Dp-finite fields I(B): positive characteristic

Will Johnson

January 15, 2022

Abstract

We partially generalize the known results on dp-minimal fields to dp-finite fields. We prove a dichotomy: if K is a sufficiently saturated dp-finite expansion of a field, then either K has finite Morley rank or K has a non-trivial $\mathrm{Aut}(K/A)$-invariant valuation ring for some small set A. In the positive characteristic case, we can even obtain a henselian valuation ring. Using this, we classify the positive characteristic dp-finite pure fields.

8 Introduction

The two main conjectures for NIP fields are

- The *henselianity conjecture*: any NIP valued field $(K, \mathcal{O})$ is henselian.
- The *Shelah conjecture*: any NIP field K is algebraically closed, real closed, finite, or admits a non-trivial henselian valuation.

These conjectures are known to imply a full classification of dp-finite fields, i.e., fields of finite dp-rank [11]. See [20] for a reference on NIP and dp-rank, and [6] for a reference on valued fields and henselianity.

In an earlier paper [13], we proved the henselianity conjecture for positive characteristic NIP fields. Continuing [13], we prove the Shelah conjecture for positive characteristic dp-finite fields. This yields the classification of positive characteristic dp-finite fields.

Our main technical result is the following statement, which holds in any characteristic.

Theorem 8.1 (= Theorem 11.29)**.** *Let $(K, +, \cdot, \ldots)$ be a sufficiently saturated dp-finite field, possibly with extra structure. Then either*

- *K has finite Morley rank, or*
- *There is an $\mathrm{Aut}(K/A)$-invariant non-trivial valuation ring on K for some small set A.*

Unfortunately, we can only obtain a henselian valuation ring in positive characteristic.

2010 Mathematical Subject Classification: 03C45

Key words and phrases: dp-rank, NIP fields, modular lattices

8.1 The story so far

We review several facts from [13]. Let $(\mathbb{M}, +, \cdot, \ldots)$ be a field, possibly with extra structure. Assume

- $\mathbb{M}$ is sufficiently saturated
- $\mathbb{M}$ has finite dp-rank
- $\mathbb{M}$ does not have finite Morley rank.

Under these assumptions, we defined a notion of *heavy* and *light* definable sets [13, Definition 4.19]. We proved the following:

Fact 8.2 ([13, Theorem 4.20])**.**

1. *A definable set $D \subseteq \mathbb{M}$ is heavy if and only if it is not light.*
2. *Light sets form an ideal.*
3. *Heaviness/lightness is definable in families.*
4. *Heaviness/lightness is preserved by affine symmetries $x \mapsto ax + b$.*

If X, Y are definable subsets of $\mathbb{M}$, we defined

$$X -_\infty Y := \{a \in \mathbb{M} : X \cap (Y + a) \text{ is heavy}\}.$$

Then we defined a *basic neighborhood* to be a set of the form $X -_\infty X$ for heavy definable $X \subseteq \mathbb{M}$. We proved:

Fact 8.3 ([13, Proposition 6.4])**.**

1. *Every basic neighborhood is heavy.*
2. *The family of basic neighborhoods is downwards directed.*

For any small model $M \preceq \mathbb{M}$, we defined the set I_M of *M-infinitesimals* to be the intersection of all M-definable basic neighborhoods. We proved:

Fact 8.4 ([13, Remark 6.8, Theorem 6.16, and Proposition 6.18])**.**

1. *I_M is an additive subgroup of $\mathbb{M}$, type-definable over M.*
2. *Every definable set containing I_M is heavy.*
3. *I_M is minimal among subgroups satisfying the previous two conditions.*
4. *I_M is a non-zero M-linear proper subspace of $\mathbb{M}$.*

Recall that a set is "A-invariant" if it is $\mathrm{Aut}(\mathbb{M}/A)$-invariant for some small set A. This is a very weak form of A-definability.

Fact 8.5 ([13, Theorem 7.5]). *Suppose $\mathbb{M}$ has positive characteristic. If there is a small set A and a non-trivial A-invariant valuation ring $\mathcal{O}$, then there is a small set A' and a non-trivial A'-invariant* henselian *valuation ring $\mathcal{O}'$.*

In order to prove the Shelah conjecture for positive characteristic dp-finite fields, it therefore suffices to produce a non-trivial A-invariant valuation ring. This is Theorem 8.1.

8.2 Constructing an invariant valuation ring

Assume $\mathbb{M}$ is unstable and highly saturated. Fix a small submodel $M_0 \preceq \mathbb{M}$. Let $\mathcal{P}$ be the poset of type-definable M_0-linear subspaces $G \subseteq \mathbb{M}$ that are *00-connected*, in the sense that $G = G^{00}$. Then $\mathcal{P}$ is always a *modular lattice*, with lattice operations given by

$$G \vee H = G + H$$
$$G \wedge H = (G \cap H)^{00}.$$

By [3, Proposition 4.5.2], $\mathcal{P}$ has *breadth* at most $r = \text{dp-rk}(\mathbb{M})$: for any $G_1, \ldots, G_{r+1} \in \mathcal{P}$, there is some i such that

$$G_1 \wedge \cdots \wedge G_{r+1} = G_1 \wedge \cdots \wedge G_{i-1} \wedge G_{i+1} \wedge \cdots \wedge G_{r+1}.$$

See [8, Chapter V] for background on lattice theory, modularity, and breadth.

In the dp-minimal case ($r = 1$), $\mathcal{P}$ is linearly ordered. Therefore, for $J \in \mathcal{P}$ the set

$$\{a \in \mathbb{M} : a \cdot J \subseteq J\}$$

is a valuation ring on $\mathbb{M}$. Taking J to be the M_0-infinitesimals I_{M_0}, we obtain a non-trivial valuation ring on $\mathbb{M}$, proving Theorem 8.1.

In higher ranks, the situation is more complicated. To begin, we need some lattice-theoretic way to detect valuations.

If M is an R-module, let $\mathrm{Sub}_R(M)$ denote the lattice of R-submodules of M.

Definition 8.6 ([16, Definition 4.1]). An *r-inflator* on $\mathbb{M}$ consists of a semisimple M_0-algebra S of length r, and a family of maps

$$f_n : \mathrm{Sub}_{\mathbb{M}}(\mathbb{M}^n) \to \mathrm{Sub}_S(S^n)$$

satisfying the following axioms:

1. Each f_n is order-preserving:

$$X \leq Y \implies f_n(X) \leq f_n(Y).$$

2. Each f_n is $GL_n(M_0)$-equivariant.

3. The f_n are compatible with $\oplus$:

$$f_{n+m}(X \oplus Y) = f_n(X) \oplus f_m(Y).$$

4. Each f_n scales lengths by a factor of r:

$$\ell_S(f_n(X)) = r \cdot \dim_{\mathbb{M}}(X).$$

If $(\mathcal{O}, \mathfrak{m})$ is a valuation ring on $\mathbb{M}$ with residue field $k = \mathcal{O}/\mathfrak{m}$, then there is a 1-inflator given by

$$\begin{aligned} f_n : \mathrm{Sub}_{\mathbb{M}}(\mathbb{M}^n) &\to \mathrm{Sub}_k(k^n) \\ X &\mapsto (X \cap \mathcal{O}^n + \mathfrak{m}^n)/\mathfrak{m}^n. \end{aligned}$$

Modulo fine print, all 1-inflators arise this way [16, Theorem 5.20]. Thus, 1-inflators on $\mathbb{M}$ are equivalent to valuations, and r-inflators are some kind of "generalized valuations."

In a later paper [16], we will give a proof of Theorem 8.1 in two steps:

Step 1 (= [16, Theorem 9.3]). *If $J \in \mathcal{P}$ satisfies a special property (Definition 11.3 below), then J canonically determines an r-inflator $\{f_n\}_{n\in\mathbb{N}}$ for some $r \le \text{dp-rk}(\mathbb{M})$.*

Step 2 (= [16, Theorem 10.12]). *Any r-inflator $\{f_n\}_{n\in\mathbb{N}}$ on $\mathbb{M}$ canonically determines a finite set of valuation rings on $\mathbb{M}$.*

The present paper is a simplified proof, avoiding all use of inflators, but losing some of the corresponding intuition.

The proof uses a certain construction of Puczyłowski [19] which associates to any modular lattice M a modular pregeometry $U(M)$, as well as a map Φ from M to the lattice $\overline{M}$ of closed sets in $U(M)$. We will review this construction in §9.4 below. For an introduction to pregeometries, see [21] or [8, §V.3.3].

Returning to our saturated dp-finite $\mathbb{M}$, let $\mathcal{P}_n$ be the lattice of 00-connected type-definable M_0-linear subspaces of $\mathbb{M}^n$. For example, $\mathcal{P}_1 = \mathcal{P}$. Fix some $J \in \mathcal{P}_1$. For each n, consider the interval $[J^n, \mathbb{M}^n] \subseteq \mathcal{P}_n$. Using Puczyłowski's construction, we get a map

$$\begin{aligned} f_n : \mathrm{Sub}_{\mathbb{M}}(\mathbb{M}^n) &\to \overline{[J^n, \mathbb{M}^n]} \\ V &\mapsto \Phi(V + J^n). \end{aligned}$$

When J satisfies a certain technical condition (Definition 11.3), this map scales lengths proportionally (Remark 11.12). In a later paper [16, Corollary 7.29, Theorem 9.3], we will build isomorphisms $\overline{[J^n, \mathbb{M}^n]} \cong \mathrm{Sub}_S(S^n)$ and verify the inflator axioms, for some semisimple algebra S.

With the maps f_n in hand, we extract a finite set of valuation rings in §11.2–11.4. The intuition from inflators is explained in Remark 11.13.

Remark 8.7. The infinitesimals I_{M_0} play a minor but critical role in the above construction. Specifically, the existence of I_{M_0} rules out the degenerate possibility $\mathcal{P}_1 = \{0, \mathbb{M}\}$.

8.3 A 00-technicality

In the lattices $\mathcal{P}_n$, the $\wedge$-operator is given by

$$A \wedge B = (A \cap B)^{00}.$$

The $(-)^{00}$ causes some technical problems.[1] Luckily, by choosing M_0 carefully, we can get rid of the $(-)^{00}$.

Theorem (= Corollary 10.7)**.** *There is a small submodel $M_0 \preceq \mathbb{M}$ such that $J = J^{00}$ for every type-definable M_0-linear subspace $J \leq \mathbb{M}$. Consequently, the lattice operations on $\mathcal{P}$ are given by*

$$G \vee H = G + H$$
$$G \wedge H = G \cap H.$$

This is a corollary of the following uniform bounding principle for dp-finite abelian groups:

Theorem (= Theorem 10.4)**.** *If H is a type-definable subgroup of a dp-finite abelian group G, then $|H/H^{00}|$ is bounded by a cardinal $\kappa(G)$ depending only on G.*

8.4 Outline

In §9, we review some abstract facts about modular lattices, including the notions of breadth and Goldie dimension (§9.2) and Puczyłowski's construction of a modular pregeometry on uniform elements (§9.4). We will also prove a subadditivity theorem for breadth (Proposition 9.19), which is probably known to experts in lattice theory.

In §10 we prove the technical fact that $|H/H^{00}|$ is uniformly bounded as H ranges over type-definable subgroups of a dp-finite abelian group. In §11, we apply these tools to construct valuation rings on dp-finite fields. §12 we verify the Shelah conjecture for positive characteristic dp-finite fields, and enumerate the consequences.

9 Modular lattices

Recall that a lattice is *modular* if the identity

$$(x \vee a) \wedge b = (x \wedge b) \vee a$$

holds whenever $a \leq b$. See [8, Chapter V] for background on modular lattices. Modularity is equivalent to the statement that for any a, b, the interval $[a \wedge b, a]$ is isomorphic as a poset to $[b, a \vee b]$ via the maps

$$[a \wedge b, a] \to [b, a \vee b]$$
$$x \mapsto x \vee b$$

[1] In the proof of Proposition 11.4, we need ($a \in G$ and $a \in H$) to imply $a \in G \wedge H$.

and

$$[b, a \vee b] \to [a \wedge b, a]$$
$$x \mapsto x \wedge a.$$

This is the "isomorphism theorem for modular lattices" [8, Theorem 348].

We will write the least and greatest elements of a lattice as $\bot$ and $\top$, when they exist. In what follows, we require "lattice homomorphisms" to preserve $\vee$ and $\wedge$, but not necessarily $\top$ and $\bot$ when they exist. A "sublattice" is a subset closed under $\vee$ or $\wedge$, but not necessarily containing $\top$ and $\bot$ when they exist.

Abusing notation, we will write $[\bot, a]$ to denote $\{x \in P : x \le a\}$, regardless of whether $\bot$ exists. Similarly, $[a, \top]$ denotes $\{x \in P : x \ge a\}$, regardless of whether $\top$ exists.

9.1 Independence and cubes

Let $(P, <)$ be a modular lattice with least element $\bot$.

Definition 9.1. A finite sequence $a_1, \ldots, a_n \in P \setminus \{\bot\}$ is *independent* if $a_k \wedge \bigvee_{i=1}^{k-1} a_i = \bot$ for $2 \le k \le n$.

Fact 9.2 ([8, Theorem 360]). *Independence is permutation invariant: if $a_1, \ldots, a_n$ is independent and π is a permutation of $[n]$, then $a_{\pi(1)}, \ldots, a_{\pi(n)}$ is independent.*

Therefore, independence is really a property of the *set* $\{a_1, \ldots, a_n\}$ rather than the *sequence* $a_1, \ldots, a_n$.

More generally, we can define a relative notion of independence *over* an element:

Definition 9.3. Let $(P, <)$ be a modular lattice and $b \in P$ be an element. A sequence $a_1, \ldots, a_n$ is *independent over* b if $a_i > b$ for each i, and

$$a_k \wedge \bigvee_{i<k} a_i = b$$

for $2 \le k \le n$.

In other words, an independent sequence over b is an independent sequence in the sublattice $[b, \top] \subseteq P$.

Definition 9.4. An *n-cube* in a modular lattice $(P, <)$ is a sublattice isomorphic to the powerset $\mathcal{P}ow([n])$. The *base* of the cube is its least element.

Equivalently, an n-cube in P is a family of elements $\{a_S\}_{S \subseteq [n]}$ such that

$$a_{S_1 \cup S_2} = a_{S_1} \vee a_{S_2}$$
$$a_{S_1 \cap S_2} = a_{S_1} \wedge a_{S_2}$$
$$S_1 \subsetneq S_2 \implies a_{S_1} < a_{S_2}.$$

The base is $a_\emptyset$.

Fact 9.5 ([8, Corollary 359]). *If $a_1, \ldots, a_n$ is independent over b, and we define*

$$a_S = \begin{cases} b & S = \emptyset \\ \bigvee_{i \in S} a_i & S \neq \emptyset, \end{cases}$$

for $S \subseteq [n]$, then $\{a_S\}_{S \subseteq [n]}$ is an n-cube with base b.

Conversely, if $\{a_S\}_{S \subseteq [n]}$ is an n-cube with base b, and we define $a_i = a_{\{i\}}$, then the sequence $a_1, \ldots, a_n$ is easily seen to be independent over b. We conclude that independent sequences and cubes are in bijection:

Proposition 9.6. *If $(P, <)$ is a modular lattice, if $b \in P$, and $n \in \mathbb{N}$, then there is a bijection between n-cubes with base b and sequences $a_1, \ldots, a_n$ independent over b.*

9.2 Goldie dimension and breadth

Let $(P, <)$ be a modular lattice with least element $\bot$.

Definition 9.7 ([9, Definition 6]). The *Goldie dimension* $\mathrm{G.dim}(P)$ is

$$\sup\{n \in \mathbb{N} \mid \text{There is an independent sequence } a_1, \ldots, a_n\},$$

or ∞ if no finite supremum exists.

Goldie dimension is also called *uniform dimension*. By Proposition 9.6, we can characterize Goldie dimension in terms of cubes:

Proposition 9.8. *The Goldie dimension $\mathrm{G.dim}(P)$ is the supremum of $n \in \mathbb{N}$ such that P has an n-cube with base $\bot$.*

Now let $(P, <)$ be any modular lattice, not necessarily with a least element $\bot$.

Lemma 9.9. *For $n \in \mathbb{N}$, the following are equivalent:*

1. *There are $a_1, \ldots, a_n \in P$ such that*

   $$a_1 \wedge \cdots \wedge a_n \neq a_1 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \cdots \wedge a_n$$

   for all i.

2. *There is $b \in P$ and a sequence $a_1, \ldots, a_n$ independent over b.*

3. *There is an n-cube in P.*

4. *There are $a_1, \ldots, a_n \in P$ such that*

   $$a_1 \vee \cdots \vee a_n \neq a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_n$$

   for all i.

Proof. We first prove (1) $\implies$ (2) $\implies$ (3) $\implies$ (4):

(1) $\implies$ (2): Let a_i be as in (1). Define

$$b = a_1 \wedge \cdots \wedge a_n$$
$$c_i = a_1 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \cdots \wedge a_n.$$

By assumption $c_i > b$. Note that $c_i \le a_k$ for $i \neq k$. Therefore

$$\bigvee_{i<k} c_i \le a_k$$
$$c_k \wedge \bigvee_{i<k} c_i \le c_k \wedge a_k = b.$$

Therefore the c_i are independent over b, proving (2).

(2) $\implies$ (3): Proposition 9.6.

(3) $\implies$ (4): Let $\{a_S\}_{S \subseteq [n]}$ be an n-cube. Define $a_i = a_{\{i\}}$ for $1 \le i \le n$. Then for any i,

$$a_1 \vee \cdots \vee a_n = a_{\{1,\dots,n\}}$$
$$a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_n = a_{\{1,\dots,i-1,i+1,\dots,n\}} \neq a_{\{1,\dots,n\}}$$

Therefore

$$a_1 \vee \cdots \vee a_n \neq a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_n,$$

proving (4).

Having proved (1) $\implies$ (4), the reverse implication (4) $\implies$ (1) follows by duality. □

Remark 9.10. The equivalence (1) $\iff$ (4) holds in any lattice [8, Exercise I.1.20], without assuming modularity.

Definition 9.11. The *breadth* $\mathrm{br}(P)$ of a modular lattice P is

$$\mathrm{br}(P) = \sup\{n \in \mathbb{N} \mid \text{There is an } n\text{-cube in } P\},$$

or ∞ if there is no finite supremum.

Note $\mathrm{br}(P) \ge n$ if the equivalent conditions of Lemma 9.9 hold.

Remark 9.12. The following are equivalent:

1. $\mathrm{br}(P) \le n$.
2. For any $a_1, \dots, a_{n+1} \in P$, there is i such that

$$a_1 \wedge \cdots \wedge a_{n+1} = a_1 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \cdots \wedge a_{n+1}.$$

3. For any finite subset $S \subseteq P$, there is $S' \subseteq S$ with

$$\bigwedge S' = \bigwedge S$$
$$|S'| \leq n$$

Indeed, (1) $\iff$ (2) holds by Lemma 9.9. Condition (2) is a special case of (3), and (2) $\implies$ (3) holds by an inductive argument.

Condition (3) is the conventional definition of "breadth" in lattice theory [1, Exercise II.5.6] (or [8, Exercise I.1.19]).

Warning. In the model theory of modules, there is an unrelated notion of "breadth" in modular lattices, due to Prest [18, p. 205].

Definition 9.13. Let $(P, <)$ be a modular lattice, and a, b be elements with $a \geq b$. Then $\mathrm{G.dim}(a/b)$ and $\mathrm{br}(a/b)$ denote the Goldie dimension and breadth of the sublattice $[b, a] \subseteq P$.

Lemma 9.14. *Let $(P, <)$ be a modular lattice. Suppose $a \geq b$.*

1. $\mathrm{G.dim}(a/b) \leq \mathrm{br}(a/b)$.

2. $\mathrm{br}(a/b) = \sup\{\mathrm{G.dim}(a/c) : c \in [b, a]\}$.

Proof. Clear from Proposition 9.8 and Definition 9.11. □

9.3 Subadditivity of breadth

Work in a modular lattice $(P, <)$.

Lemma 9.15. *If $x < y$ and b is arbitrary, then at least* one *of the following strict inequalities holds:*

$$x \wedge b < y \wedge b$$
$$x \vee b < y \vee b.$$

Proof. Otherwise, $x \wedge b = y \wedge b$ and $x \vee b = y \vee b$. Then

$$y = (b \vee y) \wedge y = (b \vee x) \wedge y = (b \wedge y) \vee x = (b \wedge x) \vee x = x,$$

where the middle equality is the modular law. □

Lemma 9.16. *Let $\{a_S\}_{S \subseteq [n]}$ be an n-cube in P. Let b be some element.*

1. *Suppose that $a_S \wedge b > a_\emptyset \wedge b$ for all $S \supsetneq \emptyset$. Then the sublattice $[\bot, b]$ has breadth at least n.*

2. *Suppose that $a_S \vee b < a_{[n]} \vee b$ for all $S \subsetneq [n]$. Then the sublattice $[b, \top]$ has breadth at least n.*

Proof. We prove (1); (2) is dual. Define $a_i = a_{\{i\}}$ for $1 \le i \le n$. By assumption, $a_i \wedge b > a_\emptyset \wedge b$. For any $2 \le k \le n$ we have

$$(a_k \wedge b) \wedge \bigvee_{i<k} (a_i \wedge b) \le b$$
$$(a_k \wedge b) \wedge \bigvee_{i<k} (a_i \wedge b) \le a_k \wedge \bigvee_{i<k} a_i = a_\emptyset.$$

Therefore

$$(a_k \wedge b) \wedge \bigvee_{i<k} (a_i \wedge b) \le a_\emptyset \wedge b.$$

So the sequence $a_1 \wedge b, a_2 \wedge b, \ldots, a_n \wedge b$ is independent over $a_\emptyset \wedge b$. This sequence lies in $[\bot, b]$ which must have breadth at least n by Lemma 9.9. □

Lemma 9.17. *Let $x \le b \le y$ be three elements of P. If there is an n-cube $\{a_S\}_{S \subseteq [n]}$ in $[x, y]$, then there is an m-cube in $[x, b]$ and an ℓ-cube in $[b, y]$ for some $m + \ell = n$.*

Proof. Passing to the sublattice $[x, y]$, we may assume $x = \bot$ and $y = \top$. Take $S_0 \subseteq [n]$ maximal such that $a_{S_0} \wedge b = a_\emptyset \wedge b$. Let $\ell = |S_0|$. Then $\{a_S\}_{S \subseteq S_0}$ is an ℓ-cube. For any $S \subsetneq S_0$, we have

$$a_S \wedge b = a_{S_0} \wedge b$$

by choice of S_0, and then

$$a_S \vee b < a_{S_0} \vee b$$

by Lemma 9.15. By Lemma 9.16.2, the lattice $[b, \top]$ has breadth at least ℓ.

Likewise, $\{a_S\}_{S \supseteq S_0}$ is an m-cube, for $m = n - \ell = |[n] \setminus S_0|$. By choice of S_0, we have

$$a_{S_0} \wedge b < a_S \wedge b$$

for any $S \supsetneq S_0$. By Lemma 9.16.1, the lattice $[\bot, b]$ has breadth at least m. □

Lemma 9.18. *If M_1, M_2 are two modular lattices, then*

$$\mathrm{br}(M_1 \times M_2) \ge \mathrm{br}(M_1) + \mathrm{br}(M_2).$$

Proof. Suppose M_i contains an n_i cube for $i = 1, 2$. Take a sublattice $C_i \subseteq M_i$ isomorphic to $\mathcal{P}ow([n_i])$. Then $C_1 \times C_2$ is isomorphic to $\mathcal{P}ow([n_1 + n_2])$, and $C_1 \times C_2$ is a sublattice of $M_1 \times M_2$. So $\mathrm{br}(M_1 \times M_2) \ge n_1 + n_2$. □

Recall the notation $\mathrm{br}(a/b)$ for the breadth of $[b, a]$.

Proposition 9.19. *Let $(P, \le)$ be a modular lattice.*

1. *If $a \ge b$, then*

$$\mathrm{br}(a/b) = 0 \iff a = b.$$

2. If $a \geq b \geq c$, then

$$\max(\mathrm{br}(a/b), \mathrm{br}(b/c)) \leq \mathrm{br}(a/c) \leq \mathrm{br}(a/b) + \mathrm{br}(b/c).$$

3. If a, b are arbitrary, then

$$\begin{aligned} \mathrm{br}(a/a \wedge b) &= \mathrm{br}(a \vee b/b) \\ \mathrm{br}(b/a \wedge b) &= \mathrm{br}(a \vee b/a) \\ \mathrm{br}(a \vee b/a \wedge b) &= \mathrm{br}(a/a \wedge b) + \mathrm{br}(b/a \wedge b). \end{aligned}$$

Proof. 1. If $a = b$, then $[b, a]$ is a singleton, so it cannot contain a 1-cube. If $a > b$, then $\{a, b\}$ is a 1-cube in $[a, b]$.

2. The inequalities

$$\begin{aligned} \mathrm{br}(a/b) &\leq \mathrm{br}(a/c) \\ \mathrm{br}(b/c) &\leq \mathrm{br}(a/c) \end{aligned}$$

hold because $[b, a]$ and $[c, b]$ are sublattices of $[c, a]$. Any n-cube in $[b, a]$ or $[c, b]$ would give an n-cube in $[c, a]$. The other inequality

$$\mathrm{br}(a/c) \leq \mathrm{br}(a/b) + \mathrm{br}(b/c)$$

holds by Lemma 9.17.

3. The equality $\mathrm{br}(a/a \wedge b) = \mathrm{br}(a \vee b/b)$ holds because of the isomorphism $[a \wedge b, a] \cong [b, a \vee b]$. The second equality holds similarly. Lastly, note that

$$\mathrm{br}(a \vee b/a \wedge b) \leq \mathrm{br}(a/a \wedge b) + \mathrm{br}(a \vee b/a) = \mathrm{br}(a/a \wedge b) + \mathrm{br}(b/a \wedge b)$$

by the previous points. By [8, Theorem 364], the interval $[a \wedge b, a \vee b]$ contains a sublattice isomorphic to $[a \wedge b, a] \times [a \wedge b, b]$. By Lemma 9.18,

$$\mathrm{br}(a \vee b/a \wedge b) \geq \mathrm{br}(a/a \wedge b) + \mathrm{br}(b/a \wedge b).$$

□

Corollary 9.20. *If $a_1, \ldots, a_n$ are independent over b, then*

$$\mathrm{br}(a_1 \vee \cdots \vee a_n/b) = \sum_{i=1}^{n} \mathrm{br}(a_i/b).$$

The analogue for Goldie dimension is as follows:

Fact 9.21 ([9, Corollary 7(b)]). *If $a_1, \ldots, a_n$ are independent over b, then*

$$\mathrm{G.dim}(a_1 \vee \cdots \vee a_n/b) = \sum_{i=1}^{n} \mathrm{G.dim}(a_i/b).$$

9.4 The pregeometry on uniform elements

In module theory, a submodule $N \subseteq M$ is *essential* if every non-zero submodule of M intersects N. A non-zero module M is *uniform* if every non-zero submodule is essential, i.e., any two non-zero submodules of M have non-zero intersection. For any module M, Dawson constructed a natural *pregeometry* on the set of uniform submodules [4]. For background on pregeometries (also called *independence systems* and *matroids*), see [21] or [8, §V.3.3].

Dawson's construction was generalized from modules to modular lattices by Puczyłowski [19]. Fix a modular lattice $(P, <)$ with least element $\bot$.

Definition 9.22 ([19, p. 305])**.** An element $a > \bot$ is *essential* if for every $b > \bot$, we have $a \wedge b > \bot$. The lattice P is *uniform* if every $a > \bot$ is essential. An element $a > \bot$ is *uniform* if the sublattice $[\bot, a]$ is uniform. The set of uniform elements in P is denoted $U(P)$.

Tracing through the definitions, $a \in P$ is uniform if for any $x, y \in P$ with $\bot < x \leq a$ and $\bot < y \leq a$, we have $x \wedge y > \bot$. Equivalently, $a > \bot$ is uniform iff $\mathrm{G.dim}(a/\bot) = 1$.

Theorem 9.23 (Puczyłowski [19])**.** *There is a pregeometry on $U(P)$ with the following properties:*

1. *A finite set $a_1, \ldots, a_n \in U(P)$ is independent with respect to the pregeometry if and only if $\{a_1, \ldots, a_n\}$ is lattice-theoretically independent (Definition 9.1).*
2. *The pregeometry is modular. In other words, the lattice of closed sets is modular.*
3. *For any $x \in P$, define*
$$\Phi(x) = \{a \in U(P) : x \wedge a > \bot\}.$$
Then $\Phi(x)$ is a closed set in the pregeometry.

Suppose in addition that $\mathrm{G.dim}(P) < \infty$*. Then*

4. *Every closed set is of the form $\Phi(x)$.*
5. *The rank of the pregeometry is* $\mathrm{G.dim}(P)$*. In particular, the rank is finite.*
6. *If $a_1, \ldots, a_n \in U(P)$ is a basis, then $a_1 \vee \cdots \vee a_n$ is essential in P.*

Proof. See Theorems 4, 5, 8, 9 in [19]. □

We call this pregeometry the *pregeometry on uniform elements.*

Definition 9.24. A *uniform basis* in $(P, <)$ is a basis in the pregeometry on uniform elements, i.e., a maximal independent set of uniform elements.

Lemma 9.25. *Let $a_1, \ldots, a_n$ be a uniform basis for P. Suppose $\bot < a'_i \leq a_i$ for each i. Then $a'_1, \ldots, a'_n$ is also a uniform basis for P.*

Proof. If a is uniform and $\bot < a' \leq a$, then a' is also uniform. The set $\{a'_1, \ldots, a'_n\}$ is therefore an independent set in the pregeometry on uniform elements. Since it has the same cardinality as a basis, it must itself be a basis. $\square$

Lemma 9.26. *Let $(P, \leq)$ be a modular lattice with least element $\bot$. Suppose* $\mathrm{G.dim}(P) < \infty$. *let a be any element of P. Then there is a uniform basis $b_1, \ldots, b_n, c_1, \ldots, c_m$ such that*

- *Each $b_i \leq a$.*
- *The sequence $a, c_1, \ldots, c_m$ is independent.*

Proof. Let $n = \mathrm{G.dim}(a/\bot)$. Let $b_1, \ldots, b_n$ be a uniform basis in the sublattice $[\bot, a]$. Then each b_i is a uniform element, and the b_i are independent. Therefore the set $\{b_1, \ldots, b_n\}$ is an independent set in $U(P)$. We can find $c_1, \ldots, c_m$ such that $\{b_1, \ldots, b_n, c_1, \ldots, c_m\}$ is a uniform basis in P. We claim that $c_1, \ldots, c_m, a$ is independent. Otherwise,

$$a \wedge \bigvee_{i=1}^{m} c_i > \bot.$$

By Theorem 9.23.6, $\bigvee_{i=1}^{n} b_i$ is essential in $[\bot, a]$. Therefore

$$\left(\bigvee_{i=1}^{n} b_i\right) \wedge \left(\bigvee_{i=1}^{m} c_i\right) = \left(\bigvee_{i=1}^{n} b_i\right) \wedge a \wedge \left(\bigvee_{i=1}^{m} c_i\right) > \bot.$$

But the sequence $b_1, \ldots, b_n, c_1, \ldots, c_m$ is independent, so by Fact 9.5,

$$\left(\bigvee_{i=1}^{n} b_i\right) \wedge \left(\bigvee_{i=1}^{m} c_i\right) = \bot,$$

a contradiction. $\square$

Remark 9.27. In the proof of Lemma 9.26, each b_i belongs to the set $\Phi(a)$ of Theorem 9.23.3, since $b_i \wedge a = b_i > \bot$. Thus $\{b_1, \ldots, b_n\}$ is an independent subset of $\Phi(a)$. Moreover, $\{b_1, \ldots, b_n\}$ is a maximal independent subset of $\Phi(a)$: if $\{b_1, \ldots, b_n, c\}$ is independent for some uniform element $c \in \Phi(a)$, then $\{b_1, \ldots, b_n, c \wedge a\}$ is an independent set of uniform elements in $[\bot, a]$, contradicting the choice of the b_i.

Therefore, the rank of the closed set $\Phi(a)$ is equal to n. The b_i were chosen to be a uniform basis in $[\bot, a]$, so by Theorem 9.23.5, $n = \mathrm{G.dim}(a/\bot)$. Therefore,

The rank of the closed set $\Phi(a)$ is equal to $\mathrm{G.dim}(a/\bot)$.

We will often use relative notions of the above construction:

Definition 9.28. Let $(P, <)$ be a modular lattice and $b \in P$ be an element. An element $a \in P$ is *uniform over b* if a is a uniform element in $[b, \top]$, i.e., $a > b$ and $[b, a]$ is a uniform lattice. A *uniform basis over b* is a uniform basis in $[b, \top]$.

10 Bounds on connected components

In this section, $(G, +, \dots)$ is a monster-model abelian group, possibly with additional structure, of finite dp-rank n.

Fact 10.1. *Let $G_0, \dots, G_n$ be type-definable subgroups of G. There is some $0 \le k \le n$ such that*

$$\left(\bigcap_{i=0}^{n} G_i\right)^{00} = \left(\bigcap_{i \neq k} G_i\right)^{00}.$$

This is [3, Proposition 4.5.2]; the n there agrees with dp-rk(G) by inspecting the proof.

Lemma 10.2. *Let H be a type-definable subgroup of G. There is a cardinal κ depending only on H and G such that if $H < H' < G$ for some type-definable subgroup H', and if H'/H is bounded, then H'/H has size at most κ. This κ continues to work in arbitrary elementary extensions.*

Proof. Naming parameters, we may assume that H (but not H') is type-definable over $\emptyset$. By Morley-Erdős-Rado there is some cardinal κ with the following property: for any sequence $\{a_\alpha\}_{\alpha<\kappa}$ of elements of G, there is some 0-indiscernible sequence $\{b_i\}_{i\in\mathbb{N}}$ such that for any $i_1 < \dots < i_n$ there is $\alpha_1 < \dots < \alpha_n$ such that

$$a_{\alpha_1} \cdots a_{\alpha_n} \equiv_\emptyset b_{i_1} \cdots b_{i_n}.$$

Let H' be a subgroup of G, containing H, type-definable over some small set A. Suppose that $|H'/H| \ge \kappa$. We claim that H'/H is unbounded. Suppose for the sake of contradiction that $|H'/H| < \lambda$ in all elementary extensions. Take a sequence $\{a_\alpha\}_{\alpha<\kappa}$ of elements of H' lying in pairwise distinct cosets of H. Let $\{b_i\}_{i\in\mathbb{N}}$ be an 0-indiscernible sequence extracted from the a_α by Morley-Erdős-Rado. Because the a_α live in pairwise distinct cosets of H and H is 0-definable, the b_i live in pairwise distinct cosets of H. By indiscernibility, there is a 0-definable set $D \supseteq H$ such that $b_i - b_j \notin D$ for $i \neq j$. Consider the $*$-type over A in variables $\{x_\alpha\}_{\alpha<\lambda}$ asserting that

1. $x_\alpha \in H'$ for every $\alpha < \lambda$
2. If $\alpha_1 < \dots < \alpha_n$, then
$$x_{\alpha_1} \cdots x_{\alpha_n} \equiv_\emptyset b_1 \cdots b_n.$$

This type is consistent. Indeed, if $\Sigma_{\alpha_1,\dots,\alpha_n}(\vec{x})$ is the sub-type asserting that

$$x_{\alpha_1}, \dots, x_{\alpha_n} \in H'$$
$$x_{\alpha_1} \cdots x_{\alpha_n} \equiv_\emptyset b_1 \cdots b_n$$

then $\Sigma_{\alpha_1,\dots,\alpha_n}(\vec{x})$ is satisfied by $(a_{\beta_1}, \dots, a_{\beta_n})$ for some well chosen β_i, by virtue of how the b_i were extracted. Moreover, the full type is a filtered union of $\Sigma_{\vec{\alpha}}(\vec{x})$'s, so it is consistent. Let $\{c_\alpha\}_{\alpha<\lambda}$ be a set of realizations. Then every c_α lies in H', but

$$c_\alpha - c_{\alpha'} \notin D \supseteq H$$

for $\alpha \neq \alpha'$. Therefore, the c_α lie in pairwise distinct cosets of H, and $|H'/H| \geq \lambda$, a contradiction. □

Lemma 10.3. *For any cardinal κ there is a cardinal $\tau(\kappa)$ with the following property: given any family $\{H_\alpha\}_{\alpha<\tau(\kappa)}$ of type-definable subgroups of G, there exist subsets $S_1, S_2 \subseteq \tau(\kappa)$ such that S_1 is finite, $|S_2| = \kappa$, and*

$$\left(\bigcap_{\alpha\in S_1} H_\alpha\right)^{00} \subseteq \bigcap_{\alpha\in S_2} H_\alpha.$$

Proof. Without loss of generality $\kappa \geq \aleph_0$. By the Erdős-Rado theorem, we can choose $\tau(\kappa)$ such that any coloring of the $n+1$-element subsets of $\tau(\kappa)$ with $n+1$ colors contains a homogeneous subset of cardinality κ^+. Now suppose we are given H_α for $\alpha < \tau(\kappa)$. Given $\alpha_1 < \cdots < \alpha_{n+1}$, color the set $\{\alpha_1, \ldots, \alpha_{n+1}\}$ with the smallest $k \in \{1, \ldots, n+1\}$ such that

$$\left(\bigcap_{i=1}^{n+1} H_{\alpha_i}\right)^{00} = \left(\bigcap_{i=1}^{k-1} H_{\alpha_i} \cap \bigcap_{i=k+1}^{n+1} H_{\alpha_i}\right)^{00}.$$

This is possible by Fact 10.1. Passing to a homogeneous subset and re-indexing, we get $\{H_\alpha\}_{\alpha<\kappa^+}$ such that every $(n+1)$-element set has color k for some fixed k. In particular, for any $\alpha_1 < \cdots < \alpha_{n+1} < \kappa^+$, we have

$$H_{\alpha_k} \supseteq \left(\bigcap_{i=1}^{n+1} H_{\alpha_i}\right)^{00} = \left(\bigcap_{i=1}^{k-1} H_{\alpha_i} \cap \bigcap_{i=k+1}^{n+1} H_{\alpha_i}\right)^{00}.$$

Thus, for any $\beta_1 < \beta_2 < \cdots < \beta_{2n+1} < \kappa^+$ we have

$$H_{\beta_{n+1}} \supseteq \left(\bigcap_{i=n-k+2}^{n} H_{\beta_i} \cap \bigcap_{i=n+2}^{2n-k+2} H_{\beta_i}\right)^{00} \supseteq \left(\bigcap_{i=1}^{n} H_{\beta_i} \cap \bigcap_{i=n+2}^{2n+1} H_{\beta_i}\right)^{00}$$

by taking

$$(\alpha_1, \ldots, \alpha_{n+1}) = (\beta_{n-k+2}, \ldots, \beta_{2n-k+2}).$$

Then, for any $\beta \in [n+1, \kappa]$,

$$H_\beta \supseteq (H_1 \cap \cdots \cap H_n \cap H_{\kappa+1} \cap H_{\kappa+n})^{00},$$

so we may take $S_1 = \{1, \ldots, n, \kappa+1, \ldots, \kappa+n\}$ and $S_2 = [n+1, \kappa]$. □

Theorem 10.4. *There is a cardinal κ, depending only on the ambient group G, such that for any type-definable subgroup $H < G$, the index of H^{00} in H is less than κ. This κ continues to work in arbitrary elementary extensions.*

Proof. Say that a subgroup $K \subseteq G$ is *ω-definable* if it is type-definable over a countable set. Note that if K is ω-definable, so is K^{00}. Moreover, if K_1, K_2 are ω-definable, then so are $K_1 \cap K_2$ and $K_1 + K_2$. Also note that if H is any type-definable group, then H is a small filtered intersection of ω-definable groups.

Up to automorphism, there are only a bounded number of ω-definable subgroups of G, so by Lemma 10.2 there is some cardinal κ_0 with the following property: if K is an ω-definable group and if K' is a bigger type-definable group, then either $|K'/K| < \kappa_0$ or $|K'/K|$ is unbounded.

Claim 10.5. If H is a type-definable group and K is an ω-definable group containing H^{00}, then $|H/(H \cap K)| < \kappa_0$.

Proof. Note that

$$H^{00} \subseteq H \cap K \subseteq H,$$

so $H/(H \cap K)$ is bounded. On the other hand, $H/(H \cap K)$ is isomorphic to $(H + K)/K$, which must then have cardinality less than κ_0. $\square_{\text{Claim}}$

Let $\kappa_1 = \tau((2^{\kappa_0})^+)$ where $\tau(-)$ is as in Lemma 10.3.

Claim 10.6. If H is a type-definable subgroup of G, then there are fewer than κ_1 subgroups of the form $H \cap K$ where K is ω-definable and $K \supseteq H^{00}$.

Proof. Otherwise, choose $\{K_\alpha\}_{\alpha \in \kappa_1}$ such that K_α is ω-definable, $K_\alpha \supseteq H^{00}$, and

$$H \cap K_\alpha \neq H \cap K_{\alpha'}$$

for $\alpha < \alpha' < \kappa_1$. By Lemma 10.3, there are subsets $S_1, S_2 \subseteq \kappa_1$ such that $|S_1| < \aleph_0$, $|S_2| = (2^{\kappa_0})^+$, and

$$\left(\bigcap_{\alpha \in S_1} K_\alpha \right)^{00} \subseteq \bigcap_{\alpha \in S_2} K_\alpha.$$

Let J be the left-hand side. Then J is an ω-definable group containing H^{00}, so $|H/(H \cap J)| < \kappa_0$ by Claim 10.5. Now for any $\alpha \in S_2$,

$$J \subseteq K_\alpha \implies H \cap J \subseteq H \cap K_\alpha \subseteq H.$$

There are at most $2^{|H/(H \cap J)|} \leq 2^{\kappa_0}$ groups between $H \cap J$ and J, so there are at most 2^{κ_0} possibilities for $H \cap K_\alpha$, contradicting the fact that $|S_2| > 2^{\kappa_0}$ and the $H \cap K_\alpha$ are pairwise distinct for distinct α. $\square_{\text{Claim}}$

Now given the claim, we see that the index of H^{00} in H can be at most $\kappa_0^{\kappa_1}$. Indeed, let $\mathcal{S}$ be the collection of ω-definable groups K such that $K \supseteq H^{00}$, and let $\mathcal{S}'$ be a subcollection containing a representative K for every possibility of $H \cap K$. By the second claim, $|\mathcal{S}'| < \kappa_1$. Every type-definable group is an intersection of ω-definable groups, so

$$H^{00} = \bigcap_{K \in \mathcal{S}} K = \bigcap_{K \in \mathcal{S}} (H \cap K) = \bigcap_{K \in \mathcal{S}'} (H \cap K).$$

Then there is an injective map

$$H/H^{00} \hookrightarrow \prod_{K \in \mathcal{S}'} H/(H \cap K),$$

and the right hand size has cardinality at most $\kappa_0^{\kappa_1}$. But $\kappa_0^{\kappa_1}$ is independent of H. □

Corollary 10.7. *Let $\mathbb{M}$ be a field of finite dp-rank. There is a cardinal κ with the following property: if $M \preceq \mathbb{M}$ is any small model of cardinality at least κ, and if J is a type-definable M-linear subspace of $\mathbb{M}$, then $J = J^{00}$. More generally, if J is a type-definable M-linear subspace of $\mathbb{M}^k$, then $J = J^{00}$.*

Note that we are not assuming J is type-definable over M.

Proof. Take κ as in the Theorem, M a small model of size at least κ, and J a type-definable M-linear subspace of $\mathbb{M}$. For any $\alpha \in \mathbb{M}^\times$, we have $(\alpha \cdot J)^{00} = \alpha \cdot J^{00}$. Restricting to $\alpha \in M^\times$, we see that $\alpha \cdot J^{00} = J^{00}$. In other words, J^{00} is an M-linear subspace itself. The quotient J/J^{00} naturally has the structure of a vector space over M. If it is non-trivial, it has cardinality at least κ, contradicting the choice of κ. Therefore, J/J^{00} is the trivial vector space, and $J^{00} = J$. For the "more generally" claim, apply Theorem 10.4 to the groups $\mathbb{M}^k$ and take the supremum of the resulting κ. □

Lemma 10.8. *Let $\mathbb{M}$ be a field of finite dp-rank, and $M \preceq \mathbb{M}$ be a small model. Let J be a non-zero type-definable M-linear subspace of $\mathbb{M}$. Then every definable set $X \supseteq J$ is heavy.*

Proof. Suppose for the sake of contradiction that X is light. Rescaling J and X, we may assume $1 \in J$. The set X remains light by Fact 8.2.4. Then $M \subseteq J \subseteq X$. Passing to an elementary extension of the pair $(\mathbb{M}, M)$, we may assume that M is mildly saturated. Then M defines a "critical coordinate configuration" in the sense of [13, Definition 4.7]. The set X remains light, because lightness is definable in families (Fact 8.2.3). By [13, Lemma 4.22] (with $Z = \mathbb{M}$), the inclusion $M \subseteq X$ implies that X is heavy, a contradiction. □

11 Invariant valuation rings

Let $(\mathbb{M}, +, \cdot, \ldots)$ be a monster-model finite dp-rank expansion of a field. Assume that $\mathbb{M}$ is not of finite Morley rank. Fix a small model M_0 large enough for Corollary 10.7 to apply. Thus, for any type-definable M_0-linear subspace $J \leq \mathbb{M}^n$, we have $J = J^{00}$.

Let $\mathcal{P}_n$ be the poset of type-definable M_0-linear subspaces of $\mathbb{M}^n$, let $\mathcal{P} = \mathcal{P}_1$, and let $\mathcal{P}^+$ be the poset of non-zero elements of $\mathcal{P}$.

We collect the basic facts about these posets in the following proposition:

Proposition 11.1.

1. *For each n, $\mathcal{P}_n$ is a bounded lattice.*

2. *For any small model $M \supseteq M_0$, the group I_M is an element of $\mathcal{P}$. In particular, $\mathcal{P}$ contains an element other than $\bot = 0$ and $\top = \mathbb{M}$.*

3. *If $J \in \mathcal{P}$ is non-zero, every definable set D containing J is heavy.*

4. *If $J \in \mathcal{P}^+$ is type-definable over M for some small model $M \supseteq M_0$, then $J \supseteq I_M$.*

5. *If $J \in \mathcal{P}_n$, then $J = J^{00}$.*

6. *$\mathcal{P}^+$ is a sublattice of $\mathcal{P}$, i.e., it is closed under intersection.*

7. *$\mathcal{P}$ has breadth r for some $0 < r \leq \text{dp-rk}(\mathbb{M})$. The breadth of $\mathcal{P}^+$ is also r, and the breadth of $\mathcal{P}^n$ is rn.*

Proof. 1. Clear—the lattice operations are given by

$$G \vee H = G + H$$
$$G \wedge H = G \cap H$$
$$\bot = 0$$
$$\top = \mathbb{M}^n.$$

2. Fact 8.4.4.

3. Lemma 10.8.

4. Fact 8.4.3.

5. By choice of M_0.

6. Let J_1, J_2 be two non-zero elements of $\mathcal{P}$. Let M be a small model containing M_0, over which both J_1 and J_2 are type-definable. Then $J_1 \cap J_2 \geq I_M > \bot$. Therefore $\mathcal{P}^+$ is closed under intersection.

7. Let $r = \text{br}(\mathcal{P})$. The bound $r \leq n$ follows by Fact 10.1 and (3) $\implies$ (1) of Lemma 9.9. Then
$$0 < \text{br}(\mathcal{P}^+) \leq \text{br}(\mathcal{P}) = r \leq n,$$
where the left inequality is sharp because $\mathcal{P}$ has at least three elements by part 2. If $r > \text{br}(\mathcal{P}^+)$, there is a r-cube in $\mathcal{P}$ which does not lie in $\mathcal{P}^+$. The base of this cube must be $\bot$, the only element of $\mathcal{P} \setminus \mathcal{P}^+$. Then $r \leq \text{G.dim}(\mathcal{P})$. However, part 6 says $\text{G.dim}(\mathcal{P}) \leq 1$, so $r \leq 1 \leq \text{br}(\mathcal{P}^+)$, a contradiction. Therefore $\text{br}(\mathcal{P}^+) = r = \text{br}(\mathcal{P})$. Finally, in $\mathcal{P}_n$, if we let $J_i = 0^{\oplus(i-1)} \oplus \mathbb{M} \oplus 0^{\oplus(n-i)}$ for $i = 1, \ldots, n$, then the sequence $J_1, \ldots, J_n$ is independent and $J_1 \vee \cdots \vee J_n = \mathbb{M}^n$. Thus
$$\text{br}(\mathbb{M}^n/0) = \sum_{i=1}^{n} \text{br}(J_i/0)$$

by Corollary 9.20. However, $\mathrm{br}(J_i/0) = r$ because of the isomorphism of lattices

$$\begin{aligned} &\mathcal{P} \to [0, J_i] \\ &X \mapsto 0^{\oplus(i-1)} \oplus X \oplus 0^{\oplus(n-i)}. \end{aligned}$$

□

In what follows, we will let r be $\mathrm{br}(\mathbb{M}/0)$.

Remark 11.2. If $r = 1$, then $\mathcal{P}$ is totally ordered and we can reuse the arguments for dp-minimal fields to immediately see that I_M is a valuation ideal. Usually we are not so lucky.

11.1 Special groups

Definition 11.3. An element $J \in \mathcal{P}_n$ is *special* if $\mathrm{G.dim}(\mathbb{M}^n/J) = \mathrm{br}(\mathbb{M}^n/J) = rn$.

For any $J \in \mathcal{P}_n$, we have

$$\mathrm{G.dim}(\mathbb{M}^n/J) \leq \mathrm{br}(\mathbb{M}^n/J) \leq \mathrm{br}(\mathcal{P}_n) = rn$$

by Lemma 9.14.1, Proposition 9.19.2, and Proposition 11.1.7. Therefore, $J \in \mathcal{P}_n$ is special if and only if $\mathrm{G.dim}(\mathbb{M}^n/J) \geq rn$.

Note that if $J \in \mathcal{P}_n$ is special, then any uniform basis over J has cardinality rn, by Theorem 9.23.5.

Proposition 11.4.

1. *There is at least one non-zero special $J \in \mathcal{P} = \mathcal{P}_1$.*

2. *Let $J \in \mathcal{P}$ be special. Let $A_1, \ldots, A_r$ be a uniform basis over J. Let $G \in \mathcal{P}$ be arbitrary. If $G \cap A_i \not\subseteq J$ for each i, then $G \supseteq J$.*

3. *If $J \in \mathcal{P}$ is special and nonzero and type-definable over a small model $M \supseteq M_0$, then*

$$I_M \cdot J \subseteq I_M \subseteq J$$

4. *If $I \in \mathcal{P}_n$ and $J \in \mathcal{P}_m$ are special, then $I \oplus J \in \mathcal{P}_{n+m}$ is special.*

5. *If $I \in \mathcal{P}_n$ is special and $\alpha \in \mathbb{M}^\times$, then $\alpha \cdot I$ is special.*

Proof. 1. By Proposition 11.1.7 the breadth of $\mathcal{P}^+$ is exactly r, so we can find an r-cube in $\mathcal{P}^+$. The base of such a cube is a non-zero special element of $\mathcal{P}$.

2. For $i = 1, \ldots, r$ take $a_i \in (G \cap A_i) \setminus J$. Then for each i, we have

$$\begin{aligned} a_i \in G \cap A_i &\subseteq (G + J) \cap A_i \supseteq J \cap A_i = J \\ a_i &\notin J \\ (\implies) (G + J) \cap A_i &\supsetneq J. \end{aligned}$$

Consequently,

$$A_1 \cap (G + J), A_2 \cap (G + J), \ldots, A_r \cap (G + J)$$

is an independent sequence over J. It follows that

$$\operatorname{br}(G/(G \cap J)) = \operatorname{br}((G + J)/J) \geq \operatorname{G.dim}((G + J)/J) \geq r.$$

On the other hand

$$\operatorname{br}(G/(G \cap J)) + \operatorname{br}(J/(G \cap J)) = \operatorname{br}((G + J)/(G \cap J)) \leq r,$$

and so $\operatorname{br}(J/(G \cap J)) = 0$. This forces $J = G \cap J$, so $J \subseteq G$.

3. The inclusion $I_M \subseteq J$ is Proposition 11.1.4. Let $A_1, \ldots, A_r$ be a uniform basis over J as in part 2. For each A_i choose an element $a_i \in A_i \setminus J$. Let M' be a small model containing M and the a_i's. We first claim that $I_{M'} \cdot J \subseteq I_{M'}$. Let ε be a non-zero element of $I_{M'}$. As $I_{M'}$ is closed under multiplication by $(M')^\times$, we have $M' \subseteq \varepsilon^{-1} \cdot I_{M'}$. In particular, $a_i \in \varepsilon^{-1} \cdot I_{M'}$ for each i. Then

$$(\varepsilon^{-1} \cdot I_{M'}) \cap A_i \not\subseteq J$$

for each i, so by part 2 we have

$$\varepsilon^{-1} \cdot I_{M'} \supseteq J.$$

In other words, $\varepsilon \cdot J \subseteq I_{M'}$. As ε was an arbitrary non-zero element of $I_{M'}$, it follows that $I_{M'} \cdot J \subseteq I_{M'}$. Now suppose that D is an M-definable basic neighborhood. Then D is an M'-definable basic neighborhood. By the above and compactness, there is an M'-definable basic neighborhood $X -_\infty X$ and a definable set $D_2 \supseteq J$ such that $(X -_\infty X) \cdot D_2 \subseteq D$. Furthermore, D_2 can be taken to be M-definable, because J is a directed intersection of M-definable sets. Having done this, we can then pull the parameters defining X into M, and assume that X is M-definable. (This uses the fact that heaviness is definable in families). Then we have an M-definable basic neighborhood $X -_\infty X$ and an M-definable set $D_2 \supseteq J$ such that $(X -_\infty X) \cdot D_2 \subseteq D$. As D was arbitrary, it follows that

$$I_M \cdot J \subseteq I_M.$$

4. The interval $[I, \mathbb{M}^n]$ in $\mathcal{P}_n$ is isomorphic to $[I \oplus J, \mathbb{M}^n \oplus J]$ in $\mathcal{P}_{n+m}$, so

$$rn = \operatorname{G.dim}(\mathbb{M}^n/I) = \operatorname{G.dim}((\mathbb{M}^n \oplus J)/(I \oplus J))$$
$$rm = \operatorname{G.dim}(\mathbb{M}^m/J) = \operatorname{G.dim}((I \oplus \mathbb{M}^m)/(I \oplus J)),$$

where the second line is true for similar reasons. By Lemma 9.21,

$$\operatorname{G.dim}((\mathbb{M}^n \oplus \mathbb{M}^m)/(I \oplus J)) \geq rn + rm.$$

On the other hand

$$\operatorname{G.dim}((\mathbb{M}^n \oplus \mathbb{M}^m)/(I \oplus J)) \leq \operatorname{br}((\mathbb{M}^n \oplus \mathbb{M}^m)/(I \oplus J)) \leq r(n + m)$$

so equality holds and $I \oplus J$ is special.

5. For any $\alpha \in \mathbb{M}^\times$, the map $X \mapsto \alpha \cdot X$ is an automorphism of $\mathcal{P}_n$. □

Corollary 11.5. *For any model M, $I_M \cdot I_M \subseteq I_M$.*

Proof. Take a non-zero special element $J \in \mathcal{P}_1$. Take a small model M' containing M and M_0, with J type-definable over M'. We first claim that $I_{M'} \cdot I_{M'} \subseteq I_{M'}$. Indeed,

$$I_{M'} \cdot I_{M'} \subseteq I_{M'} \cdot J \subseteq I_{M'}$$

by Proposition 11.4.3. Then we can shrink from M' to M using the technique of the proof of Proposition 11.4.3. Specifically, let D be any M-definable basic neighborhood. Then

$$I_{M'} \cdot I_{M'} \subseteq I_M \subseteq D.$$

By compactness, there is an M'-definable basic neighborhood D_2 such that $D_2 \cdot D_2 \subseteq D$. Using the fact that heaviness is definable in families, we can take D_2 to be M-definable. Then $I_M \cdot I_M \subseteq D_2 \cdot D_2 \subseteq D$. As D was arbitrary, $I_M \cdot I_M \subseteq I_M$. □

In [13, Remark 6.17], we defined a group topology on $(M, +)$, for which I_M is the set of topological infinitesimals. Corollary 11.5 implies that this topology is a ring topology. With much more work, one can show that the canonical topology is a field topology [15, Corollary 5.15].

Speculative Remark 11.6. Say that $J \in \mathcal{P}_1$ is *bounded* if $J \leq J'$ for some special J'. Based on the argument in Proposition 11.4.2-3, it seems that J is bounded if and only if $\alpha \cdot J \subseteq I_M$ for some $\alpha \in \mathbb{M}^\times$ and some small model M. Bounded elements should form a sublattice of $\mathcal{P}_1$.[2]

Definition 11.7. Let $I \in \mathcal{P}_n$ be special and $D \in \mathcal{P}_n$ be arbitrary. Then D *dominates* I if $D \geq I$ and $\mathrm{G.dim}(D/I) = nr$.

Lemma 11.8. *Let $J \in \mathcal{P}_n$ be special, let $A_1, \ldots, A_{nr}$ be a uniform basis in $[J, \mathbb{M}^n]$, and D be arbitrary. Then D dominates J if and only if $D \cap A_i \supsetneq J$ for each i. In particular, this condition doesn't depend on the choice of the basis $\{A_1, \ldots, A_{nr}\}$.*

Proof. Suppose D dominates J. Let $B_1, \ldots, B_{nr}$ be a uniform basis in $[J, D]$. The B_i are independent uniform elements in the larger interval $[J, \mathbb{M}^n]$, so $\{B_1, \ldots, B_{nr}\}$ is a uniform basis in $[J, \mathbb{M}^n]$. Therefore, for every i the sequence $B_1, \ldots, B_{nr}, A_i$ is *not* independent over J. Consequently

$$D \cap A_i \supseteq (B_1 + \cdots + B_{nr}) \cap A_i \supsetneq J.$$

Conversely, suppose $D \cap A_i \supsetneq J$ for each i. Then certainly $D \supseteq J$, and it remains to show $\mathrm{G.dim}(D/J) \geq nr$. Let $A'_i := D \cap A_i$. Then the sequence $A'_1, \ldots, A'_{nr}$ is independent over J. As each A'_i lies in $[J, D]$, it follows that $\mathrm{G.dim}(D/J) \geq nr$. □

[2] These ideas have been developed in [15, §8].

Lemma 11.9. *Let $I \in \mathcal{P}_n$ be special, and V be a k-dimensional $\mathbb{M}$-linear subspace of $\mathbb{M}^n$. Then*

$$\begin{aligned} \mathrm{G.dim}((V+I)/I) &= \mathrm{br}((V+I)/I) = kr \\ \mathrm{G.dim}(V/(V \cap I)) &= \mathrm{br}(V/(V \cap I)) = kr \\ \mathrm{G.dim}(\mathbb{M}^n/(V+I)) &= \mathrm{br}(\mathbb{M}^n/(V+I)) = (n-k)r. \end{aligned}$$

Moreover, there exist $A_1, \ldots, A_{kr}, B_1, \ldots, B_{(n-k)r} \in \mathcal{P}_n$ such that the following conditions hold:

1. *The set $\{A_1, \ldots, A_{kr}, B_1, \ldots, B_{(n-k)r}\}$ is a uniform basis in $[I, \mathbb{M}^n]$.*
2. *Let $\tilde{A}_i = A_i \cap V$. Then $\{\tilde{A}_1, \ldots, \tilde{A}_{kr}\}$ is a uniform basis in $[V \cap I, V]$.*
3. *Let $\tilde{B}_i = B_i + V$. Then $\{\tilde{B}_1, \ldots, \tilde{B}_{(n-k)r}\}$ is a uniform basis in $[V + I, \mathbb{M}^n]$.*

Given a $D \in \mathcal{P}_n$ dominating I, we may choose the A_i and B_i to lie in $[I, D]$.

Proof. Let W be a complementary $(n-k)$-dimensional $\mathbb{M}$-linear subspace, so that $V + W = \mathbb{M}^n$. Let $V' = V + I$ and $W' = W + I$. Then

$$\begin{aligned} nr = \mathrm{br}(\mathbb{M}^n/I) &\le \mathrm{br}(\mathbb{M}^n/V') + \mathrm{br}(V'/I) \\ &= \mathrm{br}((V' + W')/V') + \mathrm{br}(V'/I) \\ &= \mathrm{br}(W'/(W' \cap V')) + \mathrm{br}(V'/I) \\ &\le \mathrm{br}(W'/I) + \mathrm{br}(V'/I) \\ &= \mathrm{br}(W/(W \cap I)) + \mathrm{br}(V/(V \cap I)) \\ &\le \mathrm{br}(W/0) + \mathrm{br}(V/0). \end{aligned}$$

Now any $\mathbb{M}$-linear isomorphism $\phi : \mathbb{M}^k \xrightarrow{\sim} V$ induces an isomorphism of posets from $\mathcal{P}_k$ to $[0, V] \subseteq \mathcal{P}_n$, so

$$\begin{aligned} \mathrm{br}(V/0) &= \mathrm{br}(\mathcal{P}_k) = kr \\ \mathrm{br}(W/0) &= \mathrm{br}(\mathcal{P}_{n-k}) = (n-k)r, \end{aligned}$$

where the second line follows similarly. Therefore the inequalities above are all equalities, and

$$\begin{aligned} \mathrm{br}((V+I)/I) &= \mathrm{br}(V/(V \cap I)) = kr \\ \mathrm{br}(\mathbb{M}^n/(V+I)) &= \mathrm{br}(\mathbb{M}^n/V') = (n-k)r. \end{aligned}$$

By Lemma 9.26, there is a uniform basis $\{A_1, \ldots, A_m, B_1, \ldots, B_{nr-m}\}$ in $[I, \mathbb{M}^n]$ such that

- Each $A_i \subseteq V + I$.
- The sequence $(V + I), B_1, \ldots, B_{nr-m}$ is independent over I.

If we are given D dominating I, we may replace each A_i with $A_i \cap D$ and B_i with $B_i \cap D$, and assume henceforth that $A_i, B_i \subseteq D$. By Corollary 9.20,

$$\begin{aligned} nr = \operatorname{br}(\mathbb{M}^n/I) &\geq \operatorname{br}((V+I)/I) + \operatorname{br}(B_1/I) + \cdots + \operatorname{br}(B_{nr-m}/I) \\ &= kr + \operatorname{br}(B_1/I) + \cdots + \operatorname{br}(B_{nr-m}/I). \end{aligned}$$

Each B_i is strictly greater than I, so

$$nr \geq kr + nr - m,$$

and thus $m \geq kr$. On the other hand, the set $\{A_1, \ldots, A_m\}$ is a set of independent uniform elements in $[I, V+I]$, so

$$m \leq \operatorname{G.dim}((V+I)/I) \leq \operatorname{br}((V+I)/I) = kr.$$

Thus equality holds, $m = kr$, and the set $\{A_1, \ldots, A_m\}$ is a uniform basis in $[I, V+I]$. Applying the isomorphism

$$\begin{aligned} [I, V+I] &\xrightarrow{\sim} [V \cap I, V] \\ X &\mapsto X \cap V, \end{aligned}$$

the $\tilde{A}_i$ form a uniform basis in $[V \cap I, V]$. Next, let $Q = B_1 \vee \cdots \vee B_{(n-k)r}$. (Note that $nr - m = (n-k)r$.) The fact that $(V+I), B_1, \ldots, B_{(n-k)r}$ is independent over I implies that $(V+I) \cap Q = I$. Therefore, there is an isomorphism

$$\begin{aligned} [I, Q] &\xrightarrow{\sim} [V+I, V+I+Q] \\ X &\mapsto X + (V+I) = X + V. \end{aligned}$$

The elements $\{B_1, \ldots, B_{(n-k)r}\}$ are independent uniform elements in $[I, Q]$, and therefore the $\tilde{B}_i$ are a set of independent uniform elements in $[V+I, V+I+Q]$ or even in $[V+I, \mathbb{M}^n]$. It follows that

$$(n-k)r \leq \operatorname{G.dim}(\mathbb{M}^n/(V+I)) \leq \operatorname{br}(\mathbb{M}^n/(V+I)) = (n-k)r,$$

so equality holds and the $\tilde{B}_i$ are a uniform basis in $[V+I, \mathbb{M}^n]$. $\square$

Lemma 11.10. *Let $I, J \in \mathcal{P}_n$ be special. Then $I+J$ and $I \cap J$ are special. Furthermore, there exists*

- *a uniform basis $\hat{A}_1, \ldots, \hat{A}_n$ in $[I \cap J, \mathbb{M}^n]$,*
- *a uniform basis $\hat{B}_1, \ldots, \hat{B}_n$ in $[I+J, \mathbb{M}^n]$, and*
- *a uniform basis $A_1, \ldots, A_n, B_1, \ldots, B_n$ in $[I \oplus J, \mathbb{M}^{2n}]$*

related as follows:

$$\hat{A}_i = \{\vec{x} \in \mathbb{M}^n \mid (\vec{x}, \vec{x}) \in A_i\}$$
$$\hat{B}_i = \{\vec{x} - \vec{y} \mid (\vec{x}, \vec{y}) \in B_i\}.$$

Given $D \in \mathcal{P}_{2n}$ dominating $I \oplus J$, we may choose the A_i and B_i to lie in $[I \oplus J, D]$.

Proof. For any $J \in \mathcal{P}_n$, define

$$\Delta(C) = \{(\vec{x}, \vec{x}) \mid \vec{x} \in C\} \in \mathcal{P}_{2n}$$
$$\nabla(C) = \{(\vec{x}, \vec{x} + \vec{y}) \mid \vec{x} \in \mathbb{M}^n,\ \vec{y} \in C\} \in \mathcal{P}_{2n}.$$

Let $V = \Delta(\mathbb{M}^n) = \nabla(0)$. The maps $\Delta(-), \nabla(-)$ yield isomorphisms

$$\Delta : \mathcal{P}_n \xrightarrow{\sim} [0, V] \subseteq \mathcal{P}_{2n}$$
$$\nabla : \mathcal{P}_n \xrightarrow{\sim} [V, \mathbb{M}^{2n}] \subseteq \mathcal{P}_{2n}.$$

Indeed, the inverses are given by

$$\Delta^{-1} : [0, V] \xrightarrow{\sim} [0, \mathbb{M}^n]$$
$$C \mapsto \{\vec{x} \mid (\vec{x}, \vec{x}) \in C\}$$
$$\nabla^{-1} : [V, \mathbb{M}^{2n}] \xrightarrow{\sim} [0, \mathbb{M}^n]$$
$$C \mapsto \{\vec{x} - \vec{y} \mid (\vec{x}, \vec{y}) \in C\}.$$

Note that $\Delta^{-1}(V \cap (I \oplus J)) = I \cap J$ and $\nabla^{-1}(V + (I \oplus J)) = I + J$. Therefore, Δ^{-1} and ∇^{-1} restrict to isomorphisms

$$\Delta^{-1} : [V \cap (I \oplus J), V] \xrightarrow{\sim} [I \cap J, \mathbb{M}^n]$$
$$\nabla^{-1} : [V + (I \oplus J), \mathbb{M}^{2n}] \xrightarrow{\sim} [I + J, \mathbb{M}^n].$$

It follows that

$$\operatorname{G.dim}(\mathbb{M}^n/(I \cap J)) = \operatorname{G.dim}(V/(V \cap (I \oplus J))) = \operatorname{G.dim}((V + (I \oplus J))/V)$$
$$\operatorname{G.dim}(\mathbb{M}^n/(I + J)) = \operatorname{G.dim}(\mathbb{M}^{2n}/(V + (I \oplus J))/V).$$

Now $I \oplus J$ is special in $\mathcal{P}_{2n}$ by Proposition 11.4.4, and V is an n-dimensional $\mathbb{M}$-linear subspace of $\mathbb{M}^{2n}$, so by Lemma 11.9,

$$\operatorname{G.dim}(\mathbb{M}^n/(I \cap J)) = \operatorname{G.dim}((V + (I \oplus J))/V) = rn$$
$$\operatorname{G.dim}(\mathbb{M}^n/(I + J)) = \operatorname{G.dim}(\mathbb{M}^{2n}/(V + (I \oplus J))) = rn.$$

Therefore $I \cap J$ and $I + J$ are special. Furthermore, by Lemma 11.9 there exists a uniform basis $\{A_1, \ldots, A_{rn}, B_1, \ldots, B_{rn}\}$ over $I \oplus J$ such that

- The elements $\tilde{A}_i := A_i \cap V$ form a uniform basis in $[V \cap (I \oplus J), V]$.
- The elements $\tilde{B}_i := B_i + V$ form a uniform basis in $[V + (I \oplus J), \mathbb{M}^{2n}]$.

(Additionally, the A_i and B_i can be chosen below any given D dominating $I \oplus J$.) Applying Δ^{-1} and ∇^{-1} we see that the elements

$$\begin{aligned}\hat{A}_i = \Delta^{-1}(A_i \cap V) &= \{\vec{x} \mid (\vec{x}, \vec{x}) \in A_i \cap V\} \\ &= \{\vec{x} \mid (\vec{x}, \vec{x}) \in A_i\} \\ \hat{B}_i = \nabla^{-1}(A_i + V) &= \{\vec{x} - \vec{y} \mid (\vec{x}, \vec{y}) \in B_i + V\} \\ &= \{\vec{x} - \vec{y} \mid (\vec{x}, \vec{y}) \in B_i\}\end{aligned}$$

form uniform bases for $[I \cap J, \mathbb{M}^n]$ and $[I + J, \mathbb{M}^n]$, respectively. □

Question 11.11. *By Lemma 11.10 special elements of $\mathcal{P}_n$ form a sublattice. Can this be proven directly (lattice theoretically) within $\mathcal{P}_n$ without using the larger lattice $\mathcal{P}_{2n}$?*

Speculative Remark 11.12. We explain how the above picture should give an r-inflator, as in §8.2. Fix a special element J of $\mathcal{P}_1$. For every n, J^n is a special element of $\mathcal{P}_n$. Let $\mathcal{G}_n$ be the lattice of closed sets in the pregeometry on uniform elements over J^n. There should be natural maps

$$\oplus : \mathcal{G}_n \times \mathcal{G}_m \to \mathcal{G}_{n+m}$$

and a $GL_n(M_0)$ action on $\mathcal{G}_n$, induced by the analogous structure on the $\mathcal{P}_n$. Using this additional structure, one should be able to prove the following: there is a semisimple M_0-algebra S of length r, and isomorphisms

$$\mathcal{G}_n \cong \mathrm{Sub}_S(S^n)$$

respecting $\oplus$ and the $GL_n(M_0)$-actions.[3] For every n we should get a map

$$\begin{aligned}f_n : \mathrm{Sub}_{\mathbb{M}}(\mathbb{M}^n) &\to \mathcal{G}_n \cong \mathrm{Sub}_S(S^n) \\ V &\mapsto \Phi(V + J^n).\end{aligned}$$

These maps should form an r-inflator, as in Definition 8.6. By Remark 9.27, the rank of $\Phi(V + J^n)$ is $\mathrm{G.dim}((V + J^n)/J^n)$. Therefore the scaling axiom (Definition 8.6.4) says that

$$\mathrm{G.dim}((V + J^n)/J^n) = r \cdot \dim_{\mathbb{M}}(V),$$

which follows by Lemma 11.9. The other inflator axioms (Definition 8.6.1–3) should be a matter of tracing through the definition of f_n, ensuring that each step preserves the order, $GL_n(M_0)$-action, and $\oplus$-operation.[4]

[3] These claims have now been verified in [16]. The lattice $\mathcal{G}_n$ is the "flattening" of the interval $[J^n, \mathbb{M}^n] \subseteq \mathcal{P}_n$, in the sense of [16, Definition 7.2]. The family of lattices $\{[J^n, \mathbb{M}^n]\}_{n \in \mathbb{N}}$ form a "directory" [16, Definition 2.1] by [16, Proposition 2.12, Theorem 3.7]. By [16, Corollary 7.20], the family $\{\mathcal{G}_n\}_{n \in \mathbb{N}}$ is also a "directory," and a "semisimple directory" by [16, Proposition 7.3.1]. The existence of the semisimple M_0-algebra S now follows by [16, Theorem 2.7].

[4] The details have been verified in [16, Theorem 9.3].

Speculative Remark 11.13. Next, we describe how r-inflators should yield valuation rings. Fix an r-inflator $\{f_n\}_{n\in\mathbb{N}}$, where

$$f_n : \mathrm{Sub}_{\mathbb{M}}(\mathbb{M}^n) \to \mathrm{Sub}_S(S^n).$$

Say that $a \in \mathbb{M}$ *specializes* to $b \in S$ if

$$f_2(\mathbb{M} \cdot (1, a)) = S \cdot (1, b).$$

This should define a homomorphism

$$\mathrm{res} : R \to S \tag{1}$$

for some subring $R \subseteq \mathbb{M}$. In the case of 1-inflators ($r = 1$), R should be a valuation ring on $\mathbb{M}$.[5]

When $r > 1$, one might intuitively hope that R will be an intersection of r or fewer valuation rings on $\mathbb{M}$.[6] However, several things go wrong. For example, there is a 2-inflator on $\mathbb{C}$ given by

$$f_n(V) = (V + \overline{V}, V \cap \overline{V}),$$

where $\overline{V}$ is the complex conjugate of V.[7] For this inflator, the ring R is $\mathbb{R}$, which is *not* a finite intersection of valuation rings on $\mathbb{C}$.

There is a certain way to "mutate" f that improves the situation. Specifically, we can define a new inflator f' by the formula

$$f'_n(V) := f_{2n}(\{(\vec{x}, a \cdot \vec{x}) \mid \vec{x} \in V\})$$

for some constant a. By choosing a carefully (for example, $a = \sqrt{-1}$), the ring R' associated to f' will be bigger, and closer to being a finite intersection of valuation rings.[8]

Ideally, after finitely many mutations, one would arrive at an inflator f'' whose associated ring R'' is a finite intersection of valuation rings. Unfortunately, this does not happen.[9] Nevertheless, if one defines R_∞ to be the union of all the rings R' associated to mutations f' of f, it turns out that R_∞ **is** a finite intersection of valuation rings.[10] In other words, we get R to be a finite intersection of valuation rings *in the limit.*

[5] These claims have been verified in [16, Propositions 5.7, 5.19].

[6] The intuition is that if $K \models \mathrm{ACF}_0$ and $\mathcal{O}_1, \ldots, \mathcal{O}_r$ are independent valuation rings on K, then the structure $(K, +, \cdot, \mathcal{O}_1, \ldots, \mathcal{O}_r)$ has burden r, by [14, Theorem 11.5.7]. The analogue of Fact 10.1 holds in this context [3, Proposition 4.5.2], and one can probaby carry out the analogous construction of r-inflators. (See [16, Theorem 9.7] for one result in this direction.) So one expects to get r-inflators from r-fold intersections of valuation rings.

[7] See [16, Example 4.9] for a precise description. The semisimple ring S is $\mathbb{C} \times \mathbb{C}$ in this example, so S-modules are pairs of $\mathbb{C}$-vector spaces.

[8] The idea of "mutation" is made precise in [16, §10.1]. The fact that mutation "improves the situation" is [16, Lemma 10.11]. For the specific example $V \mapsto (V + \overline{V}, V \cap \overline{V})$ discussed above, see [16, §12.1].

[9] See [16, §12.3] for an example.

[10] This is verified in [16, Theorem 10.12].

In sections 11.2–11.3 below, we will follow a simpler parallel argument, which avoids the use of inflators. In Proposition 11.16 we will associate a ring $R_J \subseteq \mathbb{M}$ and an ideal $I_J \lhd R_J$ to any special J; these should correspond to the domain and kernel of the specialization map $\operatorname{res}(-)$ of (1) above. In §11.3 we will "mutate" J by replacing it with

$$J' = J \cap a_1 \cdot J \cap \cdots \cap a_n \cdot J.$$

This should correspond to mutation of inflators.[11] By considering the union of $R_{J'}$ as J' ranges over all mutations, we will obtain a finite intersection of valuation rings (Theorem 11.26).

11.2 The associated rings and ideals

Definition 11.14. Let $J \in \mathcal{P}_n$ be special, and $a \in \mathbb{M}^\times$. Say that a *contracts* J if $a = 0$ or J dominates $a \cdot J$ (i.e., $\operatorname{G.dim}(J/a \cdot J) = nr$).

Note that when $a \neq 0$, J dominates $a \cdot J$ if and only if $a^{-1} \cdot J$ dominates J.

Lemma 11.15.

1. *Let $A_1, \ldots, A_{nr}$ be a uniform basis over $J \in \mathcal{P}_n$ and a be an element of $\mathbb{M}$. If $a \cdot A_i \subseteq J$ for all i, then a contracts J. Conversely, suppose a contracts J. Then there exists $A_i' \in (J, A_i]$ such that $A_1', \ldots, A_{nr}'$ is a uniform basis over J and $a \cdot A_i' \subseteq J$ for each i.*
2. *If a contracts J and $b \in \mathbb{M}^\times$, then a contracts $b \cdot J$.*
3. *If a, b contract J then $a + b$ contracts J.*
4. *If a contracts J and $b \cdot J \subseteq J$, then $a \cdot b$ contracts J.*
5. *If a contracts both $I \in \mathcal{P}_n$ and $J \in \mathcal{P}_m$, then a contracts $I \oplus J \in \mathcal{P}_{n+m}$.*
6. *If a contracts $I, J \in \mathcal{P}_n$, then a contracts $I \cap J$ and $I + J$.*

Proof. 1. First suppose $a \cdot A_i \subseteq J$. If $a = 0$ then a contracts J by definition, so suppose $a \neq 0$. Then

$$(a^{-1} \cdot J) \cap A_i = A_i \supsetneq J$$

for each i, so by Lemma 11.8 the group $a^{-1} \cdot J$ dominates J, or equivalently, J dominates $a \cdot J$. Thus a contracts J. Conversely, suppose that a contracts J. If $a = 0$ then $a \cdot A_i \subseteq J$ so we may take $A_i' = A_i$. Otherwise, note that $a^{-1} \cdot J$ dominates J, so by Lemma 11.8,

$$A_i' := (a^{-1} \cdot J) \cap A_i \supsetneq J.$$

By Lemma 9.25, $\{A_1', \ldots, A_{nr}'\}$ is a uniform basis over J. Furthermore $A_i' \subseteq a^{-1} \cdot J$, so $a \cdot A_i' \subseteq J$.

[11] These correspondences have been verified in [16, Theorem 9.3] and [16, Proposition 10.15].

2. Multiplication by b induces an automorphism of $\mathcal{P}_n$ sending the interval $[a \cdot J, J]$ to $[a \cdot (b \cdot J), b \cdot J]$, so $\mathrm{G.dim}(J/a \cdot J) = \mathrm{G.dim}(b \cdot J/(ab) \cdot J)$.

3. Take a uniform basis $A_1, \ldots, A_{rn}$ over J. By part 1, we may shrink the A_i and assume that $a \cdot A_i \subseteq J$. Shrinking again, we may assume $b \cdot A_i \subseteq J$. Then

$$(a + b) \cdot A_i \subseteq a \cdot A_i + b \cdot A_i \subseteq J + J = J$$

so by part 1, $a + b$ contracts J.

4. Suppose a contracts J and $b \cdot J \subseteq J$. Then

$$\mathrm{G.dim}(J/a \cdot b \cdot J) \geq \mathrm{G.dim}(b \cdot J/a \cdot b \cdot J) = \mathrm{G.dim}(J/a \cdot J) = nr.$$

5. Let $A_1, \ldots, A_{rn}$ be a uniform basis in $[I, \mathbb{M}^n]$, and $B_1, \ldots, B_{rm}$ be a uniform basis in $[J, \mathbb{M}^m]$. Shrinking the A_i and B_i, we may assume $a \cdot A_i \subseteq I$ and $a \cdot B_i \subseteq J$. Note that the sequence

$$A_1 \oplus J, A_2 \oplus J, \ldots, A_{rn} \oplus J, I \oplus B_1, I \oplus B_2, \ldots, I \oplus B_{rm}$$

is a uniform basis in $[I \oplus J, \mathbb{M}^{n+m}]$. Multiplication by a collapses each of these uniform elements into $I \oplus J$ (using the fact that $a \cdot I \subseteq I$ and $a \cdot J \subseteq J$). Therefore a contracts $I \oplus J$.

6. We may assume $a \neq 0$. By the previous point, $a^{-1} \cdot (I \oplus J)$ dominates $I \oplus J$. By Lemma 11.10, $I + J$ and $I \cap J$ are special. Moreover, there is a uniform basis $A_1, \ldots, A_n, B_1, \ldots, B_n$ in $[I \oplus J, \mathbb{M}^{2n}]$ such that for

$$\hat{A}_i = \{\vec{x} \in \mathbb{M}^n \mid (\vec{x}, \vec{x}) \in A_i\}$$
$$\hat{B}_i = \{\vec{x} - \vec{y} \mid (\vec{x}, \vec{y}) \in B_i\}$$

the set $\{\hat{A}_1, \ldots, \hat{A}_n\}$ is a uniform basis over $I \cap J$ and the set $\{\hat{B}_1, \ldots, \hat{B}_n\}$ is a uniform basis over $I + J$. Furthermore Lemma 11.10 ensures that the A_i and B_i can be chosen in $[I \oplus J, a^{-1} \cdot (I \oplus J)]$. Thus $a \cdot A_i \subseteq I \oplus J$ and $a \cdot B_i \subseteq I \oplus J$. Then

$$\vec{x} \in \hat{A}_i \iff (\vec{x}, \vec{x}) \in A_i \implies (a \cdot \vec{x}, a \cdot \vec{x}) \in I \oplus J \iff a \cdot \vec{x} \in I \cap J,$$

so $a \cdot \hat{A}_i \subseteq I \cap J$. As the $\hat{A}_i$ form a uniform basis over $I \cap J$, it follows that a contracts $I \cap J$. Similarly,

$$(\vec{x}, \vec{y}) \in B_i \implies (a \cdot \vec{x}, a \cdot \vec{y}) \in I \oplus J \implies a \cdot (\vec{x} - \vec{y}) \in I + J$$

so $a \cdot \hat{B}_i \subseteq I + J$. Thus a contracts $I + J$. □

Proposition 11.16. *For any special $J \in \mathcal{P} = \mathcal{P}_1$, let R_J be the set of $a \in \mathbb{M}$ such that $a \cdot J \subseteq J$, and let I_J be the set of $a \in \mathbb{M}$ that contract J.*

1. *R_J is a subring of $\mathbb{M}$, containing M_0.*

2. *I_J is an ideal in R_J.*

3. *If $b \in \mathbb{M}^\times$ then $R_J = R_{b \cdot J}$ and $I_J = I_{b \cdot J}$.*

4. *If J is type-definable over $M \supseteq M_0$, then R_J and I_J are M-invariant.*

5. *If J is non-zero and type-definable over $M \supseteq M_0$ then $I_M \subseteq I_J$.*

6. *If J_1 and J_2 are special, then*

$$R_{J_1} \cap R_{J_2} \subseteq R_{J_1 \cap J_2}$$
$$I_{J_1} \cap I_{J_2} \subseteq I_{J_1 \cap J_2}$$

7. *$(1 + I_J) \subseteq R_J^\times$. Consequently, I_J lies inside the Jacobson radical of R_J.*

Proof. 1. Straightforward.

2. The set I_J is a subset of R_J. The fact that $I_J \lhd R_J$ is exactly Lemma 11.15.3-4.

3. For I_J this is Lemma 11.15.2. For R_J this is clear:

$$a \cdot J \subseteq J \implies (ab) \cdot J \subseteq b \cdot J.$$

4. The definitions are $\mathrm{Aut}(\mathbb{M}/M)$-invariant.

5. Let $A_1, \ldots, A_r$ be a uniform basis over J. For each i let a_i be an element of $A_i \setminus J$. Let M' be a small model containing M and the a_i's.

   Claim 11.17. Any $\varepsilon \in I_{M'}$ contracts J.

   Proof. We may assume $\varepsilon \neq 0$. Let $D = \varepsilon^{-1} \cdot I_{M'}$. By Fact 8.4.4, $a_i \cdot \varepsilon \in M' \cdot I_{M'} \subseteq I_{M'}$, and so $a_i \in D$. Thus

   $$D \cap A_i \not\subseteq J$$

   By Proposition 11.4.2, $D \supseteq J$. Then

   $$D \cap A_i \supsetneq J$$

   so by Lemma 11.8, D dominates J. By Proposition 11.1.4, $\varepsilon^{-1} \cdot J \supseteq \varepsilon^{-1} \cdot I_{M'} = D$. Thus $\varepsilon^{-1} \cdot J$ dominates J. $\square_{\text{Claim}}$

Let ε be a realization of the partial type over M' asserting that $\varepsilon \in I_{M'}$ and $\varepsilon \notin X$ for any light M'-definable set X. This type is consistent because M'-definable basic neighborhoods are heavy (Fact 8.3.1) and no heavy set is contained in a finite union of light sets (Fact 8.2.2). Then $\varepsilon \in I_{M'} \subseteq I_J$. As I_J is M-invariant, every realization of $\mathrm{tp}(\varepsilon/M)$ is in I_J. Let Y be the type-definable set of realizations of $\mathrm{tp}(\varepsilon/M)$. For any M-definable $X \supseteq Y$ we have

$$I_M \subseteq X -_\infty X \subseteq X - X.$$

Therefore $I_M \subseteq Y - Y$. But $Y - Y \subseteq I_J - I_J = I_J$.

6. If $a \in R_{J_1}$ and $a \in R_{J_2}$, then

$$a \cdot (J_1 \cap J_2) = (a \cdot J_1) \cap (a \cdot J_2) \subseteq J_1 \cap J_2$$

so $a \in R_{J_1 \cap J_2}$. The inclusion $I_{J_1} \cap I_{J_2} \subseteq I_{J_1 \cap J_2}$ is Lemma 11.15.6.

7. First note that 1 does not contract J. Indeed, $\mathrm{G.dim}(J/J) = 0 \neq r$. Thus $1 \notin I_J$. As I_J is an ideal, it follows that $-1 \notin I_J$.

Claim 11.18. If $\varepsilon \in I_J$ then $\varepsilon/(1+\varepsilon) \in I_J$.

Proof. We may assume $\varepsilon \neq 0$. Using Lemma 11.15.1 choose a uniform basis $\{A_1, \ldots, A_r\}$ over J such that $\varepsilon \cdot A_i \subseteq J$. For each i choose $a_i \in A_i \setminus J$. Then $\varepsilon \cdot a_i \in J$, so $(1+\varepsilon) \cdot a_i \in A_i \setminus J$. Let $\beta = (1+\varepsilon)/\varepsilon$. Then

$$\beta \cdot (\varepsilon \cdot a_i) \in \beta \cdot J$$
$$(\beta \cdot \varepsilon) \cdot a_i = (1+\varepsilon) \cdot a_i \in A_i \setminus J.$$

In particular

$$(\beta \cdot J) \cap A_i \not\subseteq J,$$

for every i, so $\beta \cdot J \supseteq J$ by Proposition 11.4.2. Then $(\beta \cdot J) \cap A_i \supsetneq J$ for every i, so $\beta \cdot J$ dominates J by Lemma 11.8. This means that $\beta^{-1} = \varepsilon/(1+\varepsilon)$ lies in I_J. $\square_{\text{Claim}}$

Now if $\varepsilon \in I_J$, then

$$\frac{1}{1+\varepsilon} = 1 - \frac{\varepsilon}{1+\varepsilon} \in 1 + I_J \subseteq R_J. \qquad \square$$

Remark 11.19. Proposition 11.16.6 also holds for $R_{J_1+J_2}$ and $I_{J_1+J_2}$.

Speculative Remark 11.20. In Proposition 11.16.5, not only is I_M a subset of I_J, it is a sub*ideal* in the ring R_J. One can probably prove this by first increasing M to contain a non-zero element j_0 of J. Then for any $\varepsilon \in I_M$ and $a \in R_J$, we have

$$\varepsilon \cdot a \cdot j_0 \in I_M \cdot R_J \cdot J \subseteq I_M \cdot J \subseteq I_M,$$

so $\varepsilon \cdot a \in j_0^{-1} I_M = I_M$. Thus $I_M \cdot R_J \subseteq I_M$. Then one can probably shrink M back to the original model by the usual methods.[12]

Suppose we could show that R_J was a finite intersection of valuation rings. (As discussed in Remark 11.13, this was the initial expectation.) Then the ring

$$R = \{x \in \mathbb{M} : xI_M \subseteq I_M\}$$

would also be a finite intersection of valuation rings. In fact, using the henselianity arguments from the dp-minimal case, one can show that R would be a single *henselian* valuation ring.[13] This would provide a nice strategy for proving the Shelah conjecture in general.

Unfortunately, it turns out that there are dp-finite fields in which I_M is *not* a valuation ideal [17, §10]. Therefore a different strategy is needed.

11.3 Mutation and the limiting ring

The next two lemmas provide a way to "mutate" a special group J and obtain a better special group J' for which $R_{J'}$ is closer than R_J to being a finite intersection of valuation rings.

Lemma 11.21. *Let $J \in \mathcal{P}$ be special and non-zero. Let $a_1, \ldots, a_n$ be elements of $\mathbb{M}^\times$. Let $J' = J \cap a_1 \cdot J \cap a_2 \cdot J \cap \cdots \cap a_n \cdot J$. Then J' is special and non-zero, $R_J \subseteq R_{J'}$, and $I_J \subseteq I_{J'}$.*

Proof. By Proposition 11.4.5, each $a_i \cdot J$ is special, so the intersection J' is special by Lemma 11.10. It is nonzero by Proposition 11.1.6. By Proposition 11.16.3 we have $R_J = R_{a_i \cdot J}$ and $I_J = I_{a_i \cdot J}$ for each i. Then the inclusions $R_J \subseteq R_{J'}$ and $I_J \subseteq I_{J'}$ follow by an interated application of Proposition 11.16.6. ☐

Recall that r is the breadth of $\mathcal{P}$.

Lemma 11.22. *Let $J \in \mathcal{P}$ be special and non-zero. Let $\alpha \in \mathbb{M}^\times$ be arbitrary. Let $J' = J \cap (\alpha \cdot J) \cap \cdots \cap (\alpha^{r-1} \cdot J)$. Let $q_0, q_1, \ldots, q_r$ be $r+1$ distinct elements of M_0. Then there is at least one i such that $\alpha \neq q_i$ and*

$$\frac{1}{\alpha - q_i} \in R_{J'}.$$

Proof. For each $0 \le i \le r$ let

$$\begin{aligned}
\alpha_i &:= \alpha - q_i \\
G_i &:= \{x \in \mathbb{M} \mid \alpha_i x \in J \wedge \alpha_i^2 x \in J \wedge \cdots \wedge \alpha_i^r x \in J\} \\
H_i &:= J \cap G_i = \{x \in \mathbb{M} \mid x \in J \wedge \alpha_i x \in J \wedge \cdots \wedge \alpha_i^r x \in J\}.
\end{aligned}$$

Also let

$$H = \{x \in \mathbb{M} \mid x \in J \wedge \alpha x \in J \wedge \cdots \wedge \alpha^r x \in J\}.$$

[12] The details are worked out in [17, Lemma 6.9].

[13] The details are worked out in [15, Proposition 7.7, Theorem 9.9].

Claim 11.23. $H_i = H$ for any i.

Proof. Note $\alpha = \alpha_i + q_i$. If $x \in H_i$ then

$$\alpha^n x = (\alpha_i + q_i)^n x = \sum_{k=0}^{n} \binom{n}{k} \alpha_i^k q_i^{n-k} x \in J$$

for $0 \leq n \leq r$, because $\alpha_i^k x \in J$, $q_i^{n-k} \in M_0$, and J is an M_0-vector space. Thus $H_i \subseteq H$; the reverse inclusion follows by symmetry. $\square_{\text{Claim}}$

Because the q_i are distinct, the $(r+1) \times (r+1)$ Vandermonde matrix built from the q_i is invertible. Let $f : \mathbb{M}^{r+1} \to \mathbb{M}^{r+1}$ be the $\mathbb{M}$-linear map sending $(1, q_i, \ldots, q_i^r)$ to the ith basis vector. Let $g : \mathbb{M} \to \mathbb{M}^{r+1}$ be the map

$$g(x) = (x, \alpha x, \ldots, \alpha^r x).$$

Claim 11.24. The composition

$$\mathbb{M} \xrightarrow{g} \mathbb{M}^{r+1} \xrightarrow{f} \mathbb{M}^{r+1} \twoheadrightarrow (\mathbb{M}/J)^{r+1}$$

has kernel H, and maps G_i into $0^i \oplus (\mathbb{M}/J) \oplus 0^{r-i}$.

Proof. The invertible matrix defining f has coefficients in M_0, and J is closed under multiplication by M_0, so f maps J^{r+1} isomorphically to J^{r+1}. Therefore,

$$f(g(x)) \in J^{r+1} \iff g(x) \in J^{r+1} \iff x \in H,$$

where the second $\iff$ is the definition of H. Now suppose $x \in G_i$. Then $g(x) - (x, q_i x, \ldots, q_i^r x) \in J^{r+1}$. Indeed, for any $0 \leq n \leq r$ we have

$$\alpha^n x = (\alpha_i + q_i)^n x = q_i^n x + \sum_{k=1}^{n} \binom{n}{k} q_i^{n-k} (\alpha_i^k x),$$

and the sum is an element of J by definition of G_i. As f preserves J^{r+1}, it follows that

$$f(g(x)) \equiv f(x, q_i x, \ldots, q_i^r x) = x \cdot e_i \pmod{J^{r+1}},$$

where e_i is the ith basis vector. $\square_{\text{Claim}}$

Claim 11.25. If $(x_0, x_1, \ldots, x_r) \in G_0 \times \cdots \times G_r$ has $x_0 + \cdots + x_r \in H$, then each $x_i \in H$.

Proof. For $0 \leq i \leq r$ let $p_i : \mathbb{M}^{r+1} \to \mathbb{M}/J$ be the composition of the ith projection and the quotient map $\mathbb{M} \to \mathbb{M}/J$. Claim 11.24 implies that

$$\begin{aligned} x \in H &\implies p_i(f(g(x))) = 0 \\ x \in G_j &\implies p_i(f(g(x))) = 0 \qquad \text{if } i \neq j. \end{aligned}$$

Thus

$$0 = p_i(f(g(x_0 + \cdots + x_r))) = p_i(f(g(x_i))).$$

As $p_j(f(g(x_i))) = 0$ for $j \neq i$, it follows that $p_j(f(g(x_i))) = 0$ for all j. In other words, $f(g(x_i)) \in J^{r+1}$. By Claim 11.24, $x_i \in H$. $\square_{\text{Claim}}$

Now Claim 11.25 implies that the map

$$\begin{aligned}(G_0/H) \times \cdots \times (G_r/H) &\to \mathbb{M}/H \\ (x_0, \ldots, x_r) &\mapsto x_0 + \cdots + x_r\end{aligned}$$

is injective. The image is D/H for some type-definable $D \in \mathcal{P}$, namely $D = G_0 + \cdots + G_r$. Then the interval $[H^{r+1}, G_0 \oplus \cdots \oplus G_r]$ in $\mathcal{P}_{r+1}$ is isomorphic to the interval $[H, D]$ in $\mathcal{P}_1$. Thus

$$r \geq \operatorname{br}(D/H) = \operatorname{br}(G_0/H) + \cdots + \operatorname{br}(G_r/H).$$

Therefore $G_i = H = H_i$ for at least one i. By definition of G_i and H_i, this means that

$$\alpha_i x \in J \wedge \cdots \wedge \alpha_i^r x \in J \implies x \in J \tag{2}$$

for any $x \in \mathbb{M}$. As $J \neq 0$, this implies $\alpha_i \neq 0$. Then (2) can be rephrased as

$$\alpha_i^{-1} \cdot J \cap \cdots \cap \alpha_i^{-r} \cdot J \subseteq J. \tag{3}$$

Define

$$\begin{aligned}J'' &:= J \cap \alpha_i^{-1} J \cap \cdots \cap \alpha_i^{-(r-1)} J \\ &= J \cap \alpha^{-1} J \cap \cdots \cap \alpha^{-(r-1)} J,\end{aligned}$$

where the second equality follows by the proof of Claim 11.23. By (3),

$$\alpha_i^{-1} \cdot J'' = \alpha_i^{-1} J \cap \cdots \cap \alpha_i^{-r} \subseteq J \cap \alpha_i^{-1} J \cap \cdots \cap \alpha_i^{-(r-1)} J = J''.$$

Therefore $\alpha_i^{-1} \in R_{J''}$. But

$$J' = J \cap \cdots \cap \alpha^{r-1} J = \alpha^{r-1} \cdot (J \cap \cdots \cap \alpha^{-(r-1)} J) = \alpha^{r-1} J''.$$

Thus, by Proposition 11.16.3

$$\alpha_i^{-1} \in R_{J''} = R_{J'}.$$

□

Theorem 11.26. *Let $J \in \mathcal{P}_1$ be special, non-zero, and type-definable over $M \supseteq M_0$. Then there is an M-invariant ring R_J^∞ and ideal $I_J^\infty \lhd R_J^\infty$ satisfying the following properties:*

- *R_J^∞ and I_J^∞ are M-invariant.*
- *$(1 + I_J^\infty) \subseteq (R_J^\infty)^\times$, so I_J^∞ is a subideal of the Jacobson radical of R_J^∞.*
- *The M-infinitesimals I_M are a subgroup of I_J^∞ (and therefore of the Jacobson radical).*
- *$M_0 \subseteq R_J^\infty$.*
- *R_J^∞ is a Bézout domain with at most r maximal ideals.*
- *The field of fractions of R_J^∞ is $\mathbb{M}$.*

Proof. Let P be the set of finite $S \subseteq \mathbb{M}^\times$ such that $1 \in S$. Then P is a commutative monoid with respect to the product $S \cdot S' = \{x \cdot y \mid x \in S, y \in S'\}$. For any $S \in P$ and $G \in \mathcal{P}_1$, define

$$G^S := \bigcap_{s \in S} s \cdot G.$$

Note that $(G^S)^{S'} = G^{S \cdot S'}$. If G is special and non-zero then by Lemma 11.21 G^S is special and non-zero, and there are inclusions $R_G \subseteq R_{G^S}$ and $I_G \subseteq I_{G^S}$. Define sets

$$R_J^\infty := \bigcup_{S \in P} R_{J^S}$$
$$I_J^\infty := \bigcup_{S \in P} I_{J^S}.$$

These sets are clearly M-invariant. Moreover, the unions are directed: given any S and S' we have

$$R_{J^S} \cup R_{J^{S'}} \subseteq R_{J^{S \cdot S'}}$$
$$I_{J^S} \cup I_{J^{S'}} \subseteq I_{J^{S \cdot S'}}.$$

Therefore R_J^∞ is a ring and I_J^∞ is an ideal. The fact that $(1 + I_J^\infty) \subseteq (R_J^\infty)^\times$ also follows (using Proposition 11.16.7). Taking $S = \{1\}$, we see that $I_J \subseteq I_J^\infty$. Proposition 11.16.5 says $I_M \subseteq I_J$, so $I_M \subseteq I_J^\infty$ as desired. Similarly, $M_0 \subseteq R_J \subseteq R_J^\infty$.

Claim 11.27. If $q_0, q_1, \ldots, q_r$ are distinct elements of M_0 and $\alpha \in \mathbb{M}^\times$, then at least one of $1/(\alpha - q_i)$ is in R_J^∞.

Proof. By Lemma 11.22, at least one of $1/(\alpha - q_i)$ lies in R_{J^S} for $S = \{1, \alpha, \ldots, \alpha^{r-1}\}$. $\square_{\text{Claim}}$

It follows formally that R_J^∞ is a Bézout domain with no more than r maximal ideals. Let a, b be two elements of R_J^∞. We claim that the ideal (a, b) is principal. This is clear if $a = 0$ or $b = 0$. Otherwise, let $\alpha = a/b$. As M_0 is infinite, Claim 11.27 implies that

$$\frac{b}{a - qb} = \frac{1}{\frac{a}{b} - q} \in R_J^\infty$$

for some $q \in M_0$. Then the principal ideal $(a - qb) \lhd R_J^\infty$ contains b, hence qb and thus a. Therefore $(a - qb) = (a, b)$.

Next, we show that R_J^∞ has at most r maximal ideals. Suppose for the sake of contradiction that there were distinct maximal ideals $\mathfrak{m}_0, \ldots, \mathfrak{m}_r$ in R_J^∞. As R_J^∞ is an M_0-algebra, each quotient $R_J^\infty/\mathfrak{m}_i$ is a field extending M_0. Take distinct $q_0, \ldots, q_r \in M_0$, and find an element $x \in R_J^\infty$ such that $x \equiv q_i \pmod{\mathfrak{m}_i}$ for each i, by the Chinese remainder theorem. Then $x - q_i \in \mathfrak{m}_i \subseteq R_J^\infty \setminus (R_J^\infty)^\times$ for each i. So $1/(x - q_i)$ does not lie in R_J^∞ for any $0 \le i \le r$, contrary to Claim 11.27.

Lastly, note that if x is any element of $\mathbb{M}^\times$, then $1/(x - q) \in R_J^\infty$ for some $q \in M_0$, $q \neq x$. As $q \in M_0 \subseteq R_J^\infty$, the field of fractions of R_J^∞ contains x. So the field of fractions must be all of $\mathbb{M}$. $\square$

11.4 From Bézout domains to valuation rings

Remark 11.28. Let R be a Bézout domain.

1. For each maximal ideal $\mathfrak{m}$, the localization $R_{\mathfrak{m}}$ is a valuation ring on the field of fractions of R.

2. R is the intersection of the valuation rings $R_{\mathfrak{m}}$.

See [2, VII, §2, Exercise 7a] and [2, II, §3, no. 3, Corollary 4], respectively.

Theorem 11.29. *Let $\mathbb{M}$ be a sufficiently saturated dp-finite field, possibly with extra structure. Suppose $\mathbb{M}$ is not of finite Morley rank. Then there is a small set $A \subseteq \mathbb{M}$ and a non-trivial A-invariant valuation ring.*

Proof. Take M_0 as usual in this section. By Proposition 11.4 there is a non-zero special $J \in \mathcal{P}_1$. The group J is type-definable over some small $M \supseteq M_0$. Let R be the R_J^∞ of Theorem 11.26. Then R is an M-invariant Bézout domain with at most r maximal ideals, the Jacobson radical of R is non-zero (because it contains I_M), and $\mathrm{Frac}(R) = \mathbb{M}$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ enumerate the maximal ideals of R. Let $\mathcal{O}_i$ be the localization $R_{\mathfrak{m}_i}$. By Remark 11.28, each $\mathcal{O}_i$ is a valuation ring on $\mathbb{M}$, and

$$R = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_k.$$

At least one $\mathcal{O}_i$ is non-trivial; otherwise $R = \mathbb{M}$ and has Jacobson radical 0.[14] Without loss of generality $\mathcal{O}_1$ is non-trivial. By the Chinese remainder theorem, choose $a \in R$ such that $a \equiv 1 \pmod{\mathfrak{m}_1}$ and $a \equiv 0 \pmod{\mathfrak{m}_i}$ for $i \neq 1$. We claim that $\mathcal{O}_1$ is $\mathrm{Aut}(\mathbb{M}/aM)$-invariant. If $\sigma \in \mathrm{Aut}(\mathbb{M}/aM)$, then $\sigma \in \mathrm{Aut}(\mathbb{M}/M)$ so σ preserves R setwise. It therefore permutes the finite set of maximal ideals. As $\mathfrak{m}_1$ is the unique maximal ideal not containing a, it must be preserved (setwise). Therefore σ preserves the localization $\mathcal{O}_1$ setwise. □

Remark 11.30. Stable fields do not admit non-trivial invariant valuation rings [15, Lemma 2.1]. Consequently, Theorem 11.29 can be used to give an extremely roundabout proof of Halevi and Palacín's theorem that stable dp-finite fields have finite Morley rank [12, Proposition 7.2].

12 Shelah conjecture and classification

Proposition 12.1. *Let K be a sufficiently saturated dp-finite field of positive characteristic. Then one of the following holds:*

- *K has finite Morley rank (and is therefore finite or algebraically closed).*

- *K admits a non-trivial henselian valuation.*

[14] Tracing through the proof, here is what explicitly happens. If $\varepsilon \in I_M$ then $-1/\varepsilon$ cannot be in R_J^∞, or else $\varepsilon \in I_M \subseteq I_J \subseteq I_{JS} \lhd R_{JS}$ and $-1/\varepsilon \in R_{JS}$ for large enough S, so that $-1 \in I_{JS}$, contradicting Proposition 11.16.7.

Proof. This is Fact 8.5 and Theorem 11.29. □

Lemma 12.2. *Let K be a sufficiently saturated dp-finite field of positive characteristic. Assume K is infinite. Let $\mathcal{O}_\infty$ be the intersection of all K-definable valuation rings on K. Then $\mathcal{O}_\infty$ is a henselian valuation ring on K whose residue field is algebraically closed.*

Proof. The proof for dp-minimal fields ([14, Theorem 9.5.7]) goes through without changes, using Proposition 12.1 together with [13, Theorems 2.6, 2.8]. Additionally, we must rule out the possibility that the residue field is real closed or finite. The first cannot happen because we are in positive characteristic. The second cannot happen because K is Artin-Schreier closed, a property which transfers to the residue field. □

Corollary 12.3. *Let K be a sufficiently saturated infinite dp-finite field of positive characteristic. If every definable valuation on K is trivial, then K is algebraically closed.*

Corollary 12.4. *Let K be a dp-finite field of positive characteristic. Then one of the following holds:*

- *K is finite.*
- *K is algebraically closed.*
- *K admits a non-trivial definable henselian valuation.*

Proof. Suppose K is neither finite nor algebraically closed. Let $K' \succeq K$ be a sufficiently saturated elementary extension. Then K' is neither finite nor algebraically closed. By Corollary 12.3 there is a non-trivial definable valuation $\mathcal{O} = \phi(K', a)$ on K'. The statement that $\phi(x; a)$ cuts out a valuation ring is expressed by a 0-definable condition on a, so we can take $a \in \operatorname{dcl}(K)$. Then $\phi(K, a)$ is a non-trivial valuation ring on K, henselian by [13, Theorem 2.8]. □

So the Shelah conjecture holds for dp-finite fields of positive characteristic.

By [11, Proposition 3.9, Remark 3.10, and Theorem 3.11], this implies the following classification of dp-finite fields of positive characteristic: up to elementary equivalence, they are exactly the Hahn series fields $\mathbb{F}_p((\Gamma))$ where Γ is a dp-finite p-divisible group. Dp-finite ordered abelian groups have been algebraically characterized and are the same thing as strongly dependent ordered abelian groups [5, 7, 10].

Acknowledgments. The author would like to thank

- Meng Chen, for hosting the author at Fudan University, where this research was carried out.
- Jan Dobrowolski, Yatir Halevi, and Françoise Point, who provided some helpful references.
- The UCLA model theorists, who read parts of this paper and pointed out typos.

- Two anonymous referees, who provided many helpful comments on this paper and [13].

This material is based upon work supported by the National Science Foundation under Award No. DMS-1803120. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

References

[1] Garrett Birkhoff. *Lattice Theory*, volume 25 of *Colloquium Publications*. American Mathematical Society, 1995.

[2] N. Bourbaki. *Algèbre Commutative*. Éléments de mathématique. Masson, 1985.

[3] Artem Chernikov, Itay Kaplan, and Pierre Simon. Groups and fields with NTP_2. *Proc. Amer. Math. Soc.*, 143:395–406, 2015.

[4] Jeremy E. Dawson. Independence spaces and uniform modules. *European Journal of Combinatorics*, 6:29–36, 1985.

[5] Alfred Dolich and John Goodrick. A characterization of strongly dependent ordered abelian groups. arXiv:1707.06344v1 [math.LO], 2017.

[6] Antonio J. Engler and Alexander Prestel. *Valued Fields*. Springer, 2005.

[7] Rafel Farré. Strong ordered abelian groups and dp-rank. arXiv:1706.05471v1 [math.LO], 2017.

[8] George Grätzer. *Lattice Theory: Foundation*. Birkhäuser, 2011.

[9] P. Grzeszcuk and E. R. Puczyłowski. On Goldie and dual Goldie dimension. *Journal of Pure and Applied Algebra*, 31:47–54, 1984.

[10] Yatir Halevi and Assaf Hasson. Strongly dependent ordered abelian groups and henselian fields. arXiv:1706.03376v3 [math.LO], 2017.

[11] Yatir Halevi, Assaf Hasson, and Franziska Jahnke. A conjectural classification of strongly dependent fields. *Bulletin of Symbolic Logic*, 25(2):182–195, June 2019.

[12] Yatir Halevi and Daniel Palacín. The dp-rank of abelian groups. *Journal of Symbolic Logic*, 84:957–986, September 2019.

[13] Will Johnson. Dp-finite fields I(A): the infinitesimals. *This journal*.

[14] Will Johnson. *Fun with Fields*. PhD thesis, University of California, Berkeley, 2016. Available at `https://math.berkeley.edu/~willij/drafts/will-thesis.pdf`.

[15] Will Johnson. Dp-finite fields II: the canonical topology and its relation to henselianity. arXiv:1910.05932v2 [math.LO], October 2019.

[16] Will Johnson. Dp-finite fields III: inflators and directories. arXiv:1911.04727v1 [math.LO], November 2019.

[17] Will Johnson. Dp-finite fields IV: the rank 2 picture. arXiv:2003.09130v1 [math.LO], March 2020.

[18] Mike Prest. *Model Theory and Modules*. Number 130 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1988.

[19] E. R. Puczyłowski. On some dimesions of modular lattices and matroids. In Gary F. Birkenmeier, Jae Keol Park, and Young Soo Park, editors, *International Symposium on Ring Theory*, pages 303–312. Birkhäuser, 1999.

[20] Pierre Simon. *A guide to NIP theories*. Lecture Notes in Logic. Cambridge University Press, July 2015.

[21] D. J. A. Welsh. *Matroid Theory*. Number 8 in London Mathematical Society Monographs. Academic Press, 1976.