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ORIGINAL PAPER

A Unified Petrov–Galerkin Spectral Method and Fast Solver for Distributed-Order Partial Differential Equations

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Abstract
Fractional calculus and fractional-order modeling provide effective tools for modeling and simulation of anomalous diffusion with power-law scalings. In complex multi-fractal anomalous transport phenomena, distributed-order partial differential equations appear as tractable mathematical models, where the underlying derivative orders are distributed over a range of values, hence taking into account a wide range of multi-physics from ultra-slow-to-standard-to-superdiffusion/wave dynamics. We develop a unified, fast, and stable Petrov–Galerkin spectral method for such models by employing Jacobi poly-fractonomials and Legendre polynomials as temporal and spatial basis/test functions, respectively. By defining the proper underlying distributed Sobolev spaces and their equivalent norms, we rigorously prove the well-posedness of the weak formulation, and thereby, we carry out the corresponding stability and error analysis. We finally provide several numerical simulations to study the performance and convergence of proposed scheme.

Keywords Distributed Sobolev space · Well-posedness analysis · Discrete inf-sup condition · Spectral convergence · Jacobi poly-fractonomials · Legendre polynomials

Mathematics Subject Classification 34L10 · 58C40 · 34K28 · 65M70 · 65M60

1 Introduction

Over the past decades, anomalous transport has been observed and investigated in a wide range of applications such as turbulence [11, 21, 43, 49], porous media [4, 7, 16, 59, 66, 67], geoscience [5], bioscience [45–48], and viscoelastic material [20, 40, 54, 55, 60]. The

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underlying anomalous features, manifesting in memory effects, non-local interactions, power-law distributions, sharp peaks, and self-similar structures, can be well described by fractional partial differential equations (FPDEs) [27, 41, 42, 44]. However, in many physical processes, which cannot be characterized with a certain single power-law scaling over the whole domain, distributed-order differential equations (DDEs) can serve as a framework for accommodating a distribution of power-law behavior. More specifically, distributed-order FPDEs are generating considerable interests in terms of accelerating superdiffusion, decelerating subdiffusion random processes in multi-physics anomalous phenomena. To model wave propagation in complex media like viscoelastic media, acoustics, and seismology, Bazhlekov and Bazhlekov [6] developed a subordination approach to multi-term time-fractional diffusion-wave equations. Besides, Chechkin et al. [8] proposed distributed-order temporal fractional diffusion equations for describing the (retarding) sub-diffusion random processes which are subordinated to the Wiener process. A faithful description of such anomalous transport requires exploiting distributed-order derivatives, in which the derivative order has a distribution over a range of values. The reader is referred to [14, 15, 19, 28, 36, 37, 42, 53, 58] and the references given therein for more details on the distributed-order fractional equations.

Numerical methods for FPDEs, which can exhibit history dependence and non-local features, have been recently addressed by developing finite-element methods [2, 23], spectral/spectral-element methods [10, 26, 38, 39, 50, 61], and also finite-difference and finite-volume methods [3, 12, 34]. Distributed-order FPDEs impose further complications in numerical analysis by introducing distribution functions, which require compliant underlying function spaces, as well as efficient and accurate integration techniques over the order of the fractional derivatives. In [18, 22, 29, 33, 56, 62], numerical analysis of distributed-order FPDEs was extensively investigated. More recently, Liao et al. [32] studied simulation of a distributed subdiffusion equation, approximating the distributed-order Caputo derivative using piecewise-linear and quadratic interpolating polynomials. Abbaszadeh and Dehghan [1] employed an alternating direction implicit approach, combined with an interpolating element-free Galerkin method, on distributed-order time-fractional diffusion-wave equations. Kharazmi and Zayernouri [24] developed a pseudo-spectral method of Petrov–Galerkin (PG) sense, employing nodal expansions in the weak formulation of distributed-order fractional PDEs. In [25], Kharazmi et al. also introduced distributed Sobolev space and developed two spectrally accurate schemes, namely, a PG spectral method and a spectral collocation method for distributed-order fractional differential equations. Besides, Tomovski and Sandev [57] investigated the solution of generalized distributed-order diffusion equations with fractional time-derivative, using the Fourier–Laplace transform method.

The main purpose of this study is to develop and analyze a PG spectral method to solve a $(1+d)$-dimensional fully distributed-order FPDE with two-sided derivatives of the form

$$\begin{aligned} &\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, {}_0^{\mathrm{C}}\mathcal{D}_t^{2\tau} u \,\mathrm{d}\tau + \sum_{i=1}^{d} \int_{\mu_i^{\min}}^{\mu_i^{\max}} \varrho_i(\mu_i)\,\big(c_{l_i}\, {}_{a_i}^{\mathrm{RL}}\mathcal{D}_{x_i}^{2\mu_i} u + c_{r_i}\, {}_{x_i}^{\mathrm{RL}}\mathcal{D}_{b_i}^{2\mu_i} u\big)\,\mathrm{d}\mu_i \\ &= \sum_{j=1}^{d} \int_{\nu_j^{\min}}^{\nu_j^{\max}} \rho_j(\nu_j)\,\big(\kappa_{l_j}\, {}_{a_j}^{\mathrm{RL}}\mathcal{D}_{x_j}^{2\nu_j} u + \kappa_{r_j}\, {}_{x_j}^{\mathrm{RL}}\mathcal{D}_{b_j}^{2\nu_j} u\big)\,\mathrm{d}\nu_j - \gamma u + f \end{aligned} \tag{1}$$

subject to homogeneous Dirichlet boundary conditions and the zero initial condition, where for $i, j = 1, 2, \cdots, d$,

$$
\begin{aligned}
&t \in [0,\ T], \quad x_j \in [a_j,\ b_j],\\
&2\tau^{\min} < 2\tau^{\max} \in (0,\ 2],\ 2\tau^{\min} \neq 1,\ 2\tau^{\max} \neq 1,\\
&2\mu_i^{\min} < 2\mu_i^{\max} \in (0,\ 1),\ 2\nu_j^{\min} < 2\nu_j^{\max} \in (1,\ 2],\\
&0 < \varphi(\tau) \in L^1((\tau^{\min}, \tau^{\max})),\ 0 < \varrho_i(\mu_i) \in L^1((\mu_i^{\min}, \mu_i^{\max})),\\
&0 < \rho_j(\nu_j) \in L^1((\nu_j^{\min}, \nu_j^{\max})),
\end{aligned}
$$

and the coefficients c_{l_i}, c_{r_i}, κ_{l_i}, κ_{r_i}, and γ are constant. We emphasize that (1) is reduced to fractional advection-dispersion-reaction equations when $\varphi(\tau)$, $\varrho_i(\mu_i)$, and $\rho_j(\nu_j)$ are chosen to be a Dirac delta function, where for $d = 1$ and $\rho_1(\nu_1) = \delta(\nu_1 - 1)$, the two-sided Riesz derivatives with proper coefficients recover the standard second-order dispersion terms. We briefly highlight the main contributions of this study as follows.

– We consider fully distributed fractional PDEs as an extension of existing fractional PDEs in [25, 50] by replacing the fractional operators by their corresponding distributed-order ones. We further derive the weak formulation of the problem.
– We construct the underlying function spaces by extending the distributed Sobolev space in [25] to higher dimensions in time and space, endowed with equivalent associated norms.
– We develop a PG spectral method, employing Legendre polynomials and Jacobi poly-fractonomials [65] as spatial and temporal basis/test functions, respectively. We also formulate a fast solver for the corresponding weak form of (1), following [50], which significantly reduces the computational expenses in high-dimensional problems.
– We establish the well-posedness of the weak form of the problem in the underlying distributed Sobolev spaces respecting the analysis in [51] and prove the stability of proposed numerical scheme. We additionally perform the corresponding error analysis, where the distributed Sobolev spaces enable us to obtain accurate error estimate of the scheme.

To examine the performance and convergence of the developed PG method in solving different cases, we also perform several numerical simulations.

The paper is organized as follows: in Sect. 2, we introduce some preliminaries from fractional calculus. In Sect. 3, we present the mathematical framework of the bilinear form and carry out the corresponding well-posedness analysis. We construct the PG method for the discrete weak form problem and formulate the fast solver in Sect. 4. In Sect. 5, we perform the stability and error analysis in detail. In Sect. 6, we illustrate the convergence rate and the efficiency of method via numerical examples. We conclude the paper with a summary.

2 Preliminaries on Fractional Calculus

Recalling the definitions of the fractional derivatives and integrals from [42, 65], we denote by ${}^{\mathrm{RL}}_{\ a}\mathcal{D}_x^{\sigma} g(x)$ and ${}^{\mathrm{RL}}_{\ x}\mathcal{D}_b^{\sigma} g(x)$ the left-sided and the right-sided Riemann–Liouville fractional derivatives of order $\sigma > 0$,

$$
{}^{\mathrm{RL}}_{\ a}\mathcal{D}_x^{\sigma} g(x) = \frac{1}{\Gamma(n-\sigma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x \frac{g(s)}{(x-s)^{\sigma+1-n}}\, \mathrm{d}s, \quad x \in [a, b], \tag{2}
$$

$$ {}^{\mathrm{RL}}_{\ x}\mathcal{D}^{\sigma}_{b} g(x) = \frac{(-1)^n}{\Gamma(n-\sigma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b \frac{g(s)}{(s-x)^{\sigma+1-n}}\,\mathrm{d}s, \quad x \in [a,b], \tag{3} $$

in which $g(x) \in L^1[a,b]$ and $\int_a^x \frac{g(s)}{(x-s)^{\sigma+1-n}}\,\mathrm{d}s$, $\int_x^b \frac{g(s)}{(s-x)^{\sigma+1-n}}\,\mathrm{d}s \in C^n[a,b]$, respectively, where $n = \lceil \sigma \rceil$. Besides, ${}^{\mathrm{C}}_{a}\mathcal{D}^{\sigma}_{x} g(x)$ and ${}^{\mathrm{C}}_{x}\mathcal{D}^{\sigma}_{b} g(x)$ represent the left-sided and the right-sided Caputo fractional derivatives, where

$$ {}^{\mathrm{C}}_{a}\mathcal{D}^{\sigma}_{x} f(x) = \frac{1}{\Gamma(n-\sigma)} \int_a^x \frac{g^{(n)}(s)}{(x-s)^{\sigma+1-n}}\,\mathrm{d}s, \quad x \in [a,b], \tag{4} $$

$$ {}^{\mathrm{C}}_{x}\mathcal{D}^{\sigma}_{b} f(x) = \frac{(-1)^n}{\Gamma(n-\sigma)} \int_x^b \frac{g^{(n)}(s)}{(s-x)^{\sigma+1-n}}\,\mathrm{d}s, \quad x \in [a,b]. \tag{5} $$

The relationship between the RL and the Caputo fractional derivatives is given by

$$ {}^{\mathrm{RL}}_{\ a}\mathcal{D}^{\nu}_{x} f(x) = \frac{f(a)}{\Gamma(1-\nu)(x-a)^{\nu}} + {}^{\mathrm{C}}_{a}\mathcal{D}^{\nu}_{x} f(x), \tag{6} $$

$$ {}^{\mathrm{RL}}_{\ x}\mathcal{D}^{\nu}_{b} f(x) = \frac{f(b)}{\Gamma(1-\nu)(b-x)^{\nu}} + {}^{\mathrm{C}}_{x}\mathcal{D}^{\nu}_{b} f(x), \tag{7} $$

when $\lceil \nu \rceil = 1$, see, e.g., (2.33) in [42]. In the case of homogeneous boundary conditions, we obtain ${}^{\mathrm{RL}}_{\ a}\mathcal{D}^{\nu}_{x} f(x) = {}^{\mathrm{C}}_{a}\mathcal{D}^{\nu}_{x} f(x) := {}_{a}\mathcal{D}^{\nu}_{x} f(x)$ and ${}^{\mathrm{RL}}_{\ x}\mathcal{D}^{\nu}_{b} f(x) = {}^{\mathrm{C}}_{x}\mathcal{D}^{\nu}_{b} f(x) := {}_{x}\mathcal{D}^{\nu}_{b} f(x)$. The Riemann–Liouville fractional integrals of Jacobi poly-fractonomials are analytically obtained in [64, 65] in the standard domain $\xi \in [-1,1]$ as

$$ {}^{\mathrm{RL}}_{-1}\mathcal{I}^{\sigma}_{\xi} \{(1+\xi)^{\beta} P^{\alpha,\beta}_{n}(\xi)\} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\sigma+1)} (1+\xi)^{\beta+\sigma} P^{\alpha-\sigma,\beta+\sigma}_{n}(\xi) \tag{8} $$

and

$$ {}^{\mathrm{RL}}_{\ \xi}\mathcal{I}^{\sigma}_{1} \{(1-\xi)^{\alpha} P^{\alpha,\beta}_{n}(\xi)\} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\sigma+1)} (1-\xi)^{\alpha+\sigma} P^{\alpha+\sigma,\beta-\sigma}_{n}(\xi), \tag{9} $$

where $0 < \sigma < 1$, $\alpha > -1$, $\beta > -1$, and $P^{\alpha,\,\beta}_{n}(\xi)$ denotes the standard Jacobi polynomials of order n and parameters α and β [9]. Accordingly,

$$ {}^{\mathrm{RL}}_{-1}\mathcal{D}^{\sigma}_{\xi} P_n(\xi) = \frac{\Gamma(n+1)}{\Gamma(n-\sigma+1)} P^{\sigma,-\sigma}_{n}(\xi)\,(1+\xi)^{-\sigma} \tag{10} $$

and

$$ {}^{\mathrm{RL}}_{\ \xi}\mathcal{D}^{\sigma}_{1} P_n(\xi) = \frac{\Gamma(n+1)}{\Gamma(n-\sigma+1)} P^{-\sigma,\sigma}_{n}(\xi)\,(1-\xi)^{-\sigma}, \tag{11} $$

where $P_n(\xi) := P^{0,0}_{n}(\xi)$ represents the Legendre polynomial of degree n (see [9]).

Let us define the distributed-order derivative as

$$ {}^{\mathrm{D}}\mathcal{D}^{\phi}_{t} f(t,x) := \int_{\tau^{\min}}^{\tau^{\max}} \phi(\tau)\, {}_{0}\mathcal{D}^{\tau}_{t} f(t,x)\,\mathrm{d}\tau, \tag{12} $$

where $\alpha \to \phi(\alpha)$ is a continuous mapping in $[\alpha^{\min}, \alpha^{\max}]$ [25] and $t > 0$. We note that by choosing the distribution function in the distributed-order derivatives to be the Dirac delta function $\delta(\tau - \tau_0)$, we recover a single (fixed) term fractional derivative, that is,

$$\int_{\tau^{\min}}^{\tau^{\max}} \delta(\tau - \tau_0)\,{}_0\mathcal{D}_t^{\tau} f(t, x)\mathrm{d}\tau = {}_0\mathcal{D}_t^{\tau_0} f(t, x), \tag{13}$$

where $\tau_0 \in (\tau^{\min}, \tau^{\max})$.

3 Mathematical Formulation

We introduce the underlying solution and test spaces along with their proper norms, and also provide some useful lemmas to derive the corresponding bilinear form and thus, prove the well-posedness of the problem.

3.1 Mathematical Framework

Let $C_0^\infty(\Lambda)$ represent the space of smooth functions with compact support in $\Lambda = (a, b)$. Recalling the definition of the Sobolev space for the real $\sigma \geqslant 0$ from [25, 30], the usual Sobolev space denoted by $H^\sigma(\Lambda)$ is the closure of $C_0^\infty(\Lambda)$ on the finite interval Λ, which is associated with the norm $\|\cdot\|_{H^\sigma(\Lambda)}$. For the real index $\sigma \geqslant 0$ and $\sigma \neq n - \frac{1}{2}$ on the bounded interval Λ, the following norms are equivalent [31]:

$$\|\cdot\|_{H^\sigma(\Lambda)} \cong \|\cdot\|_{{}^{\mathrm{l}}H^\sigma(\Lambda)} \cong \|\cdot\|_{{}^{\mathrm{r}}H^\sigma(\Lambda)} \cong |\cdot|^*_{H^\sigma(\Lambda)}, \tag{14}$$

where "$\cong$" denotes equivalence relation, $\|\cdot\|_{{}^{\mathrm{l}}H^\sigma(\Lambda)} = \left(\|{}_a\mathcal{D}_x^\sigma(\cdot)\|^2_{L^2(\Lambda)} + \|\cdot\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}}$, $\|\cdot\|_{{}^{\mathrm{r}}H^\sigma(\Lambda)} = \left(\|{}_x\mathcal{D}_b^\sigma(\cdot)\|^2_{L^2(\Lambda)} + \|\cdot\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}}$, and $|\cdot|^*_{H^\sigma(\Lambda)} = |({}_a\mathcal{D}_x^\sigma(\cdot), {}_x\mathcal{D}_b^\sigma(\cdot))_\Lambda|^{\frac{1}{2}}$. From Lemma 5.2 in [17], we have

$$|\cdot|^*_{{}^{\mathrm{l}}H^\sigma(\Lambda)} \cong |\cdot|^{\frac{1}{2}}_{{}^{\mathrm{l}}H^\sigma(\Lambda)}\, |\cdot|^{\frac{1}{2}}_{{}^{\mathrm{r}}H^\sigma(\Lambda)} = \|{}_a\mathcal{D}_x^\sigma(\cdot)\|^{\frac{1}{2}}_{L^2(\Lambda)}\, \|{}_x\mathcal{D}_b^\sigma(\cdot)\|^{\frac{1}{2}}_{L^2(\Lambda)}, \tag{15}$$

where $|\cdot|_{{}^{\mathrm{l}}H^\sigma(\Lambda)} = \|{}_a\mathcal{D}_x^\sigma(\cdot)\|_{L^2(\Lambda)}$, and $|\cdot|_{{}^{\mathrm{r}}H^\sigma(\Lambda)} = \|{}_x\mathcal{D}_b^b(\cdot)\|_{L^2(\Lambda)}$. According to Lemma 3.1 in [51], the norms $\|\cdot\|_{{}^{\mathrm{l}}H^\sigma(\Lambda)}$ and $\|\cdot\|_{{}^{\mathrm{r}}H^\sigma(\Lambda)}$ are equivalent to $\|\cdot\|_{{}^{\mathrm{c}}H^\sigma(\Lambda)}$ in space $C_0^\infty(\Lambda)$, where

$$\|\cdot\|_{{}^{\mathrm{c}}H^\sigma(\Lambda)} = \left(\|{}_x\mathcal{D}_b^\sigma(\cdot)\|^2_{L^2(\Lambda)} + \|{}_a\mathcal{D}_x^\sigma(\cdot)\|^2_{L^2(\Lambda)} + \|\cdot\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}}. \tag{16}$$

In the usual Sobolev space, for $u \in H^\sigma(\Lambda)$, we define

$$|u|^*_{H^\sigma(\Lambda)} = |({}_a\mathcal{D}_x^\sigma u, {}_x\mathcal{D}_b^\sigma v)_\Lambda|^{\frac{1}{2}} + |({}_x\mathcal{D}_b^\sigma u, {}_a\mathcal{D}_x^\sigma v)_\Lambda|^{\frac{1}{2}}, \quad \forall v \in H^\sigma(\Lambda),$$

where we assume $\sup\limits_{u \in H^\sigma(\Lambda)} |({}_a\mathcal{D}_x^\sigma u, {}_x\mathcal{D}_b^\sigma v)_\Lambda|^{\frac{1}{2}} + |({}_x\mathcal{D}_b^\sigma u, {}_a\mathcal{D}_x^\sigma v)_\Lambda|^{\frac{1}{2}} > 0, \quad \forall v \in H^\sigma(\Lambda)$, which excludes the solutions to $|u|^*_{H^\sigma(\Lambda)} = 0$. Denoted by ${}^{\mathrm{l}}H_0^\sigma(\Lambda)$ and ${}^{\mathrm{r}}H_0^\sigma(\Lambda)$ are the closures of $C_0^\infty(\Lambda)$ with respect to the norms $\|\cdot\|_{{}^{\mathrm{l}}H^\sigma(\Lambda)}$ and $\|\cdot\|_{{}^{\mathrm{r}}H^\sigma(\Lambda)}$in Λ, respectively.

Recalling from [25], ${}^{\mathfrak{D}}H^\varphi(\mathbb{R})$ represents the distributed Sobolev space on $\mathbb{R}$, which is associated with the following norm:

$$\|\cdot\|_{\mathfrak{D}H^{\varphi}(\mathbb{R})}=\left(\int_{\tau^{\min}}^{\tau^{\max}}\varphi(\tau)\ \|(1+|\omega|^2)^{\frac{\tau}{2}}\mathcal{F}(\cdot)(\omega)\|^2_{L^2(\mathbb{R})}\ \mathrm{d}\tau\right)^{\frac{1}{2}},\tag{17}$$

where $0<\varphi(\tau)\in L^1([\tau^{\min},\tau^{\max}])$, $0<\tau^{\min}<\tau^{\max}<1$ $(1<\tau^{\min},\tau^{\max}\leqslant 2)$. Subsequently, we denote by ${}^{\mathfrak{D}}H^{\varphi}(I)$ the distributed Sobolev space on the bounded open interval $I=(0,T)$, which is defined as ${}^{\mathfrak{D}}H^{\varphi}(I)=\{v\in L^2(I)|\ \exists\tilde{v}\in{}^{\mathfrak{D}}H^{\varphi}(\mathbb{R})\ \text{s.t.}\ \tilde{v}|_I=v\}$ with the equivalent norms $\|\cdot\|_{\mathrm{l},\mathfrak{D}H^{\varphi}(I)}$ and $\|\cdot\|_{\mathrm{r},\mathfrak{D}H^{\varphi}(I)}$ in [25], where

$$\|\cdot\|_{\mathrm{l},\mathfrak{D}H^{\varphi}(I)}=\left(\|\cdot\|^2_{L^2(I)}+\int_{\tau^{\min}}^{\tau^{\max}}\varphi(\tau)\ \|{}_0\mathcal{D}_t^{\tau}(\cdot)\|^2_{L^2(I)}\ \mathrm{d}\tau\right)^{\frac{1}{2}}$$

and

$$\|\cdot\|_{\mathrm{r},\mathfrak{D}H^{\varphi}(I)}=\left(\|\cdot\|^2_{L^2(I)}+\int_{\tau^{\min}}^{\tau^{\max}}\varphi(\tau)\ \|{}_t\mathcal{D}_T^{\tau}(\cdot)\|^2_{L^2(I)}\ \mathrm{d}\tau\right)^{\frac{1}{2}}.$$

In each realization of a physical process (e.g., sub- or super-diffusion), the distribution function $\varphi(\tau)$ can be obtained from experimental observations, while the theoretical setting of the problem remains invariant. More importantly, choice of the distributed Sobolev space and the associated norms provides a sharper estimate for the accuracy of the proposed PG method.

Let $\Lambda_1=(a_1,b_1)$ and $\Lambda_i=(a_i,b_i)\times\Lambda_{i-1}$ for $i=2,\cdots,d$. We define $\mathcal{X}_1={}^{\mathfrak{D}}H^{\rho_1}(\Lambda_1)$ with the associated norm $\|\cdot\|_{\mathfrak{D}H^{\rho_1}(\Lambda_1)}$, where

$$\|\cdot\|_{\mathfrak{D}H^{\rho_1}(\Lambda_1)}=\left(\|\cdot\|^2_{L^2(I)}+\int_{\nu_1^{\min}}^{\nu_1^{\max}}\rho_1(\nu_1)\left(\|{}_{a_1}\mathcal{D}_{x_1}^{\nu_1}(\cdot)\|^2_{L^2(\Lambda_1)}+\|{}_{x_1}\mathcal{D}_{b_1}^{\nu_1}(\cdot)\|^2_{L^2(\Lambda_1)}\right)\mathrm{d}\nu_1\right)^{\frac{1}{2}}.\tag{18}$$

Subsequently, we construct $\mathcal{X}_d$, such that

$$\begin{aligned}\mathcal{X}_2&={}^{\mathfrak{D}}H^{\rho_2}\Big((a_2,b_2);L^2(\Lambda_1)\Big)\cap L^2((a_2,b_2);\mathcal{X}_1),\\&\vdots\\\mathcal{X}_d&={}^{\mathfrak{D}}H^{\rho_d}\Big((a_d,b_d);L^2(\Lambda_{d-1})\Big)\cap L^2((a_d,b_d);\mathcal{X}_{d-1})\end{aligned}\tag{19}$$

associated with the norm

$$\|\cdot\|_{\mathcal{X}_d}=\left\{\|\cdot\|^2_{\mathfrak{D}H^{\rho_d}\left((a_d,b_d);L^2(\Lambda_{d-1})\right)}+\|\cdot\|^2_{L^2\left((a_d,b_d);\mathcal{X}_{d-1}\right)}\right\}^{\frac{1}{2}}.\tag{20}$$

Lemma 3.1 *Let $\nu_i>0$ and $\nu_i\neq n-\frac{1}{2}$ for $i=1,\cdots,d$. Then,*

$$\|\cdot\|_{\mathcal{X}_d}\cong\left\{\sum_{i=1}^{d}\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left(\|{}_{x_i}\mathcal{D}_{b_i}^{\nu_i}(\cdot)\|^2_{L^2(\Lambda_d)}+\|{}_{a_i}\mathcal{D}_{x_i}^{\nu_i}(\cdot)\|^2_{L^2(\Lambda_d)}\right)\mathrm{d}\nu_i+\|\cdot\|^2_{L^2(\Lambda_d)}\right\}^{\frac{1}{2}}.\tag{21}$$

Proof Considering (18), $\mathcal{X}_1$ is endowed with $\|\cdot\|_{\mathcal{X}_1} \cong \|\cdot\|_{{}^{\mathfrak{D}}H^{\rho_1}(\Lambda_1)}$. $\mathcal{X}_2$ is associated with $\|\cdot\|_{\mathcal{X}_2} = \{\|\cdot\|^2_{{}^{c}H^{\rho_2}((a_2,b_2);L^2(\Lambda_1))} + \|\cdot\|^2_{L^2((a_2,b_2);\mathcal{X}_1)}\}^{\frac{1}{2}}$, which is proved to be

$$\begin{aligned}
&\|u\|^2_{{}^{\mathfrak{D}}H^{\rho_2}\left((a_2,b_2);L^2(\Lambda_1)\right)}\\
&= \int_{\nu_2^{\min}}^{\nu_2^{\max}} \rho_2(\nu_2) \int_{a_1}^{b_1} \Big(\int_{a_2}^{b_2} |{}_{a_2}\mathcal{D}_{x_2}^{\nu_2} u|^2 \, \mathrm{d}x_2 + \int_{a_2}^{b_2} |{}_{x_2}\mathcal{D}_{b_2}^{\nu_2} u|^2 \, \mathrm{d}x_2\\
&\quad + \int_{a_2}^{b_2} |u|^2 \, \mathrm{d}x_2 \Big) \mathrm{d}x_1 \, \mathrm{d}\nu_2\\
&= \int_{\nu_2^{\min}}^{\nu_2^{\max}} \rho_2(\nu_2) \Big(\|{}_{x_2}\mathcal{D}_{b_2}^{\nu_2}(u)\|^2_{L^2(\Lambda_2)} + \|{}_{a_2}\mathcal{D}_{x_2}^{\nu_2}(u)\|^2_{L^2(\Lambda_2)} \Big) \mathrm{d}\nu_2 + \|u\|^2_{L^2(\Lambda_2)}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\|u\|^2_{L^2\left((a_2,b_2);\mathcal{X}_1\right)}\\
&= \int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1) \int_{a_2}^{b_2} \Big(\int_{a_1}^{b_1} |{}_{a_1}\mathcal{D}_{x_1}^{\nu_1} u|^2 \, \mathrm{d}x_1 + \int_{a_1}^{b_1} |{}_{x_1}\mathcal{D}_{b_1}^{\nu_1} u|^2 \, \mathrm{d}x_1\\
&\quad + \int_{a_1}^{b_1} |u|^2 \, \mathrm{d}x_1 \Big) \mathrm{d}x_2 \, \mathrm{d}\nu_1\\
&= \int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1) \Big(\|{}_{x_1}\mathcal{D}_{b_1}^{\nu_1} u\|^2_{L^2(\Lambda_2)} + \|{}_{a_1}\mathcal{D}_{x_1}^{\nu_1} u\|^2_{L^2(\Lambda_2)} \Big) \mathrm{d}\nu_1 + \|u\|^2_{L^2(\Lambda_2)}.
\end{aligned} \tag{23}$$

Next, providing that

$$\begin{aligned}
\|\cdot\|_{\mathcal{X}_{d-1}} \cong \Bigg\{ \sum_{i=1}^{d-1} \int_{\nu_i^{\min}}^{\nu_i^{\max}} \rho_i(\nu_i) \Big(&\|{}_{x_i}\mathcal{D}_{b_i}^{\nu_i}(\cdot)\|^2_{L^2(\Lambda_{d-1})}\\
&+ \|{}_{a_i}\mathcal{D}_{x_i}^{\nu_i}(\cdot)\|^2_{L^2(\Lambda_{d-1})} \Big) \mathrm{d}\nu_i + \|\cdot\|^2_{L^2(\Lambda_{d-1})} \Bigg\}^{\frac{1}{2}},
\end{aligned}$$

the inductive step is attained according to

$$\begin{aligned}
&\|u\|^2_{{}^{\mathfrak{D}}H^{\rho_d}\left((a_d,b_d);L^2(\Lambda_{d-1})\right)}\\
&= \int_{\Lambda_{d-1}} \Big(\int_{a_d}^{b_d} |u|^2 \, \mathrm{d}x_d + \int_{a_d}^{b_d} \int_{\nu_d^{\min}}^{\nu_d^{\max}} \rho_d(\nu_d) \big(|{}_{a_d}\mathcal{D}_{x_d}^{\nu_d} u|^2 + |{}_{x_d}\mathcal{D}_{b_d}^{\nu_d} u|^2 \big) \mathrm{d}\nu_d \, \mathrm{d}x_d \Big) \mathrm{d}\Lambda_{d-1}\\
&= \int_{\nu_d^{\min}}^{\nu_d^{\max}} \rho_d(\nu_d) \Big(\|{}_{x_d}\mathcal{D}_{b_d}^{\nu_d}(u)\|^2_{L^2(\Lambda_d)} + \|{}_{a_d}\mathcal{D}_{x_d}^{\nu_d}(u)\|^2_{L^2(\Lambda_d)} \Big) \mathrm{d}\nu_d + \|u\|^2_{L^2(\Lambda_d)},
\end{aligned} \tag{24}$$

and

$$
\begin{aligned}
&\|u\|^2_{L^2\left((a_d,b_d);\mathcal{X}_{d-1}\right)}\\
&=\int_{a_d}^{b_d}\int_{\Lambda_{d-1}}\Big(\sum_{i=1}^{d-1}\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left(|{}_{a_i}\mathcal{D}_{x_i}^{\nu_i}u|^2+|{}_{x_i}\mathcal{D}_{b_i}^{\nu_i}u|^2\right)\mathrm{d}\nu_i\Big)\mathrm{d}\Lambda_{d-1}\,\mathrm{d}x_d\\
&\quad+\int_{a_d}^{b_d}\int_{\Lambda_{d-1}}|u|^2\mathrm{d}\Lambda_{d-1}\,\mathrm{d}x_d\\
&=\sum_{i=1}^{d-1}\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left(\|{}_{x_i}\mathcal{D}_{b_i}^{\nu_i}u\|^2_{L^2(\Lambda_d)}+\|{}_{a_i}\mathcal{D}_{x_i}^{\nu_i}u\|^2_{L^2(\Lambda_d)}\right)\mathrm{d}\nu_i+\|u\|^2_{L^2(\Lambda_d)}.
\end{aligned}
\tag{25}
$$

Therefore, (21) arises from (22), (23), (24), and (25) by induction, and the proof is complete.

The following assumptions allow us to prove the uniqueness of the bilinear form by excluding the solutions to $|\left({}_{a_i}\mathcal{D}_{x_i}^{\nu_i}u, {}_{x_i}\mathcal{D}_{b_i}^{\nu_i}v\right)_{\Lambda_d}|+|\left({}_{x_i}\mathcal{D}_{b_i}^{\nu_i}u, {}_{a_i}\mathcal{D}_{x_i}^{\nu_i}v\right)_{\Lambda_d}|=0$ for $i=1,\cdots,d$ and $|({}_0\mathcal{D}_t^{\tau}u, {}_t\mathcal{D}_T^{\tau}v)_\Omega|=0$.

Assumption 1 For $u\in\mathcal{X}_d$,

$$
\sup_{u\in\mathcal{X}_d}\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left(|\left({}_{a_i}\mathcal{D}_{x_i}^{\nu_i}u, {}_{x_i}\mathcal{D}_{b_i}^{\nu_i}v\right)_{\Lambda_d}|+|\left({}_{x_i}\mathcal{D}_{b_i}^{\nu_i}u, {}_{a_i}\mathcal{D}_{x_i}^{\nu_i}v\right)_{\Lambda_d}|\right)\mathrm{d}\nu_i>0,\quad\forall v\in\mathcal{X}_d
$$

when $i=1,\cdots,d$, and $\Lambda_d^i=\prod_{\substack{j=1\\ j\neq i}}^{d}(a_j,b_j)$.

Assumption 2 For $u\in{}^{\mathrm{l},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))$, $\sup_{0\neq u\in{}^{\mathrm{l},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}\int_{\tau^{\min}}^{\tau^{\max}}\varphi(\tau)\,|({}_0\mathcal{D}_t^{\tau}u, {}_t\mathcal{D}_T^{\tau}v)_\Omega|\mathrm{d}\tau>0$, $\forall v\in{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))$.

In Lemma 3.3 in [51], it is shown that if $1<2\nu_i<2$ for $i=1,\cdots,d$ and $u,v\in\mathcal{X}_d$, then $\left({}_{x_i}\mathcal{D}_{b_i}^{2\nu_i}u,v\right)_{\Lambda_d}=\left({}_{x_i}\mathcal{D}_{b_i}^{\nu_i}u, {}_{a_i}\mathcal{D}_{x_i}^{\nu_i}v\right)_{\Lambda_d}$, and $\left({}_{a_i}\mathcal{D}_{x_i}^{2\nu_i}u,v\right)_{\Lambda_d}=\left({}_{a_i}\mathcal{D}_{x_i}^{\nu_i}u, {}_{x_i}\mathcal{D}_{b_i}^{\nu_i}v\right)_{\Lambda_d}$. Consequently, we derive,

$$
\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left({}_{x_i}\mathcal{D}_{b_i}^{2\nu_i}u,v\right)_{\Lambda_d}\mathrm{d}\nu_i=\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left({}_{x_i}\mathcal{D}_{b_i}^{\nu_i}u, {}_{a_i}\mathcal{D}_{x_i}^{\nu_i}v\right)_{\Lambda_d}\mathrm{d}\nu_i \tag{26}
$$

and

$$
\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left({}_{a_i}\mathcal{D}_{x_i}^{2\nu_i}u,v\right)_{\Lambda_d}\mathrm{d}\nu_i=\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left({}_{a_i}\mathcal{D}_{x_i}^{\nu_i}u, {}_{x_i}\mathcal{D}_{b_i}^{\nu_i}v\right)_{\Lambda_d}\mathrm{d}\nu_i. \tag{27}
$$

Additionally, in the light of Lemma 3.2 in [51], we have,

$$
\begin{aligned}
&\int_{\nu_i^{\min}}^{\nu_i^{\max}}\rho_i(\nu_i)\left(|\left({}_{a_i}\mathcal{D}_{x_i}^{\nu_i}u, {}_{x_i}\mathcal{D}_{b_i}^{\nu_i}v\right)_{\Lambda_d}|+|\left({}_{x_i}\mathcal{D}_{b_i}^{\nu_i}u, {}_{a_i}\mathcal{D}_{x_i}^{\nu_i}v\right)_{\Lambda_d}|\right)\mathrm{d}\nu_i\\
&\cong\ |u|_{\mathfrak{D}H^{\rho_i}\left((a_i,b_i);L^2(\Lambda_d^i)\right)}\ |v|_{\mathfrak{D}H^{\rho_i}\left((a_i,b_i);L^2(\Lambda_d^i)\right)}
\end{aligned}
\tag{28}
$$

for $i=1,\cdots,d$, where Assumption 1 holds.

Next, we study the property of the fractional time-derivative in the following lemmas.

Lemma 3.2 *If* $0 < 2\tau^{\min} < 2\tau^{\max} < 1$ $(1 < 2\tau^{\min} < 2\tau^{\max} < 2)$ *and* $u, v \in {}^{1,\mathfrak{D}}H^{\varphi}(I)$, *when* $u|_{t=0}(= \frac{\mathrm{d}u}{\mathrm{d}t}|_{t=0}) = 0$, *then*

$$\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left({}_0\mathcal{D}_t^{2\tau} u, v\right)_I \mathrm{d}\tau = \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left({}_0\mathcal{D}_t^{\tau} u, {}_t\mathcal{D}_T^{\tau} v\right)_I \mathrm{d}\tau, \tag{29}$$

where $I = (0, T)$, $0 < \varphi(\tau) \in L^1\left([\tau^{\min}, \tau^{\max}]\right)$.

Proof It follows from [25] that for $u, v \in H^{\tau}(I)$, when $u|_{t=0}(= \frac{\mathrm{d}u}{\mathrm{d}t}|_{t=0}) = 0$ and $v|_{t=T}(= \frac{\mathrm{d}v}{\mathrm{d}t}|_{t=T}) = 0$, we have

$$\left({}_0\mathcal{D}_t^{2\tau} u, v\right)_I = \left({}_0\mathcal{D}_t^{\tau} u, {}_t\mathcal{D}_T^{\tau} v\right)_I. \tag{30}$$

Then, (29) arises from (30).

Let $0 < 2\tau^{\min} < 2\tau^{\max} < 1$ $(1 < 2\tau^{\min} < 2\tau^{\max} \leqslant 2)$, and $\Omega = I \times \Lambda_d$, where $I = (0, T)$ and $\Lambda_d = \prod_{i=1}^{d} (a_i, b_i)$. We define

$$\begin{aligned} &{}^{1,\mathfrak{D}}H^{\varphi}\left(I; L^2(\Lambda_d)\right) \\ &:= \left\{ u \mid \|u(t,\cdot)\|_{L^2(\Lambda_d)} \in {}^{1,\mathfrak{D}}H^{\varphi}(I), u|_{t=0}\left(= \frac{\mathrm{d}u}{\mathrm{d}t}|_{t=0}\right) \right. \\ &\left. = u|_{x_i=a_i} = u|_{x_i=b_i} = 0,\ i = 1, \cdots, d \right\}, \end{aligned} \tag{31}$$

which is endowed with the norm $\|\cdot\|_{{}^{1,\mathfrak{D}}H^{\varphi}\left(I; L^2(\Lambda_d)\right)}$, where we have

$$\begin{aligned} \|u\|_{{}^{1,\mathfrak{D}}H^{\varphi}(I; L^2(\Lambda_d))} &= \left\| \|u(t,\cdot)\|_{L^2(\Lambda_d)} \right\|_{{}^{1,\mathfrak{D}}H^{\varphi}(I)} \\ &= \left(\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left\|{}_0\mathcal{D}_t^{\tau}(u)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau + \|u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \end{aligned} \tag{32}$$

Similarly, we define

$$\begin{aligned} &{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}\left(I; L^2(\Lambda_d)\right) \\ &:= \left\{ v \mid \|v(t,\cdot)\|_{L^2(\Lambda_d)} \in {}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I), v|_{t=T}\left(= \frac{\mathrm{d}v}{\mathrm{d}t}|_{t=0}\right) \right. \\ &\left. = v|_{x_i=a_i} = v|_{x_i=b_i} = 0,\ i = 1, \cdots, d \right\}, \end{aligned} \tag{33}$$

which is equipped with the norm $\|\cdot\|_{{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I; L^2(\Lambda_d))}$. Following (32),

$$\begin{aligned} \|v\|_{{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I; L^2(\Lambda_d))} &= \left\| \|v(t,\cdot)\|_{L^2(\Lambda_d)} \right\|_{{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I)} \\ &= \left(\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left\|{}_t\mathcal{D}_T^{\tau}(v)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau + \|v\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \end{aligned} \tag{34}$$

Lemma 3.3 *For* $u \in {}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))$ *and* $0 < 2\tau^{\min} < 2\tau^{\max} < 1$ $(1 < 2\tau^{\min} < 2\tau^{\max} < 2)$, $\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\,|({}_0\mathcal{D}_t^{\tau}u, {}_t\mathcal{D}_T^{\tau}v)_{\Omega}|\,\mathrm{d}\tau \leqslant \|u\|_{{}^{\mathrm{l},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}\,\|v\|_{{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}, \forall v \in {}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))$.

Proof From Lemma 3.6 in [51], we have

$$|({}_0\mathcal{D}_t^{\tau}u, {}_t\mathcal{D}_T^{\tau}v)_{\Omega}| \leqslant \Big(\|{}_0\mathcal{D}_t^{\tau}u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \Big(\|{}_t\mathcal{D}_T^{\tau}v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}.$$

Followingly, by the Hölder inequality,

$$\begin{aligned}
&\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\,|({}_0\mathcal{D}_t^{\tau}u, {}_t\mathcal{D}_T^{\tau}v)_{\Omega}|\mathrm{d}\tau \\
&= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \int_{\Lambda_d}\int_0^T |{}_0\mathcal{D}_t^{\tau}u|\,|{}_t\mathcal{D}_T^{\tau}v|\mathrm{d}t\,\mathrm{d}\Lambda_d\,\mathrm{d}\tau \\
&\leqslant \Big(\int_{\tau^{\min}}^{\tau^{\max}}\int_{\Lambda_d}\int_0^T \varphi(\tau)\,|{}_0\mathcal{D}_t^{\tau}u|^2\,\mathrm{d}t\mathrm{d}\Lambda_d\Big)^{\frac{1}{2}} \\
&\quad\cdot \Big(\int_{\tau^{\min}}^{\tau^{\max}}\int_{\Lambda_d}\int_0^T \varphi(\tau)\,|{}_t\mathcal{D}_T^{\tau}v|^2\,\mathrm{d}t\mathrm{d}\Lambda_d\Big)^{\frac{1}{2}} \\
&= \Big(\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\,\|{}_0\mathcal{D}_t^{\tau}u\|^2_{L^2(\Omega)}\mathrm{d}\tau + \|u\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\
&\quad\cdot \Big(\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\,\|{}_t\mathcal{D}_T^{\tau}v\|^2_{L^2(\Omega)}\mathrm{d}\tau + \|v\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\
&= \|u\|_{{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}\,\|v\|_{{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}.
\end{aligned} \tag{35}$$

Lemma 3.4 *For any* $u \in {}^{\mathrm{l},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))$ *and* $0 < 2\tau^{\min} < 2\tau^{\max} < 1$ $(1 < 2\tau^{\min} < 2\tau^{\max} \leqslant 2)$, *there exists a constant* $c > 0$ *and independent of* u, *such that*

$$\sup_{0 \neq v \in {}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))} \frac{\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\,|({}_0\mathcal{D}_t^{\tau}u, {}_t\mathcal{D}_T^{\tau}v)_{\Omega}|\mathrm{d}\tau}{|v|_{{}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}} \geqslant c|u|_{{}^{\mathrm{l},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))} \tag{36}$$

under Assumption 2.

Proof Following Lemma 2.4 in [13] and Lemma 3.7 in [51], for any $u \in {}^{\mathrm{l},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))$, let $\mathcal{V}_u = H(t-T)\left(u - u|_{t=T}\right)$, assuming that $\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\,|({}_0\mathcal{D}_t^{\tau}u, {}_t\mathcal{D}_T^{\tau}u|_{t=T})_{\Omega}| > 0$, where $H(t)$ is the Heaviside function. Evidently, $\mathcal{V}_u \in {}^{\mathrm{r},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))$. From the Hölder inequality, we obtain

$$
\begin{aligned}
&\|\mathcal{V}_u\|^2_{{}^{r,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}\\
&= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, \|{}_t\mathcal{D}_T^{\tau}\big(H(t-T)\left(u - u|_{t=T}\right)\big)\|^2_{L^2(\Omega)} \mathrm{d}\tau\\
&= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left\| {}_t^{\mathrm{RL}}\mathcal{I}_T^{1-\tau} \frac{\mathrm{d}}{\mathrm{d}t}\big(H(t-T)\left(u - u|_{t=T}\right)\big)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau\\
&= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left\| {}_t^{\mathrm{RL}}\mathcal{I}_T^{1-\tau} \left(\frac{\mathrm{d}\,H(t-T)}{\mathrm{d}t}\left(u - u|_{t=T}\right)\right.\right.\\
&\quad \left.\left. + H(t-T)\frac{\mathrm{d}\left(u - u|_{t=T}\right)}{\mathrm{d}t}\right)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau\\
&= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left\| {}_t^{\mathrm{RL}}\mathcal{I}_T^{1-\tau} \left(H(t-T)\frac{\mathrm{d}\left(u - u|_{t=T}\right)}{\mathrm{d}t}\right)\right\|^2_{L^2(\Omega)} \mathrm{d}\tau\\
&= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, \|{}_t\mathcal{D}_T^{\tau}\, u\|^2_{L^2(\Omega)} \mathrm{d}\tau.
\end{aligned}
\tag{37}
$$

Regarding (12) in [51], $\|\mathcal{V}_u\|^2_{{}^{r,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))} \cong \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, \|{}_0\mathcal{D}_t^{\tau}\, u\|^2_{L^2(\Omega)}\mathrm{d}\tau = \|u\|^2_{{}^{l,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}$. Hence, $\|{}_t\mathcal{D}_T^{\tau}\mathcal{V}_u\|^2_{L^2(\Omega)} \cong \|{}_0\mathcal{D}_t^{\tau} u\|^2_{L^2(\Omega)}$. Therefore,

$$
\begin{aligned}
&\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, |({}_0\mathcal{D}_t^{\tau} u, {}_t\mathcal{D}_T^{\tau}\mathcal{V}_u)_{\Omega}|\, \mathrm{d}\tau\\
&= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \int_{\Lambda_d}\int_0^T |{}_0\mathcal{D}_t^{\tau} u|\, |{}_t\mathcal{D}_T^{\tau}\mathcal{V}_u| \mathrm{d}t\, \mathrm{d}\Lambda_d\, \mathrm{d}\tau\\
&\geqslant \tilde{\beta} \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \int_{\Lambda_d}\int_0^T |{}_0\mathcal{D}_t^{\tau} u|^2 \mathrm{d}t\, \mathrm{d}\Lambda_d\, \mathrm{d}\tau\\
&= |u|^2_{{}^{l,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))},
\end{aligned}
\tag{38}
$$

where $\tilde{\beta} > 0$ and independent of u. Considering (37) and (38), we obtain

$$
\begin{aligned}
\sup_{0\neq v \in {}^{r,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))} \frac{\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, |({}_0\mathcal{D}_t^{\tau} u, {}_t\mathcal{D}_T^{\tau} v)_{\Omega}| \mathrm{d}\tau}{|v|_{{}^{r,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}} &\geqslant \frac{\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, |({}_0\mathcal{D}_t^{\tau} u, {}_t\mathcal{D}_T^{\tau}\mathcal{V}_u)_{\Omega}| \mathrm{d}\tau}{|\mathcal{V}_u|_{{}^{r,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}}\\
&\geqslant \tilde{\beta}\, |u|_{{}^{l,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))}.
\end{aligned}
\tag{39}
$$

Lemma 3.5 *If* $0 < 2\tau^{\min} < 2\tau^{\max} < 1\,(1 < 2\tau^{\min} < 2\tau^{\max} \leqslant 2)$ *and* $u, v \in {}^{l,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))$, *then*

$$
\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left({}_0\mathcal{D}_t^{2\tau} u, v\right)_{\Omega} \mathrm{d}\tau = \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \left({}_0\mathcal{D}_t^{\tau} u, {}_t\mathcal{D}_T^{\tau} v\right)_{\Omega} \mathrm{d}\tau,
\tag{40}
$$

where $0 < \varphi(\tau) \in L^1\Big([\tau^{\min}, \tau^{\max}]\Big)$.

Proof By Lemma 3.2,

$$\begin{aligned}\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\left({}_0\mathcal{D}_t^{2\tau}u, v\right)_\Omega \mathrm{d}\tau &= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\int_{\Lambda_d}\int_0^T |{}_0\mathcal{D}_t^{2\tau}u|\,|v|\,\mathrm{d}t\,\mathrm{d}\Lambda_d\,\mathrm{d}\tau \\ &= \int_{\Lambda_d}\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\int_0^T |{}_0\mathcal{D}_t^{\tau}u|\,|{}_t\mathcal{D}_T^{\tau}v|\,\mathrm{d}t\,\mathrm{d}\tau\,\mathrm{d}\Lambda_d \\ &= \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\left({}_0\mathcal{D}_t^{\tau}u, {}_t\mathcal{D}_T^{\tau}v\right)_\Omega \mathrm{d}\tau.\end{aligned} \tag{41}$$

3.2 Solution and Test Function Spaces

Take $0 < 2\tau^{\min} < 2\tau^{\max} < 1$ $(1 < 2\tau^{\min} < 2\tau^{\max} \leqslant 2)$ and $1 < 2\nu_i^{\min} < 2\nu_i^{\max} \leqslant 2$ for $i = 1, \cdots, d$. We define the solution space

$$\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega) := {}^{\mathrm{l},\mathfrak{D}}H^{\varphi}\Big(I;L^2(\Lambda_d)\Big) \cap L^2(I;\mathcal{X}_d) \tag{42}$$

associated with the norm

$$\|u\|_{\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} = \left\{ \|u\|^2_{{}^{\mathrm{l},\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))} + \|u\|^2_{L^2(I;\mathcal{X}_d)} \right\}^{\frac{1}{2}}. \tag{43}$$

Considering Lemma 3.1,

$$\begin{aligned}\|u\|_{L^2(I;\mathcal{X}_d)} &= \Big\| \, \|u(t,.)\|_{\mathcal{X}_d} \Big\|_{L^2(I)} \\ &= \left\{ \sum_{i=1}^{d} \int_{\nu_i^{\min}}^{\nu_i^{\max}} \rho_i(\nu_i)\Big(\|{}_{x_i}\mathcal{D}_{b_i}^{\nu_i}(u)\|^2_{L^2(\Omega)} + \|{}_{a_i}\mathcal{D}_{x_i}^{\nu_i}(u)\|^2_{L^2(\Omega)} \Big)\mathrm{d}\nu_i + \|u\|^2_{L^2(\Lambda_d)} \right\}^{\frac{1}{2}}.\end{aligned} \tag{44}$$

Therefore, from (32) and (44),

$$\begin{aligned}\|u\|_{\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} = &\Bigg\{ \|u\|^2_{L^2(\Omega)} + \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, \|{}_0\mathcal{D}_t^{\tau}(u)\|^2_{L^2(\Omega)}\mathrm{d}\tau \\ &+ \sum_{i=1}^{d}\int_{\nu_i^{\min}}^{\nu_i^{\max}} \rho_i(\nu_i)\Big(\|{}_{x_i}\mathcal{D}_{b_i}^{\nu_i}(u)\|^2_{L^2(\Omega)} + \|{}_{a_i}\mathcal{D}_{x_i}^{\nu_i}(u)\|^2_{L^2(\Omega)} \Big)\mathrm{d}\nu_i \Bigg\}^{\frac{1}{2}}.\end{aligned} \tag{45}$$

Similarly, we define the test space

$$\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega) := {}^{\mathrm{r},\mathfrak{D}}H^{\varphi}\Big(I;L^2(\Lambda_d)\Big) \cap L^2(I;\mathcal{X}_d) \tag{46}$$

equipped with the norm

$$\begin{aligned}\|v\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} &= \left\{ \|v\|^2_{{}^{\tau}H^{\varphi}(I;L^2(\Lambda_d))} + \|v\|^2_{L^2(I;\mathcal{X}_d)} \right\}^{\frac{1}{2}} \\ &= \left\{ \|v\|^2_{L^2(\Omega)} + \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, \|{}_t\mathcal{D}_T^{\tau}(v)\|^2_{L^2(\Omega)} \mathrm{d}\tau \right. \\ &\quad \left. + \sum_{i=1}^{d} \int_{\nu_i^{\min}}^{\nu_i^{\max}} \rho_i(\nu_i) \left(\|{}_{x_i}\mathcal{D}_{b_i}^{\nu_i}(v)\|^2_{L^2(\Omega)} + \|{}_{a_i}\mathcal{D}_{x_i}^{\nu_i}(v)\|^2_{L^2(\Omega)} \right) \mathrm{d}\nu_i \right\}^{\frac{1}{2}} \end{aligned} \tag{47}$$

by Lemmas 3.1 and 3.2. Take $\Omega = I \times \Lambda_d$ for a positive integer d. The PG spectral method reads as: find $u \in \mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$, such that

$$a(u, v) = l(v), \quad \forall v \in \mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega), \tag{48}$$

where the functional $l(v) = (f, v)_\Omega$ and

$$\begin{aligned} a(u,v) = &\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, ({}_0\mathcal{D}_t^{\tau}\, u, {}_t\mathcal{D}_T^{\tau}\, v)_\Omega \mathrm{d}\tau \\ &+ \sum_{i=1}^{d} \int_{\mu_i^{\min}}^{\mu_i^{\max}} \varrho_i(\mu_i) \left(c_{l_i} ({}_{a_i}\mathcal{D}_{x_i}^{\mu_i}\, u,\ {}_{x_i}\mathcal{D}_{b_i}^{\mu_i}\, v)_\Omega + c_{r_i} ({}_{a_i}\mathcal{D}_{x_i}^{\mu_i}\, v,\ {}_{x_i}\mathcal{D}_{b_i}^{\mu_i}\, u)_\Omega \right) \mathrm{d}\mu_i \\ &- \sum_{j=1}^{d} \int_{\nu_j^{\min}}^{\nu_j^{\max}} \rho_j(\nu_j) \left(k_{l_j} ({}_{a_j}\mathcal{D}_{x_j}^{\nu_j}\, u,\ {}_{x_j}\mathcal{D}_{b_j}^{\nu_j}\, v)_\Omega + k_{r_j} ({}_{a_j}\mathcal{D}_{x_j}^{\nu_j}\, v,\ {}_{x_j}\mathcal{D}_{b_j}^{\nu_j}\, u)_\Omega \right) \mathrm{d}\nu_j \\ &+ \gamma(u, v)_\Omega \end{aligned} \tag{49}$$

following (26), (27) and Lemma 3.5 and $\gamma, c_{l_i}, c_{r_i}, \kappa_{l_i}$, and κ_{r_i} are all constant. Besides, $0 < 2\tau^{\min} < 2\tau^{\max} < 1$ $(1 < 2\tau^{\min} < 2\tau^{\max} \leqslant 2)$, $0 < 2\mu_i^{\min} < 2\mu_i^{\max} < 1$, and $1 < 2\nu_j^{\min} < 2\nu_j^{\max} \leqslant 2$ for $i, j = 1, 2, \cdots, d$.

Remark 1 In the case $\tau < \frac{1}{2}$, additional regularity assumptions are required to ensure equivalence between the weak and strong formulations, see [23] for more details.

U_N and V_N are chosen as the finite-dimensional subspaces of $\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$ and $\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$, respectively. Then, the PG scheme reads as: find $u_N \in U_N$, such that

$$a(u_N, v_N) = l(v_N), \quad \forall v \in V_N, \tag{50}$$

where

$$\begin{aligned} a(u_N, v_N) = &\int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, ({}_0\mathcal{D}_t^{\tau}\, u_N, {}_t\mathcal{D}_T^{\tau}\, v_N)_\Omega \mathrm{d}\tau \\ &+ \sum_{i=1}^{d} \int_{\mu_i^{\min}}^{\mu_i^{\max}} \varrho_i(\mu_i) \left[c_{l_i} ({}_{a_i}\mathcal{D}_{x_i}^{\mu_i}\, u_N,\ {}_{x_i}\mathcal{D}_{b_i}^{\mu_i}\, v_N)_\Omega + c_{r_i} ({}_{a_i}\mathcal{D}_{x_i}^{\mu_i}\, u_N,\ {}_{x_i}\mathcal{D}_{b_i}^{\mu_i}\, v_N)_\Omega \right] \mathrm{d}\mu_i \\ &- \sum_{j=1}^{d} \int_{\nu_j^{\min}}^{\nu_j^{\max}} \rho_j(\nu_j) \left[k_{l_j} ({}_{a_j}\mathcal{D}_{x_j}^{\nu_j}\, u_N,\ {}_{x_j}\mathcal{D}_{b_j}^{\nu_j}\, v_N)_\Omega + k_{r_j} ({}_{a_j}\mathcal{D}_{x_j}^{\nu_j}\, u_N,\ {}_{x_j}\mathcal{D}_{b_j}^{\nu_j}\, v_N)_\Omega \right] \mathrm{d}\nu_j \\ &+ \gamma(u_N, v_N)_\Omega. \end{aligned} \tag{51}$$

Representing u_N as a linear combination of elements in U_N, the finite-dimensional problem (51) leads to a linear system, known as the Lyapunov system, introduced in Sect. 4.

3.3 Well-Posedness Analysis

The following assumption permits us to prove the uniqueness of the weak form in (48) in Theorem 3.8.

Assumption 3 For all $v \in \mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$,

$$\sup_{u\in\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, |({}_0\mathcal{D}_t^{\tau}\, u, {}_t\mathcal{D}_T^{\tau}\, v)_\Omega| \mathrm{d}\tau \;>\; 0,$$

$$\sup_{u\in\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \int_{\nu_j^{\min}}^{\nu_j^{\max}} \rho_j(\nu_j) \left(|({}_{a_j}\mathcal{D}_{x_j}^{\nu_j}\, u,\ {}_{x_j}\mathcal{D}_{b_j}^{\nu_j}\, v)_\Omega| + |({}_{x_j}\mathcal{D}_{b_j}^{\nu_j}\, u,\ {}_{a_j}\mathcal{D}_{x_j}^{\nu_j}\, v)_\Omega| \right) \mathrm{d}\nu_j \;>\; 0,$$

$$\sup_{u\in\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} |(u,v)_\Omega| \;>\; 0,$$

when $j = 1, \cdots, d$.

Lemma 3.6 (Continuity) *Let Assumption 3 hold. The bilinear form in (49) is continuous, i.e., for $u \in \mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$,*

$$\exists\beta > 0,\quad |a(u,v)| \leqslant \beta\, \|u\|_{\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \|v\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}, \quad \forall v \in \mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega). \tag{52}$$

Proof It follows from (28) and Lemma 3.3.

Theorem 3.7 *Let Assumption 3 holds. The inf-sup condition of the bilinear form (49) for any $d \geqslant 1$ holds with $\beta > 0$, that is,*

$$\inf_{0\neq u\in\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \ \sup_{0\neq v\in\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \frac{|a(u,v)|}{\|v\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \|u\|_{\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}} \geqslant \beta > 0, \tag{53}$$

where $\Omega = I \times \Lambda_d$.

Proof For $u \in \mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$ and $v \in \mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$ under Assumption 3,

$$\begin{aligned} |a(u,v)| \cong & |(u,v)_\Omega| + \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, |({}_0\mathcal{D}_t^{\tau}\, u, {}_t\mathcal{D}_T^{\tau}\, v)_\Omega| \mathrm{d}\tau \\ & + \sum_{i=1}^{d} \int_{\mu_i^{\min}}^{\mu_i^{\max}} \rho_i(\mu_i) \left(|({}_{a_i}\mathcal{D}_{x_i}^{\mu_i}\, u,\ {}_{x_i}\mathcal{D}_{b_i}^{\mu_i}\, v)_\Omega| + |({}_{x_i}\mathcal{D}_{a_i}^{\mu_i}\, u,\ {}_{a_i}\mathcal{D}_{x_i}^{\mu_i}\, v)_\Omega| \right) \mathrm{d}\mu_i \\ & + \sum_{j=1}^{d} \int_{\nu_j^{\min}}^{\nu_j^{\max}} \rho_j(\nu_j) \left(|({}_{a_j}\mathcal{D}_{x_j}^{\nu_j}\, u,\ {}_{x_j}\mathcal{D}_{b_j}^{\nu_j}\, v)_\Omega| + |({}_{x_j}\mathcal{D}_{b_j}^{\nu_j}\, u,\ {}_{a_j}\mathcal{D}_{x_j}^{\nu_j}\, v)_\Omega| \right) \mathrm{d}\nu_j. \end{aligned} \tag{54}$$

Following (28) and Theorem 4.3 in [51],

$$\sum_{i=1}^{d}\int_{v_i^{\min}}^{v_i^{\max}}\rho_i(v_i)\Big(|({}_{a_i}\mathcal{D}_{x_i}^{v_i}(u),{}_{x_i}\mathcal{D}_{b_i}^{v_i}(v))_\Omega|+|({}_{x_i}\mathcal{D}_{b_i}^{v_i}(u),{}_{a_i}\mathcal{D}_{x_i}^{v_i}(v))_\Omega|\Big)$$
$$\geqslant \tilde{C}_1\sum_{i=1}^{d}\Big[\int_{v_i^{\min}}^{v_i^{\max}}\rho_i(v_i)\Big(\|{}_{a_i}\mathcal{D}_{x_i}^{v_i}(u)\|_{L^2(\Omega)}\Big)\mathrm{d}v_i\int_{v_i^{\min}}^{v_i^{\max}}\rho_i(v_i)\Big(\|{}_{x_i}\mathcal{D}_{b_i}^{v_i}(v)\|_{L^2(\Omega)}\Big)\mathrm{d}v_i$$
$$+\int_{v_i^{\min}}^{v_i^{\max}}\rho_i(v_i)\Big(\|{}_{x_i}\mathcal{D}_{b_i}^{v_i}(u)\|_{L^2(\Omega)}\Big)\mathrm{d}v_i\int_{v_i^{\min}}^{v_i^{\max}}\rho_i(v_i)\Big(\|{}_{a_i}\mathcal{D}_{x_i}^{v_i}(v)\|_{L^2(\Omega)}\Big)\mathrm{d}v_i\Big].$$

Thus,

$$\sum_{i=1}^{d}\int_{v_i^{\min}}^{v_i^{\max}}\rho_i(v_i)\Big(|({}_{a_i}\mathcal{D}_{x_i}^{v_i}(u),{}_{x_i}\mathcal{D}_{b_i}^{v_i}(v))_\Omega|+|({}_{x_i}\mathcal{D}_{b_i}^{v_i}(u),{}_{a_i}\mathcal{D}_{x_i}^{v_i}(v))_\Omega|\Big)\mathrm{d}v_i$$
$$\geqslant \tilde{C}_1\sum_{i=1}^{d}\int_{v_i^{\min}}^{v_i^{\max}}\rho_i(v_i)\Big(\|{}_{a_i}\mathcal{D}_{x_i}^{v_i}(u)\|_{L^2(\Omega)}+\|{}_{x_i}\mathcal{D}_{b_i}^{v_i}(u)\|_{L^2(\Omega)}\Big)\mathrm{d}v_i$$
$$\times\sum_{j=1}^{d}\int_{v_j^{\min}}^{v_j^{\max}}\rho_j(v_j)\Big(\|{}_{x_j}\mathcal{D}_{b_j}^{v_j}(v)\|_{L^2(\Omega)},+\|{}_{a_j}\mathcal{D}_{x_j}^{v_j}(v)\|_{L^2(\Omega)}\Big)\mathrm{d}v_j$$
$$=\tilde{C}_1|u|_{L^2(I;\mathcal{X}_d)}\,|v|_{L^2(I;\mathcal{X}_d)}, \tag{55}$$

where $\tilde{C}_1$ is a positive constant and independent of u. Considering Lemma 3.4, there exists a positive constant $\tilde{C}_2>0$ and independent of u, such that

$$\sup_{0\neq v\in\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}\frac{\int_{\tau^{\min}}^{\tau^{\max}}\varphi(\tau)\,|({}_0\mathcal{D}_t^{\tau}(u),{}_t\mathcal{D}_T^{\tau}(v))_\Omega|\mathrm{d}\tau}{|v|_{\mathrm{r},\mathfrak{D}}{}_{H^{\varphi}(I;L^2(\Lambda_d))}}\geqslant\tilde{C}_2|u|_{\mathrm{l},\mathfrak{D}}{}_{H^{\varphi}(I;L^2(\Lambda_d))}. \tag{56}$$

Furthermore, for $u\in\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$,

$$\sup_{0\neq v\in\mathfrak{B}^{\varphi,\cdots,\rho_d}(\Omega)}\frac{\int_{\tau^{\min}}^{\tau^{\max}}\varphi(\tau)\,|({}_0\mathcal{D}_t^{\tau}(u),{}_t\mathcal{D}_T^{\tau}(v))_\Omega|\mathrm{d}\tau}{|v|_{\mathrm{r},\mathfrak{D}}{}_{H^{\varphi}(I;L^2(\Lambda_d))}}$$
$$\cong\sup_{0\neq v\in\mathfrak{B}^{\varphi,\cdots,\rho_d}(\Omega)}\frac{\int_{\tau^{\min}}^{\tau^{\max}}\varphi(\tau)\,|({}_0\mathcal{D}_t^{\tau}(u),{}_t\mathcal{D}_T^{\tau}(v))_\Omega|\mathrm{d}\tau}{|v|_{\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}} \tag{57}$$

and

$$\sup_{0\neq v\in\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}\frac{\sum_{j=1}^{d}\int_{v_j^{\min}}^{v_j^{\max}}\rho_j(v_j)\Big(|({}_{a_j}\mathcal{D}_{x_j}^{v_j}u,{}_{x_j}\mathcal{D}_{b_j}^{v_j}v)_\Omega|+|({}_{x_j}\mathcal{D}_{b_j}^{v_j}u,{}_{a_j}\mathcal{D}_{x_j}^{v_j}v)_\Omega|\Big)\mathrm{d}v_j}{\|v\|_{L^2(I;\mathcal{X}_d)}}$$
$$\cong\sup_{0\neq v\in\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}\frac{\sum_{j=1}^{d}\int_{v_j^{\min}}^{v_j^{\max}}\rho_j(v_j)\Big(|({}_{a_j}\mathcal{D}_{x_j}^{v_j}u,{}_{x_j}\mathcal{D}_{b_j}^{v_j}v)_\Omega|+|({}_{x_j}\mathcal{D}_{b_j}^{v_j}u,{}_{a_j}\mathcal{D}_{x_j}^{v_j}v)_\Omega|\Big)\mathrm{d}v_j}{\|v\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}}. \tag{58}$$

Therefore, from (55), (56), (57), and (58), we have

$$
\begin{aligned}
\sup_{0\neq v\in\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \frac{|a(u,v)|}{\|v\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}} &\geqslant \bar{\beta} \sup_{0\neq v\in\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \frac{|(u,v)_\Omega| + \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, |({}_0\mathcal{D}_t^{\tau}\, u, {}_t\mathcal{D}_T^{\tau}\, v)_\Omega| \mathrm{d}\tau}{\|v\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}} \\
&+ \frac{\sum_{j=1}^{d} \int_{v_j^{\min}}^{v_j^{\max}} \rho_j(v_j) \Big(|({}_{a_j}\mathcal{D}_{x_j}^{v_j}\, u, {}_{x_j}\mathcal{D}_{b_j}^{v_j}\, v)_\Omega| + |({}_{x_j}\mathcal{D}_{b_j}^{v_j}\, u, {}_{a_j}\mathcal{D}_{x_j}^{v_j}\, v)_\Omega| \Big) \mathrm{d}v_j}{\|v\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}} \\
&\geqslant \bar{\beta}\, \bar{C} \Big(\|u\|_{L^2(\Omega)} + |u|_{{}^{1,\mathfrak{D}}H^{\varphi}(I;L^2(\Lambda_d))} + |u|_{L^2(I;\mathcal{X}_d)} \Big),
\end{aligned} \tag{59}
$$

where $\bar{C} = \min\{\tilde{C}_2, \tilde{C}_1\}$. Accordingly,

$$
\inf_{0\neq u\in\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \sup_{0\neq v\in\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)} \frac{|a(u,v)|}{\|v\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}} \geqslant \beta\, \|u\|_{\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}, \tag{60}
$$

where $\beta = \bar{\beta}\,\bar{C}$ is a positive constant and independent.

Theorem 3.8 (Well-posedness) *For* $0 < 2\tau^{\min} < 2\tau^{\max} < 1$ $(1 < 2\tau^{\min} < 2\tau^{\max} \leqslant 2)$, $1 < 2v_i^{\min} < 2v_i^{\max} \leqslant 2$, *and* $i = 1, \cdots, d$, *there exists a unique solution to* (50), *which is continuously dependent on* $f \in \left(\mathcal{B}^{\tau,v_1,\cdots,v_d}\right)^{\star}(\Omega)$, *where* $\left(\mathcal{B}^{\tau,v_1,\cdots,v_d}\right)^{\star}(\Omega)$ *is the dual space of* $\mathcal{B}^{\tau,v_1,\cdots,v_d}(\Omega)$.

Proof In virtue of the generalized Babuška–Lax–Milgram theorem [52], the well-posedness of the weak form in (48) in $(1+d)$ dimensions is guaranteed by the continuity and the inf-sup condition, which are proved in Lemma 3.6 and Theorem 3.7, respectively.

4 Petrov–Galerkin Method

To construct a PG spectral method for the finite-dimensional weak form problem in (50), we first define the proper finite-dimensional basis/test spaces and then implement the numerical scheme.

4.1 Space of Basis (U_N) and Test (V_N) Functions

As discussed in [51], we take the spatial basis, given in the standard domain $\xi \in [-1,1]$ as $\phi_m(\xi) = \sigma_m\big(P_{m+1}(\xi) - P_{m-1}(\xi)\big)$, $m = 1, 2, \cdots$, where $P_m(\xi)$ are the Legendre polynomials of order m and $\sigma_m = 2 + (-1)^m$. Besides, employing Jacobi poly-fractonomials of the first kind [63, 65], the temporal basis functions are given in the standard domain $\eta \in [-1,1]$ as $\psi_n^{\tau}(\eta) = \sigma_n(1+\eta)^{\tau} P_{n-1}^{-\tau,\tau}(\eta)$, $n = 1, 2, \cdots$.

We also let $\eta(t) = 2t/T - 1$ and $\xi_j(s) = 2\frac{s-a_j}{b_j-a_j} - 1$ to be temporal and spatial affine mappings from $t \in [0,T]$ and $x_j \in [a_j, b_j]$ to the standard domain $[-1,1]$, respectively. Therefore,

$$
U_N = \mathrm{span}\Big\{ \Big(\psi_n^{\tau}\circ\eta\Big)(t) \prod_{j=1}^{d} \Big(\phi_{m_j}\circ\xi_j\Big)(x_j) \,:\, n = 1, 2, \cdots, \mathcal{N},\ m_j = 1, 2, \cdots, \mathcal{M}_j \Big\}.
$$

Similarly, we employ Legendre polynomials and Jacobi polyfractonomials of the second kind in the standard domain to construct the finite-dimensional test space as

$$V_N = \mathrm{span}\Big\{ \Big(\Psi^{\tau}_{r}\circ\eta\Big)(t) \prod_{j=1}^{d} \Big(\Phi_{k_j}\circ\xi_j\Big)(x_j) \,:\, r = 1,2,\cdots,\mathcal{N},\ k_j = 1,2,\cdots,\mathcal{M}_j \Big\},$$

where $\Psi^{\tau}_{r}(\eta) = \widetilde{\sigma}_r(1-\eta)^{\tau} P^{\tau,-\tau}_{r-1}(\eta)$, $r = 1,2,\cdots$ and $\Phi_k(\xi) = \widetilde{\sigma}_k\big(P_{k+1}(\xi) - P_{k-1}(\xi)\big)$, $k = 1,2,\cdots$. The coefficient $\widetilde{\sigma}_k$ is defined as $\widetilde{\sigma}_k = 2(-1)^k + 1$.

Since the univariate basis/test functions belong to the fractional Sobolev spaces (see [65]) and $0 < \varphi(\tau) \in L^1((\tau^{\min}, \tau^{\max}))$, $0 < \rho_j(\nu_j) \in L^1((\nu_j^{\min}, \nu_j^{\max}))$ for $j = 1,\cdots,d$, $U_N \subset \mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$ and $V_N \subset \mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$. Accordingly, we approximate the solution in terms of a linear combination of elements in U_N, which satisfies the initial and boundary conditions.

4.2 Implementation of the PG Spectral Method

The solution u_N of (50) can be represented as

$$u_N(x,t) = \sum_{n=1}^{\mathcal{N}} \sum_{m_1=1}^{\mathcal{M}_1} \cdots \sum_{m_d=1}^{\mathcal{M}_d} \hat{u}_{n,m_1,\cdots,m_d} \left[\psi^{\tau}_{n}(t) \prod_{j=1}^{d} \phi_{m_j}(x_j) \right] \tag{61}$$

in Ω, and also, we take $v_N = \Psi^{\tau}_{r}(t) \prod_{j=1}^{d} \Psi_{k_j}(x_j)$, $r = 1,2,\cdots,\mathcal{N}$, $k_j = 1,2,\cdots,\mathcal{M}_j$. Accordingly, by replacing u_N and v_N in (50), we obtain the following Lyapunov system:

$$\begin{aligned} &\Big(S^{\varphi}_{\tau} \otimes M_1 \otimes M_2 \cdots \otimes M_d + \sum_{j=1}^{d} (M_\tau \otimes M_1 \otimes \cdots \otimes M_{j-1} \otimes S^{\mathrm{Tot}}_{j} \otimes M_{j+1} \cdots \otimes M_d) \\ &+ \gamma M_\tau \otimes M_1 \otimes M_2 \cdots \otimes M_d \Big) \mathcal{U} = F, \end{aligned} \tag{62}$$

in which $\otimes$ represents the Kronecker product and F denotes the multi-dimensional load matrix whose entries are given as

$$F_{r,k_1,\cdots,k_d} = \int_{\Omega} f(t,x_1,\cdots,x_d) \Big(\Psi^{\tau}_{r}\circ\eta\Big)(t) \prod_{j=1}^{d} \Big(\Phi_{k_j}\circ\xi_j\Big)(x_j)\,\mathrm{d}\Omega, \tag{63}$$

and $S^{\mathrm{Tot}}_{j} = c_{l_j} S^{\sigma_j}_{l} + c_{r_j} S^{\sigma_j}_{r} - \kappa_{l_j} S^{\rho_j}_{l} - \kappa_{r_j} S^{\rho_j}_{r}$. The matrices S^{φ}_{τ} and M_τ denote the temporal stiffness and mass matrices, respectively; $S^{\sigma_j}_{l}$, $S^{\sigma_j}_{r}$, $S^{\rho_j}_{l}$, $S^{\rho_j}_{r}$, and M_j denote the spatial stiffness and mass matrices. The entries of the spatial mass matrix M_j are computed analytically, while we employ proper quadrature rules to accurately compute the entries of the temporal mass matrix M_τ as discussed in [50]. The entries of S^{φ}_{τ} are also computed based on Theorem 3.1 (spectrally/exponentially accurate quadrature rule in α-dimension) in [25]. Likewise, we present the computation of S^{Tot}_{j} in Lemma A.1 in Appendix A.

Remark 2 The choices of coefficients in the construction of finite-dimensional basis/test functions lead to symmetric mass/stiffness matrices, which help formulating the following fast solver.

4.3 Unified Fast FPDE Solver

To formulate a closed-form solution to the Lyapunov system (62), we follow [64] and develop a fast solver in terms of the generalized eigen-solutions.

Theorem 4.1 [50] *Take $\{\mathbf{e}^j_{m_j}, \lambda^j_{m_j}\}_{m_j=1}^{\mathcal{M}_j}$ as the set of general eigen-solutions of the spatial stiffness matrix S_j^{Tot} with respect to the mass matrix M_j. Besides, let $\{\mathbf{e}^\tau_n, \lambda^\tau_n\}_{n=1}^{\mathcal{N}}$ be the set of general eigen-solutions of the temporal mass matrix M_τ with respect to the stiffness matrix S^φ_τ. Then, the unknown coefficients matrix $\mathcal{U}$ is obtained as*

$$\mathcal{U} = \sum_{n=1}^{\mathcal{N}} \sum_{m_1=1}^{\mathcal{M}_1} \cdots \sum_{m_d=1}^{\mathcal{M}_d} \kappa_{n,m_1,\cdots,m_d}\, \mathbf{e}^\tau_n \otimes \mathbf{e}^1_{m_1} \otimes \cdots \otimes \mathbf{e}^d_{m_d}, \tag{64}$$

where

$$\kappa_{n,m_1,\cdots,m_d} = \frac{(\mathbf{e}_n^{\tau}\mathbf{e}^1_{m_1} \cdots \mathbf{e}^d_{m_d})F}{[(\mathbf{e}^{\tau\mathrm{T}}_n S^\varphi_\tau \mathbf{e}^\tau_n)\prod_{j=1}^{d}(\mathbf{e}^{j\mathrm{T}}_{m_j} M_j \mathbf{e}^j_{m_j})]\Lambda_{n,m_1,\cdots,m_d}}, \tag{65}$$

and

$$\Lambda_{n,m_1,\cdots,m_d} = \left[(1+\gamma\ \lambda^\tau_n) + \lambda^\tau_n \sum_{j=1}^{d}(\lambda^j_{m_j})\right].$$

Remark 3 The naive computation of all entries in (65) leads to a computational complexity of $O(\mathcal{N}^{2+2d})$, including construction of stiffness and mass matrices. By performing sum-factorization [64], the operator counts can be reduced to $O(\mathcal{N}^{2+d})$.

5 Stability and Error Analysis

The following theorems provide the finite-dimensional stability and error analysis of the proposed scheme, based on the well-posedness analysis from Sect. 3.3.

5.1 Stability Analysis

Theorem 5.1 *Let Assumption 3 hold. The PG spectral method for (51) is stable; that is*

$$\inf_{0\neq u_N\in U_N} \sup_{0\neq v\in V_N} \frac{|a(u_N, v_N)|}{\|v_N\|_{\mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}\|u_N\|_{\mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)}} \geqslant \beta > 0 \tag{66}$$

holds with $\beta > 0$ and independent of N.

Proof Regarding $U_N \subset \mathcal{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$ and $V_N \subset \mathfrak{B}^{\varphi,\rho_1,\cdots,\rho_d}(\Omega)$, (66) follows directly from Theorem 3.7.

Remark 4 The bilinear form (51) can be expanded in terms of the basis and test functions to obtain the lower limit of β, see [50, 64].

5.2 Error Analysis

Denoting by $P_{\mathcal{M}}(\Lambda)$ the space of all polynomials of degree $\leqslant \mathcal{M}$ on $\Lambda \subset \mathbb{R}$, $P^{\varphi}_{\mathcal{M}}(\Lambda) := P_{\mathcal{M}}(\Lambda) \cap {}^{\mathfrak{D}}H^{\varphi}(\Lambda)$, where $0 < \varphi(\tau) \in L^1((\tau^{\min}, \tau^{\max}))$ and ${}^{\mathfrak{D}}H^{\varphi}(\Lambda)$ is the distributed Sobolev space associated with the norm $\| \cdot \|_{{}^{\mathfrak{D}}H^{\varphi}(\Lambda)}$. In this section, we take $I_0 = (0, T)$, $I_i = (a_i, b_i)$ for $i = 1, \cdots, d$, $\Lambda_i = I_i \times \Lambda_{i-1}$, and $\Lambda_i^j = \prod_{\substack{k=1 \\ k \neq j}}^{i} I_k$. Besides, $0 < 2\tau^{\min} < 2\tau^{\max} < 1$ $(1 < 2\tau^{\min} < 2\tau^{\max} \leqslant 2)$, $1 < 2\nu_i^{\min} < 2\nu_i^{\max} \leqslant 2$ for $i = 1, \cdots, d$. Where there is no confusion, the symbols I_i, Λ_i, and Λ_i^j and the intervals of $(\tau^{\min}, \tau^{\max})$ and $(\nu_i^{\min}, \nu_i^{\max})$ will be dropped from the notations.

Theorem 5.2 [35] *Let r_1 be a real number, where $r_1 \neq \mathcal{M}_1 + \frac{1}{2}$, and $1 \leqslant r_1$. There exists a projection operator $\Pi^{\nu_1}_{r_1,\mathcal{M}_1}$ from $H^{r_1}(\Lambda_1) \cap H_0^{\nu_1}(\Lambda_1)$ to $P^{\nu_1}_{\mathcal{M}_1}(\Lambda_1)$, such that for any $u \in H^{r_1}(\Lambda_1) \cap H_0^{\nu_1}(\Lambda_1)$, we have $\|u - \Pi^{\nu_1}_{r_1,\mathcal{M}_1} u\|_{{}^cH^{\nu_1}(\Lambda_1)} \leqslant c_1 \mathcal{M}_1^{\nu_1 - r_1} \|u\|_{H^{r_1}(\Lambda_1)}$, where c_1 is a positive constant.*

Theorem 5.3 [25] *Let $r_0 \geqslant 1$, $r_0 \neq \mathcal{N} + \frac{1}{2}$. There exists an operator $\Pi^{\varphi}_{r_0,\mathcal{N}}$ from $H^{r_0}(I) \cap {}^{1,\mathfrak{D}}H^{\varphi}(I)$ to $P^{\varphi}_{\mathcal{N}}(\Lambda_1)$, such that for any $u \in H^{r_0}(I) \cap {}^{1,\mathfrak{D}}H^{\varphi}(I)$, we have*

$$\|u - \Pi^{\varphi}_{r_0,\mathcal{N}} u\|^2_{{}^1H^{\varphi}(I)} \leqslant c_0 \mathcal{N}^{-2r_0} \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau) \mathcal{N}^{2\tau} \|u\|_{H^{r_0}(I)} \mathrm{d}\tau,$$

where c_0 is a positive constant and $0 < \varphi(\tau) \in L^1((\tau^{\min}, \tau^{\max}))$.

In the following, employing Theorems 5.2 and 5.3 and also Theorem 5.3 from [51], we study the properties of higher dimensional approximation operators in the following Lemmas.

Theorem 5.4 *Let $r_1 \geqslant 1$, $r_1 \neq \mathcal{M}_1 + \frac{1}{2}$. There exists a projection operator $\Pi^{\rho_1}_{r_1,\mathcal{M}_1}(I_1)$ from $H^{r_1}(I_1) \cap {}^{1,\mathfrak{D}}H^{\rho_1}(I_1)$ to $P^{\rho_1}_{\mathcal{M}_1}(I_1)$, such that for any $u \in H^{r_1}(I_1) \cap {}^{1,\mathfrak{D}}H^{\rho_1}(I_1)$, we have*

$$\|u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1} u\|^2_{{}^{1,\mathfrak{D}}H^{\rho_1}(I_1)} \leqslant \mathcal{M}_1^{-2r_1} \int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1) \mathcal{M}_1^{2\nu_1} \|u\|_{H^{r_1}(I_1)} \mathrm{d}\nu_1,$$

where $0 < \rho_1(\nu_1) \in L^1((\nu_1^{\min}, \nu_1^{\max}))$.

Proof From Theorem 5.2 for $u \in H^{r_1} \cap {}^cH^{\nu_1}$, we have $\|u - \Pi^{\nu_1}_{r_1,\mathcal{M}_1} u\|_{{}^cH^{\nu_1}(\Lambda_1)} \leqslant \mathcal{M}_1^{\nu_1 - r_1} \|u\|_{H^{r_1}(\Lambda_1)}$. Therefore, for $u \in H^{r_1}(I_1) \cap {}^1H^{\rho_1}(I_1)$, we have

$$
\begin{aligned}
\|u-\Pi^{\rho_1}_{r_1,\mathcal{M}_1}u\|^2_{{}^{1,\mathfrak{D}}H^{\rho_1}(I_1)} &= \int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1)\,\|u-\Pi^{\nu_1}_{r_1,\mathcal{M}_1}u\|^2_{{}^{\mathfrak{c}}H^{\nu_1}(\Lambda_1)}\mathrm{d}\nu_1\\
&\leqslant \int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1)\,\mathcal{M}_1^{2\nu_1-2r_1}\,\|u\|^2_{H^{r_1}(\Lambda_1)}\mathrm{d}\nu_1\\
&= \mathcal{M}_1^{-2r_1}\int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1)\,\mathcal{M}_1^{2\nu_1}\,\|u\|_{H^{r_1}(I_1)}\mathrm{d}\nu_1.
\end{aligned}
$$

Lemma 5.5 *Let the real-valued* $1\leqslant r_1,\ r_2$ *and* $\Omega=I_1\times I_2$. *If* $u\in {}^{1,\mathfrak{D}}H^{\rho_2}(I_2,H^{r_1}(I_1))\cap H^{r_2}(I_2,{}^{1,\mathfrak{D}}H_0^{\rho_1}(I_1))$, *then*

$$
\begin{aligned}
&\|u-\Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{\mathcal{B}^{\rho_1,\rho_2}(\Omega)}\\
&\leqslant \mathcal{M}_2^{-2r_2}\int_{\nu_2^{\min}}^{\nu_2^{\max}}\rho_2(\nu_2)\Big(\mathcal{M}_2^{2\nu_2}\|u\|_{H^{r_2}(I_2,L^2(I_1))}+\mathcal{M}_2^{2\nu_2}\mathcal{M}_1^{-2r_1}\|u\|_{H^{r_2}(I_2,H^{r_1}(I_1))}\Big)\mathrm{d}\nu_2\\
&\quad+\mathcal{M}_1^{-2r_1}\int_{\nu_1^{\min}}^{\nu_1^{\max}}\rho_1(\nu_1)\Big(\mathcal{M}_1^{2\nu_1}\|u\|_{H^{r_1}(I_1,L^2(I_2))}+\mathcal{M}_1^{2\nu_1}\mathcal{M}_2^{-2r_2}\|u\|_{H^{r_1}(I_1,H^{r_2}(I_2))}\Big)\mathrm{d}\nu_1\\
&\quad+\mathcal{M}_2^{-2r_2}\|u\|_{{}^{\mathfrak{D}}H^{\rho_1}(I_1,H^{r_2}(I_2))}+\mathcal{M}_1^{-2r_1}\|u\|_{{}^{\mathfrak{D}}H^{\rho_2}(I_2,H^{r_1}(I_1))},
\end{aligned}
\tag{67}
$$

where $\|\cdot\|_{\mathcal{B}^{\rho_1,\rho_2}(\Omega)}=\left\{\|\cdot\|^2_{H^{\rho_1}(I_1,L^2(I_2))}+\|\cdot\|^2_{H^{\rho_2}(I_1,L^2(I_1))}\right\}^{\frac{1}{2}}$, $0<\rho_1(\nu_1)\in L^1([\nu_1^{\min},\nu_1^{\max}])$, *and* $0<\rho_2(\nu_2)\in L^1([\nu_2^{\min},\nu_2^{\max}])$.

Proof For $u\in {}^{1,\mathfrak{D}}H^{\rho_2}(I_2,H^{r_1}(I_1))\cap H^{r_2}(I_2,H^{\rho_1}(I_1))$, evidently $u\in H^{r_2}(I_2,H^{r_1}(I_1))$, $u\in H^{r_2}(I_2,L^2(I_1))$, and $u\in H^{r_1}(I_1,L^2(I_2))$.

Besides, from the definition of $\|\cdot\|_{\mathcal{B}^{\rho_1,\rho_2}(\Omega)}$, we have

$$
\begin{aligned}
&\|u-\Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|_{\mathcal{B}^{\rho_1,\rho_2}(\Omega)}\\
&=\left\{\|u-\Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{{}^{\mathfrak{D}}H^{\rho_1}(I_1,L^2(I_2))}+\|u-\Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{{}^{\mathfrak{D}}H^{\rho_2}(I_2,L^2(I_1))}\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Following Lemma 5.3 in [51] and Theorem 5.4, $\|u-\Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{{}^{\mathfrak{D}}H^{\rho_2}(I_2,L^2(I_1))}$ can be simplified to

$$
\begin{aligned}
&\|u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{\mathfrak{D}H^{\rho_2}(I_2,L^2(I_1))}\\
&= \|u - \Pi^{\rho_2}_{r_2,\mathcal{M}_2}u + \Pi^{\rho_2}_{r_2,\mathcal{M}_2}u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{\mathfrak{D}H^{\rho_2}(I_2,L^2(I_1))}\\
&\leqslant \|u - \Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{\mathfrak{D}H^{\rho_2}(I_2,L^2(I_1))}\\
&\quad + \|\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{\mathfrak{D}H^{\rho_2}(I_2,L^2(I_1))}\\
&\leqslant \mathcal{M}_2^{-2r_2}\int_{\nu_2^{\min}}^{\nu_2^{\max}} \rho_2(\nu_2)\,\mathcal{M}_2^{2\nu_2}\|u\|^2_{H^{r_2}(I_2,L^2(I_1))}\mathrm{d}\nu_2\\
&\quad + \|(\Pi^{\rho_2}_{r_2,\mathcal{M}_2} - \mathcal{I})(u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}u)\|^2_{\mathfrak{D}H^{\rho_2}(I_2,L^2(I_1))}\\
&\quad + \|u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}u\|^2_{\mathfrak{D}H^{\rho_2}(I_2,L^2(I_1))}\\
&\leqslant \mathcal{M}_2^{-2r_2}\int_{\nu_2^{\min}}^{\nu_2^{\max}} \rho_2(\nu_2)\,\mathcal{M}_2^{2\nu_2}\|u\|^2_{H^{r_2}(I_2,L^2(I_1))}\mathrm{d}\nu_2\\
&\quad + \mathcal{M}_2^{-2r_2}\mathcal{M}_1^{-2r_1}\int_{\nu_2^{\min}}^{\nu_2^{\max}} \rho_2(\nu_2)\,\mathcal{M}_2^{2\nu_2}\|u\|^2_{H^{r_2}(I_2,H^{r_1}(I_1))}\mathrm{d}\nu_2\\
&\quad + \mathcal{M}_1^{-2r_1}\|u\|^2_{\mathfrak{D}H^{\rho_2}(I_2,H^{r_1}(I_1))},
\end{aligned}
\tag{68}
$$

where $\mathcal{I}$ is the identity operator. Furthermore,

$$
\begin{aligned}
&\|u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{L^2(I_2,H^{\rho_1}(I_1))}\\
&= \|u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}u + \Pi^{\rho_1}_{r_1,\mathcal{M}_1}u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{\mathfrak{D}H^{\rho_1}(I_1,L^2(I_2))}\\
&\leqslant \|u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}u\|^2_{\mathfrak{D}H^{\rho_1}(I_1,L^2(I_2))}\\
&\quad + \|\Pi^{\rho_1}_{r_1,\mathcal{M}_1}u - \Pi^{\rho_1}_{r_1,\mathcal{M}_1}\Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{\mathfrak{D}H^{\rho_1}(I_1,L^2(I_2))}\\
&\leqslant \mathcal{M}_1^{-2r_1}\int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1)\,\mathcal{M}_1^{2\nu_1}\|u\|^2_{H^{r_1}(I_1,L^2(I_2))}\mathrm{d}\nu_1\\
&\quad + \|(\Pi^{\rho_1}_{r_1,\mathcal{M}_1} - \mathcal{I})(u - \Pi^{\rho_2}_{r_2,\mathcal{M}_2}u)\|^2_{\mathfrak{D}H^{\rho_1}(I_1,L^2(I_2))}\\
&\quad + \|u - \Pi^{\rho_2}_{r_2,\mathcal{M}_2}u\|^2_{\mathfrak{D}H^{\rho_1}(I_1,L^2(I_2))}\\
&\leqslant \mathcal{M}_1^{-2r_1}\int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1)\,\mathcal{M}_1^{2\nu_1}\|u\|^2_{H^{r_1}(I_1,L^2(I_2))}\mathrm{d}\nu_1\\
&\quad + \mathcal{M}_2^{-2r_2}\mathcal{M}_1^{-2r_1}\int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1)\,\mathcal{M}_1^{2\nu_1}\|u\|^2_{H^{r_1}(I_1,H^{r_2}(I_2))}\mathrm{d}\nu_1\\
&\quad + \mathcal{M}_2^{-2r_2}\|u\|^2_{\mathfrak{D}H^{\rho_1}(I_1,H^{r_2}(I_2))}.
\end{aligned}
\tag{69}
$$

Therefore, (67) can be derived immediately from (68) and (69).

Likewise, Lemma 5.4 can be easily extended to the d-dimensional approximation operator as

$$
\begin{aligned}
&\|u - \Pi_d^h u\|^2_{{}^m\mathfrak{D}H^{\rho_i}(I_i, L^2(\Lambda_d^i))} \\
&\leqslant \mathcal{M}_i^{-2r_i} \int_{\nu_i^{\min}}^{\nu_i^{\max}} \rho_i(\nu_i)\, \mathcal{M}_i^{2\nu_i} \|u\|^2_{H^{r_i}(I_i, L^2(\Lambda_d^i))} \mathrm{d}\nu_i \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{d} \mathcal{M}_j^{-2r_j} \|u\|^2_{\mathfrak{D}H^{\rho_i}(I_i, H^{r_j}(I_j, L^2(\Lambda_d^{i,j})))} \\
&\quad + \mathcal{M}_i^{-2r_i} \int_{\nu_i^{\min}}^{\nu_i^{\max}} \rho_i(\nu_i)\, \mathcal{M}_i^{2\nu_i} \sum_{\substack{j=1\\ j\neq i}}^{d} \mathcal{M}_j^{-2r_j} \|u\|^2_{H^{r_i}(I_i, H^{r_j}(I_j, L^2(\Lambda_d^{i,j})))} \mathrm{d}\nu_i \\
&\quad + \sum_{\substack{k=1\\ k\neq i}}^{d} \sum_{\substack{j=1\\ j\neq i,k}}^{d} \mathcal{M}_j^{-2r_j} \mathcal{M}_k^{-2r_k} \|u\|^2_{\mathfrak{D}H^{\rho_i}(I_i, H^{r_k, r_j}(I_k \times I_j, L^2(\Lambda_d^{i,j,k})))} \\
&\quad + \cdots + \mathcal{M}_i^{-2r_i} \int_{\nu_i^{\min}}^{\nu_i^{\max}} \rho_i(\nu_i)\, \mathcal{M}_i^{2\nu_i} \\
&\quad \cdot \prod_{\substack{j=1\\ j\neq i}}^{d} \mathcal{M}_j^{-r_j} \|u\|^2_{{}^c H^{\nu_i}(I_i, H^{r_1, \cdots, r_d}(\Lambda_d^i))} \mathrm{d}\nu_i,
\end{aligned} \tag{70}
$$

where $\Pi_d^h = \Pi^{\rho_1}_{r_1, \mathcal{M}_1} \cdots \Pi^{\rho_d}_{r_d, \mathcal{M}_d}$.

Theorem 5.6 *Let* $1 \leqslant r_i,\ I_0 = (0, T),\ I_i = (a_i, b_i),\ \Omega = I_0 \times (\prod_{i=1}^{d} I_i),\ \Lambda_k = \prod_{i=1}^{k} I_i,\ \Lambda_k^j = \prod_{\substack{i=1\\ i\neq j}}^{k} I_i,$ *and* $\frac{1}{2} < \nu_i^{\min} < \nu_i^{\max} \leqslant 1$ *for* $i = 1, \cdots, d$. *If*

$$
u \in \left(\bigcap_{i=1}^{d} H^{r_0}(I_0, {}^{\mathfrak{D}}H^{\rho_i}(I_i, H^{r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_d}(\Lambda_d^i))) \right) \cap {}^{l,\mathfrak{D}}H^{\varphi}(I_0, H^{r_1, \cdots, r_d}(\Lambda_d)),
$$

then

$$
\begin{aligned}
&\|u - \Pi^{\varphi}_{r_0, \mathcal{N}} \Pi_d^h u\|^2_{\mathcal{B}^{\tau, \nu_1, \cdots, \nu_d}(\Omega)} \\
&\leqslant \mathcal{N}^{-2r_0} \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, \mathcal{N}^{2\tau} \|u\|_{H^{r_0}(I_0, L^2(\Lambda_d))} \mathrm{d}\tau \\
&\quad + \mathcal{N}^{-2r_0} \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, \mathcal{N}^{2\tau} \sum_{j=1}^{d} \mathcal{M}_j^{-2r_j} \|u\|^2_{H^{r_0}(I_0, H^{r_j}(I_j, L^2(\Lambda_d^j)))} \mathrm{d}\tau + \cdots \\
&\quad + \mathcal{N}^{-2r_0} \int_{\tau^{\min}}^{\tau^{\max}} \varphi(\tau)\, \mathcal{N}^{2\tau} \Big(\prod_{j=1}^{d} \mathcal{M}_j^{-2r_j} \Big) \|u\|_{H^{r_0}(I_0, H^{r_1, \cdots, r_d}(\Lambda_d))} \mathrm{d}\tau \\
&\quad + \sum_{i=1}^{d} \int_{\nu_i^{\min}}^{\nu_i^{\max}} \rho_i(\nu_i) \Big\{ \mathcal{M}_i^{2\nu_i - 2r_i} \|u\|_{H^{r_i}(I_i, L^2(\Lambda_d^i \times I_0))} + \cdots \\
&\quad + \mathcal{M}_i^{2\nu_i - 2r_i} \Big(\prod_{\substack{j=1\\ j\neq i,k}}^{d} \mathcal{M}_j^{-2r_j} \Big) \|u\|_{H^{r_i}(I_i, H^{r_1, \cdots, r_d}(\Lambda_d^i, L^2(I_0)))} \Big\} \mathrm{d}\nu_i,
\end{aligned} \tag{71}
$$

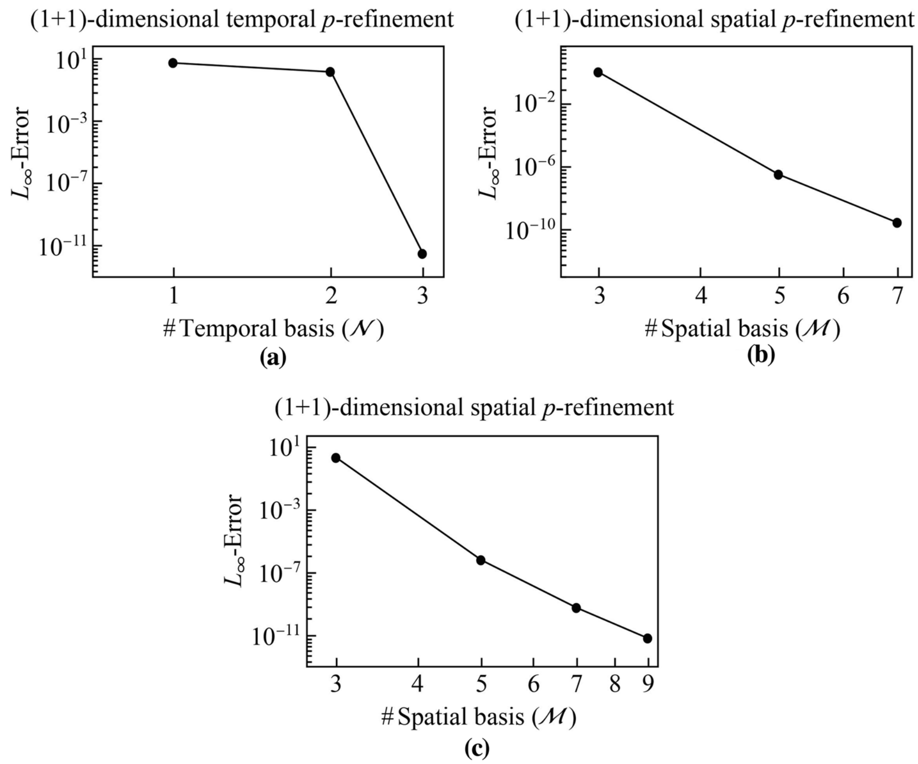

Fig. 1 Temporal/spatial p-refinement for case I with singularity of order $\alpha = 10^{-4}$. (**a**): $p_1 = 3$, $p_2 = p_3 = 2$, and expansion order of $\mathcal{N} \times 9$. (**b**): $p_1 = 2$, $p_2 = p_3 = 2$, and expansion order of $3 \times \mathcal{M}$. (**c**): $p_1 = 3$, $p_2 = p_3 = 2$, and expansion order of $4 \times \mathcal{M}$

where $\Pi_d^h = \Pi_{r_1,\mathcal{M}_1}^{\rho_1} \cdots \Pi_{r_d,\mathcal{M}_d}^{\rho_d}$ *and* β *is a real positive constant.*

Proof Directly from (45), we conclude that

$$\|u\|^2_{\mathcal{B}^{\tau,\nu_1,\cdots,\nu_d}(\Omega)} \leqslant \|u\|^2_{{}^{1}H^{\tau}(I_0,L^2(\Lambda_d))} + \sum_{i=1}^{d} \|u\|^2_{L^2(I_0,{}^{\mathfrak{D}}H^{\rho_i}(I_i,L^2(\Lambda_d^i)))}.$$

Next, it follows from Theorem 5.3 that

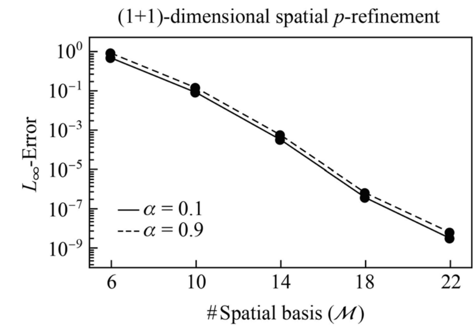

Fig. 2 Spatial p-refinement for case II, $p_1 = 3$, $\alpha = 0.1$, and $\alpha = 0.9$

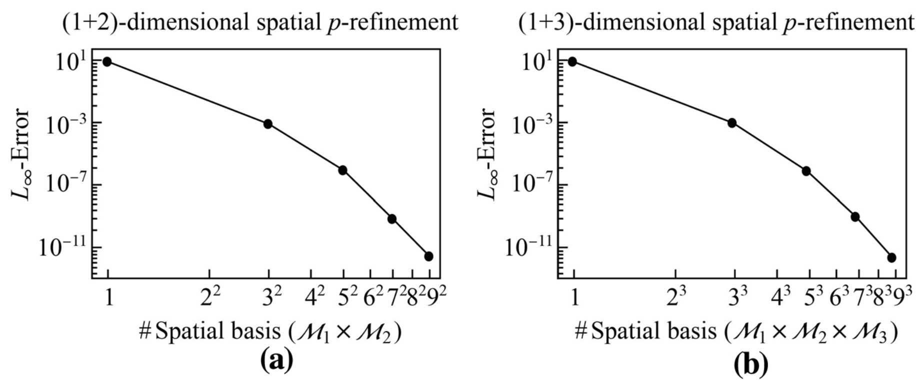

Fig. 3 Spatial p-refinement for case III with singularity of order $\alpha = 10^{-4}$. (**a**): $(1+2)$-dimensional, $p_1 = 3$, $p_{2i} = p_{2i+1} = 1$, where the expansion order is $\mathcal{N} \times \mathcal{M}_1 \times \mathcal{M}_2$. (**b**): $(1+3)$-dimensional, $p_1 = 3$, $p_{2i} = p_{2i+1} = 1$, where the expansion order is $\mathcal{N} \times \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3$

$$\begin{aligned}
&\| u - \Pi^{\varphi}_{r_0,\mathcal{N}} \Pi^h_d u \|^2_{{}^1\mathfrak{D} H^{\varphi}(I_0, L^2(\Lambda_d))} \\
&\leqslant \mathcal{N}^{-2r_0} \int_{\tau_{\min}}^{\tau_{\max}} \varphi(\tau)\, \mathcal{N}^{2\tau} \\
&\quad \cdot \Bigg[\|u\|^2_{H^{r_0}(I_0, L^2(\Lambda_d))} + \sum_{j=1}^{d} \mathcal{M}_j^{-r_j} \|u\|^2_{H^{r_0}(I_0, H^{r_j}(I_j, L^2(\Lambda_d)))} + \cdots \\
&\quad + \Bigg(\prod_{j=1}^{d} \mathcal{M}_j^{-r_j} \Bigg) \|u\|_{H^{r_0}(I_0, H^{r_1,\cdots,r_d}(\Lambda_d))} \Bigg] \mathrm{d}\tau.
\end{aligned} \tag{72}$$

Therefore, (71) is obtained immediately from (70) and (72).

Remark 5 Since the inf-sup condition holds (see Theorem 5.1), by Lemma 3.6, the error in the numerical scheme is less than or equal to a constant times the projection error. Hence, the results above imply the spectral accuracy of the scheme.

Table 1 CPU time, PG spectral method for fully distributed (1+d)-dimensional diffusion problems. $u^{\text{ext}} = t^{p_1+\alpha} \times \prod_{i=1}^{3} (1+x_i)^{p_{2i}}(1-x_i)^{p_{2i+1}}$, where $\alpha = 10^{-4}$, $p_1 = 3$, and the expansion order is 4×11^d

	$p_{2i} = p_{2i+1} = 2$			$p_{2i} = p_{2i+1} = 3$		
	$d = 1$	$d = 2$	$d = 3$	$d = 1$	$d = 2$	$d = 3$
CPU time/s	1 546.81	1 735.03	2 358.67	1 596.16	1 786.61	2 407.22
$\|e\|_{L^\infty(\Omega)}$	6.84×10^{-12}	4.45×10^{-12}	3.27×10^{-12}	6.27×10^{-12}	3.86×10^{-12}	2.71×10^{-12}

6 Numerical Tests

We provide several numerical examples to investigate the performance of the proposed scheme. We consider a $(1 + d)$-dimensional fully distributed diffusion problem with the left-sided derivative by letting $c_{l_i} = c_{r_i} = \kappa_{r_i} = 0$, $\kappa_{l_i} = 1$, $0 < 2\tau^{\min} < 2\tau^{\max} < 1$, and $1 < 2\nu_i^{\min} < 2\nu_i^{\max} \leqslant 2$ in (50) for $i = 1, \cdots, d$, where the computational domain is $\Omega = (0, 2) \times \prod_{i=1}^{d} (-1, 1)$. We report the measured L^∞ error, $\|e\|_{L^\infty} = \|u_N - u^{\text{ext}}\|_{L^\infty}$ as the maximum bound of $\|e\|_{L^2}$.

In each of the following test cases, we use the method of fabricated solutions to construct the load vector, given an exact solution u^{ext}. Here, we assume $u^{\text{ext}} = u_t \times \prod_{i=1}^{d} u_{x_i}$. We project the spatial part in each dimension, u_{x_i}, on the spatial bases, and then, construct the load vector by plugging the projected exact solution into the weak form of the problem. This helps us to take the fractional derivative of the exact solution more efficiently, while by truncating the projection with a sufficient number of terms, we make sure that the corresponding projection error does not dominantly propagate into the convergence analysis of numerical scheme.

Case I We consider a smooth solution in space with finite regularity in time as

$$u^{\text{ext}} = t^{p_1+\alpha} \times ((1 + x_1)^{p_2}(1 - x_1)^{p_3}) \tag{73}$$

to investigate the spatial/temporal p-refinement. We allow the singularity to take order of $\alpha = 10^{-4}$, while p_1, p_2, and p_3 take some integer values. We show the L^∞-error for different test cases in Fig. 1, where by tuning the fractional parameter of the temporal basis, we can accurately capture the singularity of the exact solution, when the approximate solution converges as we increase the expansion order. In each case of spatial/temporal p-refinement, we choose sufficient number of bases in the other directions to make sure that their corresponding error is of machine precision order.

Considering $\alpha = 10^{-4}$, $p_1 = 2$, $p_2 = p_3 = 2$ in (73), and the temporal order of expansion being fixed ($\mathcal{N} = 4$) in the spatial p-refinement, we get the rate of convergence as a function of the minimum regularity in the spatial direction. From Theorem 5.6, the rate of convergence is bounded by the spatial approximation error, i.e., $\|e\|_{L^2(\Omega)} \leqslant \|e\|_{L^\infty(\Omega)} \leqslant \mathcal{M}_1^{-2r_1} \cdot \int_{\nu_1^{\min}}^{\nu_1^{\max}} \rho_1(\nu_1)\, \mathcal{M}_1^{2\nu_1} \|u\|_{H^{r_1}(I_1, L^2(I_0))} \mathrm{d}\nu_1$, where $r_1 = p_2 + \frac{1}{2} - \epsilon$ is the minimum regularity of the exact solution in the spatial direction for $\epsilon < \frac{1}{2}$. Conforming to Theorem 5.6, the practical rate of convergence $\bar{r}_1 = 16.05$ in $\|e\|_{L^\infty(\Omega)}$ is greater than $r_1 \approx 2.50$.

Case II We consider $u^{\text{ext}} = t^{p_1+\alpha} \sin(2\pi x_1)$, where $p_1 = 3$, and let $\alpha = 0.1$ and $\alpha = 0.9$. We set the number of temporal basis functions, $\mathcal{N} = 4$, and show the convergence of approximate solution by increasing the number of spatial basis, $\mathcal{M}$ in Fig. 2. The main difficulty in this case is the construction of load vector. To accurately compute the integrals in the load vector, we project the spatial part of forcing function, $\sin(2\pi x_1)$, on the spatial bases. To make sure that the corresponding error is of machine-precision order and thus, not dominant, we truncate the projection at 25 terms, where there error is of order 10^{-16}. Therefore, the quadrature rule over derivative order should be performed for 25 terms rather than only a single $\sin(2\pi x_1)$ term. This will increase the computational cost.

Case III (High-dimensional *p*-refinement) We consider the exact solution of the form,

$$u^{\text{ext}} = t^{p_1+\alpha} \times \prod_{i=1}^{3}(1+x_i)^{p_{2i}}(1-x_i)^{p_{2i+1}} \tag{74}$$

with singularity of order $\alpha = 10^{-4}$, where $p_1 = 3$, and $p_{2i} = p_{2i+1} = 1$. Similar to previous cases, we set the number of temporal bases, $\mathcal{N} = 4$, and study convergence by uniformly increasing the number of spatial bases in all dimensions. Figure 3 shows the results for $(1+2)$-dimensional and $(1+3)$-dimensional problems with expansion order of $\mathcal{N} \times \mathcal{M}_1 \times \mathcal{M}_2$, and $\mathcal{N} \times \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3$, respectively. Following Case I, the computed rate of convergence $\bar{r}_1 = \bar{r}_2 = \bar{r}_3 = 16.13$ in (74) for $\alpha = 10^{-4}$ is greater than the minimum regularity of the exact solution $r \approx 2.05$, which is in agreement with Theorem 5.6.

In addition to the convergence study, we examine the efficiency of the developed method and fast solver by comparing the CPU times for $(1+1)$-, $(1+2)$-, and $(1+3)$-dimensional space-time hypercube domains in case III. The computed CPU times are obtained on an INTEL(XEON E52670) processor of 2.5 GHz, and reported in Table 1.

7 Summary

We developed a unified PG spectral method for fully distributed-order PDEs with constant coefficients on a $(1+d)$-dimensional space-time hypercube, subject to homogeneous Dirichlet initial/boundary conditions. We obtained the weak formulation of the problem, and proved the well-posedness by defining the proper underlying distributed Sobolev spaces and the associated norms. We then formulated the numerical scheme, exploiting Jacobi poly-fractonomials as temporal basis/test functions, and Legendre polynomials as spatial basis/test functions. To improve the efficiency of the proposed method in higher dimensions, we constructed a unified fast linear solver employing certain properties of the stiffness/mass matrices, which significantly reduced the computation time. Moreover, we proved the stability of the developed scheme and carried out the error analysis. Finally, via several numerical test cases, we examined the practical performance of the proposed method and illustrated the spectral accuracy.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A Entries of Spatial Stiffness Matrix

Here, we provide the computation of entries of the spatial stiffness matrix by performing an affine mapping ϑ from the standard domain $\mu_j^{\mathrm{stn}} \in [-1,1]$ to $\mu_j \in [\mu_j^{\max}, \mu_j^{\min}]$.

Lemma A.1 *The total spatial stiffness matrix S_j^{Tot} is symmetric and its entries can be exactly computed as*

$$S_j^{\mathrm{Tot}} = c_{l_j} \times S_l^{\varrho_j} + c_{r_j} \times S_r^{\varrho_j} - \kappa_{l_j} \times S_l^{\rho_j} - \kappa_{r_j} \times S_r^{\rho_j}, \tag{A1}$$

where $j = 1, 2, \cdots, d$.

Proof Regarding the definition of the stiffness matrix, we have

$$\begin{aligned} \{S_l^{\varrho_j}\}_{r,n} &= \int_{-1}^{1} \int_{\mu_j^{\min}}^{\mu_j^{\max}} \varrho_j(\mu_j)\, {}_{-1}\mathcal{D}_{\xi_j}^{\mu_j}\Big(\phi_n(x_j)\Big)\, {}_{\xi_j}\mathcal{D}_1^{\mu_j}\Big(\varPhi_r(x_j)\Big)\,\mathrm{d}x_j, \\ &= \beta_1 \int_{-1}^{1}\int_{-1}^{1} \varrho_j\big(\vartheta(\mu_j^{\mathrm{stn}})\big)\, {}_{-1}\mathcal{D}_{\xi_j}^{\mu_j^{\mathrm{stn}}}\Big(P_{n+1}(\xi_j) - P_{n-1}(\xi_j)\Big) \\ &\quad \times {}_{\xi_j}\mathcal{D}_1^{\mu_j^{\mathrm{stn}}}\Big(P_{k+1}(\xi_j) - P_{k-1}(\xi_j)\Big)\,\mathrm{d}\xi_j, \\ &= \beta_1\Big(\widetilde{S}^{\varrho_j}_{r+1,n+1} - \widetilde{S}^{\varrho_j}_{r+1,n-1} - \widetilde{S}^{\varrho_j}_{r-1,n+1} + \widetilde{S}^{\varrho_j}_{r-1,n-1}\Big), \end{aligned} \tag{A2}$$

where $\beta_1 = \widetilde{\sigma}_r\, \sigma_n \left(\frac{\mu_j^{\max} - \mu_j^{\min}}{2}\right)$ and

$$\begin{aligned} \widetilde{S}^{\varrho_j}_{r,n} &= \int_{-1}^{1}\int_{-1}^{1} \varrho_j\big(\vartheta(\mu_j^{\mathrm{stn}})\big)\, {}_{-1}\mathcal{D}_{\xi_j}^{\mu_j^{\mathrm{stn}}}\Big(P_n(\xi_j)\Big)\, {}_{\xi_j}\mathcal{D}_1^{\mu_j^{\mathrm{stn}}}\Big(P_r(\xi_j)\Big)\,\mathrm{d}\xi_j\,\mathrm{d}\mu_j^{\mathrm{stn}} \\ &= \int_{-1}^{1} \varrho_j\big(\vartheta(\mu_j^{\mathrm{stn}})\big)\, \frac{\Gamma(r+1)}{\Gamma(r-\mu_j^{\mathrm{stn}}+1)}\, \frac{\Gamma(n+1)}{\Gamma(n-\mu_j^{\mathrm{stn}}+1)} \\ &\quad \times \int_{-1}^{1} (1-\xi_j^2)^{-\mu_j^{\mathrm{stn}}}\, P_r^{-\mu_j^{\mathrm{stn}}, \mu_j^{\mathrm{stn}}}\, P_n^{\mu_j^{\mathrm{stn}}, -\mu_j^{\mathrm{stn}}}\,\mathrm{d}\xi_j\,\mathrm{d}\mu_j^{\mathrm{stn}}. \end{aligned}$$

$\widetilde{S}^{\varrho_j}_{r,n}$ can be computed accurately using Gauss–Legendre (GL) quadrature rules as

$$\begin{aligned} \widetilde{S}^{\varrho_j^{stn}}_{r,n} &= \sum_{q=1}^{\mathcal{Q}} \frac{\Gamma(r+1)}{\Gamma(r-\mu_j^{stn}|_q+1)}\, \frac{\Gamma(n+1)}{\Gamma(n-\mu_j^{stn}|_q+1)}\, \varrho_j|_q\, w_q \\ &\quad \times \int_{-1}^{1} (1-\xi_j^2)^{-\mu_j^{stn}|_q}\, P_r^{-\mu_j^{stn}|_q, \mu_j^{stn}|_q}(\xi_j)\, P_n^{\mu_j^{stn}|_q, -\mu_j^{stn}|_q}(\xi_j)\mathrm{d}\xi_j, \end{aligned} \tag{A3}$$

in which $\mathcal{Q} \geqslant \mathcal{M}_j + 2$ represents the minimum number of GL quadrature points $\{\mu_j^{stn}|_q\}_{q=1}^{\mathcal{Q}}$ for exact quadrature, and $\{w_q\}_{q=1}^{\mathcal{Q}}$ are the corresponding quadrature weights. Exploiting the property of the Jacobi polynomials where $P_n^{\alpha,\beta}(-\xi_j) = (-1)^n P_n^{\beta,\alpha}(\xi_j)$, we have $\widetilde{S}^{\varrho_j^{\mathrm{stn}}}_{r,n} = (-1)^{(r+n)}\widetilde{S}^{\varrho_j^{\mathrm{stn}}}_{n,r}$. Following [50], $\widetilde{\sigma}_r$ and σ_n are chosen, such that $(-1)^{(n+r)}$ is canceled. Accordingly, $\{S_l^{\varrho_j}\}_{n,r} = \{S_l^{\varrho_j}\}_{r,n} = \{S_r^{\varrho_j}\}_{r,n} = \{S_r^{\varrho_j}\}_{r,n}$ due to the symmetry of $S_l^{\varrho_j}$ and $S_r^{\varrho_j}$. Similarly, we get $\{S_l^{\rho_j}\}_{n,r} = \{S_l^{\rho_j}\}_{r,n} = \{S_r^{\rho_j}\}_{n,r} = \{S_r^{\rho_j}\}_{r,n}$. Eventually, we conclude that the stiffness matrix $S_l^{o_j}$, $S_r^{o_j}$, $S_l^{\rho_j}$, $S_r^{\rho_j}$, and thereby $\{S_j^{\mathrm{Tot}}\}_{n,r}$ as the sum of symmetric matrices is symmetric.

References

1. Abbaszadeh, M., Dehghan, M.: An improved meshless method for solving two-dimensional distributed order time-fractional diffusion-wave equation with error estimate. Numer. Algorithms **75**(1), 173–211 (2017)
2. Ainsworth, M., Glusa, C.: Aspects of an adaptive finite element method for the fractional Laplacian: a priori and a posteriori error estimates, efficient implementation and multigrid solver. Comput. Methods Appl. Mech. Eng. **327**, 4–35 (2017)
3. Ammi, M.R.S., Jamiai, I.: Finite difference and Legendre spectral method for a time-fractional diffusion-convection equation for image restoration. Discrete Contin. Dyn. Syst. Ser. S **11**(1), 103 (2018)
4. Ardakani, A.G.: Investigation of Brewster anomalies in one-dimensional disordered media having Lévy-type distribution. Eur. Phys. J. B **89**(3), 76 (2016)
5. Armour, K.C., Marshall, J., Scott, J.R., Donohoe, A., Newsom, E.R.: Southern Ocean warming delayed by circumpolar upwelling and equatorward transport. Nat. Geosci. **9**(7), 549–554 (2016)
6. Bazhlekova, E., Bazhlekov, I.: Subordination approach to multi-term time-fractional diffusion-wave equations. J. Comput. Appl. Math. **339**, 179–192 (2018)
7. Benson, D.A., Wheatcraft, S.W., Meerschaert, M.M.: Application of a fractional advection-dispersion equation. Water Resour. Res. **36**(6), 1403–1412 (2000)
8. Chechkin, A., Gorenflo, R., Sokolov, I.: Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equations. Phys. Rev. E **66**(4), 046129 (2002)
9. Chen, S., Shen, J., Wang, L.L.: Generalized Jacobi functions and their applications to fractional differential equations. Math. Comput. **85**(300), 1603–1638 (2016)
10. Chen, S., Shen, J., Wang, L.L.: Laguerre functions and their applications to tempered fractional differential equations on infinite intervals. J. Sci. Comput. **74**, 1–28 (2017)
11. Cheng, A., Wang, H., Wang, K.: A Eulerian–Lagrangian control volume method for solute transport with anomalous diffusion. Numer. Methods Partial Differ. Equ. **31**(1), 253–267 (2015)
12. Coronel-Escamilla, A., Gómez-Aguilar, J., Torres, L., Escobar-Jiménez, R.: A numerical solution for a variable-order reaction-diffusion model by using fractional derivatives with non-local and non-singular kernel. Phys. A Stat. Mech. Appl. **491**, 406–424 (2018)
13. Duan, B., Jin, B., Lazarov, R., Pasciak, J., Zhou, Z.: Space-time Petrov–Galerkin FEM for fractional diffusion problems. Comput. Methods Appl. Math. **18**, 1 (2017)
14. Duan, J.S., Baleanu, D.: Steady periodic response for a vibration system with distributed order derivatives to periodic excitation. J. Vib. Control **21**, 3124 (2018)
15. Eab, C., Lim, S.: Fractional Langevin equations of distributed order. Phys. Rev. E **83**(3), 031136 (2011)
16. Edery, Y., Dror, I., Scher, H., Berkowitz, B.: Anomalous reactive transport in porous media: experiments and modeling. Phys. Rev. E **91**(5), 052130 (2015)
17. Ervin, V.J., Roop, J.P.: Variational solution of fractional advection dispersion equations on bounded domains in $\mathbb{R}_d$. Numer. Methods Partial Differ. Equ. **23**(2), 256 (2007)
18. Fan, W., Liu, F.: A numerical method for solving the two-dimensional distributed order space-fractional diffusion equation on an irregular convex domain. Appl. Math. Lett. **77**, 114–121 (2018)
19. Gorenflo, R., Luchko, Y., Yamamoto, M.: Time-fractional diffusion equation in the fractional Sobolev spaces. Fract. Calc. Appl. Anal. **18**(3), 799–820 (2015)
20. Goychuk, I.: Anomalous transport of subdiffusing cargos by single kinesin motors: the role of mechano-chemical coupling and anharmonicity of tether. Phys. Biol. **12**(1), 016,013 (2015)
21. Iwayama, T., Murakami, S., Watanabe, T.: Anomalous eddy viscosity for two-dimensional turbulence. Phys. Fluids **27**(4), 045104 (2015). https://doi.org/10.1063/1.4916956
22. Jin, B., Lazarov, R., Sheen, D., Zhou, Z.: Error estimates for approximations of distributed order time fractional diffusion with nonsmooth data. Fract. Calc. Appl. Anal. **19**(1), 69–93 (2016)
23. Jin, B., Lazarov, R., Thomée, V., Zhou, Z.: On nonnegativity preservation in finite element methods for subdiffusion equations. Math. Comput. **86**(307), 2239–2260 (2017)
24. Kharazmi, E., Zayernouri, M.: Fractional pseudo-spectral methods for distributed-order fractional PDEs. Int. J. Comput. Math. **95**(6/7), 1340–1361 (2018)

25. Kharazmi, E., Zayernouri, M., Karniadakis, G.E.: Petrov–Galerkin and spectral collocation methods for distributed order differential equations. SIAM J. Sci. Comput. **39**(3), A1003–A1037 (2017)
26. Kharazmi, E., Zayernouri, M., Karniadakis, G.E.: A Petrov–Galerkin spectral element method for fractional elliptic problems. Comput. Methods Appl. Mech. Eng. **324**, 512–536 (2017)
27. Klages, R., Radons, G., Sokolov, I.M.: Anomalous Transport: Foundations and Applications. Wiley-VCH, New York (2008)
28. Konjik, S., Oparnica, L., Zorica, D.: Distributed-order fractional constitutive stress-strain relation in wave propagation modeling. Zeitsch. Angew. Math. Phys. **70**(2), 51 (2019)
29. Li, X., Rui, H.: Two temporal second-order H1-Galerkin mixed finite element schemes for distributed-order fractional sub-diffusion equations. Numer. Algorithms **79**(4), 1107–1130 (2018)
30. Li, X., Xu, C.: A space-time spectral method for the time fractional diffusion equation. SIAM J. Numer. Anal. **47**(3), 2108–2131 (2009)
31. Li, X., Xu, C.: Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation. Commun. Comput. Phys. **8**(5), 1016 (2010)
32. Liao, Hl, Lyu, P., Vong, S., Zhao, Y.: Stability of fully discrete schemes with interpolation-type fractional formulas for distributed-order subdiffusion equations. Numer. Algorithms **75**(4), 845–878 (2017)
33. Luchko, Y.: Boundary value problems for the generalized time-fractional diffusion equation of distributed order. Fract. Calc. Appl. Anal. **12**(4), 409–422 (2009)
34. Macías-Díaz, J.: An explicit dissipation-preserving method for Riesz space-fractional nonlinear wave equations in multiple dimensions. Commun. Nonlinear Sci. Numer. Simul. **59**, 67–87 (2018)
35. Maday, Y.: Analysis of spectral projectors in one-dimensional domains. Math. Comput. **55**(192), 537–562 (1990)
36. Mainardi, F., Mura, A., Gorenflo, R., Stojanović, M.: The two forms of fractional relaxation of distributed order. J. Vib. Control **13**(9/10), 1249–1268 (2007)
37. Mainardi, F., Mura, A., Pagnini, G., Gorenflo, R.: Time-fractional diffusion of distributed order. J. Vib. Control **14**(9/10), 1267–1290 (2008)
38. Mao, Z., Shen, J.: Efficient spectral-Galerkin methods for fractional partial differential equations with variable coefficients. J. Comput. Phys. **307**, 243–261 (2016)
39. Mao, Z., Shen, J.: Spectral element method with geometric mesh for two-sided fractional differential equations. Adv. Comput. Math. **44**, 1–27 (2017)
40. Mashelkar, R., Marrucci, G.: Anomalous transport phenomena in rapid external flows of viscoelastic fluids. Rheol. Acta **19**(4), 426–431 (1980)
41. Meerschaert, M.M.: Fractional Calculus, Anomalous Diffusion, and Probability. Fractional Dynamics: Recent Advances, pp. 265–284. World Scientific, Singapore (2012)
42. Meerschaert, M.M., Sikorskii, A.: Stochastic Models for Fractional Calculus, vol. 43. Walter de Gruyter, Berlin (2012)
43. Metzler, R., Jeon, J.H., Cherstvy, A.G., Barkai, E.: Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking. Phys. Chem. Chem. Phys. **16**(44), 24128–24164 (2014)
44. Metzler, R., Klafter, J.: The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. **339**(1), 1–77 (2000)
45. Naghibolhosseini, M.: Estimation of outer-middle ear transmission using DPOAEs and fractional-order modeling of human middle ear. Ph.D. thesis, City University of New York, NY (2015)
46. Naghibolhosseini, M., Long, G.R.: Fractional-order modelling and simulation of human ear. Int. J. Comput. Math. **95**, 1–17 (2017)
47. Perdikaris, P., Karniadakis, G.E.: Fractional-order viscoelasticity in one-dimensional blood flow models. Ann. Biomed. Eng. **42**(5), 1012–1023 (2014)
48. Regner, B.M.: Randomness in biological transport. UC San Diego Electronic Theses and Dissertations, UC San Diego (2014)
49. Samiee, M., Akhavan-Safaei, A., Zayernouri, M.: A fractional subgrid-scale model for turbulent flows: theoretical formulation and a priori study. arXiv:1909.09943 (2019)
50. Samiee, M., Zayernouri, M., Meerschaert, M.M.: A unified spectral method for FPDES with two-sided derivatives; part I: a fast solver. J. Comput. Phys. **385**, 225–243 (2019)
51. Samiee, M., Zayernouri, M., Meerschaert, M.M.: A unified spectral method for FPDES with two-sided derivatives; part II: stability, and error analysis. J. Comput. Phys. **385**, 244–261 (2019)
52. Shen, J., Tang, T., Wang, L.L.: Spectral Methods: Algorithms, Analysis and Applications, vol. 41. Springer Science & Business Media, New York (2011)
53. Sokolov, I., Chechkin, A., Klafter, J.: Distributed-order fractional kinetics. arXiv preprint cond-mat/0401146 (2004)

54. Suzuki, J., Zayernouri, M., Bittencourt, M., Karniadakis, G.: Fractional-order uniaxial visco-elasto-plastic models for structural analysis. Comput. Methods Appl. Mech. Eng. **308**, 443–467 (2016)
55. Suzuki, J.L., Zayernouri, M.: An automated singularity-capturing scheme for fractional differential equations. arXiv:1810.12219 (2018)
56. Tian, W., Zhou, H., Deng, W.: A class of second order difference approximations for solving space fractional diffusion equations. Math. Comput. **84**(294), 1703–1727 (2015)
57. Tomovski, Ž., Sandev, T.: Distributed-order wave equations with composite time fractional derivative. Int. J. Comput. Math. **95**, 1–14 (2017)
58. Tomovski, Ž., Sandev, T.: Distributed-order wave equations with composite time fractional derivative. Int. J. Comput. Math. **95**(6/7), 1100–1113 (2018)
59. Tyukhova, A., Dentz, M., Kinzelbach, W., Willmann, M.: Mechanisms of anomalous dispersion in flow through heterogeneous porous media. Phys. Rev. Fluids **1**(7), 074,002 (2016)
60. Varghaei, P., Kharazmi, E., Suzuki, J.L., Zayernouri, M.: Vibration analysis of geometrically nonlinear and fractional viscoelastic cantilever beams. arXiv:1909.02142 (2019)
61. Yamamoto, M.: Weak solutions to non-homogeneous boundary value problems for time-fractional diffusion equations. J. Math. Anal. Appl. **460**(1), 365–381 (2018)
62. Zaky, M.A.: A Legendre collocation method for distributed-order fractional optimal control problems. Nonlinear Dyn. **2667**, 1–15 (2018)
63. Zayernouri, M., Ainsworth, M., Karniadakis, G.E.: Tempered fractional Sturm–Liouville eigenproblems. SIAM J. Sci. Comput. **37**(4), A1777–A1800 (2015)
64. Zayernouri, M., Ainsworth, M., Karniadakis, G.E.: A unified Petrov–Galerkin spectral method for fractional PDES. Comput. Methods Appl. Mech. Eng. **283**, 1545–1569 (2015)
65. Zayernouri, M., Karniadakis, G.E.: Fractional Sturm–Liouville eigen-problems: theory and numerical approximation. J. Comput. Phys. **252**, 495–517 (2013)
66. Zhang, Y., Meerschaert, M.M., Baeumer, B., LaBolle, E.M.: Modeling mixed retention and early arrivals in multidimensional heterogeneous media using an explicit Lagrangian scheme. Water Resour. Res. **51**(8), 6311–6337 (2015)
67. Zhang, Y., Meerschaert, M.M., Neupauer, R.M.: Backward fractional advection dispersion model for contaminant source prediction. Water Resour. Res. **52**(4), 2462–2473 (2016)