A criterion for uniform finiteness in the imaginary sorts

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Abstract

Let T be a theory. If T eliminates $\exists^\infty$, it need not follow that T^{eq} eliminates $\exists^\infty$, as shown by the example of the p-adics. We give a criterion to determine whether T^{eq} eliminates $\exists^\infty$. Specifically, we show that T^{eq} eliminates $\exists^\infty$ if and only if $\exists^\infty$ is eliminated on all interpretable sets of "unary imaginaries." This criterion can be applied in cases where a full description of T^{eq} is unknown. As an application, we show that T^{eq} eliminates $\exists^\infty$ when T is a C-minimal expansion of ACVF.

1 Introduction

1.1 Conventions

"Definable" will refer to definable subsets of M^n, while "interpretable" will refer to the more general notion. "Definable" and "interpretable" will always mean definability and interpretability with parameters from M.

A *definable family* is a family of the form $\{D_a\}_{a\in Y}$, where Y is a definable set and $D_a = \{b \in X : (a,b) \in D\}$ for some definable sets X and $D \subseteq Y \times X$. We can similarly talk about interpretable families of interpretable sets. An *interpretable collection* of interpretable sets is a set of the form $\mathcal{D} = \{D_a : a \in Y\}$ for some interpretable family $\{D_a\}_{a\in Y}$. Replacing Y with a quotient, we can always ensure that the map $a \mapsto D_a$ is a bijection from Y to $\mathcal{D}$.

In particular, an "interpretable collection" is a different type of object from an "interpretable family." An interpretable collection is a set of sets, and an interpretable family is a function from a set to the universe of sets.

1.2 Elimination of $\exists^\infty$

Definition 1.1. Let X be a definable or interpretable set in an $\aleph_0$-saturated structure. Say that $\exists^\infty$ *is eliminated on* X if for every interpretable family $\{D_a\}_{a\in Y}$ of subsets of X, the following (equivalent) conditions hold:

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1. The set $\{a \in Y : |D_a| = \infty\}$ is interpretable.

2. There is an $n \in \mathbb{N}$ such that for all $a \in Y$,

$$|D_a| = \infty \iff |D_a| > n.$$

In a non-saturated structure M, we use Condition 2, which is invariant under elementary extensions, and stronger than Condition 1. In other words, we say that "$\exists^\infty$ is eliminated on X" if this holds in an $\aleph_0$-saturated elementary extension $M^* \succeq M$. This is a slight abuse of terminology.

Definition 1.2. A theory T has *uniform finiteness* if $\exists^\infty$ is eliminated on every definable set. We also say that T *eliminates* $\exists^\infty$.

In a 1-sorted theory, uniform finiteness is equivalent to elimination of $\exists^\infty$ on the home sort, by the following observation.

Observation 1.3. If $\exists^\infty$ is eliminated on X and Y, it is eliminated on $X \times Y$. In fact, $S \subseteq X \times Y$ is finite if and only if both of the projections $S \to X$ and $S \to Y$ have finite image.

Example 1.4. If $(M, \leq, +)$ is a dense o-minimal structure, then M eliminates $\exists^\infty$. Indeed, a definable set $X \subseteq M$ is infinite if and only if X has non-empty interior.

Example 1.5. If $(K, +, \cdot)$ is a p-adically closed field, such as $\mathbb{Q}_p$, then K eliminates $\exists^\infty$. In fact, a definable set $X \subseteq K$ is infinite if and only if it has non-empty interior, by work of Macintyre [6].

Example 1.6. The ordered abelian group $(\mathbb{Z}, \leq, +)$ does not eliminate $\exists^\infty$, because there is no uniform bound on the size of the finite intervals $[1, n]$.

2 When does T^{eq} eliminate $\exists^\infty$?

Uniform finiteness does not pass from T to T^{eq}. In other words, $\exists^\infty$ can be eliminated on definable sets without being eliminated on interpretable sets. This happens in $\mathbb{Q}_p$, which interprets $(\mathbb{Z}, \leq, +)$ as the value group.

In many theories, it is difficult to fully characterize interpretable sets. For example, in the theory of algebraically closed valued fields (ACVF), the classification of interpretable sets is rather complicated [1]. Moreover, this classification fails to generalize to C-minimal expansions of ACVF [2].

In Theorem 2.3, we will give a relatively simple criterion which can be used to show that T^{eq} eliminates $\exists^\infty$ without first characterizing interpretable sets. As an application, we will show that T^{eq} eliminates $\exists^\infty$ when T is a C-minimal expansion of ACVF.

Assume henceforth that T is one-sorted (but see Remark 2.8).

Definition 2.1. In a model $M \models T$, a *unary* definable set is a definable subset of $M = M^1$.

Definition 2.2. An interpretable set X is a *set of unary imaginaries* if there is an interpretable relation $R \subseteq X \times M$ such that the following map is an injection:

$$x \mapsto R_x := \{m \in M : (x, m) \in R\}.$$

In other words, X is a set of unary imaginaries if the elements of X are codes for unary definable sets, in some uniform way.

Theorem 2.3. *Suppose that $\exists^\infty$ is eliminated on every set of unary imaginaries. Then T^{eq} eliminates $\exists^\infty$.*

Proof. Let $M_0 \models T$ be a model. Let N_0 be the expansion of M_0^{eq} by a new sort $\mathbb{N} \cup \{\infty\}$ and functions

$$\begin{aligned} Y &\to \mathbb{N} \cup \{\infty\} \\ a &\mapsto |D_a| \end{aligned}$$

for every interpretable family $\{D_a\}_{a \in Y}$ in M_0^{eq}. Let $N = (M^{\text{eq}}, \mathbb{N}^* \cup \{\infty\})$ be an $\aleph_0$-saturated elementary extension of N_0.

Then $\mathbb{N}^*$ is an $\aleph_0$-saturated elementary extension of $\mathbb{N}$, M is an $\aleph_0$-saturated model of T, and every interpretable set X in M has a non-standard "size"

$$|X| \in \mathbb{N}^* \cup \{\infty\}.$$

Say that X is *pseudofinite* if $|X|$ is less than the symbol ∞. (In particular, finite sets are pseudofinite.) It suffices to show that every pseudofinite interpretable set is finite, because of the $\aleph_0$-saturation of N.

Say that an interpretable set X in M is *wild* if there is an infinite pseudofinite interpretable collection of subsets of X. Otherwise, say X is *tame*. By assumption, $\exists^\infty$ is eliminated on sets of unary imaginaries. Therefore, every pseudofinite set of unary imaginaries is finite. Equivalently, M^1 is tame.

Claim 2.4. If X is tame, so is any interpretable subset of X. If X and Y are tame, then so is $X \cup Y$.

Proof. The first statement is trivial. For the second statement, let $\mathcal{D}$ be a pseudofinite interpretable collection of subsets of $X \cup Y$. Note that $\{D \cap X : D \in \mathcal{D}\}$ is

- *pseudofinite*, because $\mathcal{D}$ is pseudofinite, and
- *finite*, because X is tame

Similarly, $\{D \cap Y : D \in \mathcal{D}\}$ is finite. Finally, the map

$$D \mapsto (D \cap X, D \cap Y)$$

yields an injection from $\mathcal{D}$ into a product of two finite sets. Thus $\mathcal{D}$ is finite. $\square_{\text{Claim}}$

Claim 2.5. Let $\pi : X \to Y$ be an interpretable map with finite fibers. If Y is tame, then so is X.

Proof. By saturation, there is a uniform upper bound k on the size of the fibers. We proceed by induction on k. The base case $k = 1$ is trivial. Suppose $k > 1$. Let $\mathcal{D}$ be a pseudofinite interpretable collection of subsets of X. Let

$$\mathcal{E} = \{\pi(D) : D \in \mathcal{D}\}$$

and

$$\mathcal{F} = \{\pi(X \setminus D) : D \in \mathcal{D}\}$$

Then $\mathcal{E}$ and $\mathcal{F}$ are both pseudofinite interpretable collections of subsets of Y. By tameness of Y, they are both finite.

It remains to show that the fibers of $\mathcal{D} \to \mathcal{E} \times \mathcal{F}$ are finite. Replacing $\mathcal{D}$ with such a fiber, we may assume that $\pi(D)$ and $\pi(X \setminus D)$ are independent of D, as D ranges over $\mathcal{D}$. Let $U = \pi(D)$ and $V = \pi(X \setminus D)$ for any/every $D \in \mathcal{D}$. Let $Y' = U \cap V$ and $X' = \pi^{-1}(Y')$. Then the map $D \mapsto D \cap X'$ is injective on $\mathcal{D}$, because every element D of $\mathcal{D}$ contains $\pi^{-1}(U \setminus V)$ and is disjoint from $\pi^{-1}(V \setminus U)$. So it suffices to show that X' is tame. Let D be some arbitrary element of $\mathcal{D}$. Then $X' \cap D$ and $X' \setminus D$ each intersect every fiber of $X' \to Y'$, by choice of X'. In particular, the two maps

$$X' \cap D \to Y'$$

$$X' \setminus D \to Y'$$

have finite fibers of size less than k. By Claim 2.4, Y' is tame, and by induction, $X' \cap D$ and $X' \setminus D$ are tame. By Claim 2.4, X' is tame. $\square_{\text{Claim}}$

Claim 2.6. Suppose that $\pi : X \to Y$ is an interpretable surjection with finite fibers. Suppose that Y is tame. Let $\mathcal{F}$ be an interpretable collection of sections of π. If $\mathcal{F}$ is pseudofinite, then $\mathcal{F}$ is finite.

Proof. A section is determined by its image. $\square_{\text{Claim}}$

Claim 2.7. Suppose X and Y are tame. Then so is $X \times Y$.

Proof. Let $\mathcal{D}$ be a pseudofinite interpretable collection of subsets of $X \times Y$. For each $a \in X$, the set $\{a\} \times Y \subseteq X \times Y$ is tame, so the collection

$$\mathcal{E}_a := \{D \cap (\{a\} \times Y) : D \in \mathcal{D}\}$$

is finite. Then

$$\pi : \coprod_{a \in X} \mathcal{E}_a \to X$$

is an interpretable surjection with finite fibers. Each element $D \in \mathcal{D}$ induces a section of π, namely, the map σ_D sending a point $a \in X$ to (the code for) $D \cap (\{a\} \times Y)$. This gives an interpretable injection from $\mathcal{D}$ to sections of π. By Claim 2.6 and the fact that X is tame, it follows that $\mathcal{D}$ is finite. $\square_{\text{Claim}}$

It follows that M^n is tame for all $n \geq 1$. Now if Y is any interpretable set, then Y is a set of codes of subsets of M^n, for some n. By tameness of M^n, it follows that if Y is pseudofinite, then Y is finite. This completes the proof of Theorem 2.3. □

Remark 2.8. Theorem 2.3 can be easily generalized to multi-sorted theories. The necessary and sufficient criterion for T^{eq} to eliminate $\exists^\infty$ is that for each sort X, $\exists^\infty$ is eliminated on (the indexing set of) any interpretable collection of subsets of X. More precisely, if $\{D_a\}_{a \in Y}$ is an interpretable family of subsets of X and the map $a \mapsto D_a$ is injective, then $\exists^\infty$ is eliminated on Y. In the language of the proof of Theorem 2.3, this implies that each sort is tame, which ensures that every definable set is tame.

3 C-minimal expansions of ACVF

As an example, we apply Theorem 2.3 to C-minimal expansions of ACVF.[1] Let T be a C-minimal expansion of ACVF, and K be a sufficiently saturated model of T. As in the proof of Theorem 2.3, work in a setting with nonstandard counting functions.

Observation 3.1. Let $B_1, \ldots, B_n$ be pairwise disjoint balls in K. Then the union $\bigcup_{i=1}^n B_i$ cannot be written as a boolean combination of fewer than n balls.

This follows from uniqueness of the swiss-cheese decomposition; see [4, Proposition 3.23, Theorem 3.26].

Lemma 3.2. *There is no pseudofinite infinite interpretable collection of pairwise disjoint balls.*

Proof. Let $\mathcal{S}$ be such a set. By compactness, there must be some sequence $\mathcal{S}_1, \mathcal{S}_2, \ldots$ such that each $\mathcal{S}_i$ is a finite set of pairwise disjoint balls, the $\mathcal{S}_i$ are uniformly interpretable (bounded in complexity), and $\lim_{i \to \infty} |\mathcal{S}_i| = \infty$.

The unions $U_i = \bigcup \mathcal{S}_i \subseteq K$ are uniformly definable (bounded in complexity), so there is some absolute bound on the number of balls needed to express U_i. But Observation 3.1 says that this number is at least $|\mathcal{S}_i|$, a contradiction. □

C-minimality implies that the value group Γ is densely o-minimal [3, Lemma 2.7(i)]. Therefore $\exists^\infty$ is eliminated in Γ, and there are no pseudofinite infinite interpretable subsets of Γ.

Lemma 3.3. *There is no pseudofinite infinite interpretable collection of balls.*

[1] An expansion T of ACVF is *C-minimal* if, in any model $M \models T$, every definable set $X \subseteq M^1$ is a finite boolean cobination of balls. See [7] for more information on C-minimality. The theory ACVF is C-minimal by Theorem 4.11 in [7]. Certain expansions of ACVF by analytic functions are shown to be C-minimal in [5].

Proof. Let $\mathcal{S}$ be such a set. Let $\mathcal{S}_0$ be the set of minimal elements of $\mathcal{S}$. For each $B \in \mathcal{S}_0$, let $\mathcal{S}_B$ denote the elements of $\mathcal{S}$ containing B. In a pseudofinite poset, every element is greater than or equal to a minimal element, so

$$\mathcal{S} = \bigcup_{B \in \mathcal{S}_0} \mathcal{S}_B.$$

The set $\mathcal{S}_0$ is pseudofinite, hence finite by Lemma 3.2. Therefore, $\mathcal{S}_B$ is infinite for some B.

Now $\mathcal{S}_B$ is a chain of balls. Let $\rho : \mathcal{S}_B \to \Gamma$ be the map sending a ball to its radius. This map is nearly injective; the fibers have size at most 2. The range of ρ is pseudofinite, hence finite. Therefore, the domain $\mathcal{S}_B$ is finite, a contradiction. □

Finally, suppose that $\exists^\infty$ is not eliminated on some set X_0 of unary imaginaries. Then there is a pseudofinite infinite set $A \subseteq X_0$. Let D_a be the unary set associated to $a \in A$. Note that $a \mapsto D_a$ is injective.

For each a, there is a unique minimal set of balls $\mathcal{B}_a$ such that D_a can be written as a boolean combination of $\mathcal{B}_a$. The correspondence $a \mapsto \mathcal{B}_a$ is an interpretable finite-to-finite correspondence from A to the set $\mathcal{B}$ of balls. Let I denote the "image" of this correspondence:

$$I := \bigcup_{a \in A} \mathcal{B}_a.$$

The set $I \subseteq \mathcal{B}$ is pseudofinite, hence finite by Lemma 3.3. The boolean algebra generated by I is finite, and contains every D_a. By injectivity of $a \mapsto D_a$, the set A is finite, a contradiction.

By Theorem 2.3, we have proven the following:

Proposition 3.4. *T^{eq} eliminates $\exists^\infty$ when T is a C-minimal expansion of ACVF.*

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