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# ON A POLYHARMONIC DIRICHLET PROBLEM AND BOUNDARY EFFECTS IN SURFACE SPLINE APPROXIMATION*

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**Abstract.** For compact domains with smooth boundaries, we present a surface spline approximation scheme that delivers rates in $L_p$ that are optimal for linear approximation in this setting. This scheme can overcome the boundary effects, observed by Johnson [*Constr. Approx.*, 14 (1998), pp. 429–438], by placing centers with greater density near the boundary. It owes its success to an integral identity employing a minimal number of boundary layer potentials, which, in turn, is derived from the boundary layer potential solution to the Dirichlet problem for the $m$-fold Laplacian. Furthermore, this integral identity is shown to be the "native space extension" of the target function.

**Key words.** surface spline, layer potential, polyharmonic, extension operator, Dirichlet problem

**AMS subject classifications.** 35C15, 35J58, 41A25, 46E35

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## 1. Introduction.

In this paper we consider three seemingly unrelated, but connected, problems. The first treats the complication of the boundary in surface spline approximation—this is a fundamental problem for kernel based approximation and is of prime importance for treating scattered data. The second seeks a linear operator that provides smooth extensions to functions defined on bounded domains. The third—the solution of the polyharmonic Dirichlet problem with boundary layer potentials—is a basic problem in potential theory and elliptic PDEs.

*Problem* 1: *Surface spline approximation.* Radial basis function (RBF) approximation involves approximating a target function $f$ by a linear combination of translates of a fixed, radially symmetric function (the RBF) $\phi : \mathbb{R}^d \to \mathbb{R}$ sampled from a finite point set $\Xi \subset \mathbb{R}^d$. The approximant takes the form $s_{f,\Xi}(x) = \sum_{\xi\in\Xi} A_\xi \phi(x-\xi)$, where the coefficients $(A_\xi)_{\xi\in\Xi} \in \mathbb{R}^\Xi$ are to be determined. (For technical reasons, one often permits the addition of an auxiliary, low-degree polynomial term—we ignore this for now, but it is expanded upon later.)

A basic family of RBFs is the family of *surface splines*, which are (up to a constant multiple) the fundamental solutions $\phi_{m,d}$ of the $m$-fold Laplacian in $\mathbb{R}^d$. We consider the approximation power of RBF approximation with surface splines over bounded regions: when $\Omega \subset \mathbb{R}^d$ is bounded, $f : \Omega \to \mathbb{R}$ and $\Xi \subset \Omega$. Specifically, we wish to determine precisely the degradation of error estimates for surface spline approximation in the presence of the boundary and how this may be overcome. A detailed explanation of these "boundary effects" can be found in section 1.1.

*Problem* 2: *Norm minimizing extension.* For a bounded region $\Omega \subset \mathbb{R}^d$ and $f \in W_2^m(\Omega)$, we seek an extension $f_e : \mathbb{R}^d \to \mathbb{C}$ that is best in the sense that it has a

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minimal $m$th seminorm

$$|f_e|_{D^{-m}L_2} := \left( \sum_{|\alpha|=m} \binom{m}{\alpha} \int_{\mathbb{R}^d} |D^\alpha f_e(x)|^2 \mathrm{d}x \right)^{1/2}. \tag{1.1}$$

This is the $m$th Sobolev seminorm, but in this context it is often called the Beppo Levi seminorm. The Beppo Levi space $D^{-m}L_2(\mathbb{R}^d) = \{f \in W_{2,loc}^m(\mathbb{R}^d) \mid |f|_{D^{-m}L_2} < \infty\}$ is a reproducing kernel semi-Hilbert space; it and the above extension have been studied in [12]. There, Duchon has shown that $f_e$ satisfies $f_e = \mu_f * \phi_{m,d} + p$, where $\mu_f$ is a distribution supported in $\overline{\Omega}$ and $p$ is a polynomial of degree at most $m-1$. In [31], Johnson demonstrates that the linear map $f \mapsto \mu_f$ is bounded from $W_2^m(\Omega)$ to $W_2^{-m}(\mathbb{R}^d)$ and from the Besov space $B_{2,1}^{m+1/2}(\Omega)$ to $B_{2,\infty}^{1/2-m}(\mathbb{R}^d)$.

The more general notion of a native space extension operator for a conditionally positive definite kernel, of which this is an example, has been introduced and studied in [41]. The mapping properties of this extension operator have been exploited in scattered data fitting problems, starting with [12] but continuing in [31, 30, 33]. Of particular relevance is the "doubling property" of certain smooth functions observed by Schaback, which leads to faster than expected "superconvergence" rates for interpolation [40, 42]. This is a consequence of the above extension having the form $\mu_f \in L_2(\Omega)$ (such conditions have heretofore been challenging to characterize—we provide a characterization in section 8).

The goal here is to identify the distribution $\mu_f$ explicitly in terms of $f$ on $\overline{\Omega}$, namely in terms of values in $\Omega$ and boundary data on $\partial\Omega$. To date the only case where this is known is when $m = 2$ on the disk $\Omega = B(0,1)$ in $\mathbb{R}^2$ [28].

*Problem* 3: *Layer potential solution of a Dirichlet problem.* For a compact region $\Omega \subset \mathbb{R}^d$, we consider the homogeneous $m$-fold Laplacian with nonhomogeneous boundary conditions:

$$\begin{cases} \Delta^m u(x) = 0 & \text{for } x \in \Omega, \\ \lambda_k u = h_k & \text{for } k = 0 \text{ to } m-1, \end{cases} \tag{1.2}$$

where the boundary differential operators are $\lambda_k := \mathrm{Tr}\Delta^{\frac{k}{2}}$ when $k$ is even and $\lambda_k := D_{\vec{n}}\Delta^{\frac{k-1}{2}}$ when $k$ is odd. (Here $\mathrm{Tr} : C(\overline{\Omega}) \to C(\partial\Omega)$ is the restriction to the boundary and $\vec{n}$ is the outer unit normal to the boundary.) Our goal is to provide a solution using $m$ boundary layer potentials

$$u(x) = \sum_{j=0}^{m-1} \int_{\partial\Omega} g_j(\alpha)\lambda_{j,\alpha}\phi_{m,d}(x-\alpha)\,\mathrm{d}\sigma(\alpha) + p(x) \tag{1.3}$$

with an extra polynomial term $p \in \Pi_{m-1}$. The kernel $\lambda_{j,\alpha}\phi_{m,d}(x-\alpha)$ of the $j$th boundary layer potential is obtained by applying the boundary operator $\lambda_j$ to $(x,\alpha) \mapsto \phi_{m,d}(x-\alpha)$ in the second variable; in other words, $\lambda_{j,\alpha}\phi_{m,d}(x-\alpha) = \big(\lambda_j \phi_{m,d}(x - \cdot)\big)(\alpha)$.

In short, we wish to find auxiliary functions $g_0, \ldots, g_{m-1}$ and $p$ given boundary data $h_0, \ldots, h_{m-1}$.

Of course, there is a well-established theory for Dirichlet problems of the sort (1.2), although solutions of the form (1.3) are not featured prominently in the literature. Many approaches make use of different types of boundary integrals (e.g., Poisson kernels as in [1, 2] or "double layer potentials" as in [35, 17]). Obtaining from these the single layer potential solution we seek (by converting such representations to the form given above) is similar to solving a Dirichlet-to-Neumann problem.

Still, it seems likely that a solution like (1.3) already exists somewhere. We have only been able to find one in the planar biharmonic case $d = 2$, $m = 2$ in [7]. (The higher order problem, $m > 2$, is complicated by the challenge of demonstrating the ellipticity of the resulting integral operator. See section 5.) In any case, the inclusion of its derivation is warranted for the sake of understanding the approximation results, the regularity of the functions $g_j$, and the necessity of the extra polynomial term (whose role in RBF approximation has been a subject of investigation; cf. [22]).

*The connection between the problems.* The solution of each of these three problems hinges on the ability to represent a function $f : \overline{\Omega} \to \mathbb{C}$ with a combination of integrals of the form

$$f(x) = \int_\Omega \Delta^m f(\alpha)\phi_{m,d}(x-\alpha)\mathrm{d}\alpha + \sum_{j=0}^{m-1}\int_{\partial\Omega} N_j f(\alpha)\lambda_{j,\alpha}\phi_{m,d}(x-\alpha)\mathrm{d}\sigma(\alpha) + p(x). \tag{1.4}$$

This representation indicates precisely the distribution $\mu_f$ used in the norm minimizing extension (Problem 2). A special example of its use is to provide the solution to the Dirichlet problem (Problem 3); in turn, the boundary layer solution of (1.2) yields almost directly the formula (1.4).

Finally, a certain discretization of the representation yields an approximation scheme which conveniently addresses the boundary effects. This scheme replaces the kernels appearing in (1.4), namely $\phi_{m,d}(x - \alpha)$ and $\lambda_{j,\alpha}\phi_{m,d}(x - \alpha)$, by new kernels, $k(x, \alpha)$ and $k_j(x, \alpha)$, where $k(x, \alpha) = \sum_{\xi\in\Xi} a(\alpha, \xi)\phi_{m,d}(x-\xi)$ and $k_j(x, \alpha) = \sum_{\xi\in\Xi} a_j(\alpha, \xi)\phi_{m,d}(x - \xi)$. The approximant

$$T_\Xi f(x) = \int_\Omega \Delta^m f(\alpha)k(x, \alpha)\mathrm{d}\alpha + \sum_{j=0}^{m-1}\int_{\partial\Omega} N_j f(\alpha)k_j(x, \alpha)\mathrm{d}\sigma(\alpha) + p(x) \tag{1.5}$$

is an RBF approximant and provides precise approximation orders for surface spline approximation (Problem 1). Moreover, on certain point sets $\Xi$ it successfully treats the boundary effects by permitting rates of convergence matching those of the boundary-free setting. Such a scheme has been introduced in [20] to treat the problem on the disk in $\mathbb{R}^2$, but earlier schemes of this sort have been used in [10, 16]. Similar localizations of the RBF were initially introduced in [15].

**1.1. Background on boundary effects for surface spline approximation.** Boundary effects for surface spline approximation (as well as other RBF methods) have been discussed in [36, section 4]. They are easily observed numerically with practical discussions in numerous later texts [18, 33, 25]. They can also be understood analytically by showing that the approximation order from finite dimensional spaces generated by $\phi_{m,d}$ is prematurely saturated. The meaning of this statement is explained below.

For $J \in \mathbb{N}$, define the space generated by $\phi_{m,d}$ and $\Xi$, augmented by $\Pi_J$ (polynomials of degree at most $J$) with the corresponding moment conditions on the coefficients being

$$S_J(\Xi, \phi_{m,d}) := \left\{ \sum_{\xi\in\Xi} A_\xi\phi_{m,d}(\cdot - \xi) + p \;\middle|\; p \in \Pi_J \quad \forall q \in \Pi_J, \sum_{\xi\in\Xi} A_\xi q(\xi) = 0 \right\}.$$

The $L_p(\Omega)$ approximation order from $S_J(\Xi, \phi_{m,d})$ is defined as $\gamma > 0$ so that

$$\operatorname{dist}(f, S_J(\Xi, \phi_{m,d}))_p := \min_{s \in S_J(\Xi, \phi_{m,d})} \|f - s\|_{L_p(\Omega)} = O(h^\gamma),$$

where $h$, the *fill distance*

$$h := h(\Xi, \Omega) := \sup_{x \in \Omega} \operatorname{dist}(x, \Xi), \tag{1.6}$$

measures the density of $\Xi$ in $\Omega$.

The first positive results concerning approximation orders in this setting were obtained by Duchon. In [13, 12] it was shown that, on domains satisfying an interior cone condition, interpolation of a function in $D^{-m}L_2(\mathbb{R}^d)$ delivers $L_p$ approximation order $\gamma_p := \min(m, m + d/p - d/2)$. More precisely, for the (unique) function $I_\Xi f \in S_{m-1}(\Xi, \phi_{m,d})$ which satisfies $I_\Xi f\,|_\Xi = f\,|_\Xi$, the estimate $\|f - I_\Xi f\|_{L_p(\Omega)} \le Ch^{\gamma_p}\|f\|_{W_2^m(\Omega)}$ holds. The order $\gamma_p$ is illustrated in Figure 1 as a dotted line.

In [34], Madych and Nelson introduced interpolation by surface splines on multi-integer grids, i.e., where centers are assumed to be $h\mathbb{Z}^d$ and the domain of $f$ is all of $\mathbb{R}^d$ (in this case, $\Xi = h\mathbb{Z}^d$ is not finite and the space $S(h\mathbb{Z}^d, \phi_{m,d})$ consists of convergent infinite linear combinations[1]). Buhmann demonstrated that interpolation in this setting enjoys substantially larger approximation orders than observed in the work of Duchon. In [6], it is shown that interpolation by functions in $S(h\mathbb{Z}^d, \phi_{m,d})$ of shifts of $\phi_{m,d}$ delivers approximation order $2m$ for sufficiently smooth functions. Other "free space" results for surface spline approximation were obtained by Dyn and Ron [16], Bejancu [4], Johnson [29], Schaback [40], and DeVore and Ron [10]—these show for various schemes that the approximation order $2m$ can be attained when the boundary can be neglected (by considering centers that are reasonably sampled throughout $\mathbb{R}^d$, or in a sufficiently large neighborhood of $\Omega$, or by considering functions which are compactly supported in $\Omega$ or come from some other (smaller) class of functions for which boundary effects are not an issue). This approximation order is illustrated in Figure 1 as a solid, horizontal line.

The inverse result of Johnson [27] shows that for $\Omega = B$, the unit ball in $\mathbb{R}^d$, $\Xi \subset (1 - \frac{1}{2}h)B$, and for any $J$, $1 \le p \le \infty$, there exists $f \in C^\infty(\overline{B})$ such that

$$\operatorname{dist}(f, S_J(\Xi, \phi_{m,d}))_p \ne o(h^{m+1/p}). \tag{1.7}$$

(This result holds regardless of the polynomial space $\Pi_J$, including $\Pi_{-1} = \{0\}$.) This upper bound on the approximation order is illustrated in Figure 1 as a dashed line.

The current state of the art for surface spline approximation with scattered centers in bounded domains comes from interpolation by functions in $S_{m-1}(\Xi, \phi_{m,d})$. We separate this into two cases, depending on the parameter $p$. For $\Omega \subset \mathbb{R}^d$ having a sufficiently smooth boundary and for sufficiently smooth $f$ (specifically, for $f$ in the Sobolev space $W_2^{m+1}(\mathbb{R}^d)$ when $p = 1$ and for $f$ in the Besov space $B_{2,1}^{m+1/p}(\mathbb{R}^d)$ when $1 < p \le 2$), the rate

$$\|f - I_\Xi f\|_p = O(h^{m+1/p}) \tag{1.8}$$

holds for $1 \le p \le 2$—this is to be found in [31]. By the upper bound (1.7), this is the best possible approximation order. On the other hand, in [30] it has been shown

[1]Because $\phi_{m,d}$ has global support, one considers linear combinations generated by a bounded, rapidly decaying "localization" $\psi = \sum_{j \in \mathbb{Z}^d} a_j \phi_{m,d}(\cdot - j)$ of shifts of $\phi_{m,d}$.

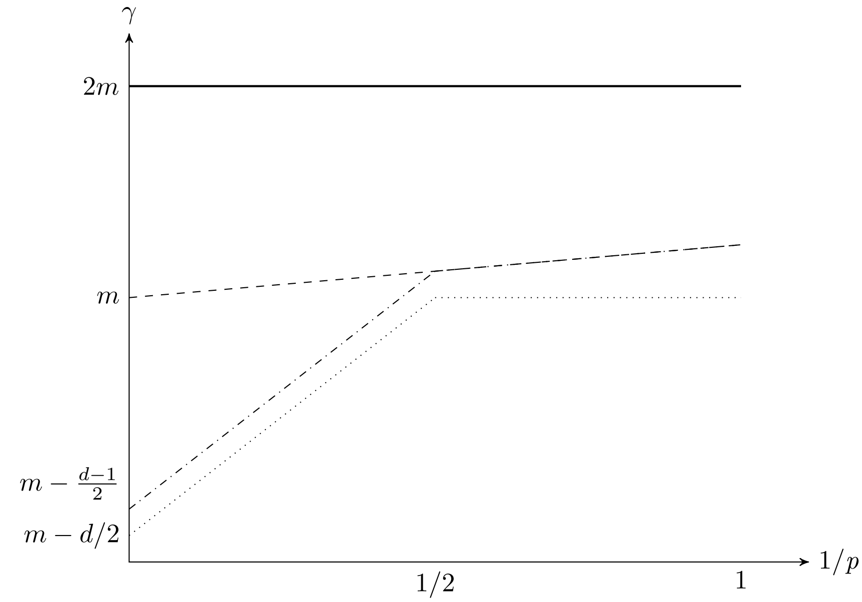


FIG. 1. *Graphs of the boundary-free $L_p$ approximation order (solid), and Johnson's upper bound on the $L_p$ approximation order in the presence of the boundary (dashed) and Duchon's $L_p$ approximation order (dots). The current best $L_p$ approximation order in the presence of a smooth boundary is the dash-dotted broken line.*

that, for $p > 2$ and for sufficiently smooth $f$ (for $f$ in the Besov space $B^{m+1/2}_{2,1}(\mathbb{R}^d)$),

$$\|f - I_\Xi f\|_p = O(h^{\gamma_p+1/2}) \tag{1.9}$$

holds. This result, for the case $p = 2$, has recently been studied again [33] using techniques from elliptic PDEs and confirming the saturation order on $B^{m+1/2}_{2,1}$.

Thus there is a gap between the best approximation order for $p > 2$ and Johnson's upper bound (1.7). This situation is reflected in Figure 1. Moreover, the classes of functions for which (1.9) and (1.8) hold—except when $p = 2$—are smaller than one would expect (in particular, for (1.8), where $1 \le p < 2$, smoothness is measured in the stronger $L_2$ norm rather than the weaker $L_p$ norm).

In this paper, we show that convergence rate $\operatorname{dist}(f, S_{m-1}(\Xi, \phi_{m,d}))_p = \mathcal{O}(h^{m+1/p})$ holds for target functions $f \in B^{m+1/p}_{p,1}(\Omega)$ when $1 < p < \infty$ and slightly smaller spaces when $p = 1, \infty$.

**1.2. Overview.** The goals of this paper are to demonstrate that the representation (1.4) holds, to study the regularity properties of auxiliary functions $g_j$, to use this to attack the boundary effects in surface spline approximation with the aid of the scheme (1.5), and to give an explicit representation of the Beppo Levi extension operator.

The basic strategy of using the solution of (1.2) to obtain (1.4) is introduced in section 2. This section contains the main theorems concerning the solution of (1.2), the validity of the identity (1.4), and the regularity of the boundary operators $N_j$ (some technical proofs are given later).

Mapping properties of the boundary layer operators used in (1.4) and in the solution of the Dirichlet problem are studied in section 3. In particular, the regularity of such operators "up to the boundary" is studied here, as well as jump conditions and transposition of the boundary operators. These results may be well known to some readers (e.g., many can be found in [1]); they are included here to keep the manuscript self-contained and because these results are used in later sections.

Section 4 treats the solution of (1.2) by a boundary integral method adapted from a technique treating the biharmonic problem used in [8, 7]. It recasts the problem initially as an integral equation which can be solved by providing a bounded inverse to an integral operator $L$ acting between reflective Banach spaces.

Section 5 uses the theory of pseudodifferential operators to analyze the problem. It calculates the principal symbol of the integral operator described in section 4 and shows that it is elliptic. This is used to determine mapping properties of $L$, as well as to show that $L$ has closed range.

Section 6 gives proofs of the main theorems (which have been stated in section 2).

In section 7 we present and study the surface spline approximation scheme which treats functions defined on bounded regions using $S_{m-1}(\Xi, \phi_{m,d})$. The section is devoted to establishing the approximation power of this scheme and to showing how oversampling near the boundary can overcome boundary effects.

In section 8 we discuss how (1.4) provides the extension which minimizes the Sobolev seminorm. This is then connected to the improved interpolation error estimates of Schaback (sometimes called superconvergence or "doubling").

### 1.3. Notation and background.

*Types of domains considered.* We consider bounded, connected, open $\Omega \subset \mathbb{R}^d$ having a $C^\infty$ outer normal, which we denote by $\vec{n} : \partial\Omega \to \mathbb{S}^{d-1}$. In a neighborhood $\mathcal{N}(\partial\Omega) := \partial\Omega + B(0, \epsilon_0)$ of the boundary of $\Omega$, we can describe $\partial\Omega$ as the zero set of a "signed distance function" $\rho : \mathcal{N}(\partial\Omega) \to (-\epsilon_0, \epsilon_0)$. This means that for all $x \in \mathcal{N}(\partial\Omega)$, there is a unique $\gamma(x) \in \partial\Omega$ with $\operatorname{dist}(x, \partial\Omega) = |x - \gamma(x)| = |\rho(x)|$ which satisfies $\rho(x) < 0$ if and only if $x \in \Omega$.

By extending the normal vector field to the neighborhood of the boundary via $\vec{n}(x) = \vec{n}(\gamma(x))$ for $x \in \mathcal{N}(\partial\Omega)$, we can smoothly extend the boundary differential operators $\lambda_j$ to $\mathcal{N}(\partial\Omega)$ as well:

$$\Lambda_j f(x) := \begin{cases} \Delta^{\frac{j}{2}} f(x) & \text{for even } j, \\ \sum_{\ell=1}^d \vec{n}_\ell(x) \frac{\partial}{\partial x_\ell} \Delta^{\frac{j-1}{2}} f(x) & \text{for odd } j. \end{cases} \tag{1.10}$$

It follows that $\lambda_j = \mathrm{Tr}\Lambda_j$.

*Normal/tangential coordinates.* Suppose $O' \subset \mathbb{R}^{d-1}$ is open and bounded, $W$ is a neighborhood of the closure of $O'$, and $\tilde{\Psi} : W \to \tilde{\Psi}(W) \subset \partial\Omega$ is a diffeomorphism. For $U' = \tilde{\Psi}(O')$, and $\epsilon$ sufficiently small, we have tangential and normal coordinates in $U = U' + B(0, \epsilon) \subset \mathcal{N}(\partial\Omega)$ via

$$\Psi : O \to U : \mathbf{x} = (x', x_d) \mapsto \tilde{\Psi}(x') + x_d \vec{n}\big(\tilde{\Psi}(x')\big). \tag{1.11}$$

Here $O = O' + B(0, \epsilon) \subset \mathbb{R}^d$. Define smooth vector fields $\mathbf{e}_j(\mathbf{x}) = \frac{\partial}{\partial x_j} \Psi(x_1, \dots, x_d)$ for $1 \le j \le d$. The Gram matrix of the Jacobian $D\Psi$ of $\Psi$ is $\mathsf{G} : O \to \mathrm{GL}(d, \mathbb{R})$, with $\mathsf{G}_{i,j} = \langle \mathbf{e}_i(\mathbf{x}), \mathbf{e}_j(\mathbf{x}) \rangle$. We denote its inverse by $\mathsf{G}^{-1} = \big(\mathsf{G}^{i,j}\big)_{i,j}$.

We note that $\rho(\Psi(x)) = x_d$. It follows that $(\nabla\rho)(\Psi(\mathbf{x})) = \vec{n}(\tilde{\Psi}(x')) = \mathbf{e}_d(\mathbf{x})$. For fixed $t \in (-\epsilon_0, \epsilon_0)$, let $\partial\Omega_t := \{u \in \mathbb{R}^d \mid \rho(u) = t\}$. When $x_d = t$, define the level set $M_t := \partial\Omega_t \cap U = \Psi(O' \times \{t\})$. Since $(\nabla\rho)(\Psi(\mathbf{x}))$ is normal to $M_t$ at $\Psi(\mathbf{x})$, we have that $\langle \mathbf{e}_j(\mathbf{x}), \mathbf{e}_d(\mathbf{x})\rangle = \delta_{j,d}$. Thus $\mathsf{G}$ and $\mathsf{G}^{-1}$ have a two block structure (with a $1\times 1$ block and another of size $d-1\times d-1$). In short, we write

$$\mathsf{G}(\mathbf{x}) = \begin{pmatrix} \mathsf{G}_{d-1}(\mathbf{x}) & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad \mathsf{G}^{-1}(\mathbf{x}) = \begin{pmatrix} \big(\mathsf{G}_{d-1}(\mathbf{x})\big)^{-1} & 0\\ 0 & 1\end{pmatrix}.$$

*Distributions.* For open $U\subset \mathbb{R}^d$, the spaces of compactly supported test functions and distributions on $U$ are denoted by $\mathcal{D}(U) = C_0^\infty(U)$ and $\mathcal{D}'(U)$. The space of $C^\infty$ functions is $\mathcal{E}(U) = C^\infty(U)$ and the space of compactly supported distributions is $\mathcal{E}'(U)$, while the space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$ and the space of tempered distributions is $\mathcal{S}'(\mathbb{R}^d)$. We identify $C^\infty(U,\mathbb{R}^m)$ and $C_c^\infty(U,\mathbb{R}^m)$ with $\big(\mathcal{E}(U)\big)^m$ and $\big(\mathcal{D}(U)\big)^m$, respectively. The duals are $(\mathcal{D}'(U))^m$ and $(\mathcal{E}'(U))^m$.

For an open set $U\subset\partial\Omega$, $\mathcal{D}(U)$, $\mathcal{E}(U)$, $\mathcal{D}'(U)$, and $\mathcal{E}'(U)$ retain the same meaning. Because $\partial\Omega$ is compact, $\mathcal{E}(\partial\Omega) = \mathcal{D}(\partial\Omega)$ and $\mathcal{E}'(\partial\Omega) = \mathcal{D}'(\partial\Omega)$. Because $\partial\Omega$ is endowed with the surface measure $\sigma$, locally integrable functions can be identified with distributions via the pairing $\langle g,\phi\rangle = \int_{\partial\Omega} g(x)\phi(x)\mathrm{d}\sigma(x)$ (valid for all $\phi\in\mathcal{D}(\partial\Omega)$). For an operator on distributions, we use $()^{\mathrm{t}}$ to indicate the transpose with respect to this pairing, so $\langle M^{\mathrm{t}}T, f\rangle = \langle T, Mf\rangle$.

*Pullback.* For open sets $U, O\subset\mathbb{R}^d$ and a smooth diffeomorphism $\Psi: O\to U$, the *pullback* of a smooth function is $\Psi^*(g) = g\circ\Psi$. The pullback extends continuously as a map between $\mathcal{E}'(U)\to\mathcal{E}'(O)$ and $\mathcal{D}'(U)\to\mathcal{D}'(O)$.

The pullback of the surface measure $\delta_{\partial\Omega} : g\mapsto \int_{\partial\Omega} g(x)\mathrm{d}\sigma(x)$ is obtained by writing $\partial\Omega$ as the zero set of $\rho:\mathbb{R}^d\to\mathbb{R}$. We have $\rho^*\delta = \delta_{\partial\Omega}$ (cf. [23, Theorem 6.1.5]). If $\Psi: O\to U$ maps $O\cap\mathbb{R}^{d-1}$ to $U\cap\partial\Omega$ (for instance if we use tangential and normal coordinates), then it follows that $\Psi^*\delta_{\partial\Omega} = (\rho\circ\Psi)^*\delta = \delta_{\mathbb{R}^{d-1}}$, the standard Lebesgue measure on $\mathbb{R}^{d-1}\times\{0\}$. Distributions of the form $f\cdot\delta_{\partial\Omega}$ supported in $U$ are transformed according to $\Psi^*\left(f\cdot\delta_{\partial\Omega}\right) = (\Psi^* f)\cdot\delta_{\mathbb{R}^{d-1}}$.

*Coordinate change.* By conjugating with $\Psi^*$, we express an operator $A:\mathcal{D}'(U)\to\mathcal{D}'(U)$ *in coordinates* on $O$ as $A^\Psi = \Psi^* A(\Psi^*)^{-1}$.

For $f\in C^\infty(U)$, let $F = f\circ\Psi$. Then $\nabla^\Psi F = \sum_{k=1}^d\sum_{j=1}^d \mathsf{G}^{jk}\frac{\partial F}{\partial x_j}\mathbf{e}_k$. The Laplace operator in coordinates is $\Delta^\Psi F(x) = \sum_{j,k=1}^d \frac{1}{\sqrt{\det\mathsf{G}}}\frac{\partial}{\partial x_j}\big(\mathsf{G}^{k,j}\sqrt{\det\mathsf{G}}\frac{\partial}{\partial x_k}F(x)\big)$.

*Operators in normal and tangential coordinates.* For $\mathbf{u} = \Psi(\mathbf{x})\in\partial\Omega_t$, the unit normal is $\mathbf{e}_d(\mathbf{x})$. The vector fields $\mathbf{e}_1|_{\partial\Omega_t},\dots,\mathbf{e}_{d-1}|_{\partial\Omega_t}$, which lie tangent to $\partial\Omega_t$, have corresponding Gram matrix $\mathsf{G}_{d-1}|_{M_t}$. The Laplace–Beltrami operator $\Delta_t$ for $\partial\Omega_t$ is given in coordinates by

$${\Delta_t}^\Psi F(\mathbf{x}) = \sum_{j,k=1}^{d-1}\frac{1}{\sqrt{\det\mathsf{G}_{d-1}(\mathbf{x})}}\frac{\partial}{\partial x_j}\left(\mathsf{G}^{k,j}(\mathbf{x})\sqrt{\det\mathsf{G}_{d-1}(\mathbf{x})}\frac{\partial}{\partial x_k}F(\mathbf{x})\right).$$

From these observations, it follows that the Laplacian can be decomposed as

$$\Delta^\Psi F = {\Delta_t}^\Psi F + \frac{\partial^2}{\partial x_d^2}F + \mu(\mathbf{x})\frac{\partial}{\partial x_d}F, \tag{1.12}$$

with $\mu(\mathbf{x}) := \frac{1}{\sqrt{\det\mathsf{G}(\mathbf{x})}}\frac{\partial}{\partial x_d}\sqrt{\det\mathsf{G}(x)} = \Psi^*(\mathrm{div}\vec{n})\in C^\infty(O)$.

For $f\in\mathcal{D}'(U)$, we get $\Psi^* D_{\vec{n}} f = \frac{\partial}{\partial x_d}\Psi_* f$. Likewise, $D_{\vec{n}}^{\mathrm{t}} f = -\nabla(\vec{n}f) = -\langle\vec{n},\nabla f\rangle - (\mathrm{div}\vec{n})f$ satisfies $\Psi^* D_{\vec{n}}^{\mathrm{t}} f = -\frac{\partial}{\partial x_d}\Psi^* f - \mu\Psi^* f$.

*Fourier transform.* For $f\in L_1(\mathbb{R}^d)$, we define $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi\rangle}\mathrm{d}x$. This is extended to tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ in the usual way. For $f\in\mathcal{S}'(\mathbb{R}^d)$ and

$g \in \mathcal{S}(\mathbb{R}^d)$, we have the usual Parseval identity $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$. The inverse Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is denoted by $f^\vee$. For distributions which are represented by integrable functions, this is given by the integral $f^\vee(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} f(\xi)e^{i\langle x,\xi\rangle}\mathrm{d}\xi$.

*Smoothness spaces.* For $1 \le p < \infty$ and $k \in \mathbb{N}$, $W_p^k(\Omega)$ denotes the standard Sobolev space over $\Omega$. When $p = \infty$, we use the standard space $C^k(\overline{\Omega})$ of functions having continuous $k$th order derivatives up to the boundary of $\Omega$. For noninteger orders, we consider two main extensions.

For $s \in (0,\infty)$, $1 \le p < \infty$, and $1 \le q \le \infty$, the Besov space $B^s_{p,q}(\Omega)$ is the real interpolation space $[W_p^m(\Omega), W_p^k(\Omega)]_{\theta,q}$, with $\theta = \frac{s-k}{m-k}$. When $p = \infty$ and $s \in (0,\infty)\setminus \mathbb{N}$, we consider $C^s(\overline{\Omega})$ the Hölder space; it is well known that $C^s(\overline{\Omega}) = B^s_{\infty,\infty}(\Omega) = [C^m(\Omega), C^k(\Omega)]_{\theta,\infty}$, where $m, k \in \mathbb{N}$ and $\theta = \frac{s-k}{m-k}$. See [47] for background and further references on Besov and Hölder spaces.

For $1 < p < \infty$ and $s \in \mathbb{R}$, we define the the Bessel potential space $H_p^s(\mathbb{R}^d)$ as

$$H_p^s(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \mid \big((1+|\cdot|^2)^{s/2}\widehat{f}\big)^\vee \in L_p(\mathbb{R}^d)\}.$$

It is the preimage under the Bessel potential $J^s = (1-\Delta)^{s/2}$ of $L_p(\mathbb{R}^d)$ and is equipped with the norm $\|f\|_{H_p^s} := \|J^s f\|_p$. When $s = k \in \mathbb{N}$, $H_p^s(\mathbb{R}^d)$ coincides with $W_p^k(\mathbb{R}^d)$. Furthermore, these are particular examples of Triebel–Lizorkin spaces, namely $H_p^s = F^s_{p,2}$. See [47, section 1.3.2] and references therein for background.

We denote the space of compactly supported distributions in $H_p^s(\mathbb{R}^d)$ (resp., $W_p^k(\mathbb{R}^d)$) by $H^s_{p,c}(\mathbb{R}^d)$ (resp., $W^k_{p,c}(\mathbb{R}^d)$). Likewise, $H^s_{p,loc}(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d) \mid (\forall \psi \in \mathcal{D}(\mathbb{R}^d))\, f\psi \in H_p^s(\mathbb{R}^d)\}$, and $W^s_{p,loc}(\mathbb{R}^d)$ has the obvious modification.

Of special importance is the fact that, for all $s \in \mathbb{R}$, pointwise multiplication by smooth functions is continuous: for every $s, p$, there are a constant $C$ and an integer $m \in \mathbb{N}$ so that if $f \in H_p^s(\mathbb{R}^d)$ and $g \in C^\infty$, then $\|fg\|_{H_s^p} \le C\|g\|_{C^m}\|f\|_{H_p^s}$ (see [47, Theorem 4.2.2]). Similarly, for a diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$, there is a constant $C$ so that for all $f \in H_p^s$ we have $\|\Phi^* f\|_{H_p^s} \le C\|f\|_{H_p^s}$. It follows that if $K \subset O$ is compact and $\Phi : U \to O$ is a diffeomorphism between open sets in $\mathbb{R}^d$, then there is a constant $C_K$ so that for all $f \in H_p^s$ with support $\mathrm{supp}(f) \subset K$ the estimate $\|\Phi^* f\|_{H_p^s} \le C_K\|f\|_{H_p^s}$ holds. The dual of $H_p^s(\mathbb{R}^d)$ is identified with $H_{p'}^{-s}(\mathbb{R}^d)$ in the sense that the pairing $\langle g, f\rangle_{H_{p'}^{-s}, H_p^s}$ is the extension by continuity of the $L_2$ pairing. (This is roughly [45, Remark 7.1.9].)

*Smoothness spaces on $\partial\Omega$.* Let $(U_j, \Phi_j : U_j \to O_j \subset \mathbb{R}^{d-1})$ be an atlas for $\partial\Omega$, and let $(\tau_j)$ be a partition of unity subordinate to $(U_j)$. For $1 < p < \infty$, we define the Bessel potential spaces $H_p^s(\partial\Omega)$ by way of the norm $\|f\|_{H_p^s(\partial\Omega)} := \sum_j \|(\Phi_j^{-1})^*(\tau_j f)\|_{H_p^s(\mathbb{R}^{d-1})}$.

**2. Multilayer representation of functions.** In this section we discuss the key identity

$$f(x) = \int_\Omega \Delta^m f(\alpha)\phi(x-\alpha)\,\mathrm{d}\alpha + \sum_{j=0}^{m-1}\int_{\partial\Omega} g_j(\alpha)\lambda_{j,\alpha}\phi(x-\alpha)\,\mathrm{d}\sigma(\alpha) + p(x) \tag{2.1}$$

(with $p \in \Pi_{m-1}$), which we later show is valid for smooth functions. The identity determines $f$ from its $m$-fold Laplacian and $m$ layer potentials

$$V_j g_j(x) := \int_{\partial\Omega} g_j(\alpha)\lambda_{j,\alpha}\phi(x-\alpha)\,\mathrm{d}\sigma(\alpha).$$

Each layer potential involves an auxiliary boundary function $g_j$ and a kernel $\lambda_{j,\alpha}\phi(x-\alpha)$ obtained from

$$\phi(x)=\phi_{m,d}(x):=C_{m,d}\begin{cases}|x|^{2m-d}\log|x| & d\,\text{is even},\\ |x|^{2m-d} & d\,\text{is odd},\end{cases} \tag{2.2}$$

the fundamental solution of $\Delta^m$ in $\mathbb{R}^d$; cf. [3, equation (2.11)].

For the remainder of the paper, we assume $d$ and $m > d/2$ have been fixed, writing $\phi$ in place of $\phi_{m,d}$. As the fundamental solution, $\Delta^m\phi(x) = 0$ for $x \neq 0$ and $\phi \in C^\infty(\mathbb{R}^d \setminus \{0\})$. By direct differentiation of (2.2), one easily sees that

$$|D^\beta\phi(x)| \le \begin{cases} C_{m,d,\beta}|x|^{2m-d-|\beta|}(\log|x|+1), & |\beta|\le 2m-d,\\ C_{m,d,\beta}|x|^{2m-d-|\beta|}, & |\beta|>2m-d\end{cases} \tag{2.3}$$

(see [21, Claim 5] and the subsequent discussion). A consequence, used throughout this paper, concerns convolution of $\phi$ with compactly supported distributions that annihilate polynomials (such convolutions are well defined, at least on the complement of the support of the distribution).

Lemma 2.1. *Let $L \ge 2m-d$. For a compactly supported distribution $F$ for which $F \perp \Pi_L$ and for $x \notin \operatorname{supp}(F)$, $|F * \phi(x)| \le C(1+|x|)^{2m-d-L-1}$ holds (with constant $C$ depending on $F$).*

For functions $u, v$, we have *Green's formula*

$$\int_\Omega u(x)\Delta^m v(x) - v(x)\Delta^m u(x)\mathrm{d}x = \sum_{j=0}^{2m-1}(-1)^j\int_{\partial\Omega}\lambda_j u(x)\lambda_{2m-j-1}v(x)\mathrm{d}\sigma(x), \tag{2.4}$$

which follows directly from the divergence theorem and holds for a general class of domains $\Omega$ (we will be satisfied by considering bounded domains with smooth boundaries) and for all functions $u, v$ in $C^{2m}(\overline{\Omega})$. A consequence of this is *Green's representation* (see [3, equation (2.11)]) for smooth functions:

$$\int_\Omega \Delta^m f(\alpha)\phi(x-\alpha)\mathrm{d}\alpha + \sum_{j=0}^{2m-1}(-1)^j\int_{\partial\Omega}(\lambda_j f)(\alpha)\ \lambda_{2m-j-1,\alpha}\phi(x-\alpha)\,\mathrm{d}\sigma(\alpha) = \begin{cases} f(x), & x\in\Omega,\\ 0, & x\in\mathbb{R}^d\setminus\overline{\Omega}.\end{cases} \tag{2.5}$$

This determines $f$ from its $m$-fold Laplacian and $2m$ boundary values $\lambda_j f$, with $j = 0, \dots, 2m-1$.

This representation is unsatisfactory for our purposes (i.e., producing an approximation operator using scattered translates of the fundamental solution $\phi$). Although we could attempt to discretize (2.4) to obtain an approximation operator similar to (1.5), the higher order derivatives of $\phi$ at the boundary are too singular, causing a degradation in the approximation power of the scheme.

To simplify the problem, we decompose $f = f_1 + f_2$ into a solution, $f_1$, of the polyharmonic Dirichlet problem (1.2) with boundary values from $f$ and a part, $f_2 = f - f_1$, which vanishes to $m$th order at the boundary and satisfies $\Delta^m f_2 = \Delta^m f$. The identity (2.1) will follow if $f_1$ can be expressed as

$$f_1(x) = \sum_{j=0}^{m-1}\int_{\partial\Omega} g_j(\alpha)\lambda_{j,\alpha}\phi(x-\alpha)\,\mathrm{d}\sigma(\alpha) + p(x) \tag{2.6}$$

and is sufficiently smooth near the boundary. We could then apply (2.5) to obtain

$$f_2 = \int_\Omega \Delta^m f(\alpha)\phi(\cdot - \alpha)\,\mathrm{d}\alpha + \sum_{k=0}^{m-1}(-1)^k \int_{\partial\Omega} \lambda_{2m-k-1} f_2(\alpha)\ \lambda_{k,\alpha}\phi(\cdot-\alpha)\,\mathrm{d}\sigma(\alpha), \tag{2.7}$$

since the lowest order boundary values of $f_2$ vanish (for $j = 0,\dots,m-1$, $\lambda_j f_2 = \lambda_j f - \lambda_j f_1 = 0$). In summary, to obtain (2.1) we show that the following hold:

[A] solutions of Dirichlet's problem are of the form (2.6);

[B] for smooth boundary data, functions of the form (2.6) are smooth near the boundary.

Item [A] is the subjection of sections 3–6, where we will demonstrate the following theorem.

THEOREM 2.2. *For functions $h_0,\dots,h_{m-1}$ with $h_k \in C^\infty(\partial\Omega)$, there is a function $u$ satisfying* (1.2) *and having the form* (2.6) *with $g_j \in C^\infty(\partial\Omega)$ for each $j$. Moreover, for each $j = 0,\dots,m-1$ and $1 < p < \infty$, we have, for $s \ge 0$,*

$$\|g_j\|_{H_p^{s+j+1-2m}(\partial\Omega)} \le C_{s,p} \max_{k=0,\dots,m-1} \|h_k\|_{H_p^{s-k}(\partial\Omega)}.$$

*Proof.* The proof of this theorem is given in section 6. ∎

If $h_k = \lambda_k f$ for some $f \in W_2^m(\Omega)$, then $h_k \in H_2^{m-k-1/2}(\partial\Omega)$ by the trace theorem. A consequence of Theorem 2.2 is a converse of sorts: a polyharmonic extension to $\mathbb{R}^d$ from the Dirichlet data.

COROLLARY 2.3. *Suppose $h_k \in H_2^{m-k-1/2}(\partial\Omega)$ for $k = 0,\dots,m-1$. Then there exist $p \in \Pi_{m-1}$ and $g_j \in H_2^{j+1/2-m}(\partial\Omega)$ for $j = 0,1,\dots,m-1$, so that $u = \sum V_j g_j + p \in W_{2,loc}^m(\mathbb{R}^d)$ and $u$ solves* (1.2).

Item [B] requires understanding the boundary behavior of the layer potential solution (2.6), which will be developed along the way.

Of course, along with the representation (2.1), we also expect the auxiliary boundary functions $g_j$ to be sufficiently regular, determined by operators (*trace operators*) applied to $f$ that map appropriate $L_p$ smoothness spaces continuously into $L_p$. This is summarized in the main theorem of this section.

THEOREM 2.4. *For $f \in C^{2m}(\overline{\Omega})$, the representation* (2.1) *holds pointwise, and for $1 < p < \infty$, the representation* (2.1) *holds a.e. for $f \in W_p^{2m}(\Omega)$. The functions $g_j$ are given by linear operators: $g_j = N_j f$. For $s \ge 0$ and $1 < p < \infty$, the operator $N_j : B_{p,1}^{s+2m-j-1+1/p}(\Omega) \to H_p^s(\partial\Omega)$ is bounded.*

*Proof.* The proof of this theorem is given in section 6. ∎

**3. Boundary layer potential operators.** We now consider the boundary layer potential operators $V_j$ defined initially on $L_1(\partial\Omega)$:

$$V_j g(x) = \int_{\partial\Omega} g(\alpha)\lambda_{j,\alpha}\phi(x-\alpha)\,\mathrm{d}\alpha. \tag{3.1}$$

In this section we make initial analytic observations of $V_j$: showing continuous extension to distributions and investigating the smoothness of functions $V_j g$ near the boundary. These are nontrivial, but well-known, properties of $V_j$, and they are necessary to mention for our later work.

**3.1. Boundary layer potentials as convolutions.** The boundary layer potential operators introduced in (3.1) can be viewed as convolutions of derivatives of $\phi$ with certain distributions supported on the boundary $\partial\Omega$. The distributions in question are derivatives of $g \cdot \delta_{\partial\Omega} \in \mathcal{E}'(\mathbb{R}^d)$ when $g \in \mathcal{D}'(\partial\Omega)$. (Specifically, this means $\langle g \cdot \delta_{\partial\Omega}, \varphi\rangle = \langle g, \varphi\,|_{\partial\Omega}\rangle$ for $\varphi \in C^\infty(\mathbb{R}^d)$.) The map $g \mapsto g \cdot \delta_{\partial\Omega}$ is continuous from $\mathcal{D}'(\partial\Omega)$ to $\mathcal{E}'(\mathbb{R}^d)$.

For $g \in \mathcal{D}'(\partial\Omega)$, we define $V_j g$ as a convolution $V_j g := \phi * \big((\Lambda_j)^{\mathrm{t}}(g \cdot \delta_{\partial\Omega})\big)$ where the formally transposed operator $\Lambda_j^{\mathrm{t}}$ is a differential operator of order $j$; namely, $\Lambda_j^{\mathrm{t}} = \Delta^{\frac{j}{2}}$ when $j$ is even, and $\Lambda_j^{\mathrm{t}} = -\sum_{\ell=1}^d \vec{n}_\ell(x)\frac{\partial}{\partial x_\ell}\Delta^{\frac{j-1}{2}} + \sum_{|\beta|\le j-1} A_\beta(x) D^\beta$ for odd $j$.

When $g$ is an integrable function, $g \cdot \delta_{\partial\Omega}$ is the measure $\varphi \mapsto \int_{\partial\Omega}\varphi(\alpha)g(\alpha)\mathrm{d}\sigma(\alpha)$. In that case, the new definition coincides with our initial one given in (3.1). The expression $\phi * \big[\Lambda_j^{\mathrm{t}}(g \cdot \delta_{\partial\Omega})\big]$, interpreted as a convolution between the tempered distribution $\phi$ and the compactly supported distribution $\big[\Lambda_j^{\mathrm{t}}(g \cdot \delta_{\partial\Omega})\big]$, is thus a tempered distribution as well. It follows that the restriction to $\mathbb{R}^d \setminus \partial\Omega$ is $V_j g(x) = \big\langle (\Lambda_j)^{\mathrm{t}}(g \cdot \delta_{\partial\Omega}), \phi(x - \cdot)\big\rangle$ and $V_j g \in C^\infty(\mathbb{R}^d \setminus \partial\Omega)$. In other words, the operators $V_j$ produce distributions on $\mathbb{R}^d$ with singular support $\partial\Omega$ that are polyharmonic in $\mathbb{R}^d\setminus\partial\Omega$.

The above convolution is an important representation of the operator $V_j$, but it does not adequately indicate the behavior of $V_j g$ near the boundary $\partial\Omega$; this is considered in the next subsection.

**3.2. Boundary regularity.** By (2.3), the kernel $\lambda_{j,\alpha}\phi(x-\alpha)$ is locally integrable on $\partial\Omega$, provided that $0 \le j \le 2m-2$ (this is the case in the construction of $f_1 = \sum_{j=0}^{m-1} V_j g_j + p$ in (2.6)—only nonsingular kernels are used). Unfortunately, this only guarantees a limited smoothness near the boundary $\partial\Omega$; by dominated convergence, for each $j = 0, \dots, 2m-2$, $V_j g \in C^{2m-j-2}(\mathbb{R}^d)$. This is enough to guarantee the existence of the boundary values $\lambda_k f_1$ for $k = 0, \dots, m-1$ required by (1.2), but it is insufficient for higher derivatives—for instance, those required by (2.7).

*Smoothness up to the boundary.* The smoothness of boundary layer potentials in the vicinity of the boundary has been treated in different forms under the heading of "transmission conditions" (cf. [5]) and earlier (cf. [1]). We follow the approach of Duduchava [14], by manipulating Green's representation, to get the following result, which illustrates that for smooth $g$, boundary layer potentials $V_j g$ have smoothness at the boundary—this is a topic we return to in the next section, where we consider the mapping properties of operators $\mathrm{Tr}\Lambda_k V_j g$.

LEMMA 3.1. *For an integer $0 \le j \le 2m-1$, let $s$ be an integer greater than $j+1$. For $g \in C^s(\partial\Omega)$, there is a function $F \in C^{2m+s-j-1}(\mathbb{R}^d)$, so that $\lambda_k F = 0$ for $k = 0, \dots, 2m+s-j-1$, $k \ne 2m-j-1$, and $\lambda_{2m-j-1}F = g$. Furthermore, there is a constant $C$ (independent of $g$), so that $\|F\|_{C^{s+2m-j-1}(\mathbb{R}^d)} \le C\|g\|_{C^s(\partial\Omega)}$.*

*Proof.* Let $L = 2m+s-j-1$. Consider the sequence of $L+1$ "boundary values," $\mathbf{r} = (r_0, \dots, r_L)$ , where each $r_J = 0$ is zero except when $J = 2m-j-1$ entry, which is $r_{2m-j-1} = g$.

We work in normal/tangential coordinates by considering a partition of unity $(\tau_i)_{i\in\mathcal{I}}$ subordinate to a cover $(U_i)_{i\in\mathcal{I}}$ with maps $\Psi_i : O_i \to U_i$ as described in section 1.3. For each $i \in \mathcal{I}$, we obtain a smooth function $f_i : O \to \mathbb{R}$ which has Dirichlet values on $O_i'$ given by $\Psi_i^* r_J : O_i' \to \mathbb{R}$. On the original domain, we then have $F = \sum \tau_i(\Psi^*)^{-1} f_i$.

Rather than working with Dirichlet maps $(\Lambda_J)^\Psi$ for $0 \le J \le L$, we work with the "standard" Dirichlet system ($\frac{d^J}{dx_d^J}$) on $O'$. By [32, Lemma 2.3], we have the relation $\frac{d^J}{dx_d^J}\varphi = \sum_{k\le J} T_{J,k}(\Lambda_k)^\Psi\varphi$, where each $T_{J,k}$ is a (tangential) differential operator of order $J-k$ on $O'$. (We can also work in reverse, obtaining the original maps $\Lambda_J$ in terms of combinations of tangential derivatives of $\frac{d^k}{dx_d^k}$; this is done below.)

For each $J \le L$, set $f_J := \sum_{k\le J} T_{J,k}\Psi^* r_k$. We produce the full collection of "jets" of order $L$ along $O'$ by defining, for $0 \le J \le L$ and $\alpha' \in \mathbb{Z}_+^{d-1}$, $u_{\alpha',J}(x',0) := D^{\alpha'} f_J(x',0)$. These satisfy the requirement of Whitney's extension theorem over $O'$ given in [23, Theorem 2.3.6], so there is an extension $f: O \to \mathbb{R}$ in $C^{2m+s-j-1}(O)$ satisfying $D^\alpha f(x) = u_\alpha$ and $\|f\|_{C^L(\mathbb{R}^d)} \le C \max \|f_J\|_\infty$.

Because $(\Lambda_J)^\Psi = \sum_{k\le J} \tilde{T}_{J,k}\frac{d^k}{dx_d^k}$ with each $\tilde{T}_{J,k}$ a differential operator of order $J-k$ (again by [32, Lemma 2.3]), we have $(\Lambda_J)^\Psi f = \Psi^* r_J$ and $\|f\|_{C^L(\mathbb{R}^d)} \le C\max\|\Psi^* r_J\|_{C^J}$. □

LEMMA 3.2. *For integers $j, s$, with $0 \le j \le 2m-1$ and $s > j+1$, let $g \in C^s(\partial\Omega)$, and let $F \in C^{2m+s-j-1}(\mathbb{R}^d)$ be the function given in Lemma* 3.1. *Then there is $G \in C^{s-j-1}(\mathbb{R}^d)$, so that*

$$V_j g(x) = (-1)^{j-1}\begin{cases}\phi * G(x) - F(x), & x \in \Omega,\\ \phi * G(x), & x \in \mathbb{R}^d \setminus \overline{\Omega}.\end{cases}$$

*Proof.* By applying Green's representation (2.5), we see that

$$\int_\Omega \Delta^m F(\alpha)\phi(x-\alpha)\mathrm{d}\alpha + (-1)^j\int_{\partial\Omega} g(\alpha)\lambda_{j,\alpha}\phi(x-\alpha)\mathrm{d}\sigma(\alpha) = \begin{cases}F(x), & x\in\Omega,\\ 0, & x \in \mathbb{R}^d\setminus\overline{\Omega}.\end{cases}$$

Let $G = \chi_\Omega \Delta^m F$ denote the extension by zero of $\Delta^m F$ outside of $\Omega$. With $L = 2m+s-j-1$, it follows that $G \in C^{L-2m}(\mathbb{R}^d)$, since $\Delta^m F \in C^{L-2m}(\mathbb{R}^d)$ and $\lambda_{2m+k}F = 0$ for $k = 0,\dots,L-2m$. □

COROLLARY 3.3. *For $j \in \mathbb{N}$, let $s$ be an integer greater than $j+1$. For $g \in C^s(\partial\Omega)$, the boundary layer potential*

$$V_j g = \int_{\partial\Omega} g(\alpha)\lambda_{j,\alpha}\phi(\cdot - \alpha)\mathrm{d}\sigma(\alpha)$$

*is in $C^{s+2m-j-2}(\overline{\Omega})$ as well as in $C^{s+2m-j-2}(\mathbb{R}^d\setminus\Omega)$. Furthermore,*

$$\|V_j g\|_{C^{s+2m-j-2}(\overline{\Omega})} \le C\|g\|_{C^s(\partial\Omega)}$$

*as well as*

$$\|V_j g\|_{C^{s+2m-j-2}(K\cap\mathbb{R}^d\setminus\Omega)} \le C_K\|g\|_{C^s(\partial\Omega)}$$

*for each compact $K \subset \mathbb{R}^d$.*

*Proof.* Since both $G*\phi$ and $F - G*\phi$ are in $C^{L-1}(\mathbb{R}^d)$, the proposition follows in case $j < 2m$. For general $j \in \mathbb{N}$, we simply observe that for $j = 2mr + j'$ (with $0 \le j' < 2m$), the identity $V_j g = \int_{\partial\Omega} g(\alpha)\lambda_{j,\alpha}\phi(\cdot-\alpha)\mathrm{d}\sigma(\alpha) = \Delta^{rm}V_{j'}g$ is valid for $x \notin \partial\Omega$. The result follows because $V_{j'}g$ is in $C^{s+2m-j'-2}(\overline{\Omega})$ (resp., is in $C^{s+2m-j-2}(\mathbb{R}^d\setminus\Omega)$), and therefore $\Delta^{rm}V_{j'}g$ is in $C^{s+2m-j-2}(\overline{\Omega})$ (resp., $C^{s+2m-j-2}(\mathbb{R}^d\setminus\Omega)$) . □

Note that increased smoothness (beyond $C^{2m-j-2}$) of $V_j g$ cannot be extended across the boundary. Indeed, Lemma 3.2 gives the following classical jump conditions.

Corollary 3.4. *For integers $j, s$, with $0 \le j \le 2m-1$ and $s > j+1$, let $g \in C^s(\partial\Omega)$. Then, for $k = 0, \dots, 2m+s-j-2$, $k \neq 2m-j-1$, we have for $x \in \partial\Omega$,*

$$\lim_{y \in \Omega \to x} \Lambda_k V_j g(y) = \lim_{y \in \mathbb{R}^d \setminus \overline{\Omega} \to x} \Lambda_k V_j g(y),$$

*while for $k = 2m-j-1$, we have*

$$\lim_{y \in \Omega \to x} \Lambda_k V_j g(y) - \lim_{y \in \mathbb{R}^d \setminus \overline{\Omega} \to x} \Lambda_k V_j g(y) = (-1)^j g(x).$$

We will return to these jump discontinuities in section 5.5.

**3.3. Boundary operators.** Corollary 3.3 implies that $V_j : C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$ is continuous (with the usual Fréchet space topologies on $C^\infty(\partial\Omega)$ and $C^\infty(\overline{\Omega})$). This permits us to define the following operators, which we call "boundary operators."

Definition 3.5. *For $j, k \in \mathbb{N}$, let $v_{k,j}^+ : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ be the operator defined for $g \in C^\infty(\partial\Omega)$ as*

$$v_{k,j}^+ g(x) := \lim_{y \in \mathbb{R}^d \setminus \overline{\Omega} \to x} \Lambda_k V_j g(y).$$

*Likewise, let $v_{k,j}^- : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ be defined as*

$$v_{k,j}^- g(x) := \lim_{y \in \Omega \to x} \Lambda_k V_j g(y).$$

*Remark* 3.6. By the local integrability of $\Lambda_{k,x}\Lambda_{j,\alpha}\phi(x-\alpha)$ when $k+j \le 2m-2$, it follows that we can take $v_{k,j}^+ = v_{k,j}^- = \mathrm{Tr}(\Lambda_k V_j \big|_{C^\infty(\partial\Omega)})$. In this case, we drop the $\pm$ notation and write $v_{k,j}$.

We note also that when $k+j \le 2m-2$, then $v_{k,j}^{\mathrm{t}} = v_{j,k}$. This follows because $\phi$ is even, so $\Lambda_{k,\alpha}\phi(x-\alpha) = \Lambda_{k,\alpha}\phi(\alpha - x)$ for all $k$. Hence $\int_{\partial\Omega} s(x) v_{k,j} g(x) \mathrm{d}\sigma(x) = \int_{\partial\Omega} g(x) v_{j,k} s(x) \mathrm{d}\sigma(x)$ with the exchange of limits justified by the local integrability of the kernel $\lambda_{k,x}\lambda_{j,\alpha}\phi(x-\alpha)$.

In contrast to the case $j+k \le 2m-2$, we note that we have for $k+j = 2m-1$, $v_{k,j}^- \neq v_{k,j}^+$. Indeed, $v_{k,j}^- g - v_{k,j}^+ g = (-1)^j g$ by the observation in Corollary 3.4.

In subsequent sections, we will express the boundary operators $v_{k,j}^\pm$ as pseudo-differential operators. The symbol classes to which they belong (determined in section 5.3) resolve their regularity.

Lemma 3.7. *Let $1 < p < \infty$, and take $s \in \mathbb{R}$. Then, for $j, k \in \mathbb{N}$, the operators $v_{k,j}^+$ and $v_{k,j}^-$ are bounded from $H_p^s(\partial\Omega)$ to $H_p^{s+2m-1-j-k}(\partial\Omega)$.*

*Proof.* The proof of this lemma is postponed until section 5.3. ∎

**4. The solution of the Dirichlet problem.** We now focus on solving the polyharmonic Dirichlet problem (1.2) using boundary layer potentials. To this end, we follow the approach taken by Chen and Zhou [7, Chapter 8] with our main points of departure being that we consider the Dirichlet problems in higher dimensions (i.e., $d \ge 2$), for higher order polyharmonic equations (i.e., $m \ge 2$), and for boundary data from Sobolev spaces $\prod_{j=0}^{m-1} H_p^{s-j}(\partial\Omega)$, with $1 < p < \infty$, rather than for data from $L_2$ Sobolev spaces $H_2^s \times H_2^{s-1}$. (Many of these changes are modest if technical. However, the change to higher order $m$ requires greater care in demonstrating ellipticity of the system—this is considered in section 5.4.)

We may seek a function of the form $T\mathbf{g} := \sum_{j=0}^{m-1} V_j g_j : \mathbb{R}^d \to \mathbb{R}$, with $\mathbf{g} = (g_j)_{j=0}^{m-1}$, which can be expressed as $T\mathbf{g} = \phi * \mu_{\mathbf{g}}$ (per section 3.1). In particular, it solves $\Delta^m T\mathbf{g} = 0$ in $\Omega$. Thus we simply require $T\mathbf{g}$ to satisfy the boundary conditions, which yields the system of integral equations

$$h_k = \lambda_k \sum_{j=0}^{m-1} V_j g_j = \sum_{j=0}^{m-1} v_{k,j} g_j \quad \text{for } k = 0, \dots, m-1, \tag{4.1}$$

where $v_{k,j} := \lambda_k V_j$ have been introduced in the previous section. Write this as $L\mathbf{g} = \mathbf{h}$, where

$$L \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{m-1} \end{pmatrix} := \begin{pmatrix} v_{0,0}g_0 + v_{0,1}g_1 + \cdots + v_{0,m-1}g_{m-1} \\ v_{1,0}g_0 + v_{1,1}g_1 + \cdots + v_{1,m-1}g_{m-1} \\ \vdots \\ v_{m-1,0}g_0 + v_{m-1,1}g_1 + \cdots + v_{m-1,m-1}g_{m-1} \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{m-1} \end{pmatrix}.$$

By the discussion in section 3.3, namely Lemma 3.7, $L$ is continuous from

$$(\mathcal{D}'(\partial\Omega))^m \to (\mathcal{D}'(\partial\Omega))^m,$$

and by Remark 3.6 it is self-transposed. Unfortunately, this system is not invertible in general. To treat this, we modify the system by augmenting it with certain polynomial side conditions. This is explained in the following subsection. Our goal is to solve this augmented system, and we do so in stages. First, we develop the problem further, so that it becomes a problem of inverting an operator on a product of reflexive Sobolev spaces. Then we show that this operator possesses an inverse of a sort: a parametrix. Finally, we use the parametrix to prove that a slightly modified version of system of integral equations (4.1) is invertible.

**4.1. The system of integral equations, some of the operators involved, and the Sobolev spaces used.** We look for solutions of the modified system

$$L^\sharp \left(\frac{\vec{A}}{\mathbf{g}}\right) := \left(\begin{array}{c|c} 0 & P^{\mathrm{t}} \\ \hline P & L \end{array}\right) \left(\frac{\vec{A}}{\mathbf{g}}\right) = \left(\frac{\vec{B}}{\mathbf{h}}\right),$$

where $\vec{A}, \vec{B} \in \mathbb{R}^N$, with $N = \frac{(m-1+d)!}{(m-1)!d!} = \dim(\Pi_{m-1})$, and $P : \mathbb{R}^N \to C^\infty(\partial\Omega, \mathbb{R}^m)$ can be represented as a Vandermonde-style matrix whose $\ell$th column consists of the basic boundary operators applied to the $\ell$th basis element for $\Pi_{m-1}$, namely $(P)_{k\ell} = \lambda_k p_\ell$. Thus $(P\vec{A})_k := \sum_{\ell=1}^N A_\ell \lambda_k p_\ell$. The operator $P^{\mathrm{t}} : \big(\mathcal{D}'(\partial\Omega)\big)^m \to \mathbb{R}^N$ is its natural transpose, namely $(P^{\mathrm{t}}\mathbf{g})_\ell = \sum_{k=0}^{m-1} \langle g_k, \lambda_k p_\ell \rangle$. The function $\sum_{\ell=1}^N A_\ell p_\ell + \sum_{j=0}^{m-1} V_j g_j$ solves the Dirichlet problem with $N$ extra "side conditions." The relevance of these extra conditions will be made clear in section 4.2.

We restrict $L^\sharp$ to various products of Bessel potential spaces and recast the problem in the context of reflexive Banach spaces. Thus we make the following definition.

Definition 4.1. *For $1 < p < \infty$ and $s \in \mathbb{R}$, let $X_{p,s} := \prod_{j=0}^{m-1} H_p^{s+j}(\partial\Omega)$ and $X_{p,s}^\sharp := \mathbb{R}^N \times X_{p,s}$. Similarly, let $Y_{p,s} := \prod_{j=0}^{m-1} H_p^{s-j}(\partial\Omega)$ and $Y_{p,s}^\sharp := \mathbb{R}^N \times Y_{p,s}$.*

*Remark* 4.2. We have defined the Bessel potential space $H_p^s(\partial\Omega)$ in section 1.3. We remark that these are smoothness spaces over the manifold $\partial\Omega$ which are reflexive, the dual of $H_p^s(\partial\Omega)$ being $H_{p'}^{-s}(\partial\Omega)$ under the bilinear form $H_p^s(\partial\Omega) \times H_{p'}^{-s}(\partial\Omega) \to$

$\mathbb{C} : (g,h) \mapsto \langle g,h\rangle$ inherited from the pairing $\langle \phi,\psi\rangle = \int_{\partial\Omega} \phi(x)\psi(x)\mathrm{d}x$ defined on test functions. From this, we naturally identify the dual of $X_{p,s}$ with $Y_{p',-s}$ and vice versa. We have the identification $X^\sharp_{p,s}$ with $Y^\sharp_{p',-s}$ via the pairing $\langle(\vec{A},\mathbf{g}),(\vec{B},\mathbf{h})\rangle_{X^\sharp_{p,s},Y^\sharp_{p',-s}} = \langle \mathbf{g},\mathbf{h}\rangle_{X_{p,s},Y_{p',-s}} + \sum_{\ell=1}^N A_\ell B_\ell$.

**4.2. Bounded invertibility of $L^\sharp$.** The problem we now face is to show that the restriction of $L^\sharp$ is boundedly invertible from $X^\sharp_{p,s}$ to $Y^\sharp_{p,s+2m-1}$. To do this, we use three lemmas which are proved in the coming subsections.

The first lemma concerns the regularity of the operator $L$. It is a direct consequence of the mapping properties of the constituent pseudodifferential operators $v_{k,j}$, and follows in a more-or-less immediate way from Lemma 3.7, which in turn shows boundedness of $L^\sharp$ from $X^\sharp_{p,s}$ to $Y^\sharp_{p,s}$.

Lemma 4.3. *For $1 < p < \infty$ and $s \in \mathbb{R}$, $L^\sharp$ is a bounded map from $X^\sharp_{p,s}$ to $Y^\sharp_{p,s+2m-1}$.*

*Proof.* The proof of this lemma follows directly from Lemma 3.7. ∎

The second lemma concerns the range of the map $L_{p,s} := L^\sharp\big|_{X^\sharp_{p,s}} : X^\sharp_{p,s} \to Y^\sharp_{p,s+2m-1}$. The following section will demonstrate that there is a near right inverse $R : Y_{p,s+2m-1} \to X_{p,s}$ (known as a *parametrix*) so that $LR = \mathrm{Id} + K$, where $K : Y_{p,s+2m-1} \to Y_{p,s+2m-1}$ is a compact operator.

Lemma 4.4. *For $1 < p < \infty$ and $s \in \mathbb{R}$, $L^\sharp(X^\sharp_{p,s})$ is closed in $Y^\sharp_{p,s+2m-1}$.*

*Proof.* The proof of this lemma is postponed until section 5.4. ∎

Lemma 4.3 follows from showing that $L$ is a pseudodifferential operator of a prescribed order, and Lemma 4.4 follows from showing that it is elliptic (elliptic pseudodifferential operators are Fredholm operators). This is the approach we take in the coming subsections.

The third lemma shows the injectivity of the operator $L^\sharp$. This is the moment where using $L$ is insufficient, and the auxiliary polynomial operators $P$ and $P^{\mathrm{t}}$ must be used.

Lemma 4.5. *$L_{p,s} : X^\sharp_{p,s} \to Y^\sharp_{p,s+2m-1}$ is $1{-}1$.*

*Proof.* The proof of this lemma is postponed until section 5.5. ∎

Together, the previous three lemmas imply the following result, which is the key to solving the polyharmonic Dirichlet problem and consequently to obtaining the desired integral representation.

Proposition 4.6. *For $1 < p < \infty$ and $s \in \mathbb{R}$, $L_{p,s} : X^\sharp_{p,s} \to Y^\sharp_{p,s+2m-1}$ is boundedly invertible.*

*Proof.* It suffices to show that $L^\sharp_{p,s}$ is invertible from $X^\sharp_{p,s}$ to $Y^\sharp_{p,s+2m-1}$; the open mapping theorem then guarantees the boundedness of the inverse.

Lemma 4.3, in conjunction with the definition of $L^\sharp$ and the duality of the spaces $X^\sharp_{p,s}$ and $Y^\sharp_{p,1-(s+2m)}$, indicates that $(L_{p,s})^{\mathrm{t}} = L_{p',1-(s+2m)}$. By Lemma 4.5, this operator is $1{-}1$, and we have that $\ker(L_{p,s})^{\mathrm{t}} = \{0\}$. Since the range of $L_{p,s}$ is closed, we have that

$$\mathrm{ran}(L_{p,s}) = \overline{\mathrm{ran}(L_{p,s})} = \big(\ker(L^{\mathrm{t}}_{p,s})\big)_\perp = \big(\ker(L_{p',1-(s+2m)})\big)_\perp = Y_{p,s+2m-1}.$$

Consequently, $L_{p,s}$ is invertible. ∎

**5. Expressing boundary layer potential operators as pseudodifferential operators.** We continue our investigation of the boundary layer potential operators $V_j$, their boundary values $v_{k,j} = \lambda_k V_j$, and the full boundary integral operator $L$. By changing variables so that portions of the boundary $U \cap \partial\Omega$ are flattened, we may express these as pseudodifferential operators. In particular, we can calculate the principal symbols of the boundary operators $v_{k,j}$, the orders of which (determined by the order of the principal symbol) determine their mapping properties, from which Lemma 4.3 follows naturally. We use this calculation to demonstrate the ellipticity of $L$, which guarantees that it has closed range.

**5.1. Background.** Before discussing pseudodifferential operators, we mention some other useful classes of operators. A continuous linear operator $K : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$ is a *smoothing* operator (alternatively a regularizing or negligible operator). An operator $A : \mathcal{D}(U) \to \mathcal{E}'(U)$ is *properly supported* if also $A^{\mathrm{t}} : \mathcal{D}(U) \to \mathcal{E}'(U)$; by duality, it is clear that such an operator is continuous also from $\mathcal{E}(U)$ to $\mathcal{D}'(U)$.

**5.1.1. Pseudodifferential operators on Euclidean domains.** We briefly highlight some aspects of the theory of pseudodifferential operators—these can be found in a variety of sources (including [19, 24, 39, 43, 44]; this is but a small sampling of resources).

DEFINITION 5.1. *Given an open subset $O$ of $\mathbb{R}^d$ for $p \in C^\infty(O \times \mathbb{R}^d)$, we say that $p$ is in the symbol class $S^N_{1,0}(X)$ if for each pair of multi-integers $\alpha, \beta$ and each compact $K \subset U$ there is a constant $C_{\alpha,\beta,K}$ so that*

$$|D_x^\beta D_\xi^\alpha (p(x,\xi))| \le C_{\alpha,\beta,K}(1+|\xi|)^{N-|\alpha|}$$

*holds for all $x \in K$.*

Given a symbol $p \in S^N_{1,0}(O)$, we can (initially) define an operator on test functions as

$$\mathrm{Op}(p)f(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} p(x,\xi)\widehat{f}(\xi)\mathrm{d}\xi.$$

The operator can be continuously extended to map $\mathcal{D}(O) \to \mathcal{E}(O)$ and $\mathcal{E}'(O) \to \mathcal{D}'(O)$ (cf. [43, Theorem 1.5]). The operator (thus extended) is a pseudodifferential operator of order $N$.

Clearly, $S^N_{1,0}(O) \subset S^{N+1}_{1,0}(O)$ and (because of mapping properties described in item 5 below) the symbol class $S^{-\infty}(U) = \bigcap_{N\in\mathbb{Z}} S^N_{1,0}(O)$ generates smoothing operators. For any $p \in S^N_{1,0}(O)$, there are a properly supported operator $P$ and a smoothing operator $R = \mathrm{Op}(r)$, with $r \in S^{-\infty}(O)$, so that $\mathrm{Op}(p) = P + R$ (cf. [19, Proposition 7.8]).

RESULT 5.2. *The following results hold for such operators:*

1. *Let $f$ be a distribution on $O$ and $P$ be a pseudodifferential operator. If $\Upsilon$ is the largest open set on which $f$ is smooth (i.e., the complement of the singular support), then $Pf$ is $C^\infty(\Upsilon)$ as well. Indeed, if $\tau$ and $\omega$ are smooth functions on $O$ for which* $\mathrm{supp}(\tau) \subsetneq \{\mathrm{x} \in \mathrm{O} \mid \omega(\mathrm{x}) = 0\}$, *then $f \mapsto \omega P(\tau f)$ is smoothing.*
2. *Given a countable sequence of symbols $(p_j)_{j=0}^\infty$ with each $p_j \in S^{N_j}_{1,0}(O)$ and $N_j$ decreasing, there is a symbol $p \in S^{N_0}_{1,0}(O)$ such that for every $M \in \mathbb{N}$, $p - \sum_{j=0}^M p_j \in S^{N_{M+1}}_{1,0}(O)$. In this case, we write $p = \sum_{j=0}^\infty p_j$.*

3. *For symbols $a \in S^M_{1,0}(O)$ and $b \in S^N_{1,0}(O)$ for which one of* $\mathrm{Op}(a)$ *and* $\mathrm{Op}(b)$ *is properly supported, the composition* $\mathrm{Op}(a)\mathrm{Op}(b)$ *is a pseudodifferential operator of order $N+M$ and has symbol*

$$(a \odot b)(x,\xi) = \sum_{|\alpha|=0}^{\infty} \frac{(-i)^{|\alpha|}}{\alpha!} D^\alpha_\xi a(x,\xi) D^\alpha_x b(x,\xi)$$

*(with convergence of the series understood as in item* 2*).*

4. *The class of pseudodifferential operators is closed under diffeomorphism, and order is preserved. Indeed, we have, for $\Phi : U \to O$ and symbol $p \in S^N_{1,0}(U)$, that the operator $(\mathrm{Op}(p))^\Phi$ is a pseudodifferential operator with symbol $p^\Phi \in S^N_{1,0}(O)$ given by*

$$p^\Phi(\Phi(x),\xi) = \sum_{\alpha \in \mathbb{N}^d} \frac{1}{\alpha!} \phi_\alpha(x,\xi) D^\alpha_\xi p(x,(D\Phi)^{\mathrm{t}}\xi).$$

*Here $\phi_\alpha(x,\xi)$ is a polynomial*[2] *in $\xi$ of degree at most $|\alpha|/2$, and $\phi_0 = 1$.*

5. *For a symbol $p \in S^N_{1,0}(O)$, the operator* $\mathrm{Op}(p)$ *maps $H^s_{p,\mathrm{c}}(O)$ boundedly to $H^{s-N}_{p,\mathrm{loc}}(O)$ for all $s \in \mathbb{R}$, $1 < p < \infty$.*

Item 1 is in [43, Chapter 2, Theorem 2.1] and in [19, Proposition 7.11]. Item 2 is in [43, Chapter 2, Theorem 3.1] and [19, Lemma 7.3]. Item 3 is in [43, Chapter 2, section 4] and [19, Theorem 7.13]. Item 4 is in [43, Chapter 2, Theorem 5.1] and [19, Theorem 8.1]. Item 5 follows from [43, Chapter 11, Theorem 2.1] .

**5.1.2. Polyhomogeneous operators and ellipticity.** A symbol $p \in S^N_{1,0}(O)$ is *positively homogeneous of order $N$* if it satisfies, for $\lambda \geq 1$ and $|\xi| \geq 1$,

$$p(x,\lambda\xi) = \lambda^N p_j(x,\xi). \tag{5.1}$$

A symbol $p \in S^N_{1,0}(O)$ is called *polyhomogeneous* if it has an (asymptotic) expansion $p = \sum_{j=0}^\infty p_j$, where each $p_j \in S^{N-j}_{1,0}(U)$ and is positively homogeneous of order $N-j$.

Definition 5.3. *Let $S^N(U)$ denote the set of polyhomogeneous symbols of order $N$. Furthermore, for $m \in \mathbb{N}$, let $S^N(U,m)$ denote the set of matrix valued symbols $p = (p_{j,k})_{j,k}$, where each $p_{j,k} \in S^N(U)$.*

In the case of a matrix valued symbol, $\mathrm{Op}(p)$ is defined as

$$\mathrm{Op}(p)\mathbf{f}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} p(x,\xi)\widehat{\mathbf{f}}(\xi)\mathrm{d}\xi,$$

where $\widehat{\mathbf{f}}(\xi) = [\widehat{f_0},\dots,\widehat{f_{m-1}}]^{\mathrm{t}}$ is the entrywise Fourier transform of $\mathbf{f} = [f_0,\dots,f_{m-1}]^{\mathrm{t}}$ and $p(x,\xi)\widehat{\mathbf{f}}(\xi)$ is a matrix-vector product. Similarly,

$$p \odot q(x,\xi) = \sum_{|\alpha|=0}^{\infty} \frac{(-i)^{|\alpha|}}{\alpha!} D^\alpha_\xi p(x,\xi) D^\alpha_x q(x,\xi)$$

involves matrix products. This class is closed under $\odot$ and addition. Differential operators have symbols which are polynomial in $\xi$, and thus their symbols are polyhomogeneous.

[2]Specifically, $\phi_\alpha(x,\xi) = [D^\alpha_y e^{i\langle(\Phi(y)-\Phi(x)-(D\Phi(x))(y-x),\xi\rangle}]\,|_{y=x}$ .

For a pseudodifferential operator $P$ with symbol $p = \sum_{j=0}^{\infty} p_j \in S^N(X,m)$, the *principal symbol* is $p_0 \in S^N(X,m)$. Although this is an equivalence class of symbols, we make the slight abuse of terminology by referring to "the" principal symbol, and we denote it by $\sigma(P) := p_0$. We note especially that the values of $\sigma(P)$ for small values of $\xi$ are unimportant, and so we generally give $\sigma(P)(x,\xi)$ only for $|\xi| \geq 1$.

*Ellipticity and parametrices.* The property that ensures the existence of a parametrix is the *ellipticity* of the symbol. We use the following definition, which is restrictive—a more robust definition would be valid for symbols in $S^N_{1,0}(O)$—but it is sufficient for our purposes.

DEFINITION 5.4. *A symbol $p \in S^N(U,m)$ is elliptic if $p_0(x,\xi)$ is nonsingular for $|\xi| \geq 1$.*

Note in particular that if $p_0$ is a positively homogeneous, scalar symbol of order $N$ which does not vanish, then there is a constant $c > 0$ so that $c|\xi|^N \leq |p_0(x,\xi)|$, and therefore $|\xi|^{-N}|p(x,\xi)|$ is bounded from below for $|\xi|$ sufficiently large. The following consequence of ellipticity is a simplification (sufficient for our purposes) of [19, Theorem 7.18].

LEMMA 5.5. *When $p \in S^N(O,m)$ is elliptic, there is a properly supported pseudodifferential operator $Q$ (with symbol $q \in S^{-N}(O,m)$, modulo $S^{-\infty}(O,m)$) so that $Q\mathrm{Op}(p) - \mathrm{Id}$ and $\mathrm{Op}(p)Q - \mathrm{Id}$ are smoothing operators.*

It follows from the construction that if $\sum_{j=0}^{N} p_j$ is the symbol of an elliptic differential operator (with $p_j \in S^{N-j}(O)$), then the parametrix $Q$ of $\mathrm{Op}(p)$ has symbol $q = \sum_{j=0}^{\infty} q_j \in S^{-N}(O)$. We can say more, however: each term $q_j$ is rational in $\xi$.

LEMMA 5.6. *Suppose $p$ is the symbol of a (scalar) elliptic differential operator of order $N$. Then its parametrix $Q = \mathrm{Op}(q)$ is polyhomogeneous, with $q = \sum_{j=0}^{\infty} q_j$. Moreover, for every $j$, $q_j$ is positively homogeneous of order $-N-j$, and for $|\xi| \geq 1$, $\xi \mapsto q_j(x,\xi)$ is a rational function.*

*Proof.* We write $p = \sum_{j=0}^{N} p_j$ so that each $p_j \in S^{N-j}$ is a homogeneous polynomial of degree $N-j$ and therefore satisfies (5.1). The terms of $q$ can be determined via the product formula $\left(\sum_{j=0}^{\infty} q_j\right) \odot \left(\sum_{j=0}^{N} p_j\right) = 1$. Namely, after rearranging terms, we have

$$\left(\sum_{j=0}^{\infty} q_j\right) \odot \left(\sum_{j=0}^{N} p_j\right) = \sum_{j=0}^{\infty} \sum_{|\alpha|+k+\ell=j} \frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha q_k(x,\xi) D_x^\alpha p_\ell(x,\xi).$$

With the aid of a cutoff function, set $q_0(x,\xi) = (p_0(x,\xi))^{-1}$ for $|\xi| \geq 1$, and note that for $|\xi| \geq 1$, this is rational and positively homogeneous of order $-N$. Each term $\frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha q_k(x,\xi) D_x^\alpha p_\ell(x,\xi)$ is a symbol of order $-(k+\ell+|\alpha|) = -j$. Proceed by induction on $j$, setting (for $|\xi| \geq 1$)

$$q_j(x,\xi) = -(p_0(x,\xi))^{-1} \sum_{k=0}^{j-1} \left( \sum_{\ell+|\alpha|=j-k} \frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha q_k(x,\xi) D_x^\alpha p_\ell(x,\xi) \right). \tag{5.2}$$

Then $D_\xi^\alpha q_k(x,\xi)$ is rational and positively homogeneous of order $-N-k-|\alpha|$, while $D_x^\alpha p_\ell(x,\xi)$ is polynomial and positively homogeneous of order $N-\ell$ in $\xi$. Thus $\frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha q_k(x,\xi) D_x^\alpha p_\ell(x,\xi)$ is positive homogeneous of order $-k-\ell-|\alpha| = -j$ for $k < j$, and so $q_j$ is positively homogeneous of order $-N-j$. It is likewise rational as a sum of rational functions. ∎

**5.2. Expression of operators in coordinates.** In this section, we express the basic operators under consideration in normal and tangential coordinates near the boundary. Namely, we consider a map $\Psi' : O' \to U'$, with $O' \subset \mathbb{R}^{d-1}$ and $U' \subset \partial\Omega$, as described in section 1.3. We calculate the effect of the diffeomorphism $\Psi : O \to U$ given in (1.11) on the Laplacian, the boundary operators $\Lambda_j$, and the fundamental solution of $\Delta^m$. Finally, we use this to analyze the boundary layer potential operators $V_j$.

*Laplace operator.* From the decomposition (1.12), the principal symbol for $\Delta^\Psi$ is

$$\sigma(\Delta^\Psi)(\mathbf{x},\xi) = \sigma\left(\Delta_t{}^\Psi + \frac{\partial^2}{\partial x_d^2}\right)(\mathbf{x},\xi) = -\left(\xi_d^2 + \sum_{j=1}^{d-1}\sum_{k=1}^{d-1} \mathsf{G}^{i,j}(\mathbf{x})\xi_j\xi_k\right),$$

and because of the positivity of the first (i.e., least) eigenvalue of $\mathsf{G}^{-1}$ we see that $\Delta^\Psi$ and $(\Delta^m)^\Psi$ are elliptic (of orders 2 and $2m$, respectively). Set

$$\mathsf{d}(\mathbf{x},\boldsymbol{\eta}) := \sum_{j=1}^{d-1}\sum_{k=1}^{d-1} \mathsf{G}^{j,k}(\mathbf{x})\xi_j\xi_k$$

for dual variable $\boldsymbol{\eta} = (\xi_1,\dots,\xi_{d-1})$. This allows us to write

$$\sigma\big((\Delta)^\Psi\big)(\mathbf{x},\xi) = -\left(\xi_d^2 + \mathsf{d}(\mathbf{x},\boldsymbol{\eta})\right).$$

At times, we will consider $\mathsf{d}|_{O'\times\mathbb{R}^d}$, and we express this restriction as $\mathsf{d}(\mathbf{y},\cdot)$, which simply means $\mathsf{d}(\mathbf{x},\cdot)$ with $x_d = 0$.

*Normal derivative.* The principal symbol of $(D_{\vec{n}}^{\mathrm{t}})^\Psi$ is $\sigma\big((D_{\vec{n}}^{\mathrm{t}})^\Psi\big)(\mathbf{x},\xi) = -i\xi_d$. We can also express the principal symbol of the differential operator $\Lambda_j^{\mathrm{t}}$, the adjoint of the operator defined in (1.10), as

$$\sigma\big((\Lambda_j^{\mathrm{t}})^\Psi)\big)(\mathbf{x},\xi) = \begin{cases} (-1)^{\frac{j}{2}}\left(\xi_d^2 + \mathsf{d}(\mathbf{x},\boldsymbol{\eta})\right)^{j/2}, & j \text{ is even,} \\ (-1)^{\frac{j+1}{2}} i\xi_d\left(\xi_d^2 + \mathsf{d}(\mathbf{x},\boldsymbol{\eta})\right)^{(j-1)/2}, & j \text{ is odd.}\end{cases} \tag{5.3}$$

**5.2.1. The fundamental solution to $\boldsymbol{\Delta^m}$ in local coordinates.** The solution operator, $f \mapsto \phi * f$, for $\Delta^m$ in $\mathbb{R}^d$ is a Fourier multiplier with symbol $\widehat{\phi}(\xi) = |\xi|^{-2m}$ (at least when considering distributions supported on $\mathbb{R}^d \setminus \{0\}$). If not for its behavior near $\xi = 0$ it would be in $S^{-2m}(\mathbb{R}^d)$. This is easily fixed by making the decomposition $\phi * f = Ef + Kf$ into a properly supported pseudodifferential operator and a smoothing operator.

Note that the formula $\Delta^m\phi * g = g = \phi * (\Delta^m g)$ is valid for test functions $g \in \mathcal{D}(\mathbb{R}^d)$ which satisfy $g \perp \Pi_{2m}$. Thus $(\Delta^m)^\Psi(E)^\Psi$ and $(E)^\Psi(\Delta^m)^\Psi$ both equal the identity, modulo addition of a smoothing operator. It follows that $E^\Psi$ is a parametrix for $(\Delta^m)^\Psi$, the $m$-fold composition of the operator $(\Delta)^\Psi$ from (1.12) on $O$ (derived from $\Delta^m$). By Lemma 5.6, $E^\Psi$ has a polyhomogeneous symbol $\sum_{j=0}^\infty e_j(\mathbf{x},\boldsymbol{\xi})$; we can express its principal symbol as

$$\sigma(E^\Psi)(\mathbf{x},\boldsymbol{\xi}) = e_0(\mathbf{x},\boldsymbol{\xi}) = (-1)^m\left(\xi_d^2 + \mathsf{d}(\mathbf{x},\boldsymbol{\eta})\right)^{-m}. \tag{5.4}$$

*Remark* 5.7. Note that $\sigma(E^\Psi)$ can also be obtained by Result 5.2, item 4.

**The boundary layer potential operator in local coordinates.** To describe the coordinate representation of the operators $g \mapsto \Lambda_k V_j g = \Lambda_k \phi * \left(\Lambda_j^{\mathrm{t}} g \cdot \delta_{\partial\Omega}\right)$, we focus on the the coordinate version of $(\Lambda_k)E(\Lambda_j^{\mathrm{t}})$, since this differs from the map $f \mapsto \Lambda_k \phi * (\Lambda_j^{\mathrm{t}} f)$ by a smoothing operator. It follows that it too is polyhomogeneous (as a product of polyhomogeneous operators): $(\Lambda_k)E(\Lambda_j^{\mathrm{t}}) = \mathrm{Op}(p)$, with $p = \sum_{\ell=0}^{\infty} p_\ell$, and $p_\ell \in S^{j+k-2m-\ell}(O)$. Writing $n = j+k$, its principal symbol is determined by combining (5.3) and (5.4):

$$\sigma\left(\left(\Lambda_k E \Lambda_j^{\mathrm{t}}\right)^{\Psi}\right)(\mathbf{x},\boldsymbol{\xi}) = (-1)^{m-\frac{n}{2}} \begin{cases} \frac{1}{\left(\xi_d^2 + \mathsf{d}(\mathbf{x},\boldsymbol{\eta})\right)^{m-\frac{n}{2}}}, & j,k \text{ are even},\\[2ex] \frac{\xi_d^2}{\left(\xi_d^2 + \mathsf{d}(\mathbf{x},\boldsymbol{\eta})\right)^{m+1-\frac{n}{2}}}, & j,k \text{ are odd},\\[2ex] \frac{i(-1)^{\ell}\xi_d}{\left(\xi_d^2 + \mathsf{d}(\mathbf{x},\boldsymbol{\eta})\right)^{m-\frac{n-1}{2}}}, & n \text{ is odd}. \end{cases}$$

**5.3. Boundary operators in local coordinates.** The expression of $(\Lambda_k E \Lambda_j^{\mathrm{t}})^{\Psi}$ as the polyhomogeneous operator $p$ permits us to write $(v_{k,j}^{\pm})^{\Psi}$ as an operator from $\mathcal{D}(O')$ to $\mathcal{E}(O')$. To this end, define $\widetilde{v_{k,j}^{\pm}} g(\mathbf{y}) := \lim_{x_d \to 0^{\pm}} (\Lambda_k E \Lambda_j^{\mathrm{t}})^{\Psi}(g \cdot \delta_{\mathbb{R}^{d-1}})(\mathbf{x})$. The fact that this is well defined for smooth $g$ is an immediate consequence of Corollary 3.3; indeed, we can write $(v_{k,j}^{\pm})^{\Psi}$ as the sum of the operator $\widetilde{v_{k,j}^{\pm}}$ and a smoothing operator. Namely,

$$(v_{k,j}^{\pm})^{\Psi} g(\mathbf{y}) = \widetilde{v_{k,j}^{\pm}} g(\mathbf{y}) + \lim_{x_d \to 0} \left[\Lambda_k{}^{\Psi} K \Lambda_j^{\mathrm{t}}{}^{\Psi}\left(g \cdot \delta_{\mathbb{R}^{d-1}}\right)\right](\mathbf{x}). \tag{5.5}$$

Note that when $j+k \le 2m-2$, $(v_{k,j}^{+})^{\Psi} g(\mathbf{y}) = (v_{k,j}^{-})^{\Psi} g(\mathbf{y})$, and so we simply write $\widetilde{v_{k,j}^{\pm}} g(\mathbf{y}) = \widetilde{v_{k,j}} g(\mathbf{y})$.

The following lemma shows that $\widetilde{v_{k,j}^{\pm}}$, and hence $v_{k,j}^{\pm}$, can be extended to distributions. It shows, roughly, that $(\Lambda_k E \Lambda_j^{\mathrm{t}})^{\Psi}$ has the "transmission property." The structure of this proof (especially Case 2) follows section 18.2 of [24]. By dealing with classical symbols, it is greatly simplified.

LEMMA 5.8. *For $j,k \in \mathbb{N}$, $\widetilde{v_{k,j}^{\pm}}$ is a polyhomogeneous operator of order $j+k-2m+1$.*

*Proof.* We split this into two cases.

*Case* 1: $j+k \le 2m-2$. In this case, we have

$$(\Lambda_k E \Lambda_j^{\mathrm{t}})^{\Psi}(g \cdot \delta_{\mathbb{R}^{d-1}})(\mathbf{x}) = (2\pi)^{-d} \int_{\mathbb{R}^{d-1}} \widehat{g}(\boldsymbol{\eta}) e^{i\mathbf{y}\cdot\boldsymbol{\eta}} \left(\int_{\mathbb{R}} p(\mathbf{x},\boldsymbol{\eta},\xi_d) e^{i x_d \cdot \xi_d} \mathrm{d}\xi_d\right) \mathrm{d}\eta$$

(the inner integral is convergent by the decay of $p$).

Letting $x_d \to 0$, we have $\widetilde{v_{k,j}} g(\mathbf{y}) = (2\pi)^{-d} \int_{\mathbb{R}^{d-1}} \widehat{g}(\boldsymbol{\eta}) e^{i\mathbf{y}\cdot\boldsymbol{\eta}} \left(\int_{\mathbb{R}} p(\mathbf{y},0,\boldsymbol{\eta},\xi_d)\mathrm{d}\xi_d\right) \mathrm{d}\eta$ after the exchange of limit and integral. The conditions on $p$ ensure that $(\mathbf{y},\boldsymbol{\eta}) \mapsto \frac{1}{2\pi}\int_{\mathbb{R}} p(\mathbf{y},0,\boldsymbol{\eta},\xi_d)\mathrm{d}\xi_d$ is a symbol in $S^{j+k-2m+1}(O')$. Indeed, for multi-indices $\alpha$ and

$\beta$, we have (by dominated convergence)

$$\begin{aligned}\left|D^{\alpha}_{\mathbf{y}}D^{\beta}_{\boldsymbol{\eta}}\int_{\mathbb{R}} p(\mathbf{y},0,\boldsymbol{\eta},\xi_d)\mathrm{d}\xi_d\right| &\le \int_{\mathbb{R}} |(D^{\alpha}_{x}D^{\beta}_{\xi}p)(\mathbf{y},0,\boldsymbol{\eta},\xi_d)|\mathrm{d}\xi_d\\ &\le C\int_{\mathbb{R}}(1+|\boldsymbol{\eta}|+|\xi_d|)^{j+k-2m-|\beta|}\mathrm{d}\xi_d\\ &= C(1+|\boldsymbol{\eta}|)^{j+k-2m+1-|\beta|},\end{aligned}$$

where in the last equation, we have used the change of variables $t=\frac{\xi_d}{1+|\boldsymbol{\eta}|}$.

A similar estimate applied to each $p_\ell$ guarantees that we can express $(v_{k,j})^{\Psi}$ as a polyhomogeneous series, namely

$$\int_{\mathbb{R}} p(\mathbf{y},0,\boldsymbol{\eta},\xi_d)\mathrm{d}\xi_d=\sum_{\ell=0}^{\infty}\int_{\mathbb{R}} p_\ell(\mathbf{y},0,\boldsymbol{\eta},\xi_d)\mathrm{d}\xi_d.$$

For $|\eta|>1$ and $\lambda>1$, $\int_{\mathbb{R}} p_\ell(\mathbf{y},0,\lambda\boldsymbol{\eta},\xi_d)\mathrm{d}\xi_d=\lambda^{j+k-2m-\ell+1}\int_{\mathbb{R}} p_\ell(\mathbf{y},0,\boldsymbol{\eta},\zeta)\mathrm{d}\zeta$ by a simple change of variables, and so each term is positively homogeneous of order $j+k-2m-\ell+1$.

*Case* 2: $j+k>2m-2$. In this case, we write $p=\sum_{\ell=0}^{N}p_\ell+p^{\flat}$, choosing $j+k-2m-N-1\le 2$. This permits us to treat $p^{\flat}$ as in Case 1; this is left to the reader. We focus on $p^{\sharp}=\sum_{\ell=0}^{N}p_\ell$.

Consider $x_d>0$ ($x_d<0$ is handled similarly). Mollify $g\cdot\delta$ as follows: for a smooth $\tau:\mathbb{R}\to\mathbb{R}$ supported in $[-1,1]$, consider $G_\epsilon(\mathbf{x})=\frac{1}{\epsilon}g(\mathbf{y})\tau(x_d/\epsilon)$. Then $(\Lambda_k E\Lambda_j^{\mathrm{t}})^{\Psi}(g\cdot\delta_{\mathbb{R}^{d-1}})=\lim_{\epsilon\to 0}(\Lambda_k E\Lambda_j^{\mathrm{t}})^{\Psi}G_\epsilon$. Because $\widehat{G_\epsilon}(\boldsymbol{\xi})=\widehat{g}(\boldsymbol{\eta})\widehat{\tau}(\epsilon\xi_d)$ is a Schwartz function, it follows that

$$\mathrm{Op}(p^{\sharp})(g\cdot\delta_{\mathbb{R}^{d-1}})=(2\pi)^{-d}\int_{\mathbb{R}^{d-1}}\widehat{g}(\eta)e^{i\langle\mathbf{y},\boldsymbol{\eta}\rangle}\int_{\mathbb{R}}\widehat{\tau}(\epsilon\xi_d)p^{\sharp}(\mathbf{x},\boldsymbol{\eta},\xi_d)e^{ix_d\xi_d}\mathrm{d}\xi_d\mathrm{d}\boldsymbol{\eta} \tag{5.6}$$

by the integrability of $\boldsymbol{\xi}\mapsto\widehat{G_\epsilon}(\boldsymbol{\xi})p^{\sharp}(\mathbf{x},\boldsymbol{\xi})$.

Note that $\widehat{\tau}$ is defined on $\mathbb{C}$ and is entire. Because each $p_\ell$ is rational in $\xi$ (for $|\xi|>1$), there is a complex region $\Omega_{R_0}:=\{\zeta\in\mathbb{C}\mid|\zeta|>R_0,\Im(\zeta)>0\}$, where for each $\ell=0,\dots,N$, $\zeta\mapsto p_\ell(\mathbf{x},\boldsymbol{\eta},\zeta)$ is defined and analytic. The inner integral $\int_{\mathbb{R}}\widehat{\tau}(\epsilon\xi_d)p^{\sharp}(\mathbf{x},\boldsymbol{\eta},\xi_d)e^{ix_d\xi_d}\mathrm{d}\xi_d$ in (5.6) can be written as

$$\int_{-R}^{R}\widehat{\tau}(\epsilon\xi_d)p^{\sharp}(\mathbf{x},\boldsymbol{\eta},\xi_d)e^{ix_d\xi_d}\mathrm{d}\xi_d-\int_{\gamma_R}\widehat{\tau}(\epsilon\zeta)p^{\sharp}(\mathbf{x},\boldsymbol{\eta},\zeta)e^{ix_d\zeta}\mathrm{d}\zeta$$

for any $R_0<R<\infty$. (Here $\gamma_R$ is the upper part of the semicircle of radius $R$ centered at 0.) Because $|e^{ix_d\zeta}\widehat{\tau}(\epsilon\zeta)|=|\int_{-1}^{1}\tau(t)e^{i(x_d-\epsilon t)\zeta}\mathrm{d}t|$, we have that $|e^{ix_d\zeta}\widehat{\tau}(\epsilon\zeta)|\le\|\tau\|_1$, provided $\epsilon<x_d$ and $\Im\zeta\ge 0$. By dominated convergence, we then have that

$$\mathrm{Op}(p^{\sharp})(g\cdot\delta_{\mathbb{R}^{d-1}})=(2\pi)^{-d}\int_{\mathbb{R}^{d-1}}\widehat{g}(\eta)e^{i\langle\mathbf{y},\boldsymbol{\eta}\rangle}\left(\int_{-R}^{R}p^{\sharp}(\mathbf{x},\boldsymbol{\eta},\xi_d)e^{ix_d\xi_d}\mathrm{d}\xi_d-\int_{\gamma_R}p^{\sharp}(\mathbf{x},\boldsymbol{\eta},\zeta)e^{ix_d\zeta}\mathrm{d}\zeta\right)\mathrm{d}\boldsymbol{\eta}.$$

Applying dominated convergence again, as we let $x_d\to 0^{+}$, we have

$$\widetilde{v_{k,j}^{\pm}}g(\mathbf{y})=(2\pi)^{-d}\int_{\mathbb{R}^{d-1}}\widehat{g}(\eta)e^{i\langle\mathbf{y},\boldsymbol{\eta}\rangle}\left(\int_{-R}^{R}p^{\sharp}(\mathbf{y},0,\boldsymbol{\eta},\xi_d)\mathrm{d}\xi_d-\int_{\gamma_R}p^{\sharp}(\mathbf{y},0,\boldsymbol{\eta},\zeta)\mathrm{d}\zeta\right)\mathrm{d}\boldsymbol{\eta}.$$

The fact that the symbol is a positively homogeneous symbol follows by taking $\lambda\ge 1$ and applying a change of variables to $\int_{-\lambda R}^{\lambda R}p_\ell(\mathbf{y},0,\lambda\boldsymbol{\eta},\xi_d)\mathrm{d}\xi_d-\int_{\gamma_{\lambda R}}p_\ell(\mathbf{y},0,\lambda\boldsymbol{\eta},\zeta)\mathrm{d}\zeta$, as in Case 1. ∎

When $j+k=n\le 2m-2$, the principal symbol, in local coordinates, is (for $|\boldsymbol{\eta}|\ge 1$)

$$\sigma\left(\widetilde{v_{k,j}}\right)(\mathbf{y},\boldsymbol{\eta})=\frac{(-1)^{m-\frac{n}{2}}}{2\pi}\begin{cases}\int_{-\infty}^{\infty}\frac{1}{\left(\xi_d^2+\mathsf{d}(\mathbf{y},\boldsymbol{\eta})\right)^{m-\frac{n}{2}}}\mathrm{d}\xi_d, & j,k \text{ both even},\\ \int_{-\infty}^{\infty}\frac{\xi_d^2}{\left(\xi_d^2+\mathsf{d}(\mathbf{y},\boldsymbol{\eta})\right)^{m+1-\frac{n}{2}}}\mathrm{d}\xi_d, & j,k \text{ both odd},\\ \int_{-\infty}^{\infty}\frac{-i\xi_d}{\left(\xi_d^2+\mathsf{d}(\mathbf{y},\boldsymbol{\eta})\right)^{m-\frac{n-1}{2}}}\mathrm{d}\xi_d, & n=j+k \text{ odd}.\end{cases}$$

After a change of variables and integrating out the $\xi_d$ variable, we are left with the simple expression

$$\sigma\left(\widetilde{v}_{k,j}\right)(\mathbf{y},\boldsymbol{\eta})=(-1)^{m-n}C_{k,j}\mathsf{d}(\mathbf{y},\boldsymbol{\eta})^{\frac{n+1}{2}-m}, \tag{5.7}$$

with $C_{k,j}=0$ when $n=j+k$ is odd. When $n=j+k$ is even, we have

$$C_{k,j}=\frac{1}{2\pi}\begin{cases}\int_{-\infty}^{\infty}\frac{1}{(\zeta^2+1)^{m-\frac{n}{2}}}\mathrm{d}\zeta=2^{1+n-2m}\mathfrak{b}_{m-n/2-1}, & j,k \text{ both even},\\ \int_{-\infty}^{\infty}\frac{\zeta^2}{(\zeta^2+1)^{m+1-\frac{n}{2}}}\mathrm{d}\zeta=2^{1+n-2m}\mathfrak{c}_{m-n/2-1}, & j,k \text{ both odd},\end{cases} \tag{5.8}$$

where $\mathfrak{b}_j:=\frac{(2j)!}{j!j!}$ and $\mathfrak{c}_{j-1}:=4\mathfrak{b}_{j-1}-\mathfrak{b}_j$ are Catalan numbers (see [21, Proposition 4.1]).

*Proof of Lemma* 3.7. The regularity of the map $v_{k,j}$ now follows from the mapping properties of pseudodifferential operators [43, Chapter 11, Theorem 2.1]) and a simple change of variables. ∎

**5.4. Ellipticity of matrix symbols and a parametrix.** In this section, we construct the global right parametrix $R$ for $L$. This is done in two stages—first by generating a local parametrix in coordinates on $O'\subset\mathbb{R}^{d-1}$ by way of Lemma 5.5 and then by carefully piecing together a number of local parametrices with the aid of a partition of unity.

**5.4.1. A local parametrix.** We consider again a map $\Psi':O'\to U'$, with $O'\subset\mathbb{R}^{d-1}$ and $U'\subset\partial\Omega$. Let $M_1:=\min_{\mathbf{y}\in O'}\lambda_1(\mathbf{y})$ and $M_{d-1}:=\max_{\mathbf{y}\in O'}\lambda_{d-1}(\mathbf{y})$ be the least and greatest eigenvalues, respectively, of the inverse Gram matrix $\mathsf{G}^{-1}(\mathbf{y})$ described in section 1.3.

The operator

$$\widetilde{L}:\begin{pmatrix}s_0\\ \vdots\\ s_{m-1}\end{pmatrix}\mapsto\begin{pmatrix}\widetilde{v}_{00} & \dots & \widetilde{v}_{0,m-1}\\ \vdots & \ddots & \vdots\\ \widetilde{v}_{m-1,0} & \dots & \widetilde{v}_{m-1,m-1}\end{pmatrix}\begin{pmatrix}s_0\\ \vdots\\ s_{m-1}\end{pmatrix} \tag{5.9}$$

is, modulo a smoothing operator, the local version of the operator $L$. Its $k,j$ entry (with indices running from $j,k=0,\dots,m-1$) is a pseudodifferential operator of order $1+j+k-2m$. Such operators, with orders that are Hankel matrices, are so-called Douglis–Nirenberg systems; cf. [11].

We consider matrix pseudodifferential operators $A$ with symbols having entries $a_{k,j}\in S^0$, since such operators have a simple notion of ellipticity. The principal symbol $\sigma(A)$ is nonsingular for large $|\xi|$ if and only if the scalar symbol $\det(\sigma(A)$ is elliptic of order 0. Thus it suffices to check that the determinant of the principal symbol is bounded from below as $|\xi| \to \infty$.

Operators with diagonal symbols are another class with a simple notion of ellipticity. Writing $a(x,\xi) = \big(a_{kj}(x,\xi)\big)$, the off-diagonal entry $a_{kj}(x,\xi)$ (with $j \neq k$) is zero, and each diagonal entry $a_{jj}$ is elliptic. Because such systems are decoupled, a parametrix of the same type exists—namely $b(x,\xi) = \big(b_{jk}(x,\xi)\big)$, with $b_{jj}$ the (scalar) parametrix of $a_{jj}$ and $b_{jk} = 0$ when $j \neq k$.

Returning to the operator $\widetilde{L}$, we make the decomposition $\widetilde{L} = \mathsf{ALS}$ with properly supported pseudodifferential operators $\mathsf{A}$ and $\mathsf{S}$ that have diagonal symbols with elliptic entries and pseudodifferential operator $\mathsf{L}$ that have a matrix symbol with entries in $S^0$ and which (as we soon shall see) is elliptic. Specifically, we require

$$\sigma(\mathsf{A}) = \begin{pmatrix} \mathsf{d}(\mathbf{y},\boldsymbol{\eta})^{(1-m)/2} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathsf{d}(\mathbf{y},\boldsymbol{\eta})^{0} \end{pmatrix}$$

and

$$\sigma(\mathsf{S}) = \begin{pmatrix} \mathsf{d}(\mathbf{y},\boldsymbol{\eta})^{-m/2} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathsf{d}(\mathbf{y},\boldsymbol{\eta})^{-1/2} \end{pmatrix}.$$

Since $M_1|\boldsymbol{\eta}|^2 \leq \mathsf{d}(\mathbf{y},\boldsymbol{\eta}) \leq M_d|\boldsymbol{\eta}|^2$, we see that each diagonal entry is elliptic. That is, for $j = 0,\dots,m-1$, the diagonal entry $\sigma(\mathsf{A})_{jj}$ is elliptic of order $1+j-m$ and $\sigma(\mathsf{S})_{jj}$ is elliptic of order $j-m$. The parametrices for operators $\mathsf{A}$ and $\mathsf{S}$ are the decoupled operators $\mathsf{B}$ and $\mathsf{T}$, respectively. These have symbols

$$\sigma(\mathsf{B}) := \begin{pmatrix} b_{0,0}(\mathbf{y},\boldsymbol{\eta}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_{m-1,m-1}(\mathbf{y},\boldsymbol{\eta}) \end{pmatrix}$$

and

$$\sigma(\mathsf{T}) := \begin{pmatrix} t_{0,0}(\mathbf{y},\boldsymbol{\eta}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t_{m-1,m-1}(\mathbf{y},\boldsymbol{\eta}) \end{pmatrix},$$

with $b_{jj}(y,\eta)$ the parametrix of $\mathsf{d}(\mathbf{y},\boldsymbol{\eta})^{(1+j-m)/2}$ (for $j = 0,\dots,m-1$) and similarly with $t_{jj}(y,\eta)$ the parametrix of $\mathsf{d}(\mathbf{y},\boldsymbol{\eta})^{(j-m)/2}$ (for $j = 0,\dots,m-1$).

The operator $\mathsf{L}$ is simply defined to be the composition $\mathsf{B}\widetilde{L}\mathsf{T}$, and its principal symbol can be computed by taking the product $\sigma(\mathsf{B})\sigma(\widetilde{L})\sigma(\mathsf{T})$. Indeed, from (5.7) and (5.8), we have $\sigma(\mathsf{L})_{j,k} \in S^0(O)$ when $j+k$ is even (otherwise it is in $S^{-1}$), and for $|\boldsymbol{\eta}| \geq 1$, we have

$$\sigma(\mathsf{L})_{j,k}(\mathbf{y},\boldsymbol{\eta}) = 2^{1+j+k-2m} \begin{cases} \mathfrak{b}_{m-(j+k)/2-1}, & j,k \text{ both even,} \\ \mathfrak{c}_{m-(j+k)/2-1}, & j,k \text{ both odd.} \end{cases}$$

Since $\det\big(\sigma(\mathsf{L})\big) = 2^{m^2}$ by [21, Proposition 5.2] (note that $\sigma(\mathsf{L})_{j,k}$ differs from $\mathsf{M}_{k,j}$ of [21] by a change of signs in the odd columns), it follows that $\mathsf{L}$ is elliptic and has a parametrix $\mathsf{R}$. A parametrix for $\widetilde{L}$ is then $\widetilde{R} = \mathsf{TRB}$, a pseudodifferential operator whose $j,k$ entry has order $2m-j-k-1$.

**5.4.2. A global parametrix.** We follow [19, Theorem 8.6] in combining local parametrices of the various $\widetilde{L}$ to obtain a global parametrix $R$ for $L$.

Let $(U_\ell, \Phi_\ell)_{\ell=1,\dots,N}$ be an atlas for $\partial\Omega$, and write $\Psi_\ell = \Phi_\ell^{-1} : O_\ell \to U_\ell$. Let $(\tau_\ell)_{\ell=1,\dots,N}$ be a smooth partition of unity for $\partial\Omega$ subordinate to $(U_\ell)_{\ell=1,\dots,N}$. Consider two families of smooth cut-off functions $(\zeta_\ell)_{\ell=1,\dots,N}$ and $(\theta_\ell)_{\ell=1\dots N}$ so that $\zeta_\ell : \partial\Omega \to [0,1]$, with $\zeta_\ell(z) = 1$, for $z \in \operatorname{supp}(\tau_\ell)$ and $\operatorname{supp}(\zeta_\ell) \subset U_\ell$ and $\theta_\ell : \partial\Omega \to [0,1]$, with $\theta_\ell(z) = 1$, for $z \in \operatorname{supp}(\zeta_\ell)$ and $\operatorname{supp}(\theta_\ell) \subset U_j$.

In each $O_\ell$, let $\widetilde{L}_\ell$ denote the operator given by (5.9). The construction in section 5.4.1 guarantees a right parametrix $\widetilde{R}_\ell$; for distributions supported in $U_\ell$, the change of coordinates $(\widetilde{R}_\ell)^{\Phi_\ell} = \Phi_\ell^* \widetilde{R}_\ell \Psi_\ell^*$ is well defined. Define the global right parametrix $R$ as

$$Rf(u) := \sum_{\ell=1}^{N} \zeta_\ell(u)\big[(\widetilde{R}_\ell)^{\Phi_\ell}(\tau_\ell f)\big](u).$$

We have $LRf = \sum_{\ell=1}^N L\big(\zeta_\ell\big[(\widetilde{R}_\ell)^{\Phi_\ell}(\tau_\ell f)\big]\big) \sim \sum_{\ell=1}^N \theta_\ell L\big(\zeta_\ell\big[(\widetilde{R}_\ell)^{\Phi_\ell}(\tau_\ell f)\big]\big)$. Note that $\widetilde{L}_\ell$ differs from $L^{\Psi_\ell}$ on $O_\ell$ by a smoothing operator. A similar statement can be made on $\partial\Omega$: for each $\ell$, $\theta_\ell L \zeta_\ell \sim \theta_\ell(\widetilde{L}_\ell)^{\Phi_\ell}\zeta_\ell$,

$$LRf \sim \sum_{\ell=1}^{N} \theta_\ell(\widetilde{L}_\ell)^{\Phi_\ell}\left(\zeta_\ell\big[(\widetilde{R}_\ell)^{\Phi_\ell}(\tau_\ell f)\big]\right) \sim \sum_{\ell=1}^{N} \theta_\ell(\widetilde{L}_\ell)^{\Phi_\ell}\left(\big[(\widetilde{R}_\ell)^{\Phi_\ell}(\tau_\ell f)\big]\right) \sim \sum_{\ell=1}^{N} \theta_\ell \tau_\ell f = f.$$

In the second equivalence (modulo a smoothing operator), we have made use of item 1 of Result 5.2 and the fact that $\operatorname{supp}(\tau_\ell)$ is contained in the zero set of $1 - \zeta_\ell$.

*Proof of Lemma* 4.4. By the existence of $R$, we have $LR_{|_{Y_{p,s+2m-1}}} = \mathrm{Id}_{Y_{p,s+2m-1}} + K$, with $K$ compact, and it follows that $\operatorname{ran}(LR_{p,s})$ is finitely complemented. Since $LR(Y_{p,s+2m-1}) \subset L(X_{p,s})$, it follows that the range of $L$ is also finitely complemented and hence closed in $Y_{p,s+2m-1}$. Because $\operatorname{ran}(P)$ is finite dimensional, $\operatorname{ran}(L) + \operatorname{ran}(P)$ is closed in $Y_{p,s+2m-1}$ and $L^\sharp(X^\sharp_{p,s})$ is closed in $Y^\sharp_{p,s+2m-1}$. ∎

**5.5. Uniqueness.** We begin by providing a uniqueness result for boundary layer potential solutions to (1.2) in the bounded domain $\Omega$, and a partial result for $\mathbb{R}^d \setminus \overline{\Omega}$. Let us introduce the bilinear form $\mathcal{B}$ via

$$\mathcal{B}(w,v) = \begin{cases} \int_\Omega \langle \nabla\Delta^{(m-1)/2} w, \nabla\Delta^{(m-1)/2} v\rangle \mathrm{d}x, & m \text{ is odd},\\ \int_\Omega \Delta^{m/2} w(x)\Delta^{m/2} v(x)\mathrm{d}x, & m \text{ is even}. \end{cases}$$

We include the result for $\Omega$ for completeness. (One finds it in [2, 32].)

LEMMA 5.9. *Suppose that $u \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ is a classical solution of* (1.2) *in $\Omega$ with homogeneous Dirichlet values (i.e., $h_k = 0$ for all $k = 0, \dots, m-1$). Then $u = 0$ in $\Omega$.*

*Proof.* Recall Green's first identity

$$\int_\Omega v(x)\Delta^m w(x)\mathrm{d}x - \mathcal{B}(w,v) = \sum_{j=0}^{m-1}(-1)^j \int_{\partial\Omega} \lambda_j v(x)\lambda_{2m-j-1} w(x)\,\mathrm{d}\sigma(x).$$

Apply this to $w = u$ and $v = u$, and observe that $\mathcal{B}(u,u) = 0$.

When $m$ is even, we see that $\Delta^{m/2}u$ vanishes a.e. in $\Omega$. When $m$ is odd, $\Delta^{(m-1)/2}u$ must be a constant a.e. in $\Omega$, but since it satisfies $\lambda_{m-1}u = 0$, it vanishes. In either case, we have that $u$ satisfies the corresponding polyharmonic Dirichlet problem of order $\lfloor m/2 \rfloor$. Repeating this argument a maximum of $\log_2 m$ times, we arrive at $|\nabla u(x)| = 0$ throughout $\Omega$ and $u|_{\partial\Omega} = 0$. ∎

The corresponding problem for the exterior is more difficult. In general, uniqueness does not hold. For example, both $u_1(x,y) = 1 + \ln(x^2+y^2)$ and $u_2(x,y) = x^2+y^2$ satisfy $\Delta^2 u = 0$ in the complement of the unit ball, and they have the same trace and normal derivative on the unit circle. To treat this, we make use of a *radiating condition*, which guarantees uniqueness for functions having controlled growth. In other words, under some additional assumptions of behavior of the function at infinity, the solution we propose will be in a unicity class for the unbounded domain; cf. [9].

The function $u$ satisfies the radiating conditions if there exists $C$ so that, for all sufficiently large $R$ (relative to $\Omega$), with the boundary operators $\lambda_j$ on $B(0,R)$, we have

$$|\lambda_j u(x)| \le C \begin{cases} R^{m-1-j} & \text{for } j = 0, \dots, m-1, \\ R^{m-d-j} & \text{for } j = m, \dots, 2m-1. \end{cases} \tag{5.10}$$

Lemma 5.10. *Suppose that $u \in C^{2m}(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^{m-1}(\mathbb{R}^d \setminus \Omega)$ is a classical solution of* (1.2) *in $\mathbb{R}^d \setminus \overline{\Omega}$ with homogeneous Dirichlet values. If $u$ satisfies* (5.10) *as $R \to \infty$, then $\Delta^{m/2} u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$ if $m$ is even and $\nabla \Delta^{(m-1)/2} u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$ if $m$ is odd.*

*Proof.* Considering Green's first identity in the set $\Upsilon_R = B(0,R) \setminus \overline{\Omega}$ (for sufficiently large $R$), we have

$$\int_{\Upsilon_R} u(x) \Delta^m u(x) \mathrm{d}x - \mathcal{B}(u,u) = \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j \int_{\partial \Upsilon_R} \lambda_j u(x) \lambda_{2m-j-1} u(x) \, \mathrm{d}\sigma(x).$$

By the homogeneous Dirichlet conditions, the boundary integrals over $\partial\Omega$ vanish. This leaves

$$\begin{aligned} \sum_{j=0}^{m-1} (-1)^j \int_{\{|x|=R\}} \lambda_j u(x) \lambda_{2m-j-1} u(x) \, \mathrm{d}\sigma(x) &\le C \sum_{j=0}^{m-1} \int_{\{|x|=R\}} R^{m-1-j} R^{1+j-d-m} \mathrm{d}\sigma(x) \\ &\le C R^{d-1} R^{-d} \xrightarrow{R\to\infty} 0. \end{aligned}$$

From this, it follows that $\mathcal{B}(u,u) = 0$. ∎

The radiating condition follows from the fact that $P^{\mathrm{t}} \mathbf{g} = 0$.[3]

*Proof of Lemma* 4.5. Fix $p, s \in \mathbb{R}$, and consider a solution $v = (A, \mathbf{g}) \in X^\sharp_{p,s}$ to the homogeneous system $L^\sharp v = 0$. This implies that $L\mathbf{g} \in \Pi_{m-1}$, and the ellipticity of $L$—in particular the fact that $\operatorname{singsupp}(\mathbf{g}) \subset \operatorname{singsupp}(L\mathbf{g}) = \emptyset$—guarantees that the entries $g_j$ of $\mathbf{g}$ are in $C^\infty(\partial\Omega)$. Because of this and Corollary 3.3, we can extend each $V_j g_j$ to $C^\infty(\overline{\Omega})$ as well as $C^\infty(\mathbb{R}^d \setminus \Omega)$, and hence the same holds for boundary layer potential $u = \sum V_j g_j + \sum A_j p_j$. Thus $u$ satisfies (1.2) with homogeneous Dirichlet boundary conditions in both components of $\mathbb{R}^d \setminus \partial\Omega$.

By Lemma 5.9, $u = 0$ in $\Omega$.

[3]This should be a familiar phenomenon for practitioners of RBF interpolation: for scattered data fitting with conditionally positive definite functions, the interpolation matrix $(\phi(\xi - \zeta))$ is augmented by various polynomial side conditions (and, simultaneously, the addition of a polynomial to keep the system square). This has the dual effect of ensuring the interpolant lies in a *native space* (a reproducing kernel semi-Hilbert space) and that the augmented system is injective.

To handle $u$ in the exterior of $\Omega$, write $u = \phi * \mu_{\mathbf{g}} + p$, with $\mu_{\mathbf{g}} = \sum_{j=0}^{m-1} \Lambda_j^{\mathrm{t}}(g_j \cdot \delta_{\partial\Omega})$. Applying the moment conditions $P^{\mathrm{t}}\mathbf{g} = 0$, we see that $\mu_{\mathbf{g}} \perp \Pi_{m-1}$. Thus, for any $q \in \Pi_{m-1}$, we have that $\phi * \mu_{\mathbf{g}} = (\phi - q) * \mu_{\mathbf{g}}$. For $x$ sufficiently far from $\partial\Omega$, let $Q_x$ be the Taylor polynomial of degree $m-1$ to $\alpha \mapsto \phi(x-\alpha)$ centered at the origin (or any other point suitably close to $\partial\Omega$). Then

$$(\phi - Q_x) * \mu_{\mathbf{g}} = \sum_{j=0}^{m-1} \int_{\partial\Omega} \lambda_{j,\alpha} \left[\phi(x-\alpha) - Q_x(\alpha)\right] g_j(\alpha) \mathrm{d}\alpha.$$

Since $\lambda_j Q_x$ is the degree $m-1-j$ Taylor polynomial to $\alpha \mapsto \lambda_{j,\alpha}\phi(x-\alpha)$, we have that

$$|\lambda_{j,\alpha} \left[\phi(x-\alpha) - Q_x(\alpha)\right]| \leq C(\mathrm{diam}(\Omega))^{m-j} \sup_{|\alpha| \leq m} \max_{\alpha \in \Omega} |D^\alpha \phi(x-\alpha)|.$$

From the remainder formula in Taylor's theorem and estimates on the derivative of the fundamental solution (2.3), we have that $\phi * \mu_{\mathbf{g}} = \mathcal{O}(|x|^{m-d}|\log(x)|)$ as $|x| \to \infty$. Repeating this for derivatives of $\phi * \mu_{\mathbf{g}}$, we observe that for $|\alpha| + d \leq m$, we have $|D^\alpha(\phi * \mu_{\mathbf{g}})(x)| = \mathcal{O}(|x|^{m-d-|\alpha|}|\log x|)$, while for $|\alpha| + d > m$, we have $|D^\alpha(\phi * \mu_{\mathbf{g}})(x)| = \mathcal{O}(|x|^{m-d-|\alpha|})$. Because $D^\alpha p(x) = \mathcal{O}(|x|^{m-1-|\alpha|})$ for $\alpha \leq m-1$, the radiating conditions (5.10) are satisfied by $u$, and Lemma 5.10 applies.

Because $\mathcal{B}(u,u) = 0$ in $\mathbb{R}^d \setminus \partial\Omega$, we have $\Lambda_m u(x) = 0$ for $x \in \mathbb{R}^d \setminus \partial\Omega$. On the other hand,

$$\Lambda_m u = \Lambda_m \left( \sum_{j=0}^{m-1} V_j g_j + p \right) = \Lambda_m V_{m-1} g_{m-1} + \sum_{j=0}^{m-1} \Lambda_m V_j g_j.$$

By Corollary 3.4, the sum $\sum_{j=0}^{m-1} \Lambda_m V_j g_j$ is continuous throughout $\mathbb{R}^d$, while for $x_0 \in \partial\Omega$,

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} \Lambda_m V_{m-1} g_{m-1}(x) - \lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^d \setminus \overline{\Omega}}} \Lambda_m V_{m-1} g_{m-1}(x) = g_{m-1}(x_0).$$

Therefore,

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} \Lambda_m u(x) - \lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^d \setminus \overline{\Omega}}} \Lambda_m u(x) = g_{m-1}(x_0) = 0.$$

The remaining auxiliary functions $g_{m-2}, g_{m-3}, \ldots, g_0$ can be treated using the same argument with the operators $\Lambda_{m+1}, \Lambda_{m+2}, \ldots, \Lambda_{2m-1}$. Finally, it follows obviously that $p = u \in \Pi_{m-1}$ vanishes. ∎

## 6. Proofs of the main results.

*Proof of Theorem* 2.2. Extend $\mathbf{h} \in Y_{p,s}$ to get $(\vec{0}, \mathbf{h})^{\mathrm{t}} \in Y_{p,s}^\sharp$. The solution is $(\vec{A}, \mathbf{g})^{\mathrm{t}} = L_{p,s}^{-1}(\vec{0}, \mathbf{h})^{\mathrm{t}}$, and one readily observes that $\mathbf{g} \in X_{p,s+1-2m}$ and $\|\mathbf{g}\|_{X_{p,s+1-2m}} \leq \|L_{p,s}^{-1}\| \|\mathbf{h}\|_{Y_{p,s}}$. As desired,

$$\|g_j\|_{W_p^{s+j+1-2m}(\partial\Omega)} \leq \|L_{p,s}^{-1}\| \max\left( \|h_0\|_{H_p^s(\partial\Omega)}, \ldots, \|h_{m-1}\|_{H_p^{s-(m-1)}(\partial\Omega)} \right)$$

for each $j = 0, \ldots, m-1$. ∎

*Proof of Corollary* 2.3. Since $\mathbf{h} \in Y_{2,m-1/2}$, $(0,\mathbf{h})^{\mathrm{t}} \in Y^{\sharp}_{2,m-1/2}$, and $(\vec{A},\mathbf{g}) = L^{-1}(0,\mathbf{h}) \in X^{\sharp}_{2,-m+1/2}$. Consider the putative solution

$$u = p + \sum_{j=0}^{m-1} V_j g_j = p + \phi * [(\Lambda_j)^{\mathrm{t}}(g_j \cdot \delta_{\partial\Omega})].$$

For $0 \le j \le m-1$, the functional $g_j \cdot \delta_{\partial\Omega}$ is in the dual of $H_2^{m-j}(\mathbb{R}^d)$: for any $f \in H_2^{m-j}(\mathbb{R}^d)$, the trace theorem guarantees $\mathrm{Tr} f \in H_2^{m-j-1/2}(\partial\Omega)$. By definition, $|\langle g_j \cdot \delta_{\partial\Omega}, f\rangle| = |\langle g_j, \mathrm{Tr} f\rangle|$, and thus

$$|\langle g_j \cdot \delta_{\partial\Omega}, f\rangle| \le \|g_j\|_{H_2^{j-m+1/2}(\partial\Omega)} \|\mathrm{Tr} f\|_{H_2^{m-j-1/2}(\partial\Omega)} \le C\|g_j\|_{H_2^{j-m+1/2}(\partial\Omega)} \|f\|_{H_2^{m-j}(\mathbb{R}^d)}.$$

Consequently, $g_j \cdot \delta_{\partial\Omega} \in H_2^{-m+j}(\mathbb{R}^d)$ and $\|g_j \cdot \delta_{\partial\Omega}\|_{H_2^{-m+j}(\mathbb{R}^d)} \le C\|g_j\|_{H_2^{-m+j+1/2}(\partial\Omega)}$ for each $j$.

It follows that $(\Lambda_j)^{\mathrm{t}}(g_j \cdot \delta_{\partial\Omega}) \in H_2^{-m}(\mathbb{R}^d)$, and, because $f \mapsto f * \phi$ is, up to a smoothing operator, a pseudodifferential operator of order $-2m$, $\sum_{j=0}^{m-1} \phi * [(\Lambda_j)^{\mathrm{t}}(g_j \cdot \delta_{\partial\Omega})] \in H^m_{2,loc}(\mathbb{R}^d)$. Thus

$$\|u\|_{H_2^m(\Omega)} \le C \max_{j=0,\dots,m-1} \|g_j\|_{H_2^{j-m+1/2}(\mathbb{R}^d)} \le C \max_{k=0,\dots,m-1} \|h_k\|_{H_2^{m-1/2-k}(\mathbb{R}^d)}.$$

The fact that $\Delta^m u = 0$ in $\Omega$ is clear from the construction. The fact that the Dirichlet conditions are satisfied follows from a limiting argument: by Theorem 2.2, the conditions hold for $\mathbf{h} \in (C^\infty)^m$; this extends to $\mathbf{h} \in Y_{2,m-1/2}$ by the density of $(C^\infty)^m$ and the continuity of the map $\mathbf{h} \mapsto u$ given by $\|u\|_{H_2^m(\Omega)} \le C\|(\vec{A},\mathbf{g})\|_{X^{\sharp}_{2,-m+1/2}} \le C\|\mathbf{h}\|_{Y_{2,m-1/2}}$. ∎

*Proof of Theorem* 2.4. We begin by considering $f \in C^\infty(\overline{\Omega})$. In this case, Theorem 2.2 ensures that there are $C^\infty$ functions $g_j$, $j = 0,\dots,m-1$, and a polynomial $p \in \Pi_{m-1}$ so that $f_1 = \sum_{j=0}^{m-1} V_j g_j + p$ solves the polyharmonic Dirichlet problem (1.2) with boundary data $h_k = \lambda_k f \in C^\infty(\partial\Omega)$ for $k = 0,\dots,m-1$.

By Corollary 3.3, the remainder $f_2 := f - f_1$ is in $C^\infty(\overline{\Omega})$, and Green's representation (2.5) gives

$$f_2(x) = \int_\Omega \Delta^m f(\alpha)\phi(x-\alpha)\mathrm{d}\alpha + \sum_{j=0}^{m-1}(-1)^{j+1}\int_{\partial\Omega}(\lambda_{2m-j-1}(f-f_1))(\alpha)\,\lambda_{j,\alpha}\phi(x-\alpha)\,\mathrm{d}\sigma(\alpha).$$

The representation of $f$ follows, and we note that

$$N_j f = g_j + (-1)^{j+1}(\lambda_{2m-j-1} f - \lambda_{2m-j-1} f_1) \tag{6.1}$$

holds.

For every $s \ge 0$, there is $C < \infty$ so that $\|\lambda_k f\|_{H_p^s(\partial\Omega)} \le C\|f\|_{B_{p,1}^{s+k+1/p}(\Omega)}$ by the trace theorem, specifically the fact that $\mathrm{Tr} : B_{p,1}^{1/p}(\Omega) \to L_p(\partial\Omega)$ is bounded (this is in [47, section 4.4.3]). In particular,

$$\|\lambda_{2m-j-1} f\|_{H_p^s(\partial\Omega)} \le C\|f\|_{B_{p,1}^{s+2m-j-1+1/p}(\Omega)} \tag{6.2}$$

holds for all $s \ge 0$ and all integers $0 \le j \le 2m-1$. It follows from Theorem 2.2 and (6.2) that for $k = 0,\dots,m-1$,

$$\|g_k\|_{H_p^s(\partial\Omega)} \le C \max_{j=0,\dots,m-1} \|\lambda_j f\|_{H_p^{s+(2m-j-k-1)}(\partial\Omega)} \le C\|f\|_{B_{p,1}^{s+2m-k-1+1/p}(\Omega)}. \tag{6.3}$$

It remains to consider $\lambda_{2m-j-1} f_1 = \lambda_{2m-j-1}(\sum_{k=0}^{m-1} V_k g_k + p)$. Employing the boundary operators, this simplifies to $\sum_{k=0}^{m-1} v^{-}_{2m-j-1,k} g_k$, since $p \in \Pi_{m-1}$ and $j \leq m-1$. It follows from Lemma 3.7 that

$$\|\lambda_{2m-j-1} f_1\|_{H^s_p(\partial\Omega)} \leq \sum_{k=0}^{m-1} \|v^{-}_{2m-j-1,k} g_k\|_{H^s_p(\partial\Omega)} \leq \sum_{k=0}^{m-1} \|g_k\|_{H_p^{(s+k-j)}(\partial\Omega)}$$

holds. We use (6.3), namely $\|g_k\|_{H_p^{(s+k-j)}(\partial\Omega)} \leq C\|f\|_{B^{s+2m-j-1+1/p}_{p,1}(\Omega)}$, to establish

$$\|\lambda_{2m-j-1} f_1\|_{H^s_p(\partial\Omega)} \leq C\,\|f\|_{B^{s+2m-j-1+1/p}_{p,1}(\Omega)}. \tag{6.4}$$

Applying the triangle inequality to (6.1) gives

$$\begin{aligned}\|N_j f\|_{H^s_p(\partial\Omega)} &\leq \|g_j\|_{H^s_p(\partial\Omega)} + \|\lambda_{2m-j-1} f\|_{H^s_p(\partial\Omega)} + \|\lambda_{2m-j-1} f_1\|_{H^s_p(\partial\Omega)} \\ &\leq C\|f\|_{B^{s+2m-j-1}_{p,1}(\Omega)},\end{aligned}$$

where the final inequality follows from the three estimates (6.3), (6.2), and (6.4).

For a general $f \in W_p^{2m}(\Omega)$, the representation (2.1) holds by the density of $C^\infty(\overline{\Omega})$ and the continuity of the operators $\Delta^m$ and $N_j$, $j = 0, \ldots, m-1$. ∎

## 7. Surface spline approximation.

**7.1. Approximation scheme.** We now develop the approximation scheme based on the integral identity introduced in section 2. The scheme and the accompanying error estimate are generalizations of the scheme in [20]. Specifically, the approximation scheme takes the form

$$T_\Xi f(x) = \int_\Omega \Delta^m f(\alpha) k(x,\alpha)\,\mathrm{d}\alpha + \sum_{j=0}^{m-1} \int_{\partial\Omega} N_j f(\alpha) k_j(x,\alpha)\,\mathrm{d}\sigma(\alpha) + p(x)$$

with Theorem 2.4 providing $p$ and $N_j f$. The challenge is to find suitable *replacement kernels* $k(x,\alpha) = \sum_\xi a(\alpha,\xi)\phi(x-\xi)$ and $k_j(x,\alpha) = \sum_\xi a_j(\alpha,\xi)\phi(x-\xi)$ so that the *error kernels*

$$E(x,\alpha) := |k(x,\alpha) - \phi(x-\alpha)|, \tag{7.1}$$

$$E_j(x,\alpha) := |k_j(x,\alpha) - \lambda_{j,\alpha}\phi(x-\alpha)|,\ j = 0,\ldots,m-1, \tag{7.2}$$

are uniformly small and decay rapidly as $|x-\alpha| \to \infty$.

It follows that the pointwise error incurred from the approximation scheme can be estimated for sufficiently smooth $f$ by

$$|f(x) - T_\Xi f(x)| \leq \int_\Omega E(x,\alpha)|\Delta^m f(\alpha)|\,\mathrm{d}\alpha + \sum_{j=0}^{m-1} \int_{\partial\Omega} E_j(x,\alpha)|N_j f(\alpha)|\,\mathrm{d}\sigma(\alpha).$$

From this, it is clear that for $f \in W_p^{2m}(\Omega)$, the $L_p$ error is bounded by

$$\|f - T_\Xi\|_{L_p(\Omega)} \leq \|\mathcal{E}\|_{p\to p}\|\Delta^m f\|_{L_p(\Omega)} + \sum_{j=0}^{m-1} \|\mathcal{E}_j\|_{p\to p}\|N_j f\|_{L_p(\partial\Omega)}, \tag{7.3}$$

where $\mathcal{E}$ and $\mathcal{E}_j$ are the integral operators induced by the error kernels $E$ and $E_j$.

In section 7.2, we state the conditions on the centers necessary for a high rate of convergence. Section 7.3 states and proves the main approximation results. A corollary showing that an increase in density of centers near the boundary yields the boundary-free approximation order is given in section 7.4.

**7.2. Error kernels.** In this section, we describe how to construct replacement kernels $k$ and $k_j$, $j = 0, \dots, m-1$, and give pointwise estimates for the corresponding error kernels $E$ and $E_j$, $j = 0, \dots, m-1$. Following this, we give operator norms for the integral operators defined by $E$ and $E_j$, which leads to components of the approximation error in (7.3).

*Interior kernel.* To construct $k(x, \alpha) = \sum_{\xi \in \Xi} a(\alpha, \xi)\phi(x - \xi)$, we require the properties of the coefficient kernel $(\alpha, \xi) \mapsto a(\alpha, \xi)$ of the following type. For $K, R > 0$ and $M \in \mathbb{N}$, the following hold:

1. $\max_{\alpha \in \overline{\Omega}} \sum_{\xi \in \Xi} |a(\alpha, \xi)| \le K$.
2. For every $p \in \Pi_M$, $\sum_{\xi \in \Xi} a(\alpha, \xi) p(\xi) = p(\alpha)$.
3. If $|\alpha - \xi| > R$, then $a(\alpha, \xi) = 0$.

We call such a coefficient kernel a stable, local polynomial reproduction of order $M$, radius $R$, and stability $K$. Local polynomial reproductions have a long history in RBF approximation and related fields—one may see their use in [49, 26, 48, 10, 20], for example.

We now recall a result guaranteeing that such local polynomial reproductions exist for regions with a Lipschitz boundary satisfying an interior cone condition[4] with aperture angle $\theta$ and radius $r$, namely for regions $\Omega$ having the property that for every $\alpha \in \overline{\Omega}$, there is $\nu_\alpha$ so that the cone

$$C(\alpha, r, \theta, \nu_\alpha) := \left\{ x \in \mathbb{R}^d \mid |x - \alpha| \le r, \left\langle \frac{x - \alpha}{|x - \alpha|}, \nu_\alpha \right\rangle \ge \cos \alpha \right\}$$

is contained in $\overline{\Omega}$.

The result we cite is the so-called *norming set* result [48, Theorem 3.14], which ensures that for every $M \in \mathbb{N}$ and $\Xi$ sufficiently dense (with $h = \max_{x \in \Omega} \operatorname{dist}(x, \Xi)$ sufficiently small, i.e., bounded above by a constant depending on $\Omega$ and $M$), appropriately rescaled cones $C$ contain subsets of $\Xi$ so that the norm of a polynomial of degree $M$ over $C$ is controlled by its values on $\Xi \cap C$. In short, there is a $\Gamma > 0$ depending on the cone parameters $r, \theta$ of $\Omega$ so that for every $p \in \Pi_M$, the uniform norm over the rescaled cone $C(\alpha) = C(\alpha, \Gamma M^2 h, \theta, \nu_\alpha)$ is controlled by the finite subset obtained from $\Xi$ (i.e., the norming set). Indeed,

$$\|p\|_{L_\infty(C(\alpha))} \le 2 \left\| p \left|_{\Xi \cap C(\alpha)} \right. \right\|_{\ell_\infty(\Xi \cap C(\alpha))}. \tag{7.4}$$

Note that beside the requirement on $h$, the geometry of $\Xi$ does not play a role in this estimate.

By (7.4), we construct a functional $\mu_\alpha$ in the dual of $\ell_\infty(\Xi \cap C(\alpha))$ representing $\delta_\alpha : p \mapsto p(\alpha)$. Namely, $\delta_\alpha p = \mu_\alpha(p\left|_{\Xi \cap C(\alpha)}\right.) = \sum_{\xi \in \Xi \cap C(\alpha)} a(\alpha, \xi) p(\xi)$ for some sequence $\big(a(\alpha, \xi)\big)_{\xi \in \Xi \cap C(\alpha)} \in \ell_1$. Because $\|\mu_\alpha\|_{\ell_\infty'} \le 2\|\delta_\alpha\|_{L_\infty'} \le 2$, we have $\sum_{\xi \in \Xi \cap C(\alpha)} |a(\alpha, \xi)| \le 2$. We extend the sequence $\big(a(\alpha, \xi)\big)_{\xi \in \Xi \cap C(\alpha)}$ by zero (i.e., $a(\alpha, \xi) = 0$ for $\xi \notin C(\alpha)$) so that $\big(a(\alpha, \xi)\big)_{\xi \in \Xi} \in \ell_1(\Xi)$.

Let $M = 2m$. Use [48, Theorem 3.14] to generate the local polynomial reproduction $a$. For the replacement kernel $k(x, \alpha) = \sum_{\xi \in \Xi} a(\alpha, \xi)\phi(x - \xi)$, the error kernel satisfies, for $x, \alpha \in \Omega$,

$$E(x, \alpha) = |\phi(x - \alpha) - k(x, \alpha)| \le C h^{2m-d} \left(1 + \frac{\operatorname{dist}(x, \alpha)}{h}\right)^{-(d+1)}$$

[4]This is a much weaker condition on $\Omega$ than we assume in this paper.

with a constant $C$ depending only on $M$ and the cone parameters $\theta$ and $\rho$. In particular, $E$ satisfies

$$\max_{x\in\Omega}\int_\Omega E(x,\alpha)\mathrm{d}\alpha \le Ch^{2m}, \qquad \max_{\alpha\in\Omega}\int_\Omega E(x,\alpha)\mathrm{d}x \le Ch^{2m}.$$

It follows that for $1 \le p \le \infty$, the integral operator $\mathcal{E} : f \mapsto \int_\Omega E(x,\alpha)f(\alpha)\mathrm{d}\alpha$ is bounded, in $L_p$ like $\|\mathcal{E}\|_{p\to p} \le Ch^{2m}$, and for $f \in W_p^{2m}(\Omega)$,

$$\left|\int_\Omega \Delta^m f(\alpha)\phi(x-\alpha)\mathrm{d}\alpha - \sum_{\xi\in\Xi} A_\xi \phi(x-\xi)\right| \le Ch^{2m}\|\Delta^m f\|_{L_p(\Omega)}, \tag{7.5}$$

where $A_\xi := \int_\Omega a(\alpha,\xi)\Delta^m f(\alpha)\mathrm{d}\alpha$.

*Boundary kernels.* To construct $k_j(x,\alpha) = \sum_{\xi\in\Xi} a_j(\alpha,\xi)\phi(x-\xi)$, we require properties similar to those of the coefficient kernels. For $K, R > 0$ and $M \in \mathbb{N}$, we seek $a_j$ so that the following hold:

1. For every $p \in \Pi_M$, $\sum_{\xi\in\Xi} a(\alpha,\xi)p(\xi) = \lambda_j p(\alpha)$.
2. If $|\alpha-\xi| > R$, then $a_j(\alpha,\xi) = 0$.
3. $\max_{\alpha\in\partial\Omega}\sum_{\xi\in\Xi}|a_j(\alpha,\xi)| \le Kh^{-j}$.

We can again use the norming set result (7.4) to build representers for the functionals $\delta_\alpha\Lambda_j$, which have norms given by Bernstein's inequality

$$|\Lambda_j p(\alpha)| \le C_\Omega (M^2/h)^j \|p\|_{L_\infty(C(\alpha))}$$

(Bernstein's inequality is given, for example, in [48, Proposition 11.6]). It follows that there is a representer $\mu_{j,\alpha}$ in the dual of $\ell_\infty(\Xi\cap C(\alpha))$ for the functional $p \mapsto \Lambda_j p(\alpha)$ in the sense that $\Lambda_j p(\alpha) = \mu_{j,\alpha}(p\left|_{\Xi\cap C(\alpha)}\right.) = \sum_{\xi\in\Xi\cap C(\alpha)} a_j(\alpha,\xi)p(\xi)$, where $\big(a_j(\alpha,\xi)\big)_{\xi\in\Xi\cap C(\alpha)}$ so that

$$\sum_{\xi\in\Xi\cap C(\alpha)} |a_j(\alpha,\xi)| = \|\mu_{j,\alpha}\| \le 2\|p\mapsto \Lambda_j p(\alpha)\| \le C_\Omega M^{2j}h^{-j}$$

(this is in [48, Theorem 11.8]). We extend by zero so that $a_j(\alpha,\xi) = 0$ for $\xi \notin C(\alpha)$.

The replacement kernels are given by $k_j(x,\alpha) = \sum_{\xi\in\Xi} a_j(\alpha,\xi)\phi(x-\xi)$. In this case, it suffices to take $M = 2m-j$. The error kernel satisfies, for every $x,\alpha\in\Omega$,

$$E_j(x,\alpha) = |\lambda_{j,\alpha}\phi(x-\alpha) - k_j(x,\alpha)| \le Ch^{2m-d-j}\left(1+\frac{\mathrm{dist}(x,\alpha)}{h}\right)^{-(d+1)}$$

with a constant $C$ depending only on $M$ and the cone parameters $\theta$ and $\rho$. In particular, $E_j$ satisfies

$$\max_{x\in\Omega}\int_{\partial\Omega} E_j(x,\alpha)\mathrm{d}\sigma(\alpha) \le Ch^{2m-j-1}, \qquad \max_{\alpha\in\partial\Omega}\int_\Omega E_j(x,\alpha)\mathrm{d}x \le Ch^{2m-j}.$$

It follows that for $1 \le p \le \infty$, the operator $\mathcal{E}_j : f \mapsto \int_{\partial\Omega} E_j(x,\alpha)f(\alpha)\mathrm{d}\sigma(\alpha)$ is bounded, in $L_p$ like $\|\mathcal{E}_j\|_{p\to p} \le Ch^{2m-j-1+1/p}$, and for $f \in W_p^{2m}(\Omega)$,

$$\left|\int_{\partial\Omega} N_j f(\alpha)\lambda_{j,\alpha}\phi(x-\alpha)\mathrm{d}\sigma(\alpha) - \sum_{\xi\in\Xi} A_{j,\xi}\phi(x-\xi)\right| \le Ch^{2m-j-1+1/p}\|N_j f\|_{L_p(\Omega)}, \tag{7.6}$$

where $A_{j,\xi} := \int_{\partial\Omega} a_j(\alpha,\xi) N_j f(\alpha) \mathrm{d}\sigma(\alpha)$.

We now can give approximation rates for the operators $T_\Xi$ for functions of *full* smoothness.

LEMMA 7.1. *Let $f \in W_p^{2m}(\Omega)$ (or $C^{2m}(\overline{\Omega})$ in case $p=\infty$). Then there are positive constants $h_0$ and $C$ (depending on $\Omega$ and $m$) so that for all $h \le h_0$,*

$$\|f - T_\Xi f\|_{L_p(\Omega)} \le C\left( h^{2m} \|\Delta^m f\|_{L_p(\Omega)} + \sum_{j=0}^{m-1} h^{2m-j-1+\frac{1}{p}} \|N_j f\|_{L_p(\partial\Omega)} \right).$$

*Proof.* The lemma follows directly from Theorem 2.4, (7.5), and (7.6). ∎

**7.3. Approximation results.** Our first result about surface spline approximation is broken into three parts, treating approximation in $L_p$, with $1<p<\infty$, treating approximation in $L_1$, and treating approximation in $L_\infty$. The error estimates follow along the lines of [20].

In each case, we make use of a $K$-functional argument to allow the operator to handle functions of lower smoothness. Let $f_e$ be the universal extension to $\mathbb{R}^d$ of the target function $f$ defined on $\Omega$ guaranteed by [38, Theorem 2.2]. Let $\eta : \mathbb{R}^d \to [0,1]$ be a compactly supported $C^\infty$ function satisfying $\int_{\mathbb{R}^d} x^\alpha \eta(x)\mathrm{d}x = \delta(|\alpha|)$ for $|\alpha| \le 2m$. This ensures that $\eta * p = p$ for all $p \in \Pi_{2m}$. We define $S_h f \in W_p^{2m}(\mathbb{R}^d)$ as $S_h f = f_e * \eta_h \in C^\infty(\mathbb{R}^d)$, where $\eta_h := h^{-d}\eta(\cdot/h)$.

In short, for a function having smoothness $m+1/p$ (this is made precise below), we have that $S_h f \in C^\infty(\Omega)$. Consequently, the following hold:

- $\|f - S_h f\|_p = \mathcal{O}(h^{m+1/p})$.
- $\|S_h f\|_{W_p^{2m}(\Omega)} = \mathcal{O}(h^{1/p-m})$.
- For every $0<s<2m$, $\|S_h f\|_{B^s_{p,1}(\Omega)} = \mathcal{O}(h^{m+1/p-s})$.

It follows that we can apply Lemma 7.1 to estimate $\|S_h f - T_\Xi S_h f\|_p$. By Theorem 2.4, we have that $\|N_j S_h f\|_{L_p(\partial\Omega)} \le C\|S_h f\|_{B_{p,1}^{2m-j-1+1/p}(\Omega)} = \mathcal{O}(h^{m+1/p-(2m-j-1+1/p)}) = \mathcal{O}(h^{j+1-m})$. Using this, we control the error terms in Lemma 7.1, i.e.,

$$h^{2m-j-1+1/p} \|N_j S_h f\|_{L_p(\partial\Omega)} = \mathcal{O}(h^{m+1/p}).$$

THEOREM 7.2 (approximation in $L_p$, $1<p<\infty$). *Let $1<p<\infty$, and suppose $f \in B^{m+1/p}_{p,1}(\Omega)$. There are positive constants $C, h_0$ so that for every $\Xi \subset \Omega$ satisfying $h \le h_0$, there is $s_f \in S_{m-1}(\Xi)$ so that*

$$\|f - s_f\|_p \le C h^{m+1/p} \|f\|_{B^{m+1/p}_{p,1}(\Omega)}.$$

*Proof.* It is an easy exercise to demonstrate the inequalities

$$\|f_e - S_h f\|_{L_p(\mathbb{R}^d)} \le C h^{m+1/p} \|f_e\|_{B^{m+1/p}_{p,1}(\mathbb{R}^d)} \le C h^{m+1/p} \|f\|_{B^{m+1/p}_{p,1}(\Omega)},$$
$$\|S_h f\|_{W_p^{2m}(\mathbb{R}^d)} \le C h^{1/p-m} \|f_e\|_{B^{m+1/p}_{p,1}(\mathbb{R}^d)} \le C h^{1/p-m} \|f_e\|_{B^{m+1/p}_{p,1}(\Omega)}$$

and, for $0<s<2m$,

$$\|S_h f\|_{B^s_{p,1}(\mathbb{R}^d)} \le C h^{m+1/p-s} \|f_e\|_{B^{m+1/p}_{p,1}(\mathbb{R}^d)} \le C h^{m+1/p-s} \|f\|_{B^{m+1/p}_{p,1}(\Omega)}.$$

By the second inequality, we have $\|\Delta^m S_h f\|_{L_p(\Omega)} \le C h^{1/p-m} \|f\|_{B^{m+1/p}_{p,1}(\Omega)}$. Theorem 2.4 and the third estimate imply that $\|N_j S_h f\|_{L_p(\partial\Omega)} \le C h^{j+1-m} \|f\|_{B^{m+1/p}_{p,1}(\Omega)}$.

We now apply $T_\Xi$ to $S_h f\,|_\Omega$ (the restriction of $S_h f$ to $\Omega$). By Lemma 7.1, $\|S_h f\,|_\Omega - T_\Xi(S_h f\,|_\Omega)\|_{L_p(\Omega)} \le C h^{2m} h^{1/p-m} \|f_e\|_{B^{m+1/p}_{p,1}(\Omega)}$ and

$$\|f - T_\Xi(S_h f\,|_\Omega)\,\|_{L_p(\Omega)} \le C h^{m+1/p} \|f\|_{B^{m+1/p}_{p,1}(\Omega)},$$

from which the theorem follows. ∎

In order to get a similar result for $L_1$, we need slightly more smoothness for the target function.[5]

THEOREM 7.3 (approximation in $L_1$). *Let $\epsilon > 0$, and suppose $f \in B^{m+1+\epsilon}_{1,1}(\Omega)$. There are positive constants $C_\epsilon, h_0$ so that for every $\Xi \subset \Omega$ satisfying $h \le h_0$, there is $s_f \in S_{m-1}(\Xi)$ so that*

$$\|f - s_f\|_1 \le C_\epsilon h^{m+1} \|f\|_{B^{m+1+\epsilon}_{1,1}(\Omega)}.$$

*Proof.* As in the previous case, one easily demonstrates the inequalities

$$\|f_e - S_h f\|_{L_1(\mathbb{R}^d)} \le C h^{m+1} \|f_e\|_{B^{m+1+\epsilon}_{1,1}(\mathbb{R}^d)} \le C h^{m+1} \|f\|_{B^{m+1+\epsilon}_{1,1}(\Omega)},$$
$$\|S_h f\|_{W^{2m}_1(\mathbb{R}^d)} \le C h^{1-m} \|f_e\|_{B^{m+1}_{1,1}(\mathbb{R}^d)} \le C h^{1-m} \|f_e\|_{B^{m+1}_{1,1}(\Omega)}$$

and, for $0 < s < 2m$ and $\epsilon > 0$,

$$\|S_h f\|_{B^{s+\epsilon}_{1,1}(\mathbb{R}^d)} \le C h^{m+1-s} \|f_e\|_{B^{m+1+\epsilon}_{1,1}(\mathbb{R}^d)} \le C h^{m+1-s} \|f\|_{B^{m+1+\epsilon}_{1,1}(\Omega)}.$$

Let $p = \frac{d-1}{d-1-\epsilon}$. By compactness, we have $\|N_j S_h f\|_{L_1(\partial\Omega)} \le C \|N_j S_h f\|_{L_p(\partial\Omega)}$. By Theorem 2.4, $\|N_j S_h f\|_{L_p(\partial\Omega)} \le C \|S_h f\|_{B^{2m-j-1+1/p}_{p,1}(\Omega)}$. Finally, an application of the embedding theorem for Besov spaces [46, Theorem 2.7.1] gives $\|S_h f\|_{B^{2m-j-1+1/p}_{p,1}(\Omega)} \le C \|S_h f\|_{B^{2m-j-1+\epsilon}_{1,1}(\Omega)}$. Together we obtain $\|N_j S_h f\|_{L_1(\partial\Omega)} \le C \|S_h f\|_{B^{2m-j-1+\epsilon}_{1,1}(\Omega)} \le C h^{j+1-m} \|f\|_{B^{m+1+\epsilon}_{1,1}(\Omega)}$.

Applying $T_\Xi$ to $S_h f\,|_\Omega$ gives

$$\|S_h f\,|_\Omega - T_\Xi(S_h f\,|_\Omega)\|_{L_1(\Omega)} \le C h^{2m} h^{1-m} \|f_e\|_{B^{m+1}_{1,\infty}(\Omega)}.$$

The triangle inequality gives $\|f - T_\Xi(S_h f\,|_\Omega)\,\|_{L_1(\Omega)} \le C h^{m+1} \|f\|_{B^{m+1+\epsilon}_{1,\infty}(\Omega)}$, from which the theorem follows. ∎

The final case follows along the same lines.

THEOREM 7.4 (approximation in $L_\infty$). *Let $\epsilon > 0$, and suppose $f \in C^{m+\epsilon}(\overline{\Omega})$. There are positive constants $C, h_0$ so that for every $\Xi \subset \Omega$ satisfying $h \le h_0$, there is $s_f \in S_{m-1}(\Xi)$ so that*

$$\|f - s_f\|_\infty \le h^m \|f\|_{C^{m+\epsilon}(\overline{\Omega})}.$$

*Proof.* In this case, we have

$$\|f_e - S_h f\|_{L_\infty(\mathbb{R}^d)} \le C h^m \|f_e\|_{C^m(\mathbb{R}^d)} \le C h^m \|f\|_{C^m(\overline{\Omega})},$$
$$\|S_h f\|_{C^{2m}(\mathbb{R}^d)} \le C h^{-m} \|f_e\|_{C^m(\mathbb{R}^d)} \le C h^{-m} \|f_e\|_{C^m(\overline{\Omega})}$$

[5] This is because of challenges in bounding pseudodifferential operators $N_j$ on spaces measuring smoothness in $L_1$. Although it may appear to be an artifact of working in this setting (after all there are many pseudodifferential operators that do not have such difficulties, e.g., constant coefficient differential operators) in the case where $\Omega$ is the disk in $\mathbb{R}^2$, it is known that the operators $N_j$ are not bounded from $W^{m+1}_1$ to $L_1$.

and, for $m < s < 2m$,

$$\|S_h f\|_{C^s(\mathbb{R}^d)} \le Ch^{m+\epsilon-s}\|f_e\|_{C^{m+\epsilon}(\mathbb{R}^d)} \le Ch^{m+\epsilon-s}\|f\|_{C^{m+\epsilon}(\overline{\Omega})}.$$

Apply Theorem 2.4 to $S_h f$ as follows. For any $2d/\epsilon < p < \epsilon$, the embedding $H_p^{\epsilon/2}(\partial\Omega) \subset L_\infty(\partial\Omega)$ holds. Applying this to $N_j S_h f$ gives

$$\|N_j S_h f\|_{L_\infty(\partial\Omega)} \le C\|N_j S_h f\|_{H_p^{\epsilon/2}(\partial\Omega)}.$$

By Theorem 2.4, we have $\|N_j S_h f\|_{L_\infty(\partial\Omega)} \le C\|f\|_{B_{p,1}^{2m-j-1+\epsilon/2}(\Omega)}$. Finally,

$$C^{2m-j-1+\epsilon}(\overline{\Omega}) \subset B_{p,1}^{2m-j-1+\epsilon/2}(\Omega)$$

holds by the compactness of $\Omega$, and therefore $\|N_j S_h f\|_{L_\infty(\partial\Omega)} \le C\|S_h f\|_{C^{2m-j-1+\epsilon}(\Omega)}$. In particular, this holds for $s = 2m - j - 1 + \epsilon$, which satisfies $m < s < 2m$.

Applying $T_\Xi$ to $S_h f\,|_{\overline{\Omega}}$, Lemma 7.1 ensures that $\|S_h f|_{\overline{\Omega}} - T_\Xi(S_h f|_{\overline{\Omega}})\|_{L_\infty(\Omega)} \le Ch^m\|f_e\|_{C^{m+\epsilon}(\overline{\Omega})}$ and $\|f - T_\Xi(S_h f|_{\overline{\Omega}})\|_{L_\infty(\Omega)} \le Ch^m\|f\|_{C^{m+\epsilon}(\overline{\Omega})}$. ☐

**7.4. Overcoming boundary effects.** We now demonstrate that the "free space" approximation order of $2m$ can be attained by increasing the density of centers in a small neighborhood of the boundary. This approach was shown to be successful in [20] and is similar to quadratic oversampling used by Rieger and Zwicknagl [37]. It works by modifying the error estimate from Lemma 7.1 with higher precision near the boundary.

We add an extra assumption about $\Xi$, namely that the sampling density of $\Xi$ near the boundary is $h^\nu$ rather than $h$. In this case, "near" means within a tube which has thickness $\propto h^\nu$.

To proceed, we fix an "oversampling factor" $\nu \ge 1$. By the smoothness and compactness of the boundary, $\Omega$ satisfies an interior cone condition. Indeed, for every aperture $0 \le \theta < \pi/2$, there is a radius $r$ so that for every $\alpha \in \partial\Omega$, the cone $C(\alpha, r, \theta, -\vec{n}_\alpha)$ lies in $\overline{\Omega}$.

It follows from [48, Theorem 3.8] that if $\Omega_{h,\nu} = \{\xi \in \Omega \mid \operatorname{dist}(\xi, \partial\Omega) \le 12h^\nu m^2\}$ satisfies the estimate $\max_{x\in\Omega_{h,\nu}} \operatorname{dist}(x, (\Xi)) \le h^\nu$, then for every $\alpha \in \partial\Omega$, the boundary cone $C(\alpha) = C(\alpha, \Gamma(2m)^2 h^\nu, \theta, -\vec{n}_\alpha)$ has the norming set property:

$$\forall p \in \Pi_{2m}, \forall \alpha \in \partial\Omega, \qquad \|p\|_{L_\infty(C(\alpha))} \le 2\left\|p\,|_{\Xi\cap C(\alpha)}\right\|_{\ell_\infty(\Xi\cap C(\alpha))}.$$

As in section 7.2, we have that $|\Lambda_j p(\alpha)| \le C_2\|p\,|_{\Xi\cap C(\alpha)}\|_{\ell_\infty}$. This is sufficient to ensure that boundary kernels $a_j : \partial\Omega \times \Xi \to \mathbb{R}$ exist so that the following three properties hold:

1. $\sum_{\xi\in\Xi} a_j(\alpha,\xi)p(\xi) = \lambda_j p(\alpha)$ for all $p \in \Pi_{2m}$.
2. $|\alpha - \xi| > \Gamma(2m)^2 h^\nu$ implies $a_j(\alpha, \xi) = 0$.
3. $\max_{\alpha\in\partial\Omega} \sum_{\xi\in\Xi} |a_j(\alpha,\xi)| \le Kh^{-\nu j}$, with $K$ depending only on $m$ and $\Omega$.

Consequently,

$$E_j(x,\alpha) = |\lambda_{j,\alpha}\phi(x-\alpha) - k_j(x,\alpha)| \le Ch^{\nu(2m-d-j)}\left(1 + \frac{\operatorname{dist}(x,\alpha)}{h^\nu}\right)^{-(d+1)}$$

with corresponding operator norm $\|\mathcal{E}_j\|_{p\to p} \le Ch^{\nu(2m-j-1+1/p)}$. This ensures the following theorem.

THEOREM 7.5. *There are positive constants $h_0$ and $C$ (depending on $\Omega$ and $m$) so that for all $\Xi$ with fill distance $h \le h_0$, and satisfying the extra condition*

$$\max_{x\in\partial\Omega} \mathrm{dist}(x,(\Xi\cap\Omega_{h,\nu})) \le h^\nu,$$

*if $f \in W_p^{2m}(\Omega)$ (or $C^{2m}(\overline{\Omega})$ in case $p=\infty$), then*

$$\|f - T_\Xi f\|_{L_p(\Omega)} \le C\left(h^{2m}\|\Delta^m f\|_{L_p(\Omega)} + \sum_{j=0}^{m-1} h^{\nu(2m-j-1+\frac{1}{p})}\|f\|_{W_p^{2m}(\Omega)}\right).$$

*Proof.* This follows from the argument used in Lemma 7.1. The details are left to the reader. ∎

This indicates how we may "oversample" $\Xi$. For $1 \le p \le \infty$, let $\nu = \frac{2mp}{mp+1}$ (or $\nu = 2$ when $p=\infty$). This is the critical exponent that delivers $L_p$ approximation order $2m$. Then the lowest order term in Theorem 7.5 is controlled by $h^{2m}$ and

$$\|f - T_\Xi f\|_{L_p(\Omega)} \le Ch^{2m}\|f\|_{W_p^{2m}(\Omega)}.$$

*Selecting points in $\Omega_{h,\nu}$.* To accomplish this practically, given a set of centers $\Xi \subset \Omega$ with fill distance $h$, we sample points $\Xi_{\partial,0}$ on $\partial\Omega$ with a density of

$$\max_{x\in\partial\Omega} \mathrm{dist}(x,\Xi_{\partial,0}) = h^\nu.$$

Extend this into $\Omega$ by choosing $2m$ layers of the form $\Xi_{\partial,j} = \{\xi^* = \xi + jh^\nu \vec{n}_\xi \mid \xi \in \Xi_{\partial,0}\}$. In that case, we have (for sufficiently small $h$) that $\cup_{j=0}^{2m}\Xi_{\partial,j}$ is a norming set for $\Omega_{h,\nu} = \{x \in \Omega \mid \mathrm{dist}(x,\partial\Omega) \le 2mh^\nu\}$.

*When is it feasible?* The centers $\Xi \subset \Omega$ have cardinality $\#\Xi \ge C\mathrm{vol}(\Omega)h^{-d}$. Since $\#\Xi_{\partial,j} \sim \#\Xi_{\partial,0} \sim Ch^{-\nu(d-1)}$, the set of additional points $\cup_{j=0}^{2m}\Xi_{\partial,j}$ has cardinality bounded by $(\#\Xi_{\partial,0})(2m+1) \le Cmh^{-\nu(d-1)}$. If we desire that the supplementary points do not exceed $Ch^{-d}$ asymptotically (meaning that the number of extra centers required to achieve approximation order $2m$ is kept on par with the number of original centers), then for fixed $d$, the $L_p$ approximation order $2m$ can be achieved for $1 \le p \le \frac{d}{(d-2)m}$ without increasing (asymptotically) the number of centers.

**8. Beppo Levi extension.** In this section, we present a boundary layer representation of Duchon's norm minimizing extension operator [12], which takes functions in $W_2^m(\Omega)$ to functions in the Beppo Levi space $D^{-m}L_2(\mathbb{R}^d)$, the space of integrable functions with the finite seminorm given in (1.1).

Since $m > d/2$, the embedding $D^{-m}L_2(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ holds; this follows from the chain of continuous embeddings $D^{-m}L_2(\mathbb{R}^d) \subset W_{2,loc}^m(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. We consider the map $\mathrm{Ext} : W_2^m(\Omega) \to D^{-m}L_2(\mathbb{R}^d) : f \mapsto f_e$ which minimizes the Beppo Levi seminorm:

$$\mathrm{Ext}f = f_e := \mathrm{argmin}\{|g|_{D^{-m}L_2(\mathbb{R}^d)} \mid g|_\Omega = f\}.$$

In [12], Duchon shows that the extension can be written as $f_e = \phi * \mu_f + \tilde{p}$, with $\tilde{p}$ a polynomial in $\Pi_{m-1}$ and $\mu_f$ a distribution supported in $\overline{\Omega}$ that annihilates $\Pi_{m-1}$. Unfortunately, not much more can be said about $\mu_f$ or $p$. (There is a general analogue to extension in reproducing kernel (semi-)Hilbert spaces for RBFs; see [41, section 9] and [42, section 3]).

In what follows, if $f : \Omega \to \mathbb{R}$, we denote its zero extension by $f_z : \mathbb{R}^d \to \mathbb{R}$ .

**8.1. Extension of functions in $W_2^m(\mathbb{R}^d)$.** For $f \in W_2^m(\mathbb{R}^d)$, Corollary 2.3 ensures that the solution to (1.2) with $h_k = \lambda_k f \in W_2^{m-k-1/2}(\partial\Omega)$ satisfies $f_1 = \sum_{j=0}^{m-1} V_j g_j + p$ and that this functions lies in $W_2^m(\Omega)$. Because $g_j \in W_2^{j+1/2-m}(\partial\Omega)$, we have that $\Lambda_j^{\mathrm{t}}(g_j \cdot \delta_j) \in W_2^{-m}(\mathbb{R}^d)$, and $f_1 = \sum_{j=0}^{m-1} \Lambda_j^{\mathrm{t}}(g_j \cdot \delta_j) * \phi + p$.

The remainder $f_2 = f - f_1$ satisfies $\lambda_j(f - f_1) = 0$ for $j = 0, \dots, m-1$. In other words, its Dirichlet data vanishes, and the zero extension of $f - f_1$, denoted by $(f - f_1)_z$, lies in $W_2^m(\mathbb{R}^d)$. Therefore, $\Delta^m(f - f_1)_z \in W_2^{-m}(\mathbb{R}^d)$, and it has support in $\overline{\Omega}$. Define $\nu_f := \Delta^m(f - f_1)_z + \sum_{j=0}^{m-1} \Lambda_j^{\mathrm{t}}(g_j \cdot \delta_j)$, and note that $f \mapsto \nu_f$ is bounded from $W_2^m(\Omega)$ to $W_2^{-m}(\mathbb{R}^d)$. Consequently, $\nu_f * \phi \in W_{2,loc}^m(\mathbb{R}^d)$.

To guarantee that $\nu_f * \phi$ resides in $D^{-m}L_2$, we need to demonstrate a polynomial annihilation property of $\nu_f$. This is done below in Lemma 8.1. The result then follows from the fact that for $|\alpha| = m$, $D^\alpha \nu_f$ is a compactly supported distribution that annihilates polynomials of degree $2m-1$, and therefore $D^\alpha \nu_f * \phi = (D^\alpha \nu_f) * \phi$. From Lemma 2.1, we have that $|D^\alpha \nu_f * \phi(x)| \le C(1 + |x|)^{-d}$, which shows that $D^\alpha \nu_f * \phi(x) \in L_2(\mathbb{R}^d)$ globally.

Lemma 8.1. *For $q \in \Pi_{m-1}$, $\langle \nu_f, q\rangle = 0$.*

*Proof.* We have that $\langle \nu_f, q\rangle = \langle (f - f_1)_z, \Delta^m q\rangle + \sum_{j=0}^{m-1} \langle g_j, \lambda_j q\rangle$. Because $q \in \Pi_{m-1}$, $\Delta^m q = 0$, and employing the side conditions $P^{\mathrm{t}}\mathbf{g} = 0$ shows that the final sum vanishes. ∎

We are now ready to prove the main theorem for this section.

Theorem 8.2. *For $f \in W_2^m(\overline{\Omega})$, $f_e = \nu_f * \phi + p$.*

*Proof.* We write $f_e = \mu_f * \phi + \tilde{p}$ and let $F = (\nu_f - \mu_f) * \phi + p - \tilde{p}$. Note that $F \in D^{-m}L_2(\mathbb{R}^d)$. For $\mu_f * \phi + \tilde{p}$, this is clear, while for $\nu_f * \phi + p$, it has been shown above. Observe that $\Delta^m F(x) = 0$ for $x \in \mathbb{R}^d \setminus \partial\Omega$. Indeed, $F = 0$ inside $\Omega$, because this is where both extension operators equal $f$.

We focus on $\mathbb{R}^d \setminus \overline{\Omega}$, where $F$ is smooth, thanks to the fact that $\nu_f$ and $\mu_f$ are both supported in $\overline{\Omega}$. Here $F \in W_{2,loc}^m(\mathbb{R}^d)$ satisfies the $m$-fold Laplace equation $\Delta^m F(x) = 0$, with homogeneous Dirichlet conditions $\lambda_j F = 0$, for $j = 0, \dots, m-1$. The polynomial annihilation property $(\nu_f - \mu_f) \perp \Pi_{m-1}$ in conjunction with Lemma 2.1 implies that $D^\beta F(x) \le C(1 + |x|)^{m-1-|\beta|}$, which means that $F = 0$ in $\mathbb{R}^d \setminus \Omega$. Since $F \in C(\mathbb{R}^d)$, this implies that $F = 0$ throughout $\mathbb{R}^d$.

Finally, this implies that $(\nu_f - \mu_f) * \phi \in \Pi_{m-1}$. Since $\nu_f - \mu_f$ is supported in $\overline{\Omega}$, $\widehat{\nu_f - \mu_f}$ is entire, and it is simultaneously supported at $\{0\}$. Thus $\nu_f = \mu_f$ and $p = \tilde{p}$. ∎

**8.2. Extension of functions in $W_2^{2m}(\Omega)$.** If $f : \Omega \to \mathbb{R}$ has greater smoothness, we can say more about the distribution $\nu_f$. Using the extended representation given by Theorem 2.4, we have $\nu_f * \phi = \left(\left(\Delta^m f\right)_z + \sum_{j=0}^{m-1}\left(\Lambda_j^{\mathrm{t}}(N_j f \cdot \delta_{\partial\Omega})\right)\right) * \phi$.

We now demonstrate the relevance of this to interpolation. First, we recall Duchon's interpolation error estimate [13, Proposition 3], which involves two key observations: for a compact $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and $f \in W_2^m(\Omega)$, the interpolation error satisfies

$$\|f - I_\Xi f\|_p \le C h^{m-d(\frac{1}{2}-\frac{1}{p})_+} |f_e - I_\Xi f|_{D^{-m}L_2(\mathbb{R}^d)} \le C h^{m-d(\frac{1}{2}-\frac{1}{p})_+} |f|_{D^{-m}L_2(\mathbb{R}^d)}. \tag{8.1}$$

The first inequality is a "zeros estimate" [13, Proposition 2], and the second follows from the fact that $I_\Xi$ is the orthogonal projection onto $S_{m-1}(\Xi)$ with respect to the $D^{-m}L_2(\mathbb{R}^d)$ inner product (this was described earlier in section 1.1).

Johnson's result shows that if $f \in W_2^{2m}(\Omega)$, we should not necessarily expect better rates. However, in [40], it is shown that if $f_e$ satisfies some basic conditions, specifically if $f_e$ is in the range of the dual of the embedding $D^{-m}L_2 \to L_2(\Omega)$, then an improved error estimate is possible.

Let us consider this in a Hilbert space setting: we write $\mathcal{N} := D^{-m}L_2(\mathbb{R}^d)/\Pi_{m-1}$ as the natural quotient Hilbert space by modding out the kernel of $|\cdot|_{D^{-m}L_2}$. There is a natural embedding $E : \mathcal{N} \to L_2(\Omega)/\Pi_{m-1} : f + \Pi_{m-1} \mapsto f + \Pi_{m-1}$. We identify the dual of $\mathcal{H} = L_2(\Omega)/\Pi_{m-1}$, with $\mathcal{H}' = L_2(\Omega) \cap \Pi_{m-1}^{\perp}$, and then calculate the adjoint $E^* : \mathcal{H}' \to \mathcal{N}$ via

$$\langle E^*g, f + \Pi_{m-1}\rangle_{\mathcal{N}} = \langle g, E(f + \Pi_{m-1})\rangle_{\mathcal{H}' \times \mathcal{H}} = \int_{\Omega} f(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} f(x)g_z(x)\mathrm{d}x,$$

where $g_z$ is the zero extension of $g$. Consequently,

$$\langle E^*g, f + \Pi_{m-1}\rangle_{\mathcal{N}} = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi)\widehat{g_z}(\xi)\mathrm{d}\xi$$

by Parseval's relation. For any $G \in E^*g$, the $D^{-m}L_2$ inner product is

$$\langle E^*g, f + \Pi_{m-1}\rangle_{\mathcal{N}} = \langle G, f\rangle_{D^{-m}L_2} = (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^{2m} G(\xi)\widehat{f}(\xi)\mathrm{d}\xi.$$

Because $f \in D^{-m}L_2(\mathbb{R}^d)$ is arbitrary, it follows that $E^*g = g_z * \phi + \Pi_{m-1}$.

If the native space extension of a function $f \in W_2^{2m}(\Omega)$ has the form $f_e = g_z * \phi + p \in E^*g$, with $g \in L_2(\Omega) \cap \Pi_{m-1}^{\perp}$, then the error in the $D^{-m}L_2(\mathbb{R}^d)$ seminorm, which appears in (8.1), satisfies

$$|f_e - I_\Xi f|^2_{D^{-m}L_2(\mathbb{R}^d)} \le \langle f_e, f_e - I_\Xi f\rangle_{D^{-m}L_2(\mathbb{R}^d)} = \langle g, E(f - I_\Xi f)\rangle_{\mathcal{H}' \times \mathcal{H}} \le \|g\|_2 \|f - I_\Xi f\|_2.$$

Applying (8.1) again gives $|f_e - I_\Xi f|_{D^{-m}L_2(\mathbb{R}^d)} \le Ch^m \|g\|_2$. Combining once more with (8.1) gives

$$\|f - I_\Xi f\|_p \le Ch^{2m - d(\frac{1}{2} - \frac{1}{p})_+} \|g\|_2.$$

This is the surface spline version of Schaback's "doubling trick" (introduced in [40, Theorem 5.1] and further discussed in [41, section 15] and [42]). A challenge is to identify those $f \in W_2^{2m}(\Omega)$ for which $f_e \in E^*g$ for some $g \in L_2(\Omega)$. This is resolved with the following observation.

Corollary 8.3. *If $\Omega$ is bounded with a smooth boundary and if $f \in W_2^{2m}(\Omega)$, then $f_e \in E^*(g)$ for some $g \in L_2(\Omega)$ if and only if $f$ is in the joint kernel $\cap_{j=0}^{m-1} \ker N_j$. If this is the case, then $f_e \in E^*(\Delta^m f)$ and*

$$\|f - I_\Xi f\|_{L_p(\Omega)} \le Ch^{2m - d(\frac{1}{2} - \frac{1}{p})_+} \|\Delta^m f\|_{L_2(\Omega)}.$$

This invites some questions about surface spline interpolation/approximation:

1. For functions in $W_p^{2m}(\Omega)$ (or $C^{2m}(\overline{\Omega})$ when $p = \infty$), does the interpolation error (measured in $L_p$) decay like $\mathcal{O}(h^{2m})$? Similarly, for functions in $B_{p,1}^{m+1/p}(\Omega)$ (or $C^m(\overline{\Omega})$ when $p = \infty$), does the interpolation error decay like

$\mathcal{O}(h^{m+1/p})$? (This is sometimes referred to as Johnson's conjecture, based on his work in [27, 31, 30].) A result like this would follow for $p=\infty$, for instance, if the Lebesgue constant $\|I_\Xi\|_{\infty\to\infty}$ were bounded.

2. Is there a saturation result similar to [27] for general $\Omega$? Specifically, can it be shown that approximation rate $o(h^{m+1/p})$ is not attained unless $f \in \ker N_{m-1}$? If so, can such a result be refined for functions in $\cap_{j=k}^{m-1} \ker N_j$, showing that in this case $\operatorname{dist}(f, S_{m-1}(\Xi)) = o(h^{2m-k+1/p})$ does not hold unless also $f \in \ker N_{k-1}$? (There is some numerical evidence for a result like this in [25].)

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