if } \sum_{i=0}^{N} a_i x^i \neq 0, \\ \mathrm{val}(\alpha) & \text{if } \sum_{i=0}^{N} a_i x^i = 0, \end{cases}$$

$$\mathrm{val}(\alpha, \sum_{i=-\infty}^{-N} a_i x^i) = \begin{cases} \mathrm{val}(\alpha) & \text{if } \alpha \neq 0, \\ \mathrm{val}(a_{-N}) & \text{if } \alpha = 0 \text{ and } a_{-N} \neq 0, \end{cases}$$

and

$$\mathrm{val}(\sum_{i=-\infty}^{N} a_i x^i) = \begin{cases} \mathrm{val}(a_0, a_1, \dots, a_N) & \text{if } \sum_{i=0}^{N} a_i x^i \neq 0, \\ \mathrm{val}(a_N) & \text{if } N < 0. \end{cases}$$

We define val of the zero element in these three complexes to be ∞. It is also useful to pick a notation for the standard notion of degree for non-zero elements of $\overline{C}^{\pi}_*(Y) = \Lambda[\![x^{-1}, x]$:

$$\mathrm{Deg}(\sum_{i=-\infty}^{N} a_i x^i) = N \qquad \text{if } a_N \neq 0.$$

The following lemma is a straightforward consequence of our analysis of the previous subsection on the behavior of cobordism maps with respect to val:

Lemma 2.37. *For any positive real number δ, there is a positive constant ϵ such that if the perturbations π, π' and the secondary perturbation on W are ϵ-admissible, then the maps $\widehat{C}_W$ and $\check{C}_W$ do not decrease* val *by more than δ. Moreover, for any $z \in \overline{C}^{\pi}_*(Y)$, the difference $|\mathrm{val}(\overline{C}_W(z)) - \mathrm{val}(z)|$ is at most δ and* $\mathrm{Deg}(\overline{C}_W(z)) = \mathrm{Deg}(z)$.

Note that the claim $\mathrm{Deg}(\overline{C}_W(z)) = \mathrm{Deg}(z)$ holds for any choices of admissible perturbations and we do not need any assumption on ϵ.

We end this subsection with some speculations about equivariant instanton Floer homology groups. An immediate consequence of the claimed properties in Remark 2.35 is that the exact triangle (2.20) is a topological invariant of Y and does not depend on the choices of the perturbation term and the Riemannian metric on Y. By dropping the powers of λ in our definitions, we can similarly define the analogue of the exact triangle (2.20) with rational coefficients.

The three flavors of monopole Floer homology $\widehat{\mathrm{HM}}(Y)$, $\widecheck{\mathrm{HM}}(Y)$ and $\overline{\mathrm{HM}}(Y)$ are also 3-manifold invariants which fit into an exact triangle of $\mathbf{Q}[x]$-modules[6][KM07]. Moreover, for integral homology spheres, the invariant $\overline{\mathrm{HM}}(Y)$ is always isomorphic to $\mathbf{Q}[x^{-1}, x]\!] = \mathrm{Hom}(\mathbf{Q}[\![x^{-1}, x], \mathbf{Q})$. These similarities motivate the following question. An affirmative answer to this question would be in the sprit of Witten's conjecture relating Donaldson invariants and Seiberg-Witten invariants [Wit94, GNY11, FL02].

Question 2.38. *Is there a relationship between the instanton invariants $\widehat{\mathrm{I}}(Y)$, $\check{\mathrm{I}}(Y)$ and the monopole invariants $\widehat{\mathrm{HM}}(Y)$, $\widecheck{\mathrm{HM}}(Y)$? What about the exact triangle* (2.20) *and its monopole counterpart* [KM07, Section 3.1]*?*

As it is pointed out in Remark 3.25, Fróyshov's instanton h-invariant can be reformulated using the exact triangle (2.20). This definition of h-invariant is similar to the definition of Monopole h-invariant [Frø96, KM07, Frø10]. As a follow up to Question 2.38, one can ask:

Question 2.39. *Is there a relationship between instanton and monopole h-invariants?*

3 Homology Cobordism Invariants

3.1 Definition of Γ_Y

We are ready to give the definition of Γ_Y. For any k, define:

$$\Gamma_Y(k) := \max(\left(\lim_{|\pi|_{\mathcal{P}} \to 0} \inf_{\substack{z \in \overline{C}^{\pi}_*(Y),\, w \in \check{C}^{\pi}_*(Y), \\ \check{d}(w) = i(z),\, \mathrm{Deg}(z) = -k}} (\mathrm{val}(z) - \mathrm{val}(w)) \right), 0) \tag{3.1}$$

Here the limit is taken over a sequence of perturbations $\{\pi_i\}$ where π_i is ϵ_i-admissible and ϵ_i converges to 0. We use the convention that the infimum of an empty set is equal to ∞. In particular, if there is no $z \in \overline{C}^{\pi}_*(Y)$ with $i(z) = 0$ and $\mathrm{Deg}(z) = -k$, then the infimum in the above expression is equal to ∞. In the following proposition, we show that the definition of $\Gamma_Y(k)$ is independent of the choice of the sequence of perturbations $\{\pi_i\}$. Later in this subsection, we give equivalent definitions of $\Gamma_Y(k)$ which is more in line with the original definition of the h-invariant in [Frø02].

[6] In the context of monopole Floer homology, $\mathbf{Q}[x]$ should be regarded as the cohomology ring of BS^1. On the other hand, equivariant instanton Floer homologies (with rational coefficients) are modules over the cohomology ring of $\mathrm{BSO}(3)$, which is again isomorphic to $\mathbf{Q}[x]$.

Proposition 3.2. *For any integer k, $\Gamma_Y(k)$ is well-defined and depends only on k and the homology cobordism class of Y.*

Proof. Suppose Y and Y' are integral homology spheres and $W : Y \to Y'$ is a homology cobordism. Suppose π and π' are ϵ-admissible perturbations for the Chern-Simons functional of Y and Y'. We extend these perturbations to W^+ using a secondary ϵ-admissible perturbations. Thus we can associate the chain complexes $\widehat{C}^{\pi}_*(Y)$, $\check{C}^{\pi}_*(Y)$, $\overline{C}^{\pi}_*(Y)$ to Y, the chain complexes $\widehat{C}^{\pi'}_*(Y')$, $\check{C}^{\pi'}_*(Y')$, $\overline{C}^{\pi'}_*(Y')$ to Y' and the chain maps $\widehat{C}_W$, $\check{C}_W$ and $\overline{C}_W$ to W.

Let $z \in \overline{C}^{\pi}_*(Y)$ and $w \in \check{C}^{\pi}_*(Y)$ be chosen such that $\mathrm{Deg}(w) = -k$ and $\check{d}(w) = i(z)$. Let $w' = \check{C}_W(w)$ and $z' = \overline{C}_W(z)$. Proposition 2.31 and Lemma 2.37 assert that:

$$\mathrm{Deg}(z') = -k, \qquad i(z') = \check{d}(w' + L(z)),$$

where L is defined in (2.34). We fix a positive constant δ. Using Lemma 2.37, we can conclude that there is a positive constant ϵ_0 such that if $\epsilon \leqslant \epsilon_0$, then:

$$\mathrm{val}(w') \geqslant \mathrm{val}(w) - \delta, \qquad |\mathrm{val}(z') - \mathrm{val}(z)| \leqslant \delta. \tag{3.3}$$

We can apply a similar argument to show that if ϵ_0 is small enough, then:

$$\mathrm{val}(L(z)) \geqslant \mathrm{val}(z) - \delta. \tag{3.4}$$

Identities (3.3) and (3.4) imply that:

$$\begin{aligned} \mathrm{val}(w' + L(z)) &\geqslant \min(\mathrm{val}(w'), \mathrm{val}(L(z))) \\ &\geqslant \min(\mathrm{val}(w), \mathrm{val}(z)) - \delta \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \mathrm{val}(z') - \mathrm{val}(w' + L(z)) &\leqslant \mathrm{val}(z) - \min(\mathrm{val}(w), \mathrm{val}(z)) + 2\delta \\ &\leqslant \max(\mathrm{val}(z) - \mathrm{val}(w), 0) + 2\delta \end{aligned}$$

By taking infimum over all pairs of (w, z) as above we have:

$$\begin{aligned} \inf_{\substack{z' \in \overline{C}^{\pi'}_*(Y'),\, w' \in \check{C}^{\pi'}_*(Y'), \\ \check{d}(w') = i(z'),\, \mathrm{Deg}(z') = -k}} (\mathrm{val}(z') - \mathrm{val}(w')) \leqslant \\ \max(\inf_{\substack{z \in \overline{C}^{\pi}_*(Y),\, w \in \check{C}^{\pi}_*(Y), \\ \check{d}(w) = i(z),\, \mathrm{Deg}(z) = -k}} (\mathrm{val}(z) - \mathrm{val}(w)), 0) + 2\delta \end{aligned} \tag{3.5}$$

By reversing the cobordism W, we obtain a similar inequality where the roles of Y and Y' are reversed. Thus the limit in (3.1) converges to a finite number for any sequence of holonomy perturbations $\{\pi_i\}$ with $|\pi_i|_{\mathcal{P}}$ being convergent to zero. Moreover, this limit is independent of the chosen sequence, and only depends on the homology cobordism class of Y and the integer k. □

Next, we attempt to unravel the definition of Γ_Y. Fix a positive integer k, and let $\alpha \in C^\pi_*(Y)$ be chosen such that:

$$d\alpha = 0, \qquad D_1 U^{k-1}(\alpha) \neq 0, \qquad D_1 U^j(\alpha) = 0 \quad \text{for any } j < k-1. \tag{3.6}$$

We can form a pair:

$$z = \sum_{i=-\infty}^{-k} D_1 U^{-i-1}(\alpha) x^i \in \overline{C}^\pi_*(Y) \qquad w = (\alpha, \sum_{i=-\infty}^{-1} a_i x_i) \in \breve{C}^\pi_*(Y) \tag{3.7}$$

where the constants $a_i \in \Lambda$ are chosen arbitrarily. Then z has degree $-k$ and $i(z) = \breve{d}(w)$. In fact, any pair of z with degree $-k$ and w, satisfying $\breve{d}(w) = i(z)$, are given as in (3.7) for an appropriate choice of α and $\{a_i\}$. In particular, the definition of Γ_Y at a positive integer can be rewritten as:

$$\Gamma_Y(k) = \lim_{|\pi|_{\mathcal{P}} \to 0} \inf_{\alpha} (\mathrm{val}(D_1 U^{(k-1)}(\alpha)) - \mathrm{val}(\alpha))) \tag{3.8}$$

where the infimum is taken over all $\alpha \in C^\pi_*(Y)$ that satisfy (3.6). Given any positive constant δ, there is ϵ such that if π is an ϵ-admissible perturbation, then the difference $\mathrm{val}(D^1 U^{(k-1)}(\alpha)) - \mathrm{val}(\alpha)$ is greater than $-\delta$. This is the reason that we do not need to take the maximum of the expression in (3.8) and 0. The possibility to drop the maximum in (3.1) for positive values of k can be also explained using the fact that for any z with negative degree, we have $L(z) = 0$ where L is defined in (2.34).

The infimum in (3.8) can be taken over an even smaller set. We firstly need to introduce a new terminology:

Definition 3.9. Let i be an integer and r be a real number. Let π be an admissible perturbation of the Chern-Simons functional of an integral homology sphere Y. Let $\alpha_1, \ldots, \alpha_k$ be the critical points of the perturbed Chern-Simons functional whose Floer gradings are equal to i mod 8. For each α_l, there is up to homotopy, a unique path z_l from the trivial connection Θ to α_l such that $\mathrm{index}(z)$ is equal to $-i-3$. Let also r_l denote the topological energy of z_l, which is an integer lift of $-\mathrm{CS}(\alpha_l)$. (See the identity in (2.1).) A *homogenous* element of $C^\pi_*(Y)$ with *weight* (r, i) is defined to be an element of the following form:

$$\lambda^r (s_1 \lambda^{r_1} \alpha_1 + s_2 \lambda^{r_2} \alpha_2 + \cdots + s_k \lambda^{r_k} \alpha_k)$$

where s_i are rational numbers. The set of all homogenous elements of $C^\pi_*(Y)$ with weight (r, i) is denoted by $C^\pi_{(r,i)}(Y)$.

Definition 3.10. Let r, i and α_l be given as above and r_l be defined as in in Definition 3.9. Given:

$$\alpha = \sum_{l=1}^{k} \sum_{j=0}^{\infty} s_{l,j} \lambda^{r_{l,j}} \alpha_l \in C^\pi_i(Y)$$

with $s_{l,j} \in \mathbf{Q}$, define:

$$\mathrm{P}_{r,i}(\alpha) := \sum_{l=1}^{k} \sum_{j=0}^{\infty} s_{l,j} \mathrm{P}_{r,i}(\lambda^{r_{l,j}} \alpha_l) \in C^\pi_i(Y)$$

where $\mathrm{P}_{r,i}(\lambda^{r_{l,j}}\alpha_l)$ is equal to $\lambda^{r_{l,j}}\alpha_l$ if $r_{l,j} = r + r_l$, and is equal to 0 otherwise. We extend $\mathrm{P}_{r,i}$ to a map on $C^\pi_*(Y)$ by requiring $\mathrm{P}_{r,i}(\beta) = 0$ if β is an element with Floer grading j such that $j \not\equiv i \bmod 8$. The map $\mathrm{P}_{r,i}$ is called *projection to homogenous elements of weight* (r, i). Similarly, we define a projection map $\mathrm{P}_r : \Lambda \to \Lambda$ which maps an element a of Λ to $s\lambda^r$ where s is the rational coefficient of λ^r in a.

The following lemma is a straightforward consequence of additivity of indices and topological energies:

Lemma 3.11. *Any homogenous element of weight (r, i) of $C^\pi_*(Y)$ is also a homogenous element of $(r + 1, i + 8)$. We also have:*

(i) d maps a homogenous element of weight (r, i) to a homogenous element of weight $(r, i - 1)$, and we have $d \circ \mathrm{P}_{r,i} = \mathrm{P}_{r,i-1} \circ d$;

(ii) U maps a homogenous element of weight (r, i) to a homogenous element of weight $(r, i - 4)$, and we have $U \circ \mathrm{P}_{r,i} = \mathrm{P}_{r,i-4} \circ U$;

(iii) D_1 maps a homogenous element of weight $(r, 1)$ to a rational multiple of λ^r, and we have $D_1 \circ \mathrm{P}_{r,1} = \mathrm{P}_r \circ D_1$;

(iv) D_2 maps λ^r to a homogenous element of weight $(r, -4)$, and we have $D_2 \circ \mathrm{P}_r = \mathrm{P}_{r,-4} \circ D_2$.

Suppose $\alpha \in C^\pi_*(Y)$ satisfies (3.6), and $\mathrm{val}(D_1U^{k-1}(\alpha)) = r_0$. We define α_0 to be $\lambda^{-r_0} \cdot \mathrm{P}_{(r_0,4(k-1)+1)}(\alpha)$, which is a homogenous element of weight $(0, 4(k - 1) + 1)$. By Lemma 3.11, α_0 satisfies the identities in (3.6) and $\mathrm{val}(D_1U^{k-1}(\alpha_0)) = 0$. Since the projection maps $\mathrm{P}_{r,i}$ do not decrease val, we can conclude:

$$\mathrm{val}(D_1U^{k-1}(\alpha_0)) - \mathrm{val}(\alpha_0) \leqslant \mathrm{val}(D_1U^{k-1}(\alpha)) - \mathrm{val}(\alpha)$$

This inequality implies that to find the value of $\Gamma_Y(k)$, it suffices to take the infimum in (3.8) over the following set:

$$\mathcal{L}^\pi_k := \{\alpha \in C^\pi_*(Y) \mid \alpha \in C^\pi_{(0,4k-3)}(Y),\ d\alpha = 0,\ (k-1) = \min\{j \mid D_1U^j(\alpha) \neq 0\}\} \tag{3.12}$$

We can give a similar description for $\Gamma_Y(k)$ in the case that k is non-positive. Let:

$$\alpha \in C^\pi_*(Y) \qquad \mathfrak{A} = \{a_0, a_1, \ldots, a_{-k}\} \subset \Lambda \tag{3.13}$$

be given such that:

$$d\alpha = \sum_{i=0}^{-k} U^i D_2(a_i). \tag{3.14}$$

Then for any arbitrary sequence $\{b_i\}_{i=-\infty}^{-1}$ the pair:

$$z = \sum_{i=0}^{-k} a_i x^i + \sum_{i=-\infty}^{-1} D_1U^{-i-1}(\alpha)x^i \in \overline{C}^\pi_*(Y) \qquad w = (\alpha, \sum_{i=-\infty}^{-1} b_i x^i) \in \breve{C}^\pi_*(Y) \tag{3.15}$$

satisfies the identity $i(z) = \check{d}(w)$, and any such pair with z being of degree $-k$ is given as in (3.15). Then we have:

$$\Gamma_Y(k) := \max\left(0, \lim_{|\pi|_{\mathcal{P}} \to 0} \inf_{(\alpha, \{a_0, a_1, \ldots, a_{-k}\})} (\mathrm{val}(a_0, a_1, \ldots, a_{-k}) - \mathrm{val}(\alpha))\right) \tag{3.16}$$

where the infimum is taken over all pairs $(\alpha, \mathfrak{A})$ which satisfy (3.14).

We can work with a smaller set of pairs $(\alpha, \mathfrak{A})$ in (3.16). Let $(\alpha, \mathfrak{A})$ be as in (3.13) satisfying (3.14). We also assume that a_{i_0} has the minimum val among the elements of $\mathfrak{A}$, which is equal to r_0. We define:

$$\alpha_0 := \lambda^{-r_0} \mathrm{P}_{r_0, -3-4i_0}(\alpha) \qquad b_i = \begin{cases} \lambda^{-r_0} \mathrm{P}_{r_0 + \frac{i_0 - i}{2}}(a_i) & i \equiv i_0 \mod 2 \\ 0 & i \not\equiv i_0 \mod 2 \end{cases}$$

Our assumption on i_0 implies that $b_i = 0$ if $i > i_0$. Projection of (3.14) to homogenous elements of weight $(r_0, -4 - 4i_0)$ shows that:

$$d\alpha_0 = \sum_{i=0}^{-k} U^i D_2(b_i). \tag{3.17}$$

We also have $\mathrm{val}(b_0, b_1, \ldots, b_{-k}) = 0$ and $\mathrm{val}(\alpha_0)$ is not less than $\mathrm{val}(\alpha) - r_0$. Consequently, we have:

$$(\mathrm{val}(b_0, b_1, \ldots, b_{-k}) - \mathrm{val}(\alpha_0)) \leqslant (\mathrm{val}(a_0, a_1, \ldots, a_{-k}) - \mathrm{val}(\alpha))$$

This analysis shows that when $k \leqslant 0$, it suffices to take the infimum in (3.8) over the following set:

$$\mathcal{L}_k^\pi := \{(\alpha, \{a_0, a_1, \ldots, a_{-k}\}) \mid \alpha \in C^\pi_{(0,4k-3)}(Y),\ d\alpha = \sum_{i=0}^{-k} U^i D_2(a_i),\ a_i \text{ is a rational multiple of } \lambda^{\frac{-k-i}{2}} \text{ if } i \equiv k \bmod 2, \text{ and is zero otherwise.}\} \tag{3.18}$$

Remark 3.19. The author does not know any example where the value of Γ_Y at a non-positive integer is a finite positive number. However, one can easily construct chain complexes over Λ with the similar formal properties as $\widetilde{C}^\pi_*(Y)$ such that the analogue of Γ_Y takes non-trivial values at non-positive integers. For instance, let (C_*, d) be a $\mathbf{Z}/8\mathbf{Z}$-graded complex generated by two generators α and β in degrees 5 and 4 such that:

$$d\alpha = \lambda^{r_1}\beta \qquad d\beta = 0$$

We also define $U : C_* \to C_{*-4}$ and $D_1 : C_* \to \Lambda$ to be trivial maps. Let also $D_2 : \Lambda \to C_*$ be defined as follows:

$$D_2(1) = \lambda^{r_2}\beta$$

where r_2 is a real number smaller than r_1. As in (3.1), we can associated a map $\Gamma : \mathbf{Z} \to \overline{\mathbf{R}}^{\geqslant 0}$. The value of this function at 0 is equal to the positive number $r_1 - r_2$ (and this is the only non-trivial value of Γ). We hope that the understanding the behavior of Γ_Y with respect to various topological constructions, such as surgery along knots, would be helpful to study whether there are complexes as above with non-trivial values of Γ_Y at non-positive integers.

Motivated by the definition of $\mathcal{L}_k^\pi$ in (3.12) and (3.18), we also give the following definition:

Definition 3.20. Let π be an admissible perturbation for the Chern-Simons functional of an integral homology sphere Y. We say $w \in \breve{C}^{\pi}_*(Y)$ and $z \in \overline{C}^{\pi}_*(Y)$ form a *special pair of degree* N if the following conditions are satisfied:

(i) $\mathrm{Deg}(z) = N$;

(ii) $w = (\alpha, 0)$ and $z = \sum_{j=-\infty}^{N} a_j x^j$;

(iii) if $N < 0$, then $\alpha \in \mathcal{L}^{\pi}_{-N}$ and if $N \geqslant 0$, then $(\alpha, \{a_0, \dots, a_N\}) \in \mathcal{L}^{\pi}_{-N}$.

(iv) For $j \leqslant -1$, we have $a_j = D_1 U^{-j-1}(\alpha)$.

In particular, any special pair of degree N satisfies the identity $i(z) = \breve{d}(w)$. The above discussion shows that to compute the value of Γ_Y at $-N$, it suffices to take the infimum in (3.1) over all special pairs (z, w) of degree N.

Example 3.21. In the case that $Y = S^3$, $\Sigma(2,3,5)$ or $-\Sigma(2,3,5)$, we fix the metrics on Y which are induced by the standard metric on the 3-dimensional sphere, the universal cover of Y. Then all critical points are non-degenerate and all moduli spaces are regular. Therefore, we can use the trivial perturbation π_0 to compute Γ_Y. In the case that $Y = S^3$, the set $\mathcal{L}^{\pi_0}_k$ is empty for any positive k and consists of elements of the form $(0, \{a_0, \dots, a_{-k}\})$ for negative values of k. This implies that:

$$\Gamma_{S^3}(k) = \begin{cases} \infty & k > 0 \\ 0 & k \leqslant 0 \end{cases}$$

The complex $C^{\pi_0}_*(\Sigma(2,3,5))$ is generated by two flat connections α and β with Floer gradings 1 and 5. Then we have:

$$D_1(\alpha) = \lambda^{\frac{1}{120}} \qquad U(\beta) = 8\lambda^{\frac{2}{5}}\alpha \qquad D_2 = 0$$

The above identities after evaluating λ at 1 are verified in [Frø02]. The calculation of the index and the Chern-Simons functional which determines the powers of λ in the above identities can be found, for example, in [FS90]. For an exposition of the method of [FS90], we refer the reader to [DX17, Subsection 3.4] where the same conventions as here for the definition of the Chern-Simons functional and Floer grading is used. These calculations show that the complex $C^{\pi_0}_*(-\Sigma(2,3,5))$ is generated by two flat connections α^* and β^* with Floer gradings 4 and 0. We also have:

$$D_2(1) = \lambda^{\frac{1}{120}}\alpha^* \qquad U(\alpha^*) = 8\lambda^{\frac{2}{5}}\beta^* \qquad D_1 = 0$$

These identities show that:

$$\Gamma_{\Sigma(2,3,5)}(k) = \begin{cases} \infty & k > 2 \\ \frac{49}{120} & k = 2 \\ \frac{1}{120} & k = 1 \\ 0 & k \leqslant 0 \end{cases}$$

and $\Gamma_{-\Sigma(2,3,5)} = \Gamma_{S^3}$.

3.2 Properties of Γ_Y

In this subsection, we review some of the basic properties of Γ_Y.

Proposition 3.22. *Γ_Y is a non-decreasing function with values in $\overline{\mathbf{R}}^{\geqslant 0}$.*

Proof. It is obvious from the definition that Γ_Y takes values in $\overline{\mathbf{R}}^{\geqslant 0}$. To show monotonicity of Γ_Y, fix an admissible perturbation π of the Chern-Simons functional and let (z_0, w_0) be a special pair of degree $-k$. In particular, z_0 has degree $-k$, $w_0 = (\alpha, 0)$ for an appropriate $\alpha \in C^\pi_*(Y)$ and $\check{d}(w_0) = i(z_0)$. We define $z_1 := x \cdot z_0$ and $w_1 := x \circ w_0$. Then z_1 is an element of $\overline{C}^\pi_*(Y)$ with degree $-k+1$. We also have $i(z_1) = \check{d}(w_1)$ because i and $\check{d}$ are $\Lambda[x]$-module homomorphisms.

It is clear that $\mathrm{val}(z_1) \leqslant \mathrm{val}(z_0)$. Moreover, for a given δ, there is ϵ such that for any ϵ-admissible perturbation:

$$\mathrm{val}(w_1) \geqslant \mathrm{val}(w_0) - \delta.$$

These inequalities imply that:

$$\mathrm{val}(z_1) - \mathrm{val}(w_1) \leqslant \mathrm{val}(z_0) - \mathrm{val}(w_0) + \delta$$

Consequently, taking infimum over all choices of of (z_0, w_0) and letting $|\pi|_{\mathcal{P}}$ go to 0 allow us to conclude that $\Gamma_Y(k-1) \leqslant \Gamma_Y(k)$. □

Proposition 3.23. *Suppose h denotes Fróyshov's invariant of an integral homology sphere Y. Then $\Gamma_Y(k)$ is a finite number if and only if $k \leqslant 2h(Y)$.*

Before giving the proof of the above proposition, we give an interpretation for h in terms of some of the terminology introduced in this paper:

Lemma 3.24. *For any integral homology Y and any admissible perturbation π, the integer $2h(Y)$ is equal to the largest value of k such that the set $\mathcal{L}^\pi_k$ is non-empty.*

Proof. For any admissible perturbation π, let $(\mathfrak{C}^\pi_*(Y), \mathfrak{d})$ be the standard Floer chain complex defined over rational numbers. The definition of this complex is similar to $C^\pi_*(Y)$ with the difference that we use rational numbers as the coefficient ring and we do not include any power of λ in the definition of the differential $\mathfrak{d}$. By dropping the powers of λ from the definition of D_1, D_2 and U, we can also define:

$$\mathfrak{D}_1 : \mathfrak{C}^\pi_1(Y) \to \mathbf{Q} \qquad \mathfrak{D}_2 : \mathbf{Q} \to \mathfrak{C}^\pi_4(Y) \qquad \mathfrak{U} : \mathfrak{C}^\pi_*(Y) \to \mathfrak{C}^\pi_{*-4}(Y)$$

Using these operators, we define:

$$\mathcal{K}^\pi_k := \{\alpha \mid \alpha \in \mathfrak{C}^\pi_{4k-3},\ \mathfrak{d}\alpha = 0,\ (k-1) = \min_{\mathfrak{D}^1\mathfrak{U}^j(\alpha)\neq 0}(j)\}$$

for positive values of k, and:

$$\mathcal{K}^\pi_k := \{(\alpha, \{s_0, s_1, \ldots, s_{-k}\}) \,|\alpha \in \mathfrak{C}^\pi_{4k-3}(Y),\ , s_i \in \mathbf{Q},\ \mathfrak{d}\alpha = \sum_{i=0}^{-k} \mathfrak{U}^i\mathfrak{D}_2(s_i),$$
$$s_i \neq 0 \text{ only if } i \equiv k \bmod 2.\}$$

Evaluation of λ at 1 gives a bijective map from $\mathcal{L}^\pi_k$ to $\mathcal{K}^\pi_k$ for any integer k. It is also shown in [Frø02, Proposition 4] that the set $\mathcal{K}^\pi_k$ is non-empty only if $k \leqslant 2h(Y)$.[7] Thus the largest k with $\mathcal{L}^\pi_k$ being non-empty is equal to $2h(Y)$. □

Proof of Proposition 3.23. Lemma 3.24 shows that if $k > 2h(Y)$, then $\mathcal{L}^\pi_k$ is empty for any admissible perturbation π. This lemma also implies that if $k \leqslant 2h(Y)$, then $\mathcal{L}^\pi_k$ is non-empty for any given ϵ-admissible perturbation π. Therefore, the infimum in (3.8) or (3.16) is finite for any given π. Now we can follow the argument in the proof of Proposition 3.2 and obtain a uniform upper bound for the infimum of (3.8) or (3.16) for any other ϵ-admissible perturbation. Thus $\Gamma_Y(k)$ is a finite number. □

Remark 3.25. Combining Lemma 3.24 and the discussion of the previous section shows that $h(Y)$ is given by the following identity:

$$\begin{aligned} h(Y) &= \max\{k \mid \exists z \in \bar{\mathrm{I}}_*(Y) \text{ with } \mathrm{Deg}(z) = -k \text{ and } i_*(z) \neq 0\} \\ &= \min\{k \mid \forall z \in \bar{\mathrm{I}}_*(Y) \text{ with } \mathrm{Deg}(z) = -k \text{ and } z \in \mathrm{image}(j_*)\} - 1 \end{aligned}$$

This definition of h is similar to the standard definition of monopole h-invariant [KM07, Section 39] and the correction term in Heegaard Floer homology [OS03].

Proposition 3.26. *For any integral homology sphere Y and any integer k, either $\Gamma_Y(k) = \infty$, $\Gamma_Y(k) = 0$, or there is an irreducible flat connection α such that $\Gamma_Y(k)$ is equal to $\mathrm{CS}(\alpha)$ mod $\mathbf{Z}$.*

Proof. Suppose π is an admissible perturbation of the Chern-Simons functional of Y. Firstly let k be a positive number and $\alpha \in \mathcal{L}^\pi_k$. The condition $\alpha \in C^\pi_{(0,4k-3)}(Y)$ implies that:

$$\alpha = s_1\lambda^{r_1}\alpha_1 + s_2\lambda^{r_2}\alpha_2 + \cdots + s_j\lambda^{r_j}\alpha_j$$

where $\alpha_1, \ldots, \alpha_k$ are the critical points of $\mathrm{CS} + f_\pi$ with Floer grading $4k-3$, $s_1, \ldots, s_k$ are rational numbers. The exponent r_i is introduced in Definition 3.9 and is equal to $-\mathrm{CS}(\alpha_i)$ mod $\mathbf{Z}$. Since $\alpha \in C^\pi_{(0,4k-3)}(Y)$, we have $\mathrm{val}(D_1U^{k-1}(\alpha)) = 0$. We also have:

$$\mathrm{val}(\alpha) = \min_{s_i \neq 0}(r_i).$$

In particular, we have:

$$\inf_{\alpha \in \mathcal{L}^\pi_k} \Big(\mathrm{val}(D_1U^{k-1}(\alpha)) - \mathrm{val}(\alpha)\Big) \in \{-r_1, \ldots, -r_k\}.$$

Therefore, there is an irreducible critical point α_i of $\mathrm{CS} + f_\pi$ with index $4k-3$ such that the above infimum is equal to $\mathrm{CS}(\alpha_i)$ mod $\mathbf{Z}$. As we let $\pi \to 0$, the values of the Chern-Simons functional on the critical points of $\mathrm{CS} + f_\pi$ converge to the finite set of values of the Chern-Simons functional on irreducible $\mathrm{SU}(2)$ flat connections [KM11b, Lemma 3.8]. □

[7] In [Frø02], the cohomological convention is used in the definition of Floer chain complexes. The reader should take that into account for comparing [Frø02] and the present article.

Remark 3.27. It is natural to ask whether there is an integral homology sphere Y such that Γ_Y takes irrational values. Proposition 3.26 implies that if Γ_Y takes an irrational value, then the value of the Chern-Simons functional of an $\mathrm{SU}(2)$-flat connection is irrational. Currently, it is unknown whether there is such a flat connection on a 3-manifold Y. For example, the values of the Chern-Simons functional for any plumbed 3-manifold takes rational values on $\mathrm{SU}(2)$-flat connections. Consequently, if Γ_Y is not rational valued, then Y is not a linear combination of plumbed 3-manifolds.

Definition 3.28. For any integral homology sphere Y, $\tau(Y)$ is defined to be:

$$\tau(Y) := \inf_z \{\mathcal{E}(z)\} \tag{3.29}$$

where the infimum is taken over all paths z along $\mathbf{R} \times Y$ from an irreducible flat connection α to the trivial connection Θ such that the moduli space of (unperturbed) ASD connections $\mathcal{M}_z(\alpha, \Theta)$ is non-empty.

Given a path z from α to Θ as in (3.29), $\mathcal{E}(z)$ is equal to $\mathrm{CS}(\alpha)$ mod $\mathbf{Z}$. Thus $\tau(Y)$ takes values in a discrete set because the Chern-Simons functional takes only finitely many values on the set of flat $\mathrm{SU}(2)$-connections. This implies that $\tau(Y)$ can be realized by the topological energy of an ASD connection from an irreducible flat connection α to the trivial connection. In particular, this constant is positive and we have:

$$\tau(Y) \geqslant \min\{r \mid r \in \mathbf{R}^{>0},\, r \equiv \mathrm{CS}(\alpha) \mod \mathbf{Z} \text{ for a flat connection } \alpha\} \tag{3.30}$$

We can use $\tau(Y)$ to give a constraint for the values of Γ_Y at positive integers:

Proposition 3.31. *For any integral homology sphere Y, we have $\Gamma_Y(1) \geqslant \tau(Y)$. Consequently, for any positive integer k, we have $\Gamma_Y(k) \geqslant \tau(Y)$.*

The proof of the above proposition needs some preparation. Firstly we start with a standard exponential decay result about instantons on tubes:

Lemma 3.32. *Let α be a non-degenerate $\mathrm{SU}(2)$-flat connection on an integral homology sphere Y. There are constants ϵ_0, ϵ_1, C_l and δ such that the following holds:*

(i) Suppose A is an ASD connection on $(0,\infty) \times Y$ such that the L^2-distance between $A|_{\{t\}\times Y}$, for $t \in (0,1)$, and α is less than ϵ_0 and $\|F(A)\|_{L^2((0,\infty)\times Y)} < \epsilon_1$. Then A is gauge equivalent to a connection of the form $\alpha + a$ where a is a 1-form on $(0,\infty) \times Y$ with values in $\mathfrak{su}(2)$ and:

$$|\nabla^l a|(t,y) \leqslant C_l e^{-\delta \cdot t} \|F(A)\|_{L^2((0,1)\times Y)}. \tag{3.33}$$

for $t \in (\frac{1}{2}, \infty)$.

(ii) Suppose A is an ASD connection on $(-T,T) \times Y$, for $T > 1$, such that the L^2-distance between $A|_{\{t\}\times Y}$, for $t \in (-T, -T+1)$, and α_0 is less than ϵ_0 and $\|F(A)\|_{L^2((-T,T)\times Y)} < \epsilon_1$. Then A is gauge equivalent to a connection of the form $\alpha + a$ where a is a 1-form on $(-T,T) \times Y$ with values in $\mathfrak{su}(2)$ and:

$$|\nabla^l a|(t,y) \leqslant C_l e^{-\delta(T-|t|)} \big(\|F(A)\|_{L^2((-T,-T+1)\times Y)} + \|F(A)\|_{L^2((T-1,T)\times Y)}\big). \tag{3.34}$$

for $t \in (-T + \frac{1}{2}, T - \frac{1}{2})$.

Proof. This lemma is essentially proved in [Don02, Theorem 4.2, Proposition 4.3 and Proposition 4.4]. In [Don02], it is assumed that Y is $\mathrm{SU}(2)$-non-degenerate. We weaken this assumption by requiring that only the given flat connection α is non-degenerate and make the additional assumption that the L^2 distance of $A|_{\{t\}\times Y}$ and α for appropriate values of t is bounded by ϵ_0. Since α is non-degenerate, the L^2-distance of α and any other flat connection is greater than a positive number κ. Let ϵ_0 be equal to $\frac{\kappa}{2}$. We claim that for any positive constant η smaller than ϵ_0 and any positive integer k, there is a constant ϵ_1 such that for any ASD connection A as in part (i) (resp. part (ii)) of the lemma, the L^2_k-distance between the connections $A|_{\{t\}\times Y}$ and α is less than η for any $t \in (\frac{1}{2}, \infty)$ (resp. $t \in (-T + \frac{1}{2}, T - \frac{1}{2})$).

We prove this claim for part (i). The proof for the other case is similar. Suppose there is a sequence of ASD connections A_i on $(0,\infty)\times Y$ such that $\|F(A_i)\|_{L^2((0,\infty)\times Y)} \to 0$, the L^2-distance of $A_i|_{\{t\}\times Y}$ for $t \in (0,1)$ is less than ϵ_0, and there is $t_i \in [1,\infty)$ such that the L^2_k-distance between $A_i|_{\{t\}\times Y}$ and the space of flat connections is equal to η. By Uhlenbeck compactness theorem, the connections $A_i|_{(t_i-1,t_i-1)\times Y}$, after passing to a subsequence and changing gauge, are C^∞_{loc}-convergent to a flat connection on $(-1,1)\times Y$. However, this is a contradiction because the L^2_k distance between $A_i|_{\{t\}\times Y}$ and any flat connection is at least ϵ_0. This verifies the claim. Given this claim, the arguments of [Don02] can be applied to prove the lemma without any change. □

Lemma 3.35. *Suppose $\{\pi_i\}_i$ is a sequence of admissible perturbations of the Chern-Simons functional of Y such that $|\pi_i|_{\mathcal{P}} \to 0$. Suppose $A_i \in \mathcal{M}^{\pi_i}_{z_i}(\alpha_i, \Theta)$ where α_i is an irreducible critical point of $\mathrm{CS} + f_{\pi_i}$ and z_i is a path with index 1. We also assume that there is a constant ν such that f_{π_i} vanishes for connections whose L^2-distance to Θ is less than ν. Then there exists an irreducible flat connection α_0, a path z_0 from α_0 to Θ, and a (non-perturbed) ASD connection $A_0 \in \mathcal{M}_{z_0}(\alpha_0, \Theta)$ such that:*

$$\mathcal{E}(A_0) \leqslant \limsup_i \mathcal{E}(A_i) \tag{3.36}$$

Proof. The connection A can be constructed as a limit of a sequence of connections associated to the connections $\{A_i\}$. The argument is an adaptation of [Don02, Theorem 5.4]. We divide the proof into several steps:

Step 1: *The sequence $\|F(A_i)\|_{L^2(\mathbf{R}\times Y)}$ is bounded.*

It suffices to show that the topological energies of paths between critical points of $\mathrm{CS} + f_{\pi_i}$ with index less than a fixed integer N are uniformly bounded by a constant K which does not depend on i. In the case that we are concerned only with the trivial perturbation, this is standard. The case of non-trivial perturbations $\{\pi_i\}$ can be reduced to the case of the trivial perturbation using the following trick.

Each $\mathrm{SU}(2)$-flat connection α has a path-connected open neighborhood such that the index and the topological energy of paths in this neighborhood are uniformly bounded. We cover the space of flat connections by finitely many such open neighborhoods. If $|\pi_i|_{\mathcal{P}}$ is small enough, then each end point of the path z_i belong to one of the chosen open sets. Now we can use additivity of indices and topological energies with respect to concatenation of paths to verify the claim.

Step 2: *There is a positive constant ϵ and for any connection A_i in the above sequence, there is a*

constant T_i such that $A_i|_{(T_i,\infty)\times Y}$ is an ASD connection and:

$$\int_{(T_i,\infty)\times Y} |F(A_i)|^2 \mathrm{dvol} = \epsilon \tag{3.37}$$

Suppose ϵ_0 and ϵ_1 are given as in Lemma 3.32. As in the proof of Lemma 3.32, we assume that ϵ_0 is smaller than half of the L^2-distance of the trivial connection and the space of irreducible flat $\mathrm{SU}(2)$-connections of Y. Let $\epsilon' := \min(\epsilon_0, \epsilon_1, \frac{\nu}{2})$ and T_i' be the largest real number such that the L^2 distance between $A|_{\{T_i'\}\times Y}$ and Θ is at least ϵ'. Then the connection $A_i|_{(T_i',\infty)\times Y}$ is an ASD connection. We claim that there is a constant N_0, independent of i, such that:

$$\int_{(T_i',\infty)\times Y} |F(A_i)|^2 \mathrm{dvol} \geqslant \frac{\epsilon'}{N_0}. \tag{3.38}$$

Given i, if the above inequality does not hold for $N_0 \geqslant 1$, then part (i) of Lemma 3.32 implies that there is a_i such that A_i is gauge equivalent to $\alpha + a_i$ and:

$$|\nabla^l a_i|(t,y) \leqslant C_l e^{-\delta\cdot t} \|F(A_i)\|_{L^2((0,1)\times Y)}.$$

In particular, we have:

$$|a_i|(T_i', y) \leqslant C_0 \frac{\epsilon'}{N_0}$$

If N_0 is large enough in compare to C_0, then the L^2 norm of a_i is less than $\frac{\kappa}{2}$, which is a contradiction. Thus the inequality holds for an appropriate value of N_0 and we define $\epsilon := \frac{\epsilon'}{N_0}$. There is also a unique value of T_i greater than T_i' such that (3.37) holds.

Step 3: *There is a connection $A_0 \in \mathcal{M}_{z_0}(\alpha_0, \Theta)$ satisfying* (3.36).

Let A_i' be given by translating A_i in the $\mathbf{R}$ direction by the parameter T_i. Then we have:

$$\int_{(0,\infty)\times Y} |F(A_i')|^2 \mathrm{dvol} = \epsilon \tag{3.39}$$

Using Lemma 3.32, the connection A_i' is gauge equivalent to a connection of the form $\alpha + a_i'$ where a_i' satisfies the inequalities in (3.33). Thus the Arzela-Ascoli theorem implies that there is a subsequence[8] of the connections which is C^∞-convergent on $(0,\infty)\times Y$. We can also employ the Uhlenbeck compactness theorem to show that there is an ASD connection A_0' and a finite subset S of $\mathbf{R}\times Y$ such that the sequence A_i', after passing to a subsequence and changing gauge, is L_1^p-convergent[9] on compact subsets of $\mathbf{R}\times Y\backslash S$ to an ASD connection A_0'. Moreover, we have:

$$\mathcal{E}(A_0') \leqslant \limsup_i \mathcal{E}(A_i)$$

[8]Here and in what follows, we always denote a subsequence with the same notation as the original sequence.

[9]The weaker L_1^p-convergence instead of C^∞-convergence is due to non-locality of holonomy perturbations. For a discussion related to adapting the Uhlenbeck compactness theorem to the ASD equation perturbed by holonomy perturbations, see [Kro05].

These two observations show that A_i' is strongly convergent (resp. weakly convergent) on $(0,\infty)\times Y$ (resp. $\mathbf{R}\times Y$) to A_0'. In particular, A_0' is asymptotic to Θ on the outgoing end of $\mathbf{R}\times Y$. Since A_0' is an ASD connection, it is convergent to a flat connection on the incoming end [Don02, Theorem 4.18]. If this flat connection is irreducible, then we are done. In the case that $A_0' \in \mathcal{M}_{z_0'}(\Theta,\Theta)$, the term $\|F(A_0')\|_{L^2((-\infty,-h)\times Y)}$ is strictly less than $\liminf_i \|F(A_i')\|_{L^2((-\infty,-h)\times Y)}$ for any value of h. Otherwise, the incoming flat connection of A_i' is convergent to the trivial connection which is a contradiction. The connection A_0' is non-trivial and hence its energy is at least $8\pi^2$, the energy of a single instanton.

Let h be chosen large enough such that S is disjoint from $(-\infty,-h)\times Y$ and the distance between $A_i|_{\{t\}\times Y}$ and Θ is less than ϵ_0 for $t\in(-h-1,-h)$. Define:

$$\eta = \min(\epsilon_1, \frac{\liminf_i \|F(A_i')\|_{L^2((-\infty,-h)\times Y)} - \|F(A_0')\|_{L^2((-\infty,-h)\times Y)}}{2})$$

where ϵ_1 is given by Lemma 3.32. For large values of i, we may pick a constant S_i such that:

$$\int_{(S_i,-h)\times Y} |F(A_i')|^2 = \eta + \int_{(-\infty,-h)\times Y} |F(A_0')|^2$$

The constants S_i are convergent to $-\infty$ and we define a new sequence of connections A_i'' by translating A_i' in the $\mathbf{R}$ direction by the parameter S_i. Another application of Uhlenbeck compactness implies that A_i'', after passing to a subsequence and changing gauge, is convergent to an ASD equation which satisfies:

$$\mathcal{E}(A_0') + \mathcal{E}(A_0'') \leqslant \limsup_i \mathcal{E}(A_i)$$

Moreover, part (ii) of Lemma 3.32 implies that A_0'' is asymptotic to the trivial connection on the outgoing end. If A_0'' is asymptotic to an irreducible flat connection on the incoming end, then we are done. Otherwise, we repeat the above process. This process terminates because the energy of a non-trivial ASD connection asymptotic to the trivial connection on both ends is at least $8\pi^2$. □

Proof of Proposition 3.31. Let $\{\pi_i\}$ be a sequence of admissible perturbations such that $|\pi_i|_{\mathcal{P}} \to 0$. Since the trivial connection is a non-degenerate critical point of the (non-perturbed) Chern-Simons functional of Y, we may also assume that f_{π_i} is trivial for connections whose L^2-distance to the trivial connection is less than a fixed constant κ. If $\alpha_i \in \mathcal{L}_1^{\pi_i}(Y)$, then the difference $\mathrm{val}(D_1(\alpha_i)) - \mathrm{val}(\alpha_i)$ is equal to the topological energy of an element A_i of $\mathcal{M}_{z_i}^{\pi_i}(\alpha_i',\Theta)$ for an irreducible flat connection α_i'. Therefore, we can pick the connections A_i such that $\mathcal{E}(A_i) \to \Gamma_Y(1)$. Applying Lemma 3.35 to this sequence of connections implies that there is an irreducible flat connection α_0 and an ASD connection $A_0 \in \mathcal{M}_{z_0}(\alpha_0,\Theta)$ such that $\Gamma_Y(1) \geqslant \mathcal{E}(A_0)$ which verifies our claim. □

Definition 3.40. Suppose Y is an $\mathrm{SU}(2)$-non-degenerate integral homology sphere. We define:

$$\tau'(Y) := \inf_{z'}\{\mathcal{E}(z')\} \tag{3.41}$$

where the infimum is taken over all paths z' such that z' is obtained by concatenation of paths $z_1,\ldots,z_k$ satisfying the following properties. There are flat connections $\alpha_0,\ldots,\alpha_k$ such that α_0 and α_k are irreducible, z_i is a path from α_{i-1} to α_i, the sum of the indices of paths z_i is equal to $4-8n$ for a non-negative integer n and there is an ASD connection $A_i \in \mathcal{M}_{z_i}(\alpha_{i-1},\alpha_i)$.

Given a path z' from α_- to α_+ as in (3.41), $\mathcal{E}(z')$ is equal to $\mathrm{CS}(\alpha_0) - \mathrm{CS}(\alpha_k)$ mod $\mathbf{Z}$. Therefore, $\tau'(Y)$ takes values in a discrete set. This implies that $\tau'(Y)$ can be realized by a tuple $(A_1, \ldots, A_k)$ of ASD connections as in the above definition. We also have:

$$\tau'(Y) \geqslant \min\{r \mid r \in \mathbf{R}^{>0},\ r \equiv \mathrm{CS}(\alpha_-) - \mathrm{CS}(\alpha_+) \mod \mathbf{Z} \text{ for irreducible flat connections } \alpha_\pm\}$$

Lemma 3.42. *Suppose Y is an $\mathrm{SU}(2)$-non-degenerate integral homology sphere. Suppose $\{\pi_i\}_i$ is a sequence of perturbations of the Chern-Simons functional of Y such that $|\pi_i|_{\mathcal{P}} \to 0$, the critical points of $\mathrm{CS} + f_{\pi_i}$ are the same as the critical points of the unperturbed Chern-Simons functional and $f_{\pi_i} = 0$ in a neighborhood of flat connections. Suppose $A_i \in \mathcal{M}^{\pi_i}_{z_i}(\alpha_{i,-}, \alpha_{i,+})$ where $\alpha_{i,-}$ and $\alpha_{i,+}$ are irreducible flat connections and z_i is a path from $\alpha_{i,-}$ to $\alpha_{i,+}$ with index 4. Then there are distinct flat connections $\alpha_0, \ldots, \alpha_k$, a path $z_{0,j}$ from α_{j-1} to α_j , and a (non-perturbed) ASD connection $A_j \in \mathcal{M}_{z_{0,j}}(\alpha_{j-1}, \alpha_j)$ such that α_0 and α_k are irreducible, the sum of the indices of $z_{0,1}, \ldots, z_{0,k}$ is equal to $4 - 8n$ for a non-negative integer n and:*

$$\mathcal{E}(z_{0,1}) + \cdots + \mathcal{E}(z_{0,k}) \leqslant \limsup_i \mathcal{E}(z_i). \tag{3.43}$$

Proof. The proof is similar to that of Proposition 3.35. Since Y is $\mathrm{SU}(2)$-non-degenerate, there are only finitely many $\mathrm{SU}(2)$-flat connections. Using this observation and the fact that the index of connections A_i are at most 4, we may assume that $A_i \in \mathcal{M}^{\pi_i}_z(\alpha_-, \alpha_+)$ for a fixed path z and irreducible flat connections α_-, α_+. Following the argument of Proposition 3.35, we can construct nontrivial ASD connections $A_j \in \mathcal{M}_{z_{0,j}}(\alpha_{j-1}, \alpha_j)$ where α_j is a flat connection, $\alpha_0 = \alpha_-$, $\alpha_k = \alpha_+$, $z_{0,j}$ is a path from α_{j-1} to α_j and the inequality (3.43) is satisfied. □

Proposition 3.44. *If Y is an $\mathrm{SU}(2)$-non-degenerate integral homology sphere, then for any positive integer i we have $\Gamma_Y(i+1) \geqslant \Gamma_Y(i) + \tau'(Y)$.*

Proof. We fix a sequence of admissible perturbations π_i such that $|\pi_i|_{\mathcal{P}} \to 0$, the critical points of $\mathrm{CS} + f_{\pi_i}$ are the same as the critical points of the unperturbed Chern-Simons functional and $f_{\pi_i} = 0$ in a neighborhood of flat connections. For each admissible perturbation π_i and any $\alpha \in \mathcal{L}^{\pi_i}_{k+1}(Y)$, we have $U(\alpha) \in \mathcal{L}^{\pi_i}_k(Y)$. Furthermore, there are flat connections $\alpha_{i,-}$, $\alpha_{i,+}$ with Floer gradings $4k-3$, $4(k-1)-3$ and $A_i \in \mathcal{M}^{\pi_i}_{z_i}(\alpha_{i,-}, \alpha_{i,+})$ with index 4 such that:

$$\mathrm{val}(U(\alpha)) - \mathrm{val}(\alpha) \geqslant \mathcal{E}(A_i).$$

By taking infimum over all $\alpha \in \mathcal{L}^{\pi_i}_{k+1}(Y)$, we conclude that :

$$\inf_{\alpha \in \mathcal{L}^{\pi_i}_{k+1}(Y)} \Big(\mathrm{val}(D_1 U^{k-1}(\alpha)) - \mathrm{val}(\alpha)\Big) \geqslant \inf_{\alpha' \in \mathcal{L}^{\pi_i}_k(Y)} \Big(\mathrm{val}(D_1 U^{k-2}(\alpha')) - \mathrm{val}(\alpha')\Big) + \inf_{A_i} \mathcal{E}(A_i).$$

where the infimum is taken over all index 4 connections $A_i \in \mathcal{M}^{\pi_i}_{z_i}(\alpha_{i,-}, \alpha_{i,+})$ where $\alpha_{i,-}$, $\alpha_{i,+}$ are irreducible flat connections. By taking the limit of the above inequality as $i \to \infty$ and using Lemma 3.42, we can conclude that $\Gamma_Y(i+1) \geqslant \Gamma_Y(i) + \tau'(Y)$. □

Next, we use the argument in the proof of Theorem 3.2 to obtain a more general result. The constant η in the following theorem is introduced in Definition 3.46:

Theorem 3.45. *Suppose $W : Y \to Y'$ is a cobordism of integral homology spheres with $b_1(W) = b^+(W) = 0$. Then $\Gamma_{Y'}(k) \leqslant \Gamma_Y(k) - \eta(W)$ for any positive integer k. For a non-positive k, we have the weaker inequality $\Gamma_{Y'}(k) \leqslant \max(\Gamma_Y(k) - \eta(W), 0)$.*

Definition 3.46. Let W be a cobordism from an integral homology sphere Y to another integral homology sphere Y'. Let:

$$\eta(W) := \inf_A \{\mathcal{E}(A)\} \tag{3.47}$$

where the infimum is taken over all ASD connections A on W which is asymptotic to irreducible flat connections on Y and Y'. Here we allow A to be a broken ASD connection. That is to say, A might have one of the following forms for irreducible flat connections α and α':

(i) $A \in \mathcal{M}_z(W, \alpha, \alpha')$;

(ii) $A = (A_0, A_1)$ where $A_0 \in \mathcal{M}_{z_0}(\mathbf{R} \times Y, \alpha, \Theta)$ and $A_1 \in \mathcal{M}_{z_1}(W, \Theta, \alpha')$;

(iii) $A = (A_0, A_1)$ where $A_0 \in \mathcal{M}_{z_0}(W, \alpha, \Theta)$ and $A_1 \in \mathcal{M}_{z_1}(\mathbf{R} \times Y', \Theta, \alpha')$;

(iv) $A = (A_0, A_1, A_2)$ where $A_0 \in \mathcal{M}_{z_0}(\mathbf{R} \times Y, \alpha, \Theta)$, $A_1 \in \mathcal{M}_{z_1}(W, \Theta, \Theta)$ and $A_2 \in \mathcal{M}_{z_2}(\mathbf{R} \times Y', \Theta, \alpha')$.

The constant $\eta(W)$ is a non-negative number. The set of possible values for the energy of ASD connections on W is a discrete subset of non-negative integers. Therefore, if $\eta(W)$ is finite, then it can be realized by the energy of a (possibly broken) ASD connection on W which is asymptotic to non-trivial connections on both ends. We have:

$$\begin{aligned}\eta(W) \geqslant \min\{r \mid r \in \mathbf{R}^{\geqslant 0},\ r \equiv \mathrm{CS}(\alpha) - \mathrm{CS}(\alpha') \mod \mathbf{Z} \\ \text{for irreducible flat connections } \alpha,\ \alpha' \text{ on } Y,\ Y'\}\end{aligned}$$

We also make the observation that if $\eta(W) = 0$, then there is a flat connection extending non-trivial flat connections on Y and Y'. Consequently, if W is simply connected, such a flat connection does not exist, and hence $\eta(W) > 0$.

Lemma 3.48. *Let $\{\pi_i\}$, $\{\pi_i'\}$ be sequences of ϵ_i-admissible perturbations for the Chern-Simons functionals of Y, Y' such that f_{π_i} and $f_{\pi_i'}$ vanish in fixed neighborhoods of the trivial connections on Y and Y'. Let π_i and π_i' be extended to an ϵ_i-admissible perturbation $\overline{\pi}_i$ on the cobordism $W : Y \to Y'$, which satisfies $b_1(W) = b^+(W) = 0$. Let $A_i \in \mathcal{M}^{\overline{\pi}_i}_{z_i}(W, \alpha_i, \alpha_i')$ be chosen such that α_i, α_i' are respectively irreducible critical points of $\mathrm{CS} + f_{\pi_i}$, $\mathrm{CS} + f_{\pi_i'}$, and z_i is a path along W with index 0. If $\epsilon_i \to 0$, then there is a (possibly broken) ASD connection A_0 on W which is asymptotic to irreducible flat connections α_0, α_0' on Y, Y' and:*

$$\mathcal{E}(A_0) \leqslant \limsup_i \mathcal{E}(A_i). \tag{3.49}$$

Proof. Firstly as in Step 1 of the proof of Lemma 3.35 we can show that the terms $\|F(A_i)\|_{L^2(W^+)}$ are uniformly bounded. Thus the Uhlenbeck compactness theorem implies that the connections A_i, after passing to a subsequence and changing gauge, are weakly convergent to an ideal instanton (A_0, S) on

W^+, where $S \subset W^+$ is a finite subset and A_0 is an ASD connection. The weakly convergence of A_i implies that A_i, after possibly changing the gauge, is L^p_1 convergent on compact subsets of $W^+ \backslash S$ to A_0, and:

$$\mathcal{E}(A_0) \leqslant \limsup_i \mathcal{E}(A_i)$$

If the connection A_0 is asymptotic to non-trivial flat connections on both ends, then there is nothing left to prove. If it is asymptotic to the trivial connection on one of the ends, say the outgoing end, then we can argue as in Step 3 of Lemma 3.35 to find an instanton A_1 in a moduli space of the form $\mathcal{M}_{z_1}(\mathbf{R} \times Y', \Theta, \alpha')$ where α' is an irreducible flat connection on Y'. The pair (A_0, A_1) also satisfy the analogue of the inequality in (3.49). The other cases can be treated similarly. □

Proof of Theorem 3.45. We follow a similar argument as in Theorem 3.2. Let $w = (\alpha, 0)$ and z form a special pair of degree $-k$. Then the pair $w' = (\varphi(\alpha) + L(z), 0)$ and $z' = \overline{C}_W(z)$ satisfies the identity $\check{d}(w') = i(z')$. Here $\varphi : C^\pi_*(Y) \to C^{\pi'}_*(Y')$ is the cobordism map associated to W. Recall that $L(z) = 0$ if k is a positive integer. The first inequality in (3.3) can be modified as follows:

$$\mathrm{val}(w') \geqslant \mathrm{val}(w) + \mathcal{E}(A)$$

for a connection $A \in \mathcal{M}^{\overline{\pi}}_z(\alpha, \alpha')$ of index 0 where α, α' are irreducible critical points of $\mathrm{CS} + f_{\pi_i}$, $\mathrm{CS} + f_{\pi'_i}$ and z is a path over W from α to α' with index 0. This the inequality in (3.5) can be improved as follows:

$$\inf_{\substack{z' \in \overline{C}^{\pi'}_*(Y'),\, w' \in \check{C}^{\pi'}_*(Y'),\\ \check{d}(w') = i(z'),\, \mathrm{Deg}(z') = -k}} (\mathrm{val}(z') - \mathrm{val}(w')) \leqslant$$
$$\max\Big(\inf_{\substack{z \in \overline{C}^{\pi}_*(Y),\, w \in \check{C}^{\pi}_*(Y),\\ \check{d}(w) = i(z),\, \mathrm{Deg}(z) = -k}} (\mathrm{val}(z) - \mathrm{val}(w)) - \inf_A \mathcal{E}(A), 0 \Big) + \delta$$

where the second infimum on the right hand side is over all elements of the moduli spaces $\mathcal{M}^{\overline{\pi}}_z(\alpha, \alpha')$ with α, α' being irreducible critical points and z being a path along W of index 0 from α to α'. By taking the limit of the above inequalities as the norms of perturbations π, π' and $\overline{\pi}$ converge to zero and using Lemma 3.48, we can conclude:

$$\Gamma_{Y'}(k) \leqslant \max(\Gamma_Y(k) - \eta(W), 0).$$

In the case that k is a positive integer, the above argument can be modified to obtain the desired stronger inequality using the fact that $L(z) = 0$. □

Example 3.50. There are negative definite cobordisms $W : S^3 \to \Sigma(p, q, pqk{+}1)$ and $W' : \Sigma(p, q, pqk{+}1) \to S^3$ with $b_1 = 0$ [CG88]. Therefore, Theorem 3.45 implies that $\Gamma_{\Sigma(p,q,pqk+1)} = \Gamma_{S^3}$.

Corollary 3.51. *Suppose $W : Y \to Y'$ is a homology cobordism and Γ_Y (and hence $\Gamma_{Y'}$) has a finite positive value in its image. Then the inclusions of Y and Y' in W induce non-trivial maps of fundamental groups. In particular, if $h(Y)$ is non-trivial, then there is no simply connected homology cobordism from Y to another integral homology sphere.*

Proof. Let $W : Y \to Y'$ be a homology cobordism and the fundamental group of Y or Y' map trivially to that of W. On one hand, $\Gamma_Y = \Gamma_{Y'}$. On the other hand, $\eta(W) > 0$ and Γ_Y, $\Gamma_{Y'}$ satisfy the inequality given in Theorem 3.45. Thus a positive value in the image of Γ_Y is a contradiction.

If $h(Y) \neq 0$, then either $h(Y) > 0$ or $h(-Y') > 0$. Propositions 3.23 and 3.31 imply that either Γ_Y or $\Gamma_{-Y'}$ has a finite positive values in its image. Since W can be also regarded as a cobordism from $-Y'$ to $-Y$, it cannot be a simply connected. □

Remark 3.52. In his groundbreaking work [Tau87], Taubes proves that if Y is an integral homology sphere and $W : Y \to Y$ is a simply connected homology cobordism (or more generally a definite cobordism), then Y cannot bound a simply connected negative definite smooth manifold with a non-standard intersection form. As it is stated in Theorem 5 and will be proved in Section 5, if Y bounds a manifold X with non-standard negative definite form (without any assumption on $\pi_1(X)$), then $\Gamma_Y(1)$ is a finite positive number. Therefore, Theorem (3.45) gives a new proof of Taubes' result.

Taubes' method (gauge theory on manifolds with periodic ends) can be adapted to prove the second half of Corollary 3.51. This was firstly pointed to the author by Chris Scaduto. The author learnt later from Masaki Taniguchi that this method is also used in [Tan17] and a proof of the second half of Corollary 3.51 is implicit there. Given a simply connected homology cobordism $W : Y \to Y$, we can form a 4-manifold M with a periodic end and a cylindrical end in the following way. For each non-negative integer i, let W_i be a copy of W. We fix a metric on W which is cylindrical in a neighborhood of the boundary components corresponding to a fixed metric on Y. For each positive integer i we identify the outgoing end of W_{i+1} with the incoming end of W_i. We also glue a copy of $[0, \infty) \times Y$ to the outgoing end of W_0. Applying the method of [Tau87] to the moduli spaces of ASD connections on M, which are asymptotic to the trivial connection on the cylindrical end and have finite energy, shows that $h(Y)$ has to vanish.

Corollary 3.51 implies that the answer to Question 1.5 is negative if the Rokhlin homomorphism $\mu : \Theta^3_{\mathbf{Z}} \to \mathbf{Z}/2\mathbf{Z}$ is replaced with Γ_Y. In general, there are integral homology spheres with non-trivial Rokhlin invariant whose Γ_Y does not take any positive value. For example, $\Gamma_{\Sigma(2,3,7)}$ does not take any finite positive value by Example 3.50. However, $\mu(\Sigma(2,3,7))$ is non-trivial. In [Dae], we shall show that Corollary 3.51 can be extended to other families of integral homology spheres including $\Sigma(2,3,7)$.

4 Relation to Fintushel and Stern's R-invariant

For $n \geqslant 3$, suppose $a_1, \ldots, a_n$ are relatively prime positive integers. Throughout this section, we denote the Seifert fibered homology sphere $\Sigma(a_1, a_2, \ldots a_n)$ by Y unless otherwise is specified. The following elementary results about topology of Seifert fibered homology spheres is standard. (See, for example, [FS85].) The 3-manifold Y admits a standard S^1-action and the quotient space is S^2. Suppose $W = Y \times D^2/S^1$ where the action of S^1 is induced by the Seifert action on Y and the standard action on the 2-dimensional disc D^2. Then W has n singular points, one for each special fiber of Y. A neighborhood of the i^{th} singular point is given by a cone over the lens space $L(a_i, b_i)$ where the constants b_i satisfy the

following identity:

$$\sum_{i=1}^{n} \frac{b_i}{a_i} = \frac{1}{a}$$

and $a = a_1 \cdot a_2 \dots a_n$. Let W_0 denote the complement of regular neighborhoods of the singular points of W. Then W_0 is a 4-manifold such that:

$$\partial W_0 = Y \sqcup -L(a_1, \beta_1) \sqcup \dots \sqcup -L(a_n, \beta_n).$$

There is an obvious projection map from $L = Y \times D^2$ to W, that induces a $\mathrm{U}(1)$-bundle on W_0. We will write L_0 for this $\mathrm{U}(1)$-bundle on W_0. Then $w_0 = c_1(L_0)$ generates $H^2(W_0; \mathbf{Z})$ and its restriction to the lens space boundary component $L(a_i, b_i)$ is b_i times the standard generator of the second cohomology of this lens space. Moreover, W_0 is negative definite and $c_1(L_0)^2 = -\frac{1}{a}$. In particular, if we add cylindrical ends to W_0 and form W_0^+, then standard Hodge theory implies that there is an abelian ASD connection B on the bundle L_0. We assume that the metric on W_0^+ is compatible with the standard spherical metrics on the lens space ends. The main goal of this section is to prove Theorem 6. At the outset, we mention how $R(a_1, \dots, a_n)$ enters into the proof of this theorem. Suppose θ is the trivial connection on the the trivial line bundle $\underline{\mathbf{C}}$ over W_0. Then we can form the path[10] z_0 along (W_0, w_0) which is represented by the ASD connection $B \oplus \theta$. Then the index formula [MMR94, Tau93] imply that the expected dimension of the moduli space $\mathcal{M}_{z_0}(W_0, w_0)$, which contains the reducible connection $B \oplus \theta$, is equal to $R(a_1, \dots, a_n)$ of (1.7). Suppose β_i is the unique positive integer less than a_i such that $1 + \beta_i \frac{a}{a_i}$ is divisible by a_i. Then the formula for $R(a_1, \dots, a_n)$ can be simplified to $2b - 3$ where b is given by [NZ85]:

$$b = \frac{1}{a} + \sum_{i=1}^{n} \frac{\beta_i}{a_i}. \tag{4.1}$$

We fix an ϵ-admissible perturbation π of the Chern-Simons functional of Y. As in Subsection 2.2, we can fix a secondary perturbation $\overline{\pi}$ of the ASD equation on W_0^+ which is compatible with the chosen perturbation of the Chern-Simons functional of Y and the trivial perturbations of Chern-Simons functional of the lens space boundary components of W_0. Moreover, we can assume that the norm of the perturbation term $\overline{\pi}$ is less than ϵ and all moduli spaces with dimension less than 8 consist of regular elements [Don02]. The following proposition can be used to prove half of Theorem 6.

Lemma 4.2. *Suppose b is an integer greater than 1 and n_0 denotes $\lfloor \frac{b}{2} \rfloor - 1$. Define $\gamma_0 \in C_*^{\pi}(Y)$ as follows:*

$$\gamma_0 := \begin{cases} \sum_{\alpha} \#\mathcal{M}_z^{\overline{\pi}}(W_0, w_0; \alpha) \lambda^{\mathcal{E}(z)} \alpha & \text{b is even,} \\ \sum_{\alpha} \#(\mathcal{M}_z^{\overline{\pi}}(W_0, w_0; \alpha) \cap V(\Sigma_0)) \lambda^{\mathcal{E}(z)} \alpha & \text{b is odd.} \end{cases} \tag{4.3}$$

[10] Previously, we defined paths along a cobordism W, which are given by equivalence classes of $\mathrm{SU}(2)$-connections on W with respect to an appropriate equivalence relation. More generally, we can define paths for a pair (W, w) where W is a 4-dimensional cobordism and $w \in H^2(W, \mathbf{Z})$. The cohomology class w determines a $\mathrm{U}(1)$-bundle on W and we fix a connection B on this bundle. Similar to the case of $\mathrm{SU}(2)$-connections, we define an equivalence relation on $\mathrm{U}(2)$-connections whose central parts are equal to B. Any equivalence class of this relation is called a path along (W, w). For more details see [Don02, Section 5.6] and [KM11b, Subsection 3.10].

where z is a path along (W_0, w_0) that restricts to a generator α of $C^{\pi}_*(Y)$ on Y and to the same flat connections on the lens space boundary components as $B \oplus \theta$. Moreover, Σ_0 is an embedded surface representing a generator of $H_2(W)$ and $V(\Sigma_0)$ denotes a divisor representing the homology class $\mu(\Sigma_0)$ on the configuration spaces of connections on W_0 [DK90]*. Then γ_0 satisfies the following properties:*

(i) $d(\gamma_0) = 0$*;*

(ii) $D_1 \circ U^k(\gamma_0) = 0$ *where* $k < n_0$*;*

(iii) $D_1 \circ U^{n_0}(\gamma_0) \neq 0$*. Moreover, if δ is an arbitrary positive number, then by choosing ϵ small enough, we have:*

$$\mathrm{val}(D_1 \circ U^{n_0}(\gamma_0)) - \mathrm{val}(\gamma_0) \leqslant \frac{1}{4a} + \delta. \tag{4.4}$$

Proof. We firstly assume that b is even. The coefficient of β in $d(\gamma_0)$ can be identified with the number of the boundary points of the compactification of the moduli space:

$$\mathcal{M}^{\overline{\pi}}_z(W_0, w_0, \beta)$$

where z is a path along (W_0, w_0) such that the above moduli space is 1-dimensional, the restriction of z to Y is β, and z has the same restriction to the lens space boundary components as $B \oplus \theta$. Note that the compactified moduli space does not have any boundary component coming from the instantons which are broken along one of the lens space boundary components because any flat connection on a lens space is reducible and the index of any such broken instanton is at least two.

For a non-negative integer k, if $D_1 \circ U^k(z_0)$ is non-zero, then there is a $(k+2)$-tuple of ASD connections $(A_0, A_1, \ldots, A_k, A_{k+1})$ such that A_0 is an ASD connection contributing to the sum in (4.3), A_i, for $1 \leqslant i \leqslant k$, is an ASD connection on $\mathbf{R} \times Y$ between irreducible flat connections α_{i-1} and α_i contributing to $U(\alpha_{i-1})$, and A_{k+1} is an ASD connection on $\mathbf{R} \times Y$ from α_k to Θ contributing to $D_1(\alpha_k)$. In particular, the index of A_0 is 0, the index of A_i, for $1 \leqslant i \leqslant k$, is 4 and the index of A_{k+1} is 1. Therefore, the sum of the indices of these ASD connections is $4k+1$. We can glue the connections A_i to obtain a connection which has the same asymptotic flat connections as $B \oplus \theta$ on the boundary of W_0. Therefore, the additivity of ASD index with respect to composition of cobordisms implies that:

$$\begin{aligned} 4k+1 &= 8\Big(\sum_{i=0}^{k+1} \mathcal{E}(A_i) - \mathcal{E}(B \oplus \theta)\Big) + 2b - 3 \\ &= 8 \sum_{i=0}^{k+1} \mathcal{E}(A_i) + \frac{2}{a} + 2b - 3 \end{aligned}$$

In particular, if $k < n_0$, then the sum of the topological energies of the connections A_i is at most $-\frac{1}{2} + \frac{1}{4a}$.[11] Since the connections A_i are solutions of perturbed ASD equation, this is impossible if ϵ is small enough. This verifies the claim in part (ii). In the case that $k = n_0$, the above identity implies that the sum of $\mathcal{E}(A_i)$ is equal to $\frac{1}{4a}$. Therefore, the inequality in (4.4) holds for a small enough ϵ, if we show that $D_1 \circ U^{n_0}(\gamma_0)$ does not vanish.

[11] In fact, this inequality can be improved to $-1 + \frac{1}{4a}$ because k and n_0 have the same parity.

Fix a positive integer k smaller than $\frac{b}{2}$. In order to address the last part of the proposition, we introduce $\gamma_k \in C^\pi_*(Y)$ which is defined similar to γ_0. Fix n_0 distinct points $x_1, \ldots, x_{n_0}$ on W_0. For any $1 \leqslant i \leqslant n_0$, a standard construction in Donaldson theory allows us to form a co-dimension 4 divisor $V(x_i)$ in the configuration space of irreducible connections on W_0, which is a geometric representative for $\mu(\text{point})$ [KM95]. Define:

$$\gamma_k := \sum_\alpha \#\big(\mathcal{M}^{\overline{\pi}}_z(W_0, w_0; \alpha) \cap V(x_1) \cap \cdots \cap V(x_k)\big)\lambda^{\mathcal{E}(z)}\alpha \tag{4.5}$$

Here we may assume that the divisors are chosen such that the intersection (4.5) is transversal. In particular, a path z contributes to the above sum, if the dimension of the moduli space $\mathcal{M}^{\overline{\pi}}_z(W_0, w_0; \alpha)$ is equal to $4k$. A similar argument as in the case of γ_0 shows that γ_k is a cycle. Moreover, we can show that γ_{k+1} and $U(\gamma_k)$ differ by a co-boundary element in $C^\pi_*(Y)$. To see this, we allow x_{k+1} to move off the boundary component Y of W_0 and consider the associated 1-parameter family of moduli spaces. Cutting this moduli space by the divisors $V(x_1), \ldots, V(x_k)$ and studying its ends give the desired relation between γ_{k+1} and $U(\gamma_k)$.[12] In particular, $D_1 U^{n_0-k}(\gamma_k)$ is independent of k.

Consider the moduli space $\mathcal{M}^{\overline{\pi}}_{\gamma_0}(W_0, w_0)$ which contains the class of the reducible connection $B \oplus \theta$. The divisors $V(x_i)$ is defined only on the complement of the reducible connection. In fact, a neighborhood of the reducible connection is a cone over the projective space $\mathbf{CP}^{b-2}$ and the restriction of $V(x_i)$ to the boundary of this cone represents the cohomology class $-h^2$ where h is the generator of $H^2(\mathbf{CP}^{b-2})$ [DK90, Subsection 5.1.2]. Let $\mathcal{M}'$ be the complement of this neighborhood of the reducible element of $\mathcal{M}^{\overline{\pi}}_{\gamma_0}(W_0, w_0)$ and N' denotes the boundary of this neighborhood. Therefore, we can form the 1-dimensional moduli space:

$$Z = \mathcal{M}' \cap V(x_1) \cap \cdots \cap V(x_{n_0}).$$

We compactify Z in the standard way to form a compact 1-manifold whose boundary is given by $N' \cap V(x_1) \cap \cdots \cap V(x_j)$ and

$$\coprod_{\gamma \# \gamma' = \gamma_0} (\mathcal{M}_\gamma(W_0, w_0; \alpha) \cap V(x_1) \cap \cdots \cap V(x_j)) \times \breve{\mathcal{M}}_{\gamma'}(\alpha, \Theta) \tag{4.6}$$

Since the count of points in $N' \cap V(x_1) \cap \cdots \cap V(x_j)$ does not vanish, the count of elements in (4.6) is also non-zero. The latter count is equal to $D_1(\gamma_{n_0})$ by definition. Thus we conclude that $D_1 \circ U^{n_0}(\gamma_0)$ is nonzero. Analogous arguments can be used to prove similar claims in the case that b is odd. The only required new ingredient is that the restriction of the cohomology class $V(\Sigma_0)$ to $\mathbf{CP}^{b-2}$ is a non-zero multiple of h. (In fact, it is equal to $-\langle c_1(L_0), \Sigma_0 \rangle h$ [DK90, Subsection 5.1.2].) □

Proposition 4.7. *For Y as above and $1 \leqslant i \leqslant \lfloor \frac{b}{2} \rfloor$, we have:*

$$\Gamma_Y(i) = \frac{1}{4a}.$$

Proof. Lemma 4.2 asserts that $\Gamma_Y(i) \leqslant \frac{1}{4a}$. It is shown in [FS90] that the value of the Chern-Simons functional for $\mathrm{SU}(2)$-flat connections on Y has the form $\frac{l}{4a}$ where l is an integer[13]. (See also [Auc94].)

[12] The essential points here are the fact that any flat connection on a lens space is reducible and we also do not face any reducible connection on W_0 in our analysis of the relevant moduli spaces.

[13] In fact, the 4-manifold W_0 plays a key role in this part, too. The main point is that any $\mathrm{SU}(2)$ flat connection on Y extends to a flat connection on W_0.

Since $\Gamma_Y(i)$ is positive by Proposition 3.31, Proposition 3.26 implies that $\Gamma_Y(i) \geqslant \frac{1}{4a}$. This completes the proof. □

Theorem 6 is a generalization of Proposition 4.7 to the case that we consider a connected sum of Seifert fibered spaces with positive R-invariants:

Proof of Theorem 6. Let M be the standard cobordism from the disjoint union of 3-manifolds Y_i to $Y_1\# \ldots \#Y_k$ obtained by gluing 1-handles. For each i, we follow the above construction to construct a cobordism $W_{i,0}$ from a disjoint union of lens spaces to Y_i. We can glue $W_{1,0}, \ldots, W_{k,0}$ to M and obtain $\widetilde{M}$ a cobordism from a disjoint union of lens spaces to $Y_1\# \ldots \#Y_k$. For each i, we also constructed a $\mathrm{U}(1)$-bundle $L_{i,0}$ on $W_{i,0}$. We extend the bundle $L_{1,0}$ on $W_{1,0}$ trivially to $\widetilde{M}$ and denote the resulting bundle by $\widetilde{L}$. By applying the same argument as in the proof of Lemma 4.2 where W_0 and L_0 are replaced with $\widetilde{M}$ and $\widetilde{L}$, we can show that for $1 \leqslant i \leqslant \lfloor \frac{R(a_1,\ldots,a_n)+3}{4} \rfloor$:

$$\Gamma_Y(i) \leqslant \frac{1}{4a_1a_2 \ldots a_n}.$$

The reverse inequality is also a consequence of Lemma 4.8 below. □

Lemma 4.8. *For integral homology spheres Y and Y', we have:*

$$\Gamma_{Y\#Y'}(1) \geqslant \min\{\tau(Y), \tau(Y')\} \tag{4.9}$$

This lemma strengthens Lemma 3.31 for integral homology spheres which are homology cobordant to connected sum of two integral homology spheres.

Proof. Consider the standard cobordism N from $Y\#Y'$ to $Y \sqcup Y'$ given by gluing a 3-handle. Let ϵ_i be a sequence of positive real numbers converging to 0. We fix ϵ_i-admissible perturbations π, π' and $\pi^{\#}$ on Y, Y' and $Y\#Y'$ and extend them to an ϵ_i-admissible perturbation $\overline{\pi}$ on N. Suppose $\gamma = \sum_{i=1}^{k} s_i\lambda^{r_i}\alpha_i$ is an element of $\mathcal{L}_1^{\pi^{\#}}(Y\#Y')$. For each i, there is a path z_i of index 1 along N from the connection α_i to the trivial connections Θ and Θ' on $Y \sqcup Y'$. Let $\mathcal{M}_{z_i}^{\overline{\pi},+}(N, \alpha_i, \Theta, \Theta')$ be the standard compactification of the 1-dimensional moduli space $\mathcal{M}_{z_i}^{\overline{\pi}}(N, \alpha_i, \Theta, \Theta')$. Clearly, we have:

$$\sum_{i=1}^{k} s_i\lambda^{r_i+\mathcal{E}(z_i)} \cdot \#\partial(\mathcal{M}_{z_i}^{\overline{\pi},+}(N, \alpha_i, \Theta, \Theta')) = 0.$$

We can also analyze the boundary components of the moduli space $\mathcal{M}_{z_i}^{\overline{\pi},+}(N, \alpha_i, \Theta, \Theta')$ to show that the above identity implies that:

$$F \circ d(\gamma) + D_1(\gamma) + \sum_i s_i\lambda^{r_i+\mathcal{E}(z_i)} \sum_{x_1\#x_2=z_i} \mathcal{M}_{x_1}^{\overline{\pi}}(N, \alpha_i, \beta, \Theta) \times \breve{\mathcal{M}}_{x_2}^{\pi}(\mathbf{R} \times Y, \beta, \Theta) +$$

$$\sum_i s_i\lambda^{r_i+\mathcal{E}(z_i)} \sum_{y_1\#y_2=z_i} \mathcal{M}_{y_1}^{\overline{\pi}}(N, \alpha_i, \Theta, \beta') \times \breve{\mathcal{M}}_{y_2}^{\pi'}(\mathbf{R} \times Y', \beta', \Theta) = 0 \tag{4.10}$$

Here $F : C_*^{\pi^\#}(Y\#Y') \to \Lambda$ is induced by counting solutions of the perturbed ASD equation on N corresponding to paths of index 0 along N which restrict to an irreducible connection on $Y\#Y'$ and trivial connections on Y and Y'. The path x_1 (respectively, y_1) along N has index 0 and restricts to the trivial connection on the end Y' (respectively, Y) and to irreducible connections on the remaining ends. The path x_2 along $\mathbf{R} \times Y$ (respectively, y_2 along $\mathbf{R} \times Y'$) has index 1 and restricts to an irreducible connection on the incoming end and to the trivial connection on the outgoing end. Since $d(\gamma) = 0$ and $D_1(\gamma) \neq 0$, (4.10) implies that either there is a path x_2 along $\mathbf{R} \times Y$ such that the moduli space $\mathcal{M}^{\pi}_{x_2}(\mathbf{R} \times Y, \beta, \Theta)$ contains an element A_i with:

$$\mathrm{val}(D_1(\gamma)) - \mathrm{val}(\gamma) \geqslant \mathcal{E}(A_i) - \delta$$

or there is a path y_2 along $\mathbf{R} \times Y'$ such that the moduli space $\mathcal{M}^{\pi'}_{y_2}(\mathbf{R} \times Y', \beta', \Theta')$ contains an element A'_i with:

$$\mathrm{val}(D_1(\gamma)) - \mathrm{val}(\gamma) \geqslant \mathcal{E}(A'_i) - \delta.$$

Here δ is a constant which converges to 0 as $i \to \infty$. Now by letting i go to ∞ and using Lemma 3.35, we can verify (4.9). □

Remark 4.11. Theorem 6 and Lemma 4.8 give some instances of relations among $\Gamma_{Y\#Y'}$, Γ_Y and $\Gamma_{Y'}$. In [DST], we study this relationship more systematically.

Corollary 4.12 ([Fur90])**.** *Suppose $\{Y_i = \Sigma(a_{i,1}, \ldots, a_{i,n_i})\}_{i\in I}$ is a collection of Seifert fibered homology spheres with positive R-invariants such that the positive integers $a_i := a_{i,1} \cdot a_{i,2} \ldots a_{i,n_i}$ are distinct. Then the integral homology spheres Y_i determine linearly independent elements of $\Theta^3_{\mathbf{Z}}$.*

Proof. If there is a linear relation among Y_i we can assume that we have a relation of the following form:

$$n_1 Y_{i_1} + \cdots + n_k Y_{i_k} = m_1 Y_{j_1} + \cdots + m_l Y_{j_l}$$

such that all integers n_r and m_s are positive, and i_r and j_s are all distinct elements of I. Evaluating the invariant Γ at 1 for the 3-manifolds given above implies that:

$$\frac{1}{4\max\{a_{i_1}, \ldots, a_{i_k}\}} = \frac{1}{4\max\{a_{j_1}, \ldots, a_{j_l}\}}$$

which is in contradiction with the assumption that the integers a_i are distinct. □

Example 4.13. If p, q are coprime positive integers and k is another positive integer, then $R(\Sigma(p, q, pqk - 1)) = 1$. Therefore, Proposition 4.7 implies that:

$$\Gamma_{\Sigma(p,q,pqk-1)}(1) = \frac{1}{4pq(pqk - 1)}.$$

Corollary 4.12 implies that $\{\Sigma(p, q, pqk - 1)\}_{k\in\mathbf{Z}^{>0}}$ spans a $\mathbf{Z}^\infty$ subgroup of $\Theta^3_{\mathbf{Z}}$. This result is proved by Furuta [FS90] and Fintushel and Stern [Fur90].

Example 4.14. Let K_k be the pretzel knot $P(-2k - 1, 4k + 1, 4k + 3)$. This knot has Alexander polynomial 1, and hence it is topologically slice. On the other hand, it is shown in [End95] that the family of knots $\{K_k\}$ are linearly independent as elements of the the smooth concordance group $\mathcal{C}$ and

generate a $\mathbf{Z}^\infty$ subgroup of $\mathcal{C}$. The proof of this result in [End95] can be reformulated in terms of Γ_Y in the following way. The branched double cover of S^3 along K_k is the Brieskorn homology sphere $\Sigma(2k+1, 4k+1, 4k+3)$. As it is observed in [End95], $R(\Sigma(2k+1, 4k+1, 4k+3)) = 1$ and hence:

$$\Gamma_{\Sigma(2k+1,4k+1,4k+3)}(1) = \frac{1}{4(2k+1)(4k+1)(4k+3)}.$$

Therefore, Corollary 4.12 implies that $\{\Sigma(2k+1, 4k+1, 4k+3)\}_{k\in\mathbf{Z}^{>0}}$ spans a $\mathbf{Z}^\infty$ subgroup of $\Theta^3_{\mathbf{Z}}$. This shows the claimed linear independence of the knots K_k.

Example 4.15. If p, q, k are as in Example 4.13, then $R(p, q, pqk + 1) = -1$. Therefore, Proposition 4.7 does not say anything about Γ_Y when $Y = \Sigma(p, q, pqk + 1)$. This is in line with Example 3.50 where it is shown that $\Gamma_Y = \Gamma_{S^3}$.

Example 4.16. Suppose p and q are coprime numbers, and a_1, a_2, ..., a_{2n+1} is a sequence of positive integers with $a_1 = p$, $a_2 = q$ and for $i \geqslant 3$:

$$a_i = k_i a_1 a_2 \dots a_{i-1} - 1.$$

Here k_i is an arbitrary positive integer. Then $R(a_1, \dots, a_{2n+1}) = n$. In particular, the R-invariant can take arbitrarily large values. As it is discussed in the introduction, if $R(a_1, \dots, a_{2n+1}) \geqslant 5$, then Propositions 3.44 and 4.7 imply that $\Sigma(a_1, \dots, a_{2n+1})$ is not homology cobordant to an $\mathrm{SU}(2)$-non-degenerate integral homology sphere.

Corollary 3 provides another family of 3-manifolds that similar techniques can be used to obtain information about Γ_Y. Next, we give the proof of this corollary which recasts the arguments of [HK12].

Proof of Corollary 3. There is a negative definite cobordism with trivial first integral homology from $\Sigma(p, q, 2pq - 1)$ to $Y_{p,q}$ [HK12, Lemma 3.6]. Thus Theorem 3 and Example 4.13 imply that:

$$\Gamma_{Y_{p,q}}(1) \leqslant \Gamma_{\Sigma(p,q,2pq-1)}(1) = \frac{1}{4pq(2pq-1)}.$$

The study of the Chern-Simons functional of $Y_{p,q}$ in [HK12] shows that $\tau(Y_{p,q}) \geqslant \frac{1}{4pq(4pq-1)}$, which can be used to verify the other inequality in (1.8). The proof of the more general inequalities in (1.9) is similar. (See the proof of Theorem 6.) □

5 4-manifolds and Reducible Connections

The following theorem is the counterpart of [Frø02, Theorem 3 and Proposition 1] which generalizes Donaldson's celebrated diagonalizability theorem [Don83]:

Proposition 5.1. *Suppose Y is a homology sphere, and X is a 4-manifold with boundary Y such that $b_1(X) = 0$, and the intersection form Q of X on $H^2(X, \mathbf{Z})/\mathrm{Tor}$ is negative-definite. Suppose a*

cohomology class $e \in H^2(X, \mathbf{Z})$ is fixed such that $Q(e)$ is an even integer, $|Q(e)| \geqslant 2$, $|Q(e)| \leqslant |Q(e')|$ for any e' with $e' \equiv e \mod 2$, and:

$$\sum (-1)^{(\frac{e+e'}{2})^2} \neq 0. \tag{5.2}$$

where the sum is over all pairs $\{e', -e'\}$ such that $e' \in H^2(X, \mathbf{Z})$, $e' \equiv e \mod 2$, and $Q(e) = Q(e')$. Then $\Gamma_Y(n_0) \leqslant -\frac{1}{4}Q(e)$ where $n_0 = -\frac{1}{2}Q(e)$. The equality holds only if the fundamental group of X admits a representation to $\mathrm{SU}(2)$ with non-trivial restriction to the boundary.

Proof. We follow the proof of [Frø02, Proposition 1]. Since the proof is also similar to the proof of Lemma 4.2, we only sketch the main steps. Suppose L is the complex line bundle on X which represents the cohomology class e. We also fix a metric with cylindrical ends on X. The bundle L admits an ASD connection B such that $\|F(B)\|_{L^2(X)}$ is finite. Let ϵ_i be a sequence of positive numbers converging 0, and for each i, fix an ϵ_i-admissible perturbation π_i of the Chern-Simons functional of Y. We also pick a compatible perturbation $\overline{\pi}_i$ of the ASD equation on X as in the previous section. Define:

$$\gamma_i := \sum_z \#\mathcal{M}_z^{\overline{\pi}_i}(X, e; \alpha) \lambda^{\mathcal{E}(z)} \alpha$$

where the sum is over all paths along (X, e) of index 0 which restricts to a generator α of $C_*^{\pi_i}(Y)$.

Following the same argument as in the proof of Lemma 4.2, we can show that $d(\gamma_i) = 0$ and $D_1 U^k(\gamma_i) = 0$ for $k < n_0 - 1$. Moreover, the inequality in (5.2) implies that $D^1 U^{n_0-1}(\gamma_i)$ is a non-zero multiple of $\lambda^{-\frac{1}{4}e^2}$. To be a bit more detailed, we consider the moduli space that contains the ASD connection $B \oplus \theta$ and as in Lemma 4.2, we cut down the complement of a small neighborhood of the reducible connections with $n_0 - 1$ codimension 4 divisors representing $\mu(\mathrm{pt})$. This determines a 1-dimensional moduli space. Clearly, the signed count of the boundary points of this moduli space is zero. This count has contribution from $D^1 U^{n_0-1}(\gamma_i)$ and reducible connections. The count associated to the reducible connections is a non-zero multiple of (5.2), and hence it is non-zero. Consequently, $D^1 U^{n_0-1}(\gamma_i)$ is non-zero. The dimension formula implies that this non-zero number is a multiple of $\lambda^{-\frac{1}{4}Q(e)}$. There is also an element A_i of the moduli space of the form $\mathcal{M}_z^{\overline{\pi}_i}(X, e; \alpha)$ such that $\mathrm{val}(\gamma)$ is equal to $\mathcal{E}(A_i)$. In particular, we have:

$$\mathrm{val}(D_1 U^{n_0-1}(\gamma_i)) - \mathrm{val}(\gamma_i) = -\frac{1}{4}Q(e) - \mathcal{E}(A_i)$$

By letting $i \to \infty$, we have:

$$\Gamma_Y(n_0) \leqslant -\frac{1}{4}Q(e) - \limsup_i \mathcal{E}(A_i)$$

The analogue of Lemma 3.48 for 4-manifolds with one boundary component implies that there is a (possibly broken) ASD connection A_0 on X which is asymptotic to an irreducible flat connection on Y such that:

$$\Gamma_Y(n_0) \leqslant -\frac{1}{4}Q(e) - \mathcal{E}(A_0)$$

In particular, $\Gamma_Y(n_0) \leqslant -\frac{1}{4}Q(e)$ and if the equality holds then A_0 has to be flat. Therefore, there is an $\mathrm{SU}(2)$-representation of $\pi_1(X)$, which extends a non-trivial representation of $\pi_1(Y)$. □

Proposition 5.3. *Suppose Y and X are given as in Proposition 5.1. Suppose also a non-negative integer m, a homology class $\Xi \in H_2(X, \mathbf{Z})$ and a cohomology class $e \in H^2(X, \mathbf{Z})$ are fixed such that $Q(e) \equiv m$ mod 2, $|Q(e)| \geqslant 2$, $|Q(e)| \leqslant |Q(e')|$ for any e' with $e' \equiv e$ mod 2 and:*

$$\sum (-1)^{Q(\frac{e+e'}{2})} (\Xi \cdot e')^m \neq 0. \tag{5.4}$$

where the sum is over all pairs $\{e', -e'\}$ such that $e' \in H^2(X, \mathbf{Z})$, $e' \equiv e \mod 2$, and $Q(e) = Q(e')$. Then $\Gamma_Y(n_0) \leqslant -\frac{1}{4}Q(e)$ where $n_0 = -\frac{Q(e)+m}{2}$. The equality holds only if the fundamental group of X admits a representation to $\mathrm{SU}(2)$ with non-trivial restriction to the boundary.

Proof. Essentially the same proof as in Proposition 5.1 verifies this claim. We just need to modify the definition of γ_i as follows:

$$\gamma_i := \sum_z \#(\mathcal{M}_z^{\overline{\pi}_i}(X, e; \alpha) \cap V(\Sigma_1) \cap \ldots V(\Sigma_m)) \lambda^{\mathcal{E}(\gamma)} \alpha$$

where Σ_j is an embedded surface[14] in X representing the homology class Ξ and $V(\Sigma_j)$ is the standard codimension 2 divisor in the configuration space of irreducible connections on X representing $\mu(\Xi)$ [DK90]. □

Proof of Theorem 5. Let e be a cohomology class which represents a non-zero element of with minimum $|Q(e)|$. In particular, $Q(e) = -m(\mathcal{L})$ where $m(\mathcal{L})$ is defined in (1.2). If $e' \in H^2(X, \mathbf{Z})$ is another cohomology class that $e \equiv e'$ mod 2 and $Q(e) = Q(e')$, then either $e - e'$ or $e + e'$ is a torsion element. In particular, in the case that $Q(e)$ is an even number, the condition in (5.2) holds and we can conclude that $\Gamma_Y(-\frac{1}{2}Q(e)) \leqslant -\frac{1}{4}Q(e)$. In the case that $Q(e)$ is an odd number, we set $m = 1$ and Ξ to be a cohomology class that $\Xi \cdot e = 1$ and apply Proposition 5.3 to conclude that $\Gamma_Y(-\frac{Q(e)+1}{2}) \leqslant -\frac{1}{4}Q(e)$. This inequality completes the proof. □

A Evaluating a Function at Cohomology classes

Suppose M is a compact topological space, $f : M \to \mathbf{R}$ is a continuous function and $\sigma \in H_*(M, \mathbf{Z})$. Let $M(r) := f^{-1}((-\infty, r])$ and define:

$$f(\sigma) := \inf\{r \mid \sigma \in \mathrm{image}(i_* : H_*(M(r), \mathbf{Z}) \to H_*(M, \mathbf{Z}))\}$$

We say $f(\sigma)$ is the *evaluation* of f at the homology class σ.

In the case that M is a smooth manifold and f is a smooth function, we can reformulate the above definition using the language of Morse homology. We firstly assume that f is a Morse function and fix a Riemannian metric on M such that f is a Morse-Smale function. Suppose $(C_*(f), d)$ is the Morse

[14] As it is customary in the definition of Donaldson invariants, we assume that these surfaces intersect generically to avoid the issue of instanton bubbles.

complex associated to f. Then $C_*(f)$ is generated by critical points of f, and d is defined using downward gradient flow lines of f. Moreover, the homology of the complex $(C_*(f), d)$, denoted by $H_*(f)$, is naturally isomorphic to $H_*(M, \mathbf{Z})$. For each homology class σ, we have:

$$f(\sigma) = \min_{\sigma=[\sum_i a_i\alpha_i]} \max_i \{f(\alpha_i)\} \tag{A.1}$$

Here the minimum is over all linear combinations $\sum_i a_i\alpha_i$ of critical points of f which represents the homology class σ. The above identity is a consequence of the fact that for any r, the sub-complex generated by critical points α of f with $f(\alpha) \leqslant r$ has the same homology as the singular homology of $M(r)$. Clearly, the above min-max formula is independent of the choice of the metric because it is equal to $f(\sigma)$. Alternatively, one can use standard continuation maps to show the independence of the left hand side of (A.1) from the choice of the metric.

Example A.2. If f is a self-indexing Morse function, then for any $\sigma \in H_i(M, \mathbf{Z})$, we have $f(\sigma) = i$.

The above definition can be modified easily for the functions which are not necessarily Morse. Given a function f, we fix a smooth function $\pi_i : M \to \mathbf{R}$ for each positive integer i such that $f + \pi_i$ is a Morse function and $|\pi_i|_{C^0} \to 0$. Then we have:

$$f(\sigma) = \lim_{i\to\infty} \min_{\substack{\sum_i a_i\alpha_i \in C_*(f+\pi_i) \\ \sigma=[\sum_i a_i\alpha_i]}} \max_i \{f(\alpha_i)\} \tag{A.3}$$

In particular, the term on the min-max formula on the right hand side of (A.3) is independent of the choice of the perturbations π_i. One can also observe from the definition of $f(\sigma)$ that this number is equal to a critical value of f.

The definition of Γ_Y has some formal similarities with the expression on the left hand side of (A.3), where M is replaced with the configuration space $\mathcal{B}(Y)$ and f is replaced with the S^1-valued Chern-Simons functional CS. Another similarity is that Γ_Y, modulo integers, essentially takes values in the set of critical values of the Chern-Simons functional, i.e., the values of the Chern-Simons functional on $\mathrm{SU}(2)$-flat connections. However, one should not go too far in this analogy. To have a better finite dimensional approximation to the construction of this paper one should consider a manifold M with an $\mathrm{SO}(3)$ action which has one fixed point and otherwise it is free. We also need to fix an $\mathrm{SO}(3)$-invariant function on M. Working with an appropriate model for the homology of M, we can produce a chain complex with coefficients in Λ which has the form in (2.6). Then repeating the construction of Subsections 2.3 and 3.1 would provide a better finite dimensional approximation to Γ_Y.

References

[Auc94] David R. Auckly, *Topological methods to compute Chern-Simons invariants*, Math. Proc. Cambridge Philos. Soc. **115** (1994), no. 2, 229–251. MR1277058 ↑38

[CDX17] L. Culler, A. Daemi, and Y. Xie, *Surgery, Polygons and $SU(N)$-Floer Homology*, To appear in Journal of Topology, ArXiv e-prints (December 2017), available at 1712.09910. ↑14

[CG88] Tim D. Cochran and Robert E. Gompf, *Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and property P*, Topology **27** (1988), no. 4, 495–512. MR976591 ↑34

[Dae] Aliakbar Daemi, *Chern-simons functional and spin cobordisms*, In preparation. ↑35

[DK90] Simon Donaldson and Peter Kronheimer, *The geometry of four-manifolds*, Oxford University Press, 1990. ↑9, 37, 38, 43

[Don02] S. K. Donaldson, *Floer homology groups in Yang-Mills theory*, Cambridge Tracts in Mathematics, vol. 147, Cambridge University Press, Cambridge, 2002. With the assistance of M. Furuta and D. Kotschick. MR1883043 (2002k:57078) ↑6, 8, 9, 10, 11, 13, 15, 29, 31, 36

[Don83] ———, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), no. 2, 279–315. MR710056 ↑1, 3, 41

[Don87] Simon Donaldson, *The orientation of Yang-Mills moduli spaces and* 4*-manifold topology*, J. Differential Geom. **26** (1987), no. 3, 397–428. MR910015 (88j:57020) ↑10, 14

[DST] Aliakbar Daemi, Kouki Sato, and Sato Taniguchi, *Chern-simons functional and the homology cobordism group II*, In preparation. ↑6, 19, 40

[DX17] Aliakbar Daemi and Yi Xie, *Sutured Manifolds and Polynomial Invariants from Higher Rank Bundles*, To appear in Geometry & Topology, ArXiv e-prints (January 2017), available at 1701.00571. ↑25

[End95] Hisaaki Endo, *Linear independence of topologically slice knots in the smooth cobordism group*, Topology and its Applications **63** (1995), no. 3, 257 –262. ↑40, 41

[FL02] P. M. N. Feehan and T. G. Leness, *A general SO(3)-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, ArXiv Mathematics e-prints (March 2002), available at math/0203047. ↑20

[Flo88] Andreas Floer, *An instanton-invariant for* 3*-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240. MR956166 ↑1, 8, 10

[Frø02] Kim A. Frøyshov, *Equivariant aspects of Yang-Mills Floer theory*, Topology **41** (2002), no. 3, 525–552. MR1910040 ↑1, 3, 9, 10, 13, 20, 25, 27, 41, 42

[Frø10] ———, *Monopole Floer homology for rational homology 3-spheres*, Duke Math. J. **155** (2010), no. 3, 519–576. MR2738582 ↑3, 20

[Frø96] ———, *The Seiberg-Witten equations and four-manifolds with boundary*, Math. Res. Lett. **3** (1996), no. 3, 373–390. MR1397685 ↑3, 20

[FS85] Ronald Fintushel and Ronald J. Stern, *Pseudofree orbifolds*, Ann. of Math. (2) **122** (1985), no. 2, 335–364. MR808222 ↑1, 4, 35

[FS90] Ronald Fintushel and Ronald J Stern, *Instanton homology of seifert fibred homology three spheres*, Proc. London Math. Soc **61** (1990), no. 3, 109–137. ↑1, 25, 38, 40

[Fur90] Mikio Furuta, *Homology cobordism group of homology* 3*-spheres*, Invent. Math. **100** (1990), no. 2, 339–355. MR1047138 ↑1, 4, 40

[GNY11] Lothar Göttsche, Hiraku Nakajima, and Kota Yoshioka, *Donaldson = Seiberg-Witten from Mochizuki's formula and instanton counting*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 1, 307–359. MR2827729 ↑20

[HHL] Kristen Hendricks, Jennifer Hom, and Tye Lidman, *Applications of involutive Heegaard Floer homology*, Journal of the Institute of Mathematics of Jussieu, 1–38. ↑3

[HK12] Matthew Hedden and Paul Kirk, *Instantons, concordance, and Whitehead doubling*, J. Differential Geom. **91** (2012), no. 2, 281–319. MR2971290 ↑1, 5, 41

[HLL18] J. Hom, A. Levine, and T. Lidman, *Knot Concordance in Homology Cobordisms*, ArXiv e-prints (January 2018), available at 1801.07770. ↑4

[HM1705] Kristen Hendricks and Ciprian Manolescu, *Involutive Heegaard Floer homology*, Duke Math. J. **166** (201705), no. 7, 1211–1299. ↑3

[Kir95] Rob Kirby, *Problems in low-dimensional topology*, Proceedings of georgia topology conference, part 2, 1995, pp. 35–473. ↑4

[KM07] Peter Kronheimer and Tomasz Mrowka, *Monopoles and three-manifolds*, New Mathematical Monographs, vol. 10, Cambridge University Press, Cambridge, 2007. MR2388043 (2009f:57049) ↑1, 3, 6, 15, 20, 27

[KM11a] ———, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. **113** (2011), 97–208. MR2805599 ↑17

[KM11b] ———, *Knot homology groups from instantons*, J. Topol. **4** (2011), no. 4, 835–918. MR2860345 (2012m:57060) ↑10, 11, 27, 36

[KM95] P. B. Kronheimer and T. S. Mrowka, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **41** (1995), no. 3, 573–734. MR1338483 (96e:57019) ↑38

[Kro05] Peter Kronheimer, *Four-manifold invariants from higher-rank bundles*, J. Differential Geom. **70** (2005), no. 1, 59–112. MR2192061 (2006g:57059) ↑10, 30

[Lin18] Francesco Lin, *A Morse-Bott approach to monopole Floer homology and the triangulation conjecture*, Mem. Amer. Math. Soc. **255** (2018), no. 1221, v+162. MR3827053 ↑3

[Man16] Ciprian Manolescu, *Pin(2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture*, J. Amer. Math. Soc. **29** (2016), no. 1, 147–176. MR3402697 ↑3

[MMR94] John W. Morgan, Tomasz Mrowka, and Daniel Ruberman, *The L^2-moduli space and a vanishing theorem for Donaldson polynomial invariants*, Monographs in Geometry and Topology, II, International Press, Cambridge, MA, 1994. MR1287851 ↑36

[NZ85] Walter D. Neumann and Don Zagier, *A note on an invariant of Fintushel and Stern*, Geometry and topology (College Park, Md., 1983/84), 1985, pp. 241–244. MR827273 ↑36

[OS03] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261. MR1957829 ↑3, 27

[OS04] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158. MR2113019 (2006b:57016) ↑1, 6

[OS05] ———, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33. MR2141852 (2006e:57041) ↑17

[PC17] Juanita Pinzón-Caicedo, *Independence of satellites of torus knots in the smooth concordance group*, Geom. Topol. **21** (2017), no. 6, 3191–3211. MR3692965 ↑5

[Tan17] Masaki Taniguchi, *Instantons for 4-manifolds with periodic ends and an obstruction to embeddings of 3-manifolds*, arXiv e-prints (July 2017), arXiv:1707.08555, available at 1707.08555. ↑4, 35

[Tau87] Clifford Henry Taubes, *Gauge theory on asymptotically periodic 4-manifolds*, J. Differential Geom. **25** (1987), no. 3, 363–430. MR882829 ↑4, 35

[Tau90] ———, *Casson's invariant and gauge theory*, J. Differential Geom. **31** (1990), no. 2, 547–599. MR1037415 ↑10

[Tau93] ———, *L^2 moduli spaces on 4-manifolds with cylindrical ends*, Monographs in Geometry and Topology, I, International Press, Cambridge, MA, 1993. MR1287854 ↑36

[Wit94] Edward Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), no. 6, 769–796. MR1306021 ↑20

Aliakbar Daemi, DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY IN ST. LOUIS, ONE BROOKINGS DRIVE, ROOM 207A, ST. LOUIS, MO 63130

E-mail address: adaemi@wustl.edu