ON SINGULAR ABREU EQUATIONS
IN HIGHER DIMENSIONS

By

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Abstract. We study the solvability of the second boundary value problem of a class of highly singular, fully nonlinear fourth order equations of Abreu type in higher dimensions under either a smallness condition or radial symmetry.

1 Introduction and statements of the main results

In this paper, which is a sequel to [6], we study the solvability of the second boundary value problem of a class of highly singular, fully nonlinear fourth order equations of Abreu type for a uniformly convex function u:

$$(1.1)\qquad \begin{cases} \sum_{i,j=1}^n U^{ij} w_{ij} = F(\cdot, u, Du, D^2u) & \text{in } \Omega, \\ w = (\det D^2 u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases}$$

Here and throughout, $U = (U^{ij})_{1\le i,j\le n}$ is the cofactor matrix of the Hessian matrix

$$D^2 u = (u_{ij})_{1\le i,j\le n} \equiv \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{1\le i,j\le n};$$

$\varphi \in C^{3,1}(\overline{\Omega})$, $\psi \in C^{1,1}(\overline{\Omega})$ with $\inf_{\partial\Omega} \psi > 0$. The left-hand side of (1.1) usually appears in Abreu's equation [1] in the problem of finding Kähler metrics of constant scalar curvature in complex geometry.

This type of equation arises from studying approximation of convex functionals such as the Rochet–Choné model in product line design [9] whose Lagrangians depend on the gradient variable, subject to a convexity constraint. Carlier–Radice [2]

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studied equations of the type (1.1) when F does not depend on the Hessian variable. When the function F depends on the Hessian variable, (1.1) was studied in [6] in two dimensions, including the case $F = -\Delta u$.

Note that (1.1) consists of a Monge–Ampère equation for u in the form of $\det D^2 u = w^{-1}$ and a linearized Monge-Ampère equation for w in the form of

$$\sum_{i,j=1}^{n} U^{ij} w_{ij} = F(\cdot, u, Du, D^2 u)$$

because the coefficient matrix (U^{ij}) comes from linearization of the Monge–Ampère operator:

$$U^{ij} = \frac{\partial \det D^2 u}{\partial u_{ij}}.$$

The solvability of second boundary problems such as (1.1) is usually established via a priori fourth order derivative estimates and degree theory. Two of the key ingredients for the a priori estimates are to establish (see [6]):

(i) positive lower and upper bounds for the Hessian determinant $\det D^2 u$; and
(ii) global Hölder continuity for w from global Hölder continuity of the linearized Monge–Ampère equation with the right-hand side having low integrability.

By Theorem 1.7 in combination with Lemma 1.5 in [8], any integrability more than $n/2$ right-hand side of the linearized Monge-Ampère equation suffices for the global Hölder continuity and $n/2$ is the precise threshold. The reason to restrict the analysis in [6] to two dimensions even for the simple case $F = -\Delta u$ is that either Δu is just a measure or it belongs to $\Delta u \in L^{1+\varepsilon_0}(\Omega)$ where $\varepsilon_0 > 0$ can be arbitrary small. The condition $n/2 < 1 + \varepsilon_0$ with small ε_0 naturally leads to $n = 2$.

In all dimensions, once we have the global Hölder continuity of w together with the lower and upper bounds on $\det D^2 u$, we can apply the global $C^{2,\alpha}$ estimates for the Monge–Ampère equation in [10, 13] to conclude that $u \in C^{2,\alpha}(\overline{\Omega})$. We update this information to $U^{ij} w_{ij} = F(\cdot, u, Du, D^2 u)$ to have a second order uniformly elliptic equation for w with global Hölder continuous coefficients and bounded right-hand side. This gives second order derivative estimates for w. Now, fourth order derivative estimates for u easily follow.

In this paper, we consider the higher dimensional case of (1.1), focusing on the right-hand side being of p-Laplacian type. In this case, the first two equations of (1.1) arise as the Euler–Lagrange equation of the convex functional

$$J_p(u) := \int_\Omega \Big(\frac{|Du|^p}{p} - \log \det D^2 u \Big) dx. \tag{1.2}$$

When $p = 2$, that is, (1.1) with $F = -\Delta u$, the a priori lower bound on $\det D^2 u$ in [6] breaks down when $n \geq 3$.

Key to this analysis in [6] is the fact that $\operatorname{trace}(U) = \Delta u$ in dimensions $n = 2$. With this crucial fact, one can use

$$U^{ij}\Big(w + \frac{1}{2}|x|^2\Big)_{ij} = -\Delta u + \operatorname{trace}(U) \geq 0$$

and then apply the maximum principle to conclude that $w + \frac{1}{2}|x|^2$ attains its maximum on $\partial\Omega$ from which the upper bound on w follows, which in turn implies the desired lower bound on $\det D^2 u$.

If $n \geq 3$, the ratio $\operatorname{trace}(U)/\Delta u$ can be in general as small as we want; in fact, this is the case, say, when one eigenvalue of $D^2 u$ is 1 while all other $n-1$ eigenvalues are a small constant.

Here, we use a new technique to solve (1.1) when $F = -\gamma \operatorname{div}(|Du|^{p-2}Du)$ where $p \geq 2$ and γ is small. More generally, our main result states as follows.

Theorem 1.1. *Assume $n \geq 3$. Let Ω be an open, smooth, bounded and uniformly convex domain in $\mathbb{R}^n$. Let $\psi \in C^{2,\beta}(\overline{\Omega})$ with $\inf_{\partial\Omega} \psi > 0$ and let $\varphi \in C^{4,\beta}(\overline{\Omega})$ where $\beta \in (0,1)$. Let $F(\cdot, z, \mathbf{p}, \mathbf{r}) : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ be a smooth function such that:*

(i) *it maps compact subsets of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ into compact subsets of $\mathbb{R}$, and*

(ii) *$F(x, u(x), Du(x), D^2u(x)) \leq 0$ in Ω for all C^2 convex functions u.*

If $\gamma > 0$ is a small constant depending only on $\beta, \varphi, \psi, n, F$ and Ω, then there is a uniformly convex solution $u \in C^{4,\beta}(\overline{\Omega})$ to the following second boundary value problem:

$$\begin{cases} \sum_{i,j=1}^n U^{ij} w_{ij} = \gamma F(\cdot, u, Du, D^2u) & \text{in } \Omega, \\ w = (\det D^2 u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The solution is unique provided that F additionally satisfies

$$\int_\Omega [F(\cdot, u, Du, D^2u) - F(\cdot, v, Dv, D^2v)](u - v)\,dx \geq 0 \quad \text{for all } u, v \in C^2(\overline{\Omega}) \text{ with } u = v \text{ on } \partial\Omega. \tag{1.4}$$

Remark 1.2. It would be very interesting to remove the smallness of γ in Theorem 1.1.

Our next result is concerned with radial solutions for the p-Laplacian right-hand side.

Theorem 1.3. *Assume that $\Omega = B_1(0) \subset \mathbb{R}^n$ and let φ and ψ be constants with $\psi > 0$. Let $p \in (1, \infty)$. Let $\beta = p - 1$ if $p < 2$ and $\beta \in (0, 1)$ if $p \geq 2$. Let $f \in \{-1, 1\}$. Consider the second boundary value problem:*

$$\begin{cases} \sum_{i,j=1}^n U^{ij} w_{ij} = f div\ (|Du|^{p-2}Du) & in\ \Omega, \\ w = (\det D^2 u)^{-1} & in\ \Omega, \\ u = \varphi & on\ \partial\Omega, \\ w = \psi & on\ \partial\Omega. \end{cases} \tag{1.5}$$

(i) *Let $f = -1$. Then there is a unique radial, uniformly convex solution $u \in C^{3,\beta}(\overline{\Omega})$ to* (1.5).

(ii) *Let $f = 1$ and let $p \in (1, n]$. In the case $p = n$, we assume further that $\psi > \frac{1}{n}$. Then there is a unique radial, uniformly convex solution $u \in C^{3,\beta}(\overline{\Omega})$ to* (1.5).

(iii) *Let $f = 1$ and let $p > n$. Suppose that $\psi \geq M(n, p)$ for some sufficiently large constant $M > 0$. Then there is a radial, uniformly convex solution $u \in C^{3,\beta}(\overline{\Omega})$ to* (1.5).

Remark 1.4. Regarding the p-Laplacian right-hand side, even in the two dimensions, the analysis in [6] left open the case $F = -\mathrm{div}\,(|Du|^{p-2}Du)$ when $p \in (1, 2)$. The missing ingredient was the lower bound for $\det D^2 u$ in the a priori estimates. If this is obtained, then one can use the recent result in [7] to establish the solvability of (1.1); see the proof of Theorem 1.3 in [7].

Remark 1.5. The size condition on ψ in Theorem 1.3 (ii) is optimal. We can see this in two dimensions as follows. If $f \equiv 1$, $n = p = 2$ and $0 < \psi \leq 1/2$, then there are no uniformly convex solutions $u \in C^4(\overline{\Omega})$ to (1.5). Indeed, if such a uniformly convex solution u exists, then the first and the last equation of (1.5) imply that

$$w(x) = \psi + \frac{1}{2}(|x|^2 - 1).$$

However, since $\psi \leq 1/2$, there is $x \in \Omega$ such that $w(x) \leq 0$, which is a contradiction to the uniform convexity of u and $w = (\det D^2 u)^{-1}$.

When $n = p = 2$, we can remove the symmetry conditions in Theorem 1.3.

Proposition 1.6. *Let Ω be an open, smooth, bounded and uniformly convex domain in $\mathbb{R}^n$ where $n=2$. Assume $f \geq 0$ and $f \in L^\infty(\Omega)$. Assume that $\varphi \in W^{4,q}(\Omega)$, $\psi \in W^{2,q}(\Omega)$ where $q > n$ with*

$$\inf_{x \in \partial\Omega} \left(\psi(x) - \frac{\|f\|_{L^\infty(\Omega)}}{2} |x|^2 \right) > 0. \tag{1.6}$$

Then there is a uniformly convex solution $u \in W^{4,q}(\Omega)$ to the following second boundary value problem:

$$
\begin{cases}
\sum_{i,j=1}^n U^{ij} w_{ij} = f\Delta u & \text{in } \Omega,\\
w = (\det D^2 u)^{-1} & \text{in } \Omega,\\
u = \varphi & \text{on } \partial\Omega,\\
w = \psi & \text{on } \partial\Omega.
\end{cases}
\tag{1.7}
$$

If f is a nonnegative constant, $\varphi \in C^\infty(\overline{\Omega})$, and $\psi \in C^\infty(\overline{\Omega})$ then there is a solution $u \in C^\infty(\overline{\Omega})$.

The key ingredient in the proof of Theorem 1.1 is the solvability and uniform estimates in $W^{4,p}(\Omega)$ for $p > n$ of (1.1) when

$$F \sim -(\Delta u)^{\frac{1}{n-1}} (\det D^2 u)^{\frac{n-2}{n-1}},$$

which reduces to $F \sim -\Delta u$ in two dimensions. This result, and its slightly more general version in Proposition 1.7, can be of independent interest.

Proposition 1.7. *Let Ω be an open, smooth, bounded and uniformly convex domain in $\mathbb{R}^n$. Assume that $\varphi \in W^{4,q}(\Omega)$, $\psi \in W^{2,q}(\Omega)$ with $\inf_{\partial\Omega} \psi > 0$ where $q > n$. Let $k \in \{1, \dots, n-1\}$. Assume that $0 \le f, g \le 1$. Then there is a uniformly convex solution $u \in W^{4,q}(\Omega)$ to the following second boundary value problem:*

$$
\begin{cases}
\sum_{i,j=1}^n U^{ij} w_{ij} = -(\Delta u)^{\frac{1}{n-1}} (\det D^2 u)^{\frac{n-2}{n-1}} f - [S_k(D^2 u)]^{\frac{1}{k(n-1)}} (\det D^2 u)^{\frac{n-2}{n-1}} g & \text{in } \Omega,\\
w = (\det D^2 u)^{-1} & \text{in } \Omega,\\
u = \varphi & \text{on } \partial\Omega,\\
w = \psi & \text{on } \partial\Omega.
\end{cases}
\tag{1.8}
$$

If $f \equiv 1$ and $g \equiv 1$, $\varphi \in C^{4,\beta}(\overline{\Omega})$, and $\psi \in C^{2,\beta}(\overline{\Omega})$, then there is a solution $u \in C^{4,\beta}(\overline{\Omega})$.

In Proposition 1.7 and what follows, for a symmetric $n \times n$ matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$, let us denote its elementary symmetric functions $S_k(A)$ where $k = 0, 1, \dots, n$ by

$$S_0(A) = 1, \quad S_k(A) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} \quad (k \ge 1).$$

The rest of the paper is devoted to proving Theorems 1.1 and 1.3, and Propositions 1.6 and 1.7.

2 Proofs of the main results

In this section, we prove Theorems 1.1 and 1.3, and Propositions 1.6 and 1.7. As in [6], it suffices to prove appropriate fourth order derivative a priori estimates.

For certain fixed parameters β (in Theorem 1.1), p (in Theorem 1.3) and k, q (in Propositions 1.6 and 1.7), we call a positive constant **universal** if it depends only on n, Ω, ψ, φ and those fixed parameters. We use c, C, C_1, $C_2, \ldots,$ to denote universal constants and their values may change from line to line.

Proof of Proposition 1.7. For simplicity, we denote

$$\begin{aligned}F(x) = &-(\Delta u(x))^{\frac{1}{n-1}}(\det D^2 u(x))^{\frac{n-2}{n-1}} f(x)\\ &-[S_k(D^2u(x))]^{\frac{1}{k(n-1)}}(\det D^2 u(x))^{\frac{n-2}{n-1}} g(x).\end{aligned}$$

We establish a priori estimates for a solution $u \in W^{4,q}(\Omega)$. Since $U^{ij}w_{ij} \leq 0$, by the maximum principle, the function w attains its minimum value on the boundary $\partial\Omega$. Thus

$$w \geq \inf_{\partial\Omega} \psi := C_1 > 0.$$

On the other hand, we note that for each $k \in \{1, \ldots, n-1\}$,

$$\Delta u \geq [S_k(D^2u)]^{\frac{1}{k}}, \tag{2.1}$$

and furthermore,

$$\text{trace}\,(U^{ij}) = S_{n-1}(D^2 u) \geq (\Delta u)^{\frac{1}{n-1}}(\det D^2 u)^{\frac{n-2}{n-1}}. \tag{2.2}$$

Indeed, (2.2) is equivalent to $(\det D^2u)\text{trace}(D^2u^{-1}) \geq (\Delta u)^{\frac{1}{n-1}}(\det D^2 u)^{\frac{n-2}{n-1}}$, or

$$[\text{Trace}(D^2u^{-1})]^{n-1} \geq \frac{\Delta u}{\det D^2 u}. \tag{2.3}$$

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of D^2u. Then (2.3) reduces to

$$\left(\sum_{j=1}^n \frac{1}{\lambda_j}\right)^{n-1} \geq \frac{\sum_{i=1}^n \lambda_i}{\prod_{i=1}^n \lambda_i} = \sum_{i=1}^n \prod_{j\neq i}^n \frac{1}{\lambda_j}.$$

This is obvious by the expansion of the left-hand side.

It follows from (2.1) and (2.2) and $0 \leq f, g \leq 1$ that

$$U^{ij}(w + |x|^2)_{ij} \geq 0.$$

By the maximum principle, the function $w + |x|^2$ attains its maximum value on the boundary $\partial\Omega$. Thus

$$w + |x|^2 \leq \max_{\partial\Omega}(\psi + |x|^2) \leq C_2 < \infty.$$

Therefore $w \leq C_2$. As a consequence,

$$C_1 \leq w \leq C_2.$$

From the second equation of (1.8), we can find a universal constant $C > 0$ such that

$$C^{-1} \leq \det D^2 u \leq C \quad \text{in } \Omega. \tag{2.4}$$

By constructing a suitable barrier, we find that Du is universally bounded in $\overline{\Omega}$:

$$\|Du\|_{L^\infty(\Omega)} \leq C. \tag{2.5}$$

From $\varphi \in W^{4,q}(\Omega)$ with $q > n$, we have $\varphi \in C^3(\overline{\Omega})$ by the Sobolev embedding theorem. By assumption, Ω is bounded, smooth and uniformly convex. From $u = \varphi$ on $\partial\Omega$ and (2.4), we can apply the global $W^{2,1+\varepsilon_0}$ estimates for the Monge–Ampère equation, which follow from the interior $W^{2,1+\varepsilon_0}$ estimates in De Philippis–Figalli–Savin [3] and Schmidt [12] and the global estimates in Savin [11] (see also [4, Theorem 5.3]), to conclude that

$$\|D^2 u\|_{L^{1+\varepsilon_0}(\Omega)} \leq C_1^* \tag{2.6}$$

for some universal constants $\varepsilon_0 > 0$ and $C_1^* > 0$.

Thus, from (2.6) and (2.1), we find that

$$\|F\|_{L^{(n-1)(1+\varepsilon_0)}(\Omega)} \leq C_3$$

for a universal constant $C_3 > 0$. Note that for all $n \geq 2$ and all $\varepsilon_0 > 0$,

$$(n-1)(1+\varepsilon_0) > n/2.$$

From $\psi \in W^{2,q}(\Omega)$ with $q > n$, we have $\psi \in C^1(\overline{\Omega})$ by the Sobolev embedding theorem. Now, we apply the global Hölder estimates for the linearized Monge–Ampère equation in [8, Theorem 1.7 and Lemma 1.5] to $U^{ij} w_{ij} = F$ in Ω with boundary value $w = \psi \in C^1(\partial\Omega)$ on $\partial\Omega$ to conclude that $w \in C^\alpha(\overline{\Omega})$ with

$$\|w\|_{C^\alpha(\overline{\Omega})} \leq C(\|\psi\|_{C^1(\partial\Omega)} + \|F\|_{L^{(n-1)(1+\varepsilon_0)}(\Omega)}) \leq C_4 \tag{2.7}$$

for universal constants $\alpha \in (0,1)$ and $C_4 > 0$. Now, we note that u solves the Monge–Ampère equation

$$\det D^2 u = w^{-1}$$

with the right-hand side being in $C^\alpha(\overline{\Omega})$ and boundary value $\varphi \in C^3(\partial\Omega)$ on $\partial\Omega$. Therefore, by the global $C^{2,\alpha}$ estimates for the Monge–Ampère equation [13, 10], we have $u \in C^{2,\alpha}(\overline{\Omega})$ with universal estimates

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C_5 \quad \text{and} \quad C_5^{-1} I_n \leq D^2 u \leq C_5 I_n. \tag{2.8}$$

Here and throughout, we use I_n to denote the $n \times n$ identity matrix. As a consequence, the second order operator $U^{ij}\partial_{ij}$ is uniformly elliptic with Hölder continuous coefficients. Now, we observe from the definition of F and (2.8) that

$$\|F\|_{L^\infty(\Omega)} \le C_6. \tag{2.9}$$

Thus, from the equation $U^{ij}w_{ij} = F$ with boundary value $w = \psi$ where $\psi \in W^{2,q}(\Omega)$, we conclude that $w \in W^{2,q}(\Omega)$ and therefore $u \in W^{4,q}(\Omega)$ with universal estimate

$$\|u\|_{W^{4,q}(\Omega)} \le C_7.$$

It remains to consider the case $f \equiv 1$ and $g \equiv 1$, $\varphi \in C^{4,\beta}(\overline{\Omega})$, and $\psi \in C^{2,\beta}(\overline{\Omega})$. In this case, we need to establish a priori estimates for $u \in C^{4,\beta}(\overline{\Omega})$. As above, instead of (2.9), we have

$$\|F\|_{C^{\frac{\alpha}{n-1}}(\overline{\Omega})} \le C_7. \tag{2.10}$$

Thus, from the equation $U^{ij}w_{ij} = F$ with boundary value $w = \psi$ where $\psi \in C^{2,\beta}(\overline{\Omega})$, we conclude that $w \in C^{2,\gamma}(\overline{\Omega})$ where $\gamma := \min\{\frac{\alpha}{n-1}, \beta\}$ and therefore $u \in C^{4,\gamma}(\overline{\Omega})$ with the universal estimate $\|u\|_{C^{4,\gamma}(\overline{\Omega})} \le C_8$. With this estimate, we can improve (2.10) to

$$\|F\|_{C^{\beta}(\overline{\Omega})} \le C_9. \tag{2.11}$$

As above, we find that $u \in C^{4,\beta}(\overline{\Omega})$ with the universal estimate $\|u\|_{C^{4,\beta}(\overline{\Omega})} \le C_{10}$.$\square$

Proof of Theorem 1.1. Without loss of generality, we can assume that $\inf_{\partial\Omega} \psi = 1$. We consider the following second boundary value problem for a uniformly convex function u:

$$\begin{cases} U^{ij}w_{ij} = -(\Delta u)^{\frac{1}{n-1}}(\det D^2 u)^{\frac{n-2}{n-1}} f_\gamma(\cdot, u, Du, D^2u) & \text{in } \Omega, \\ w = (\det D^2 u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

for some $\gamma \in (0, 1)$ to be chosen later, where

$$f_\gamma(\cdot, u, Du, D^2u) = \min\left\{\frac{-\gamma F(\cdot, u, Du, D^2u)}{(\Delta u)^{\frac{1}{n-1}}(\det D^2u)^{\frac{n-2}{n-1}}}, 1\right\}.$$

By our assumption (ii) on F, when u is a C^2 convex function, we have $0 \le f_\gamma \le 1$. By Proposition 1.7 (with $g \equiv 0$), (2.12) has a solution $u \in W^{4,q}(\Omega)$ for all $q < \infty$. Thus, the first equation of (2.12) holds pointwise a.e.

As in the proof of Proposition 1.7 (see (2.8)), we have the following a priori estimates

$$\|u\|_{C^{2,\beta}(\overline{\Omega})} \le C_1 \quad \text{and} \quad C_1^{-1} I_n \le D^2 u \le C_1 I_n \tag{2.13}$$

for some $C_1 > 0$ depending only on β, φ, ψ, n and Ω. Hence, using assumption (i) on F, we find that

$$\frac{-\gamma F(\cdot, u, Du, D^2 u)}{(\Delta u)^{\frac{1}{n-1}} (\det D^2 u)^{\frac{n-2}{n-1}}} < \frac{1}{2}$$

if $\gamma > 0$ is small, depending only on β, φ, ψ, n, F and Ω.

Thus, if $\gamma > 0$ is small, depending only on β, φ, ψ, n, F and Ω, then

$$f_\gamma = \min\left\{\frac{-\gamma F(\cdot, u, Du, D^2 u)}{(\Delta u)^{\frac{1}{n-1}} (\det D^2 u)^{\frac{n-2}{n-1}}}, 1\right\} = \frac{-\gamma F(\cdot, u, Du, D^2 u)}{(\Delta u)^{\frac{1}{n-1}} (\det D^2 u)^{\frac{n-2}{n-1}}}$$

in Ω and hence the first equation of (2.12) becomes

$$U^{ij} w_{ij} = \gamma F(\cdot, u, Du, D^2 u).$$

Using this equation together with (2.13) and $\varphi \in C^{4,\beta}(\overline{\Omega})$ and $\psi \in C^{2,\beta}(\overline{\Omega})$, we easily conclude that $u \in C^{4,\beta}(\overline{\Omega})$. Thus, there is a uniformly convex solution $u \in C^{4,\beta}(\overline{\Omega})$ to (1.3).

Assume now F additionally satisfies (1.4). Then arguing as in the proof of [6, Lemma 4.5] replacing f_δ there by γF, we obtain the uniqueness of the $C^{4,\beta}(\overline{\Omega})$ solution to (1.3). □

Remark 2.1. Clearly, Theorem 1.1 and its proof apply to dimensions $n = 2$.

Proof of Proposition 1.6. We establish a priori estimates for a solution $u \in W^{4,q}(\Omega)$ to (1.7). As in the proof of Proposition 1.7, it suffices to obtain the lower and upper bounds on $\det D^2 u$.

Observe that

$$U^{ij} w_{ij} = f \Delta u \ge 0.$$

By the maximum principle, the function w attains its maximum value on the boundary $\partial\Omega$. Thus

$$w \le \sup_{\partial\Omega} \psi < \infty.$$

By the second equation of (1.7), this gives a bound from below for $\det D^2 u$:

$$\det D^2 u \ge C^{-1}.$$

On the other hand, we have

$$\sum_{i,j=1}^{2} U^{ij}\Big(w - \frac{\|f\|_{L^\infty(\Omega)}}{2}|x|^2\Big)_{ij} = (f - \|f\|_{L^\infty(\Omega)})\Delta u \le 0.$$

By the maximum principle, the function $w - \frac{\|f\|_{L^\infty(\Omega)}}{2}|x|^2$ attains its minimum value on the boundary $\partial\Omega$. Thus, using (1.6), we find that

$$w - \frac{\|f\|_{L^\infty(\Omega)}}{2}|x|^2 \ge \inf_{x\in\partial\Omega}\Big(w(x) - \frac{\|f\|_{L^\infty(\Omega)}}{2}|x|^2\Big) > 0.$$

This gives a positive lower bound for w, that is, $w \ge C^{-1} > 0$. Using the second equation of (1.7), we obtain a bound from above for $\det D^2u$:

$$\det D^2u \le C.$$

□

Proof of Theorem 1.3. Recall that $f \in \{-1, 1\}$ and $\beta = p-1$ if $1 < p < 2$ and $\beta \in (0,1)$ if $p \ge 2$.

We first observe the following reduction of smoothness without any symmetry assumptions. Suppose that one has a uniformly convex solution $u \in C^2(\overline{\Omega})$ to (1.5) with positive lower and upper bounds on $\det D^2u$:

$$C^{-1} \le \det D^2u \le C \tag{2.14}$$

for some $C > 0$ and such that $w \in C^\beta(\overline{\Omega})$; then $u \in C^{3,\beta}(\overline{\Omega})$. Indeed, using (2.14) together with the global $C^{2,\alpha}$ estimates [13, 10] for the Monge–Ampère equation $\det D^2u = w^{-1}$ with boundary data $\varphi \in C^{3,1}(\overline{\Omega})$ and right-hand side $w^{-1} \in C^\beta(\overline{\Omega})$, we have $u \in C^{2,\beta}(\overline{\Omega})$ with estimates

$$\|u\|_{C^{2,\beta}(\overline{\Omega})} \le C_1 \quad \text{and} \quad C_1^{-1}I_n \le D^2u \le C_1 I_n. \tag{2.15}$$

As a consequence, the second order operator $U^{ij}\partial_{ij}$ is uniformly elliptic with Hölder continuous coefficients with exponent $\beta \in (0,1)$. Note that $|Du|^{p-2}Du$ is Hölder continuous with exponent β. Using the first equation of (1.5), we see that the $C^{1,\beta}(\overline{\Omega})$ estimate for w follows from [5, Theorem 8.33]. Hence, we have the $C^{3,\beta}(\overline{\Omega})$ estimates for u.

Now, we look for radial, uniformly convex solutions $u \in C^2(\overline{\Omega})$ to (1.5). Assume that the convex function u is of the form

$$u(x) = v(r)$$

where

$$v : [0,\infty) \to \mathbb{R} \quad \text{and} \quad r = |x|.$$

Let us denote

$$' = \frac{d}{dr} \quad \text{and} \quad g(r) := v'(r).$$

The requirement that $u \in C^2(\overline{\Omega})$ forces

$$g(0) = v'(0) = 0.$$

The next reduction in the proof of our theorem is the following claim.

Claim. *The existence of radial, uniformly convex solutions $u \in C^2(\overline{\Omega})$ to* (1.5) *with positive lower and upper bounds on* $\det D^2u$ *and a Hölder continuous w is equivalent to finding $g(1) > 0$ satisfying the integral equation*

$$\int_0^{g(1)} e^{\frac{f}{p}s^p} s^{n-1}ds = \frac{1}{n\psi}\left(1 + f\int_0^{g(1)} e^{\frac{f}{p}s^p} s^{p-1}ds\right). \tag{2.16}$$

To prove the claim, we compute

$$\det D^2u = v''\left(\frac{v'}{r}\right)^{n-1}, \quad w = (\det D^2u)^{-1} = \frac{1}{v''}\left(\frac{r}{v'}\right)^{n-1} \equiv W(r).$$

Since D^2u and $(D^2u)^{-1}$ are similar to diag $(v'', \frac{v'}{r}, \dots, \frac{v'}{r})$ and diag $(\frac{1}{v''}, \frac{r}{v'}, \dots, \frac{r}{v'})$, we can compute

$$U^{ij}w_{ij} = \frac{v''(v')^{n-1}}{r^{n-1}}\left(\frac{W''}{v''} + (n-1)\frac{W'}{v'}\right) = \frac{[W'(v')^{n-1}]'}{r^{n-1}}.$$

Note that v'' and v' are all nonnegative. Therefore,

$$0 \le v'(r) \le v'(1) \quad \text{for all } 0 \le r \le 1. \tag{2.17}$$

On the other hand, we have

$$\operatorname{div}(|Du|^{p-2}Du) = (p-1)(v')^{p-2}v'' + \frac{n-1}{r}(v')^{p-1} = \frac{[(v')^{p-1}r^{n-1}]'}{r^{n-1}}.$$

The first equation of (1.5) gives

$$\frac{[W'(v')^{n-1}]'}{r^{n-1}} = f\frac{[(v')^{p-1}r^{n-1}]'}{r^{n-1}}$$

which implies that, for some constant C,

$$W'(v')^{n-1} = f(v')^{p-1}r^{n-1} + C.$$

Since $v'(0) = 0$, we find that $C = 0$. Thus

$$W' = f(v')^{p-1}\left(\frac{r}{v'}\right)^{n-1} = f(v')^{p-1}v''W.$$

It follows that

$$[\log W]' = \left[\frac{f}{p}(v')^p\right]'$$

and hence, recalling $W(1) = \psi$,

$$\begin{aligned}\log W(r) &= \log W(1) + \frac{f}{p}[(v'(r))^p - (v'(1))^p] \\ &= \log\psi + \frac{f}{p}[(v'(r))^p - (v'(1))^p].\end{aligned} \tag{2.18}$$

Therefore, in terms of $g = v'$, we have after exponentiation

$$e^{\frac{f}{p}[g(r)]^p}[g(r)]^{n-1}g'(r) = \frac{1}{\psi}e^{\frac{f}{p}[g(1)]^p}r^{n-1}, \tag{2.19}$$

which is equivalent to

$$\int_0^{g(r)} e^{\frac{f}{p}s^p}s^{n-1}ds = \frac{1}{n\psi}e^{\frac{f}{p}[g(1)]^p}r^n. \tag{2.20}$$

Clearly, (2.20) leads to a solution to (1.5) in terms of $g(1), n, p$ and ψ provided $g(1) > 0$ satisfies the compatibility condition at $r = 1$:

$$\int_0^{g(1)} e^{\frac{f}{p}s^p}s^{n-1}ds = \frac{1}{n\psi}e^{\frac{f}{p}[g(1)]^p}. \tag{2.21}$$

Because

$$e^{\frac{f}{p}[g(1)]^p} = 1 + f\int_0^{g(1)} e^{\frac{f}{p}s^p}s^{p-1}ds,$$

the compatibility condition (2.21) can be rewritten as in (2.16).

Assume that $g(1) = v'(1) > 0$ has already been found, in terms of n, p and ψ. We now establish positive lower and upper bounds on $\det D^2u$ and that $w \in C^\beta(\overline{\Omega})$. Indeed, from $0 \leq g(r) \leq g(1)$, we can easily estimate

$$e^{\frac{-1}{p}[g(1)]^p}\frac{[g(r)]^n}{n} \leq \int_0^{g(r)} e^{\frac{f}{p}s^p}s^{n-1}ds \leq e^{\frac{1}{p}[g(1)]^p}\frac{[g(r)]^n}{n}.$$

Hence (2.20) gives

$$C^{-1}r \leq g(r) \leq Cr$$

for some C that depends only on $g(1) > 0$, n, p and ψ. Thus, from (2.19), we find that v'' and $\frac{v'(r)}{r}$ are bounded from below and above by positive constants. Therefore, we have positive lower and upper bounds on $\det D^2u = v''(\frac{v'}{r})^{n-1}$. Moreover, $v'(r) = |Du(x)| \in C^\alpha(\overline{\Omega})$ for all $\alpha \in (0, 1)$. Using (2.18), we also find that W, and hence w, is in $C^\alpha(\overline{\Omega})$. In particular, $w \in C^\beta(\overline{\Omega})$.

We have reduced our theorem to the existence and uniqueness of $g(1) > 0$ solving (2.16) which we now address.

(i) Recall that $f = -1$. Note that (2.16) becomes

$$\int_0^{g(1)} e^{\frac{-1}{p}s^p} s^{n-1} ds = \frac{1}{n\psi}\left(1 - \int_0^{g(1)} e^{\frac{-1}{p}s^p} s^{p-1} ds\right).$$

Clearly, there is a unique $g(1) > 0$ solving the above integral equation. Hence, there is a unique radial, uniformly convex solution $u \in C^{3,\beta}(\overline{\Omega})$ to (1.5).

(ii) Recall that $f = 1$ and $p \in (1, n]$. Note that (2.16) becomes

$$H(g(1) = I(g(1)) \tag{2.22}$$

where

$$H(t) := \int_0^t e^{\frac{1}{p}s^p} s^{n-1} ds \quad \text{and} \quad I(t) := \frac{1}{n\psi}\left(1 + \int_0^t e^{\frac{1}{p}s^p} s^{p-1} ds\right) \equiv \frac{1}{n\psi} e^{\frac{t^p}{p}}. \tag{2.23}$$

Consider first the case $p = n$. Then

$$H(t) = e^{\frac{t^n}{n}} - 1 \quad \text{and } I(t) = \frac{1}{n\psi} e^{\frac{t^n}{n}}.$$

Therefore, from (2.22) we find an explicit formula for $g(1)$ from the equation

$$e^{\frac{1}{n}[g(1)]^n} = \frac{n\psi}{n\psi - 1},$$

showing existence and uniqueness of a solution $g(1) > 0$ to (2.16) when $\psi > \frac{1}{n}$. As a result, there is a unique radial, uniformly convex solution $u \in C^{3,\beta}(\overline{\Omega})$ to (1.5). Moreover, $\psi > \frac{1}{n}$ is also the optimal condition for the existence of a radial solution to (1.5).

Now we consider the case $p \in (1, n)$ and $\psi > 0$. We show that (2.22) has a unique solution $g(1) > 0$ and hence there is a unique radial, uniformly convex solution $u \in C^{3,\beta}(\overline{\Omega})$ to (1.5). Indeed, since $1 < p < n$, the integrand of $H(t)$ grows faster than that of $I(t)$. Since $H(0) = 0 < I(0) = \frac{1}{n\psi}$, the function $H(t)$ will cross $I(t)$ for the first time from below at some point $t_0 > 0$. Thus $g(1) = t_0 > 0$ is a solution of (2.22). To show the uniqueness of $g(1)$, we show that if $t > t_0$ then $H(t) > I(t)$. Indeed, using the definition of t_0, we find that $H'(t_0) \geq I'(t_0)$. This means that

$$e^{\frac{1}{p}t_0^p} t_0^{n-1} \geq \frac{1}{n\psi} e^{\frac{1}{p}t_0^p} t_0^{p-1},$$

or, equivalently,

$$t_0^{n-p} \geq \frac{1}{n\psi}.$$

Thus, if $s > t_0$ then $s^{n-p} > \frac{1}{n\psi}$, that is,

$$e^{\frac{1}{p}s^p}s^{n-1} > \frac{1}{n\psi}e^{\frac{1}{p}s^p}s^{p-1},$$

and hence, for any $t > t_0$, we have

$$H(t) = H(t_0) + \int_{t_0}^{t} e^{\frac{1}{p}s^p}s^{n-1}ds > I(t_0) + \frac{1}{n\psi}\int_{t_0}^{t} e^{\frac{1}{p}s^p}s^{p-1}ds = I(t).$$

(iii) Recall that $f = 1$ and $p > n$. Assume that

$$\psi \geq M(n,p) := 1 + \frac{e^{1/p}}{n}\left(\int_0^1 e^{\frac{1}{p}s^p}s^{n-1}ds\right)^{-1}.$$

Then, there is a solution $g(1) > 0$ to (2.22) where H and I are defined as in (2.23). Indeed, in this case, we have $1 > \frac{e^{1/p}}{n\psi}[H(1)]^{-1} = \frac{I(1)}{H(1)}$. Therefore $I(1) < H(1)$ while $I(0) > H(0)$. Thus, (2.22) has a solution $g(1) \in (0,1)$. Consequently, there is a radial, uniformly convex solution $u \in C^{3,\beta}(\overline{\Omega})$ to (1.5). $\square$

Remark 2.2. When $p > n$, radial solutions in Theorem 1.3 (iii) are not unique in general. This corresponds to multiple crossings of H and I defined in (2.23). For example, this is in fact the case of $n = 2$, $p = 4$ and $\psi = 1$. We can plot the graphs of H and I using Maple to find that, on $[0,2]$, they cross twice at $t_1 \in (1, 6/5)$ and $t_2 \in (3/2, 2)$.

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