Michigan Math. J. 70 (2021), 837–856

Equimultiplicity Theory of Strongly F-Regular Rings

Thomas Polstra & Ilya Smirnov

Abstract. We explore the equimultiplicity theory of the F-invariants Hilbert–Kunz multiplicity, F-signature, Frobenius Betti numbers, and Frobenius Euler characteristic in strongly F-regular rings. Techniques introduced in this paper provide a unified approach to the study of localization of these invariants and detection of singularities.

1. Introduction

The most intrinsic feature of a ring R of prime characteristic $p > 0$ is the Frobenius endomorphism given by taking the p-powers, $x \mapsto x^p$. Let $F_*^e R$ be the R-module obtained by restricting scalars along the eth Frobenius endomorphism. For simplicity, we assume that $(R, \mathfrak{m}, k)$ is local and F-finite, meaning that R is a local ring and $F_*^e R$ is a finitely generated R-module for each $e \in \mathbb{N}$. At the root of prime characteristic commutative algebra and algebraic geometry is Kunz's fundamental result characterizing flatness of the Frobenius endomorphism.

Theorem 1.1 ([Kun69]). *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic p. Then R is regular if and only if $F_*^e R$ is a free R-module for some (equivalently, all) $e \in \mathbb{N}$.*

Motivated by Kunz's theorem, it is natural to study nonregular prime characteristic rings by studying algebraic, geometric, and homological properties of the family of R-modules $\{F_*^e R\}_{e \in \mathbb{N}}$, which distinguish R from a regular local ring. We consider the following measurements:

(1) $\mu(F_*^e R)$, the minimal number of generators of $F_*^e R$ as an R-module;
(2) $a_e(R)$, the largest rank of a free summand of $F_*^e R$;
(3) $\beta_i^e(R) := \dim_k(\operatorname{Tor}_i^R(k, F_*^e R))$, the ith Betti number of $F_*^e R$;
(4) $\chi_i^e(R) := \sum_{j=0}^{i} (-1)^j \beta_{i-j}^e(R)$.

The asymptotic ratio of these numbers, as compared with the rank of $F_*^e R$, produces several interesting and important numerical invariants unique to rings of prime characteristic:

(1) Hilbert–Kunz multiplicity $\mathrm{e}_{\mathrm{HK}}(R) = \lim_{e\to\infty} \mu(F_*^e R)/\operatorname{rank}(F_*^e R)$ [Mon83].
(2) F-signature $\mathrm{s}(R) = \lim_{e\to\infty} a_e(R)/\operatorname{rank}(F_*^e R)$ [HL02; SvdB97; Tuc12].
(3) The ith Frobenius Betti number $\beta_i^F(R) = \lim_{e\to\infty} \beta_i^e(R)/\operatorname{rank}(F_*^e R)$ [AL08].

Received July 1, 2019. Revision received October 11, 2019.
Polstra was supported in part by NSF Postdoctoral Research Fellowship DMS #1703856.

(4) The ith Frobenius Euler characteristic $\chi_i^F(R) = \lim_{e\to\infty} \chi_i^e(R)/\operatorname{rank}(F_*^e R)$ [dSPY18].

This paper concerns the equimultiplicity theory of these numerical invariants, a topic initiated by the second author in [Smi19]. Specifically, we are interested in understanding when the above measurements are unchanged under localization. Our main result in this direction is the following:

THEOREM A. *Let $(R, \mathfrak{m})$ be an F-finite and strongly F-regular local ring of prime characteristic p, and let $P \in \operatorname{Spec}(R)$. Let $\{\nu_e(R)\}_{e\in\mathbb{N}}$ be one of the following sequences of numbers*:

- $\{\mu(F_*^e R)\}_{e\in\mathbb{N}}$,
- $\{a_e(R)\}_{e\in\mathbb{N}}$,
- $\{\beta_i^e(R)\}_{e\in\mathbb{N}}$,
- $\{\chi_i^e(R)\}_{e\in\mathbb{N}}$.

Let $\nu(R) = \lim_{e\to\infty} \nu_e(R)/\operatorname{rank}(F_*^e R)$. Then the following are equivalent:

(1) $\nu(R) = \nu(R_P)$;
(2) *For each $e \in \mathbb{N}$, $\nu_e(R) = \nu_e(R_P)$.*

In the scenario that $\{\nu_e(R)\}_{e\in\mathbb{N}}$ is the sequence of numbers $\{\mu(F_*^e R)\}_{e\in\mathbb{N}}$, Theorem A is a significant improvement of [Smi19, Corollary 5.18], where the same theorem was proven under the additional assumption that R/P is a regular local ring.

It has been known for some time that the Hilbert–Kunz multiplicity, F-signature, and Frobenius Betti numbers serve as measurements of singularities; see [WY00; HY02; HL02; AL03], and [AL08], respectively. The Frobenius Euler characteristic was developed in [dSPY18] as a tool to prove that the functions $\beta_i^F : \operatorname{Spec}(R) \to \mathbb{R}$ sending $P \mapsto \beta_i^F(R_P)$ are upper semicontinuous, and it was unclear from those techniques whether or not the Frobenius Euler characteristic could be used to detect regular rings. Prior to this paper, only the first Frobenius Euler characteristic was proven to serve as a measurement of singularity; see [Li08, Main Theorem (iv)]. In the present paper, we show that the Frobenius Euler characteristic does indeed serve as a measurement of singularities under an additional hypothesis.

THEOREM B. *Let $(R, \mathfrak{m})$ be an F-finite and strongly F-regular local ring of prime characteristic p. Then the following are equivalent*:

(1) *R is a regular local ring*;
(2) $\chi_i^e(R) = (-1)^i \operatorname{rank}(F_*^e R)$ *for every $e \in \mathbb{N}$*;
(3) $\chi_i^e(R) = (-1)^i \operatorname{rank}(F_*^e R)$ *for some $e \in \mathbb{N}_{\geq 1}$*;
(4) $\chi_i^F(R) = (-1)^i$.

We conjecture that Theorem B can be proven under weaker hypotheses. It seems likely that, similarly to the Hilbert–Kunz multiplicity and Frobenius Betti numbers, we would only need to assume that the completion of the local ring R at the maximal ideal has no low-dimensional components in order to know that $\chi_i^F(R) = (-1)^i$ implies that R is regular.

The F-signature of a local ring $(R, \mathfrak{m}, k)$ can be studied through splitting ideals, a notion originating in [AE06]. For each $e \in \mathbb{N}$, the eth splitting ideal is

$$I_e = \{r \in R \mid \phi(F_*^e r) \in \mathfrak{m}, \forall \phi \in \operatorname{Hom}_R(F_*^e R, R)\},$$

and the F-signature of R is realized as the limit $\mathrm{s}(R) = \lim_{e\to\infty} \lambda(R/I_e)/p^{e \dim(R)}$. To better understand the behavior of F-signature under localizations, we consider relative splitting ideals: for each ideal $I \subseteq R$ and $e \in \mathbb{N}$, let

$$I_e(I) = \{r \in R \mid \phi(F_*^e r) \in I, \forall \phi \in \operatorname{Hom}_R(F_*^e R, R)\}.$$

Observe that $I_e(\mathfrak{m}) = I_e$. If $I \subseteq R$ is an $\mathfrak{m}$-primary ideal, then we can define the F-signature relative to the ideal I as $\mathrm{s}(I) = \lim_{e\to\infty} \lambda(R/I_e(I))/p^{e \dim(R)}$ (the limit, as we will observe, exists). Relative splitting ideals allow us to understand the behavior of F-signature in the context of Theorem B and we prove the following limit formula for F-signature, which is of independent interest. The corresponding formula for the Hilbert–Samuel multiplicity is a classic and very useful result of Lech [Lec57]; the version for the Hilbert–Kunz multiplicity can be found in [Smi19, Proposition 5.4] and was extensively used therein.

THEOREM C. *Let $(R, \mathfrak{m})$ be an F-finite local domain of prime characteristic p. Suppose that $I \subseteq R$ is an ideal such that R/I is Cohen–Macaulay of dimension h and $\underline{x} = x_1, \ldots, x_h$ is a parameter sequence on R/I. Then*

$$\lim_{n_1,\ldots,n_h\to\infty} \frac{1}{n_1 \cdots n_h} \mathrm{s}(I, (x_1^{n_1}, \ldots, x_h^{n_h})) = \sum_P \mathrm{e}(x_1, \ldots, x_h; R/P)\, \mathrm{s}(IR_P),$$

where the sum is taken over all prime ideals $P \supseteq I$ such that $\dim(R/I) = \dim(R/P)$.

Section 2 contains background results and basic properties of splitting ideals relative to an ideal. The proofs of Theorems A and B can be found in Section 3. We also use Section 3 to further explore the behavior of splitting ideals. For example, see Theorem 3.9 for a proof that $\operatorname{depth}(R/P) = \operatorname{depth}(R/I_e(P))$ whenever P is a prime ideal of a strongly F-regular local ring satisfying $\mathrm{s}(R) = \mathrm{s}(R_P)$. Section 4 is devoted to proving Theorem C.

2. Preliminary Results

2.1. Hilbert–Kunz Multiplicity

Monsky's introduction of Hilbert–Kunz multiplicity is a continuation of Kunz's work on prime characteristic rings in [Kun69; Kun76].

DEFINITION 2.1. Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p, and let I be an $\mathfrak{m}$-primary ideal. Denote $I^{[p^e]} = (x^{p^e} \mid x \in I)$. Then the Hilbert–Kunz multiplicity of I is

$$\mathrm{e}_{\mathrm{HK}}(I)=\lim_{e\to\infty}\frac{\lambda(R/I^{[p^e]})}{p^{e\dim R}}.$$

The Hilbert–Kunz multiplicity of a local ring $(R,\mathfrak{m},k)$ is the Hilbert–Kunz multiplicity of the maximal ideal $\mathfrak{m}$ and is denoted by $\mathrm{e}_{\mathrm{HK}}(R)$. If R is an F-finite domain, then $\operatorname{rank}(F_*^eR)=p^{e\dim R}[k:k^{p^e}]$ by [Kun76, Proposition 2.3], and therefore

$$\mathrm{e}_{\mathrm{HK}}(R)=\lim_{e\to\infty}\frac{\lambda(R/\mathfrak{m}^{[p^e]})}{p^{e\dim(R)}}=\lim_{e\to\infty}\frac{\mu(F_*^eR)}{\operatorname{rank}(F_*^eR)}.$$

Hence the definition of Hilbert–Kunz multiplicity presented in the Introduction agrees with Definition 2.1.

2.2. *F-Signature and Splitting Ideals*

The following definition, shown to us by Kevin Tucker, is a natural generalization of the splitting ideals and presents a natural extension of F-signature.

DEFINITION 2.2. Let R be an F-finite ring, and let $\mathfrak{a}$ be an ideal. The eth splitting ideal of $\mathfrak{a}$ is defined as

$$I_e(\mathfrak{a})=\{r\in R\mid \phi(F_*^er)\in\mathfrak{a},\forall\phi\in\operatorname{Hom}_R(F_*^eR,R)\}.$$

We record the following basic properties concerning splitting ideals, many of which mimic the behavior of the standard splitting ideals $I_e=I_e(\mathfrak{m})$.

LEMMA 2.3. *Suppose $(R,\mathfrak{m})$ is an F-finite local ring of prime characteristic p and Krull dimension d. Let $\mathfrak{a}\subset R$ be an ideal. Then the sequence of ideals $\{I_e(\mathfrak{a})\}$ satisfies the following properties*:

(1) $I_e(\mathfrak{a})$ *is an ideal*;
(2) $\mathfrak{a}^{[p^e]}\subseteq I_e(\mathfrak{a})$;
(3) $I_e(\mathfrak{a})^{[p]}\subseteq I_{e+1}(\mathfrak{a})$;
(4) $\phi(F_*^{e_0}I_{e+e_0}(\mathfrak{a}))\subseteq I_e(\mathfrak{a})$ *for all* $e,e_0\in\mathbb{N}$, *and* $\phi\in\operatorname{Hom}_R(F_*^{e_0}R,R)$;
(5) *If* $\mathfrak{a}$ *is* $\mathfrak{m}$*-primary, then the limit* $\mathrm{s}(\mathfrak{a}):=\lim_{e\to\infty}\frac{\lambda(R/I_e(\mathfrak{a}))}{p^{ed}}$ *exists, and* $\lambda(R/I_e(\mathfrak{a}))=\mathrm{s}(\mathfrak{a})p^{ed}+O(p^{e(d-1)})$. *The value* $\mathrm{s}(\mathfrak{a})$ *is referred to as the F-signature of* $\mathfrak{a}$;
(6) *If W is a multiplicative set, then* $I_e(\mathfrak{a})R_W=I_e(\mathfrak{a}R_W)$;
(7) $I_e(\mathfrak{a}:J)=I_e(\mathfrak{a}):J^{[p^e]}$ *for all ideals* J;
(8) *If P is a prime ideal, then $I_e(P)$ is P-primary*;
(9) *If* $x\in R$ *is a regular element of* $R/\mathfrak{a}$, *then x is a regular element of* $R/I_e(\mathfrak{a})$ *for every* $e\in\mathbb{N}$;
(10) *If R is a regular local ring, then* $I_e(\mathfrak{a})=\mathfrak{a}^{[p^e]}$ *for every* $e\in\mathbb{N}$;
(11) *If* $\mathfrak{b}\subseteq R$ *is an ideal and* $\mathfrak{a}\subseteq\mathfrak{b}$, *then* $I_e(\mathfrak{a})\subseteq I_e(\mathfrak{b})$;

Proof. The proofs of (1)–(4) are straightforward and are left to the reader. Statement (5) then follows by [PT18, Corollary 4.5]. To prove (6), it is enough to observe that $\operatorname{Hom}_R(F^e_* R, R)_W \cong \operatorname{Hom}_{R_W}(F^e_* R_W, R_W)$. For (7), we note that $a \in I_e(\mathfrak{a} : J)$ if and only if for all $\phi \in \operatorname{Hom}_R(F^e_* R, R)$, we have $\phi(F^e_* a) \in (\mathfrak{a} : J)$ or, equivalently, $\phi(F^e_* J^{[p^e]} a) = J\phi(F^e_* a) \subseteq \mathfrak{a}$. Statements (8) and (9) easily follow from (7). Observation (10) follows from Theorem 1.1; if $F^e_* R$ is free, then we easily see that $F^e_* I_e(\mathfrak{a}) = \mathfrak{a} F^e_* R$, from which it follows that $I_e(\mathfrak{a}) = \mathfrak{a}^{[p^e]}$. Property (11) is trivial. □

COROLLARY 2.4. *Let $(R, \mathfrak{m})$ be an F-finite and F-pure local ring of prime characteristic p. If $J \subsetneq I$ are ideals, then $I_e(J) \subsetneq I_e(I)$. Moreover, if R is strongly F-regular and J, I are $\mathfrak{m}$-primary, then $\mathrm{s}(J) > \mathrm{s}(I)$.*

Proof. Without loss of generality, we may assume that $I = (J, x)$ and $x \notin J$. Then $I_e(J) : x^{p^e} = I_e(J : x) \subseteq I_e(\mathfrak{m}) \neq R$, so $I_e(J) \subsetneq I_e(J) + (x^{p^e}) \subseteq I_e(J, x)$.

For the second part, observe that

$$\lambda\left(\frac{I_e(J, x)}{I_e(J)}\right) \geq \lambda\left(\frac{I_e(J) + (x^{p^e})}{I_e(J)}\right) = \lambda(R/(I_e(J) : x^{p^e})) \geq \lambda(R/I_e(\mathfrak{m})).$$

Therefore $\mathrm{s}(J) - \mathrm{s}((J, x)) \geq \mathrm{s}(R) > 0$. □

Similarly to the usual F-signature, the F-signature of an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ is realized as the limit of normalized Hilbert–Kunz multiplicities of the ideals $I_e(\mathfrak{a})$.

THEOREM 2.5. *Let $(R, \mathfrak{m})$ be an F-finite and reduced local ring of prime characteristic p, and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Then*

$$\mathrm{s}(\mathfrak{a}) = \lim_{e \to \infty} \frac{\mathrm{e}_{\mathrm{HK}}(I_e(\mathfrak{a}))}{p^{e \dim(R)}}.$$

Proof. The statement is equivalent to saying that

$$\lim_{e \to \infty} \frac{1}{p^{e \dim R}} |\lambda(R/I_e(\mathfrak{a})) - \mathrm{e}_{\mathrm{HK}}(I_e(\mathfrak{a}))| = 0,$$

which is the content of [Tuc12, Corollary 3.7]. □

3. Equimultiplicity of F-Invariants

We are interested in understanding when the invariants Hilbert–Kunz multiplicity, F-signature, Frobenius Betti numbers, and Frobenius Euler characteristics are unchanged under localization. The second author began this study in [Smi19], where the following was proven.

THEOREM 3.1 ([Smi19, Corollary 5.16]). *Let $(R, \mathfrak{m})$ be an excellent weakly F-regular local ring of prime characteristic p, and let $P \subset R$ be a prime ideal such that R/P is a regular local ring. Then the following are equivalent:*

(1) $\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(R_P)$,
(2) *for each $e \in \mathbb{N}$,* $\lambda(R/\mathfrak{m}^{[p^e]})/p^{e \dim(R)} = \lambda(R_P/P^{[p^e]}R_P)/p^{e \operatorname{ht}(P)}$.

The techniques surrounding Theorem 3.1 involve a careful and challenging analysis of the behavior of the ideals $\{P^{[p^e]}\}$ and $\{(P,\underline{x})^{[p^e]}\}$, where $\underline{x}$ is a regular system of parameters modulo P. Using elementary techniques, we recover the theorem without assuming that R/P is a regular local ring, but we do replace the assumption of weak F-regularity with the conjecturally equivalent assumption that R is strongly F-regular. Our techniques stem from a novel, yet simple, observation that if a module M is a direct summand of $F_*^{e_0}R$ for some $e_0 \in \mathbb{N}$ and R is strongly F-regular, then asymptotically there will be many direct summands of F_*^eR isomorphic to M as $e \to \infty$. To make this precise, we begin with some notation.

NOTATION 3.2. Let R be a ring, and let $N \subseteq M$ be finitely generated R-modules. Let $\operatorname{rank}_N(M)$ denote the maximal number of N-summands appearing in all possible direct sum decompositions of M.

The following lemma should be compared with [SvdB97, Proposition 3.3.1].

LEMMA 3.3. *Let $(R,\mathfrak{m})$ be an F-finite and strongly F-regular local ring. Suppose M is a finitely generated R-module such that* $\operatorname{rank}_M(F_*^{e_0}R) > 0$ *for some $e_0 \in \mathbb{N}$. Then*

$$\liminf_{e\to\infty} \frac{\operatorname{rank}_M(F_*^eR)}{\operatorname{rank}(F_*^eR)} > 0.$$

Proof. Suppose that $F_*^{e_0}R \cong M \oplus N$. For each $e \in \mathbb{N}$, write $F_*^eR \cong R^{\oplus a_e(R)} \oplus M_e$. Then $F_*^{e+e_0}R \cong F_*^{e_0}R^{\oplus a_e(R)} \oplus F_*^{e_0}M_e$, and it follows that $M^{\oplus a_e(R)}$ is a direct summand of $F_*^{e+e_0}R$. In particular, $\operatorname{rank}_M(F_*^{e+e_0}R) \geq a_e(R)$, and therefore

$$\liminf_{e\to\infty} \frac{\operatorname{rank}_M(F_*^eR)}{\operatorname{rank}(F_*^eR)} \geq \liminf_{e\to\infty} \frac{a_{e-e_0}(R)}{\operatorname{rank}(F_*^eR)} = \frac{\mathrm{s}(R)}{\operatorname{rank}(F_*^{e_0}R)} > 0. \qquad \square$$

3.1. F-Signature and Splitting Ideals

We are prepared to present a proof of Theorem A for F-signature. But we first have the following:

REMARK 3.4. To make full use of Lemma 3.3 in the following theorem, we remind the reader that the maximal rank of a free summand of a finitely generated module M over a local ring R is invariant of a choice of a direct sum decomposition. This is because R is a direct summand of M if and only if $\widehat{R}$ is a direct summand of $\widehat{M}$ and a complete local ring satisfies the Krull–Schmidt condition.

THEOREM 3.5. *Let $(R,\mathfrak{m})$ be a strongly F-regular and F-finite local ring. Suppose $P \subset R$ is a prime ideal. Then $\mathrm{s}(R) = \mathrm{s}(R_P)$ if and only if $a_e(R) = a_e(R_P)$ for every $e \in \mathbb{N}$.*

Proof. If $a_e(R) = a_e(R_P)$ for every $e \in \mathbb{N}$, then it is trivial that $\mathrm{s}(R) = \mathrm{s}(R_P)$ since $\operatorname{rank}_R(F_*^eR) = \operatorname{rank}_{R_P}(F_*^eR_P)$.

Suppose that $a_{e_0}(R_P) > a_{e_0}(R)$ and write $F_*^{e_0} R \cong R^{\oplus a_{e_0}} \oplus M_{e_0}$. Then $(M_{e_0})_P$ has a free R_P-summand. For each $e \in \mathbb{N}$, by Remark 3.4 we may write

$$F_*^e R \cong R^{\oplus a_e(R)} \oplus M_{e_0}^{\oplus \operatorname{rank}_{M_{e_0}}(F_*^e R)} \oplus N_e.$$

Localizing at the prime P, we see that $a_e(R_P) \geq a_e(R) + \operatorname{rank}_{M_{e_0}}(F_*^e R)$ and

$$\begin{aligned} \mathrm{s}(R_P) &= \lim_{e\to\infty} \frac{a_e(R_P)}{\operatorname{rank}(F_*^e R)} \geq \lim_{e\to\infty} \frac{a_e(R)}{\operatorname{rank}(F_*^e R)} + \liminf_{e\to\infty} \frac{\operatorname{rank}_{M_{e_0}}(F_*^e R)}{\operatorname{rank}(F_*^e R)} \\ &= \mathrm{s}(R) + \liminf_{e\to\infty} \frac{\operatorname{rank}_{M_{e_0}}(F_*^e R)}{\operatorname{rank}(F_*^e R)}. \end{aligned}$$

Therefore $\mathrm{s}(R_P) > \mathrm{s}(R)$ by Lemma 3.3. □

The following theorem states that the splitting ideals of R and that of a localization R_P can be effectively compared whenever the Frobenius splitting numbers of R and R_P agree.

THEOREM 3.6. *Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p, and let P be a prime ideal. Then the following are equivalent*:

(1) $a_e(R) = a_e(R_P)$,
(2) $I_e((P, I)) = I_e(P) + I^{[p^e]}$ *for all ideals I*,
(3) $I_e(\mathfrak{m}) = I_e(P) + \mathfrak{m}^{[p^e]}$.

Proof. Write $F_*^e R \cong R^{\oplus a_e(R)} \oplus M_e$. By definition, $F_*^e I_e(P) = P^{\oplus a_e(R)} \oplus \{\eta \in M_e \mid \phi(\eta) \in P, \forall \phi \in \operatorname{Hom}_R(M_e, R)\}$. Hence $a_e(R) = a_e(R_P)$ if and only if $\operatorname{Hom}_R(M_e, R) = \operatorname{Hom}_R(M_e, P)$. It follows that $\operatorname{Hom}_R(M_e, R) = \operatorname{Hom}_R(M_e, P + I)$, so $F_*^e(I_e(P) + I^{[p^e]}) = (P, I)^{\oplus a_e(R)} \oplus (M_e + IM_e) = (P, I)^{\oplus a_e(R)} \oplus M_e = F_*^e I_e(P + I)$. Thus (1) implies (2).

Since (2) trivially implies (3), it is left to show that the last condition implies the first. Suppose that $I_e(\mathfrak{m}) = (I_e(P), \mathfrak{m}^{[p^e]})$. Then

$$\begin{aligned} F_* I_e(\mathfrak{m}) &= \mathfrak{m}^{\oplus a_e(R)} \oplus M_e \\ &= \mathfrak{m}^{\oplus a_e(R)} \oplus (\{\eta \in M_e \mid \phi(\eta) \in P, \forall \phi \in \operatorname{Hom}_R(M_e, R)\} + \mathfrak{m} M_e). \end{aligned}$$

Then by Nakayama's lemma we get that

$$M_e = \{\eta \in M_e \mid \phi(\eta) \in P, \forall \phi \in \operatorname{Hom}_R(M_e, R)\},$$

that is, $\operatorname{Hom}_R(M_e, R) = \operatorname{Hom}_R(M_e, P)$, and therefore $a_e(R) = a_e(R_P)$. □

Theorems 3.5 and 3.6 imply the following:

COROLLARY 3.7. *Let $(R, \mathfrak{m})$ be an F-finite and strongly F-regular local ring of prime characteristic p, and let P be a prime ideal. Then $\mathrm{s}(R) = \mathrm{s}(R_P)$ if and only if $I_e(\mathfrak{m}) = I_e(P) + \mathfrak{m}^{[p^e]}$ for every $e \in \mathbb{N}$.*

The techniques surrounding Theorem 3.5 provide a novel proof that the F-signature of a local ring is 1 if and only if R is a regular local ring.

THEOREM 3.8 ([HL02, Corollary 16]). *Let* $(R, \mathfrak{m})$ *be an* F*-finite local ring. Then* $\mathrm{s}(R) = 1$ *if and only if* R *is a regular local ring.*

Proof. Having positive F-signature implies that R is strongly F-regular. (The converse also holds; see [AL03, Main Theorem].) Hence R is a domain, so $R_{(0)}$ is a regular ring, and therefore $\mathrm{s}(R_{(0)}) = \mathrm{s}(R)$. Then invoking Theorem 3.5, we have that $a_e(R) = a_e(R_{(0)}) = \mathrm{rank}(F^e_* R)$. Therefore $F^e_* R$ is a free R-module, and R is a regular local ring by Theorem 1.1. □

The advantage of the proof of Theorem 3.8 is that it directly uses Kunz's theorem, whereas the proof of [HL02, Corollary 16] invokes the fact that R must be regular if $\mathrm{e}_{\mathrm{HK}}(R) = 1$ [WY00; HY02]. We may also adapt our approach to give a somewhat novel proof that the Hilbert–Kunz multiplicity of a formally unmixed local ring is 1 if and only if R is a regular local ring; see Theorem 3.11.

THEOREM 3.9. *Let* $(R, \mathfrak{m})$ *be an* F*-finite and strongly* F*-regular local ring of prime characteristic* p*. Suppose that* $P \in \mathrm{Spec}(R)$, $\mathrm{s}(R) = \mathrm{s}(R_P)$ *and* $\underline{x} = x_1, \dots, x_h$ *is a sequence of elements in* R*. Then the following are equivalent*:

(1) $\underline{x}$ *is a regular sequence on* R/P;
(2) $\underline{x}$ *is a regular sequence on* $R/I_e(P)$ *for each* $e \in \mathbb{N}$;
(3) $\underline{x}$ *is a regular sequence on* $R/I_e(P)$ *for some* $e \in \mathbb{N}$.

In particular, $\mathrm{depth}(R/P) = \mathrm{depth}(R/I_e(P))$ *for all* $e \in \mathbb{N}$.

Proof. Let $x_1, \dots, x_h$ be a regular sequence on R/P. To show that $x_1, \dots, x_h$ is a regular sequence on $R/I_e(P)$ it is equivalent to check that for every $0 \le i \le h-1$,

$$(I_e(P), x_1^{p^e}, \dots, x_i^{p^e}) : x_{i+1}^{p^e} = (I_e(P), x_1^{p^e}, \dots, x_i^{p^e}).$$

By Theorems 3.5 and 3.6 we have that $(I_e(P), x_1^{p^e}, \dots, x_i^{p^e}) = I_e(P, x_1, \dots, x_i)$, and by (7) of Lemma 2.3 we have that $(I_e(P, x_1, \dots, x_i)) : x_{i+1}^{p^e} = I_e((P, x_1, \dots, x_i) : x_{i+1})$. But $x_1, \dots, x_h$ is a regular sequence on R/P, and therefore by a second application of Theorems 3.5 and 3.6 we see that

$$I_e((P, x_1, \dots, x_i) : x_{i+1}) = I_e(P, x_1, \dots, x_i) = (I_e(P), x_1^{p^e}, \dots, x_i^{p^e}).$$

Now suppose that for some $e \in \mathbb{N}$, $x_1, \dots, x_h$ is a regular sequence on $R/I_e(P)$. Then for each $0 \le i \le h-1$, we have by (7) of Lemma 2.3, Theorem 3.5, and Theorem 3.6 that

$$\begin{aligned} &I_e((P, x_1, \dots, x_i) : x_{i+1}) \\ &\quad = (I_e(P, x_1, \dots, x_i) : x_{i+1}^{p^e}) = (I_e(P), x_1^{p^e} \dots, x_{i-1}^{p^e}) : x_i^{p^e} \\ &\quad = (I_e(P), x_1^{p^e} \dots, x_{i-1}^{p^e}) = I_e((P, x_1, \dots, x_{i-1})). \end{aligned}$$

By Corollary 2.4 we must have that $(P, x_1, \dots, x_i) : x_{i+1} = (P, x_1, \dots, x_i)$ for each $0 \le i \le h-1$, and therefore $x_1, \dots, x_h$ is indeed a regular sequence on R/P. □

Let $(R, \mathfrak{m})$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$. Observe by (9) of Lemma 2.3 that $\operatorname{depth}(R/I_e(P)) \geq 1$ for every $e \in \mathbb{N}$ and $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. However, it does not follow that $\operatorname{depth}(R/I_e(P)) = \operatorname{depth}(R/P)$ if we do not assume that $\mathrm{s}(R) = \mathrm{s}(R_P)$.

EXAMPLE 3.10. Consider the regular local ring S of prime characteristic 2 obtained by localizing $\mathbb{F}_2[x, y, z, w]$ at the maximal ideal (x, y, z, w) and let $R = S/(xy - zw)$. Then R is a strongly F-regular isolated singularity. Consider the height 1 prime ideal $P = (x, z)$. By the techniques surrounding Fedder's criterion [Fed83] (c.f. [Gla96, Theorem 2.3]), for each $e \in \mathbb{N}$, we have that

$$I_e(R) = \frac{P^{[2^e]} :_S (xy - zw)^{2^e - 1}}{(xy - zw)}.$$

Observe that R/P is a regular local ring of dimension 2; yet we can check that $I_1(R) = (xz, x^2, z^2)$ and $R/I_1(R)$ has depth 1.

3.2. Hilbert–Kunz Multiplicity

Now we prove Theorem A for the Hilbert–Kunz multiplicity.

THEOREM 3.11. *Let $(R, \mathfrak{m})$ be a strongly F-regular and F-finite local ring of prime characteristic p and Krull dimension d, and let $P \in \operatorname{Spec}(R)$. Then the following are equivalent*:

(1) $\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(R_P)$;
(2) $\lambda(R/\mathfrak{m}^{[p^e]})/p^{ed} = \lambda(R_P/P^{[p^e]})/p^{e\,\mathrm{ht}(P)}$ *for every* $e \in \mathbb{N}$;
(3) $\mu(F^e_* R) = \mu(F^e_* R_P)$ *for every* $e \in \mathbb{N}$;
(4) $F^e_* R/PF^e_* R$ *is a free* R/P*-module for every* $e \in \mathbb{N}$.

Proof. Conditions (2) and (3) are equivalent by [Kun76, Proposition 2.3], and conditions (2) and (3) clearly imply (1). To show that condition (1) implies condition (3), suppose that $\mu(F^{e_0}_* R) > \mu(F^{e_0}_* R_P)$. If $F^{e_0}_* R \cong R^{\oplus a_{e_0}(R)} \oplus M_{e_0}$, then $\mu(M_{e_0}) > \mu((M_{e_0})_P)$. Let $b_e = \operatorname{rank}_{M_{e_0}}(F^e_* R)$. By Remark 3.4 we may write $F^e_* R \cong R^{\oplus a_e} \oplus (M_{e_0})^{\oplus b_e} \oplus N_e$, and it follows that

$$\mathrm{e}_{\mathrm{HK}}(R) = \lim_{e\to\infty} \frac{1}{\operatorname{rank}(F^e_* R)}(a_e(R) + b_e\mu(M_{e_0}) + \mu(N_e))$$

and

$$\mathrm{e}_{\mathrm{HK}}(R_P) = \lim_{e\to\infty} \frac{1}{\operatorname{rank}(F^e_* R)}(a_e(R) + b_e\mu((M_{e_0})_P) + \mu((N_e)_P)).$$

Therefore

$$\begin{aligned}
&\mathrm{e}_{\mathrm{HK}}(R_P)\\
&\quad\leq \lim_{e\to\infty} \frac{1}{\operatorname{rank}(F^e_* R)}(a_e(R) + b_e(\mu(M_{e_0}) - 1) + \mu(N_e))\\
&\quad\leq \lim_{e\to\infty} \frac{1}{\operatorname{rank}(F^e_* R)}(a_e(R) + b_e\mu(M_{e_0}) + \mu(N_e)) - \liminf_{e\to\infty} \frac{b_e}{\operatorname{rank}(F^e_* R)}
\end{aligned}$$

$$= \mathrm{e}_{\mathrm{HK}}(R) - \liminf_{e\to\infty} \frac{b_e}{\operatorname{rank}(F^e_* R)},$$

a value strictly less than $\mathrm{e}_{\mathrm{HK}}(R)$ by Lemma 3.3.

Now suppose that $\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(R_P)$. To show that $F^e_* R/PF^e_* R$ is a free R/P-module, observe first that, by Nakayama's lemma, $\mu_{R_P}(F^e_* R_P) = \mu_{R_P}(F^e_* R_P/PF^e_* R_P)$. Therefore

$$\begin{aligned}\mu_R(F^e_* R/PF^e_* R) &= \mu_R(F^e_* R) = \mu_{R_P}(F^e_* R_P) \\ &= \mu_{R_P}(F^e_* R_P/PF^e_* R_P) \le \mu_R(F^e_* R/PF^e_* R).\end{aligned}$$

Therefore, as an R/P-module, we have that $F^e_* R/PF^e_* R$ is generated by $\operatorname{rank}_{R/P}(F^e_* R/PF^e_* R)$ elements and must be free.

Conversely, if $F^e_* R/PF^e_* R$ is a free R/P-module for every $e \in \mathbb{N}$, then $\mu_R(F^e_* R/PF^e_* R) = \mu_{R_P}(F^e_* R_P/PF^e_* R_P)$, and therefore

$$\mu_R(F^e_* R) = \mu_R(F^e_* R/PF^e_* R) = \mu_{R_P}(F^e_* R_P/PF^e_* R_P) = \mu_{R_P}(F^e_* R_P). \quad \square$$

The following corollary is an analogue of Theorem 3.9 for the Hilbert–Kunz multiplicity.

COROLLARY 3.12. *Let $(R, \mathfrak{m})$ be a strongly F-regular and F-finite local ring of prime characteristic p. Suppose that $P \in \operatorname{Spec}(R)$ and $\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(R_P)$. Then for each sequence of elements $\underline{x} = x_1, \dots, x_h$, the following are equivalent*:

(1) *$\underline{x}$ is a regular sequence on R/P;*
(2) *$\underline{x}$ is a regular sequence on $R/P^{[p^e]}$ for each $e \in \mathbb{N}$;*
(3) *$\underline{x}$ is a regular sequence on $R/P^{[p^e]}$ for some $e \in \mathbb{N}$.*

In particular, $\operatorname{depth}(R/P) = \operatorname{depth}(R/P^{[p^e]})$ *for every $e \in \mathbb{N}$.*

Proof. For any finitely generated R-module M, a sequence of elements $\underline{x}$ is a regular sequence on M if and only if $\underline{x}$ is a regular sequence on $F^e_* M$. The corollary is immediate by Theorem 3.11 since the modules $F^e_*(R/P^{[p^e]}) \cong F^e_* R/PF^e_* R$ are free R/P-modules. $\square$

Corollary 3.12 is an improvement of an observation that can be made from [Smi19, Proposition 3.1 and Corollary 5.19]: if $(R, \mathfrak{m})$ is weakly F-regular, $P \in \operatorname{Spec}(R)$ satisfies $\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(R_P)$, and R/P is regular, then $R/P^{[p^e]}$ is Cohen–Macaulay for $e \in \mathbb{N}$.

We utilize Theorem 3.11 and results of [AE08] and provide a novel proof that the Hilbert–Kunz multiplicity of a local ring is 1 if and only if the ring is regular. We recall that a ring is unmixed if it is equidimensional and has no embedded components.

THEOREM 3.13 ([WY00]). *Let $(R, \mathfrak{m})$ be a formally unmixed local F-finite ring. Then $\mathrm{e}_{\mathrm{HK}}(R) = 1$ if and only if R is a regular local ring.*

Proof. The assumption on Hilbert–Kunz multiplicity implies that R is strongly F-regular; see [AE08, Corollary 3.6]. By Theorem 3.11 applied to $P = (0)$,

$\mu(F_*R) = \operatorname{rank} F_*R$, so F_*R is a free R-module, and R is regular by Theorem 1.1. □

3.3. Frobenius Betti Numbers and Frobenius Euler Characteristic

We now turn our attention to the behavior of Frobenius Betti numbers and Frobenius Euler characteristics under localizations.

DEFINITION 3.14. Let $(R, \mathfrak{m})$ be an F-finite local domain of prime characteristic p. For each $e \in \mathbb{N}$, let $\Omega_i^e(R)$ be the ith syzygy in the minimal free resolution of $F_*^e R$. The ith Frobenius Betti number of R is

$$\beta_i^F(R) = \lim_{e\to\infty} \frac{\mu(\Omega_i^e(R))}{\operatorname{rank}(F_*^e R)},$$

and the ith Frobenius Euler characteristic of R is

$$\chi_i^F(R) = \lim_{e\to\infty} \sum_{j=0}^{i} (-1)^j \frac{\mu(\Omega_{i-j}^e(R))}{\operatorname{rank}(F_*^e R)} = \sum_{j=0}^{i} (-1)^j \beta_{i-j}^F(R).$$

We refer the reader to [Li08; AL08; dSHB17; dSPY18] for basics on Frobenius Betti numbers and to [dSPY18] for basics on Frobenius Euler characteristic. Our study begins with a simple application of the Auslander–Buchsbaum formula.

LEMMA 3.15. *Let $(R, \mathfrak{m})$ be an F-finite local ring. The following are equivalent*:

(1) *R is a regular local ring*;

(2) *$F_*^e R$ has finite projective dimension as an R-module for every $e \geq 1$*;

(3) *$F_*^e R$ has finite projective dimension for some $e \in \mathbb{N}$.*

Proof. It is easy to see that $\operatorname{depth}(R) = \operatorname{depth}(F_*^e R)$ for every $e \in \mathbb{N}$. Hence by the Auslander–Buchsbaum formula, if the projective dimension of $F_*^e R$ is finite, then $F_*^e R$ is a free R-module, and the lemma follows from Theorem 1.1. □

LEMMA 3.16. *Let $(R, \mathfrak{m})$ be an F-finite local domain of prime characteristic p. Then*

$$\operatorname{rank}_R(\Omega_i^e(R)) = \chi_{i-1}^e(R) + (-1)^i \operatorname{rank}(F_*^e R).$$

Moreover, if R is not regular, then $\beta_i^e(R) > \operatorname{rank}_R(\Omega_i^e(R)) = \chi_{i-1}^e(R) + (-1)^i \operatorname{rank}(F_*^e R)$.

Proof. Rank is additive on exact sequences, and there are long exact sequences

$$0 \to \Omega_i^e(R) \to R^{\oplus \beta_{i-1}^e(R)} \to \cdots \to R^{\oplus \mu(F_*^e R)} \to F_*^e R \to 0.$$

By Lemma 3.15, if R is not regular, then $\Omega_i^e(R)$ is not free, and hence $\beta_i^e(R) = \mu(\Omega_i^e(R)) > \operatorname{rank}(\Omega_i^e(R))$. □

LEMMA 3.17. *Let $(R, \mathfrak{m})$ be a local F-finite domain and let $e, i \in \mathbb{N}$ with $e \geq 1$. Then $\chi_i^e(R) \geq (-1)^i \operatorname{rank}(F_*^e R)$ with equality if and only if R is a regular local ring.*

Proof. For $i = 0$, the lemma follows from Theorem 1.1. If $i \geq 1$ and $e \in \mathbb{N}$, then $\chi_i^e(R) = \beta_i^e(R) - \chi_{i-1}^e(R)$. Applying Lemma 3.16, we arrive at

$$\chi_i^e(R) \geq \chi_{i-1}^e(R) + (-1)^i \operatorname{rank}(F_*^e R) - \chi_{i-1}^e(R) = (-1)^i \operatorname{rank}(F_*^e R)$$

with equality if and only if R is a regular local ring. □

LEMMA 3.18. *Let $(R, \mathfrak{m})$ be a local F-finite domain, and let $P \in \operatorname{Spec}(R)$. Then $\beta_i^e(R) = \beta_i^e(R_P)$ if and only if $\chi_i^e(R) = \chi_i^e(R_P)$ and $\chi_{i-1}^e(R) = \chi_{i-1}^e(R_P)$. In particular, if $\beta_1^e(R) = \beta_1^e(R_P)$, then $\mu(F_*^e R) = \mu(F_*^e R_P)$.*

Proof. Observe first that

$$\beta_i^e(R) = \chi_i^e(R) + \chi_{i-1}^e(R)$$

and

$$\beta_i^e(R_P) = \chi_i^e(R_P) + \chi_{i-1}^e(R_P).$$

Suppose that $\beta_i^e(R) = \beta_i^e(R_P)$. The function $\chi_i^e\colon \operatorname{Spec}(R) \to \mathbb{R}$ is upper semicontinuous ([dSPY18, Proposition 3.1]), and therefore $\chi_i^e(R) \geq \chi_i^e(R_P)$. If $\beta_i^e(R) = \beta_i^e(R_P)$, then

$$\chi_{i-1}^e(R_P) \geq \chi_{i-1}^e(R),$$

but the function $\chi_{i-1}^e\colon \operatorname{Spec}(R) \to \mathbb{R}$ is also upper semicontinuous, and therefore equality must hold. □

Similarly to Lemma 3.3, if R is strongly F-regular and a module M appears as a direct summand of $\Omega_i^{e_0}(R)$ for some $e_0 \in \mathbb{N}$, then M appears as a direct summand of $\Omega_i^e(R)$ asymptotically many times as $e \to \infty$.

LEMMA 3.19. *Let $(R, \mathfrak{m})$ be an F-finite and strongly F-regular local ring, and let M be a finitely generated R-module. If $\operatorname{rank}_M(\Omega_i^{e_0}(R)) > 0$ for some $e_0 \in \mathbb{N}$, then*

$$\liminf_{e\to\infty} \frac{\operatorname{rank}_M(\Omega_i^e(R))}{\operatorname{rank}(F_*^e R)} > 0.$$

Proof. Suppose M is a direct summand of $\Omega_i^{e_0}(R)$. Observe that $F_*^{e+e_0} R$ has a direct summand $F_*^{e_0} R^{\oplus a_e}$. It readily follows that $\Omega_{e+e_0}(R)$ has $\Omega_{e_0}(R)^{\oplus a_e(R)}$ as a direct summand, and therefore $\operatorname{rank}_M(\Omega_{e+e_0}) \geq a_e(R)$. In particular,

$$\liminf_{e\to\infty} \frac{\operatorname{rank}_M(\Omega_e(R))}{\operatorname{rank}(F_*^e R)} \geq \frac{s(R)}{\operatorname{rank}(F_*^{e_0} R)} > 0.$$ □

We are now prepared to prove Theorem A for Frobenius Betti numbers and Frobenius Euler characteristics. We first present a proof of Theorem A for Frobenius Betti numbers.

THEOREM 3.20. *Let $(R, \mathfrak{m})$ be an F-finite strongly F-regular local ring and let $P \in \operatorname{Spec}(R)$. Then for each integer $i \geq 0$, $\beta_i^F(R) = \beta_i^F(R_P)$ if and only if $\beta_i^e(R) = \beta_i^e(R_P)$ for every $e \in \mathbb{N}$.*

Proof. Clearly, if $\beta_i^e(R) = \beta_i^e(R_P)$ for every integer e, then $\beta_i^F(R) = \beta_i^F(R_P)$. Suppose there exists an integer e_0 such that $\mu(\Omega_i^{e_0}(R_P)) < \mu(\Omega_i^{e_0}(R))$. For each $e \in \mathbb{N}$, let $b_e = \operatorname{rank}_{\Omega_i^{e_0}(R)}(\Omega_i^e(R))$. Then we can write $\Omega_i^e(R) \cong \Omega_i^{e_0}(R)^{\oplus b_e} \oplus M_e$. Localizing at P, we have

$$\Omega_i^e(R)_P \cong \Omega_i^e(R_P) \oplus F_P,$$

where F_P is a free R_P-module. It readily follows that

$$\mu(\Omega_i^e(R_P)) \leq \mu(\Omega_i^e(R)_P) \leq \mu(\Omega_i^e(R)) - b_e.$$

Therefore $\beta_i^F(R_P) \leq \beta_i^F(R) - \liminf_{e\to\infty} \frac{b_e}{\operatorname{rank}(F_*^e R)}$, which is strictly less than $\beta_i^F(R)$ by Lemma 3.19. □

Following the proof of Theorem 3.11, we recover [AL08] for strongly F-regular rings.

COROLLARY 3.21. *Let $(R, \mathfrak{m})$ be an F-finite strongly F-regular local ring. Then for each integer $i \geq 0$, $\beta_i^F(R) = 0$ if and only if R is a regular local ring.*

Proof. By Lemma 3.15, $\beta_i^F(R_{(0)}) = 0 = \beta_i^F(R)$. Therefore $\beta_i^e(R) = \beta_i^e(R_{(0)}) = 0$, and the claim follows from Lemma 3.15. □

Finally, we complete our proof of Theorem A by establishing an equimultiplicity criterion for Frobenius Euler characteristic.

THEOREM 3.22. *Let $(R, \mathfrak{m})$ be an F-finite strongly F-regular local ring and let $P \in \operatorname{Spec}(R)$. Then for each integer $i \geq 0$, $\chi_i^F(R) = \chi_i^F(R_P)$ if and only if $\chi_i^e(R) = \chi_i^e(R_P)$ for every $e \in \mathbb{N}$.*

Proof. Without loss of generality, we may assume that R is not regular. By Lemma 3.16, $\chi_i^e(R) = \chi_i^e(R_P)$ if and only if $\operatorname{rank}(\Omega_{i+1}^e(R)) = \operatorname{rank}(\Omega_{i+1}^e(R_P))$, and $\chi_i^F(R) = \chi_i^F(R_P)$ if and only if

$$\lim_{e\to\infty} \frac{\operatorname{rank}(\Omega_i^e(R))}{\operatorname{rank}(F_*^e R)} = \lim_{e\to\infty} \frac{\operatorname{rank}(\Omega_i^e(R_P))}{\operatorname{rank}(F_*^e R)}.$$

Suppose there exists an integer e_0 such that $\operatorname{rank}(\Omega_{i+1}^{e_0}(R)) \neq \operatorname{rank}(\Omega_{i+1}^{e_0}(R_P))$. Therefore $\Omega_{i+1}^{e_0}(R)_P$ has a nonzero free summand. Let $b_e = \operatorname{rank}_{\Omega_{i+1}^{e_0}(R)}(\Omega_{i+1}^e(R))$; by Lemma 3.19 $\liminf_{e\to\infty} \frac{b_e}{\operatorname{rank}(F_*^e R)} > 0$. Then for each integer $e \in \mathbb{N}$, the R_P-module $\Omega_{i+1}^e(R)_P$ contains a free summand of rank b_e. In particular, we have that $\operatorname{rank}(\Omega_{i+1}^e(R_P)) \leq \operatorname{rank}(\Omega_{i+1}^e(R)_P) - b_e$. Therefore

$$\begin{aligned}\lim_{e\to\infty} \frac{\operatorname{rank}(\Omega_{i+1}^e(R_P))}{\operatorname{rank}(F_*^e R)} &\leq \lim_{e\to\infty} \frac{\operatorname{rank}(\Omega_{i+1}^e(R))}{\operatorname{rank}(F_*^e R)} - \liminf_{e\to\infty} \frac{b_e}{\operatorname{rank}(F_*^e R)} \\ &< \lim_{e\to\infty} \frac{\operatorname{rank}(\Omega_{i+1}^e(R))}{\operatorname{rank}(F_*^e R)}.\end{aligned}$$

□

As with F-signature, Hilbert–Kunz multiplicity, and Frobenius Betti numbers, we now know that the Frobenius Euler characteristic can be used to detect regular rings, provided that we know the ring being studied is strongly F-regular.

THEOREM 3.23. *Let $(R, \mathfrak{m})$ be an F-finite strongly F-regular local ring. The following are equivalent*:

(1) *R is a regular local ring*;
(2) $\chi_i^e(R) = (-1)^i \operatorname{rank}(F_*^e R)$ *for every* $e \in \mathbb{N}$;
(3) $\chi_i^e(R) = (-1)^i \operatorname{rank}(F_*^e R)$ *for some* $e \in \mathbb{N}$;
(4) $\chi_i^F(R) = (-1)^i$.

Proof. The equivalence of (1), (2), and (3) is the content of Lemma 3.17, and (4) is trivially implied by condition (2). Now an argument with the generic point as in Theorem 3.8 shows that (4) implies (2) by Theorem 3.22. □

4. An Associativity Formula for F-Signature

Our proof of Theorem C begins with two technical lemmas.

LEMMA 4.1. *Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p. Suppose that $I \subset R$ is an ideal such that R/I is Cohen–Macaulay of dimension h and $x_1, x_2, \dots, x_h$ is a parameter sequence on R/I. Then for all sequences of natural numbers $n_1, n_2, \dots, n_h$, we have that*

(1)
$$\lambda\left(\frac{I_e(I + (x_1^{n_1}, x_2^{n_2}, \dots, x_h^{n_h}))}{I_e(I + (x_1^{n_1+1}, x_2^{n_2}, \dots, x_h^{n_h}))}\right) \geq \lambda\left(\frac{I_e(I + (x_1^{n_1-1}, x_2^{n_2}, \dots, x_h^{n_h}))}{I_e(I + (x_1^{n_1}, x_2^{n_2}, \dots, x_h^{n_h}))}\right)$$

and

(2)
$$\frac{1}{n_1+1}\lambda\left(\frac{R}{I_e(I + (x_1^{n_1+1}, x_2^{n_2}, \dots, x_h^{n_h}))}\right) \geq \frac{1}{n_1}\lambda\left(\frac{R}{I_e(I + (x_1^{n_1}, x_2^{n_2}, \dots, x_h^{n_h})}\right).$$

Proof. We may pass to $I' = I + (x_2^{n_2}, \dots, x_h^{n_h})$ and assume that $\dim R/I = 1$.

We claim there exist short exact sequences

$$0 \to \frac{I_e(I + (x^{n-1}))}{I_e(I + (x^n))} \xrightarrow{\cdot x^{p^e}} \frac{I_e(I + (x^n))}{I_e(I + (x^{n+1}))} \to \frac{I_e(I + (x^n))}{I_e(I + (x^{n+1})) + x^{p^e} I_e(I + (x^{n-1}))} \to 0.$$

Observe that if such short exact sequences exist, then the first inequality is obvious since the length of the left piece of a short exact sequence is no more than the length of the middle term. The second inequality is equivalent to the inequality

$$n\,\lambda\left(\frac{I_e(I + (x^n))}{I_e(I + (x^{n+1}))}\right) = n\left(\lambda\left(\frac{R}{I_e(I + (x^{n+1}))}\right) - \lambda\left(\frac{R}{I_e(I + (x^n))}\right)\right)$$

$$\geq \lambda(R/I_e(I+(x^n))),$$

which follows from (1) since we can filter $\lambda(R/I_e(I+(x^n)))$ as

$$\lambda(R/I_e(I+(x^n))) = \sum_{i=0}^{n-1}\lambda\left(\frac{I_e(I+(x^i))}{I_e(I+(x^{i+1}))}\right).$$

To show that the above short exact sequences exist, we first notice that

$$x^{p^e}I_e(I+(x^{n-1})) \subseteq I_e(xI+(x^n)) \subseteq I_e(I+(x^n)).$$

Indeed, if $u \in I_e(I+(x^{n-1}))$ and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$, then

$$\phi(F^e_* x^{p^e}u) = x\phi(F^e_* u) \in x(I+(x^{n-1})) \subseteq (I+(x^n)).$$

Therefore there are right exact sequences

$$\frac{I_e(I+(x^{n-1}))}{I_e(I+x^n))} \xrightarrow{\cdot x^{p^e}} \frac{I_e(I+(x^n))}{I_e(I+(x^{n+1}))} \to \frac{I_e(I+(x^n))}{I_e(I+(x^{n+1}))+x^{p^e}I_e(I+(x^{n-1}))} \to 0.$$

To show injectivity of the first map, observe that an element $u \in I_e(I+(x^{n-1}))$ satisfies $x^{p^e}u \in I_e(I+(x^{n+1}))$ if and only if $u \in I_e(I+(x^{n+1})) : x^{p^e}$. By (7) of Lemma 2.3 we have that

$$I_e(I+(x^{n_1+1})) : x^{p^e} = I_e((I+(x^{n+1})) : x) = I_e(I+(x^n)),$$

where the second equality follows by standard observations on parameter ideals in the Cohen–Macaulay ring R/I. □

The following technical lemma is very much in the spirit of [PT18, Theorem 4.3].

LEMMA 4.2. *Let $(R,\mathfrak{m})$ be an F-finite local domain of prime characteristic p and of Krull dimension d. Suppose that $I \subset R$ is an ideal such that R/I is Cohen–Macaulay of dimension h and $\underline{x} = x_1, \ldots, x_h$ is a parameter sequence on R/I. Then there exists a constant $C \in \mathbb{R}$ such that for all $e, n_1, n_2, \ldots, n_h \in \mathbb{N}$,*

$$\left|\frac{1}{p^{ed}}\lambda(R/I_e(I+(x_1^{n_1},\ldots,x_h^{n_h})) - \mathrm{s}(I+(x_1^{n_1},\ldots,x_h^{n_h}))\right| \leq \frac{Cn_1\cdots n_h}{p^e}.$$

Proof. Denote by $\underline{N} = (n_1, n_2, \ldots, n_h)$ a Cartesian product of natural numbers, let $N = n_1 n_2 \cdots n_h$, and for each $\underline{N}$, let $\underline{x}^{\underline{N}}$ be the sequence of elements $x_1^{n_1}, x_2^{n_2}, \ldots, x_h^{n_h}$. We claim that there exists a constant C, depending only on $\lambda(R/(I+(\underline{x})))$, such that for all $\underline{N}$,

$$\left|\frac{1}{p^{ed}}\lambda(R/I_e(I+(\underline{x})^{\underline{N}})) - \mathrm{s}(I+(\underline{x}^{\underline{N}}))\right| \leq \frac{CN}{p^e}.$$

We will first show that there exists a constant C such that for all $\underline{N}$ and $e \in \mathbb{N}$,

$$\frac{1}{p^{ed}}\lambda(R/I_e(I+(\underline{x})^{\underline{N}})) \leq \mathrm{s}(I+(\underline{x}^{\underline{N}})) + \frac{CN}{p^e}.$$

The R-module F_*R is finitely generated and torsion-free, so there exists a short exact sequence

$$0 \to F_*R \xrightarrow{\psi} R^{\oplus \operatorname{rank}(F_*R)} \to T \to 0,$$

where T is a finitely generated torsion R-module. By (4) of Lemma 2.3

$$\psi(F_*I_{e+1}(I + (\underline{x}^{N}))) \subseteq I_e(I + (\underline{x}^{N}))^{\oplus \operatorname{rank}(F_*R)},$$

and therefore there are right exact sequences

$$F_*R/(I_{e+1}(I + (\underline{x}^{N})) \xrightarrow{\psi} R^{\oplus \operatorname{rank}(F_*R)}/I_e(I + (\underline{x}^{N}))^{\oplus \operatorname{rank}(F_*R)} \to T_e \to 0,$$

where T_e is the homomorphic image of $T/I_e(I + (\underline{x}^{N}))T$. Therefore

$$\begin{aligned} &\operatorname{rank}(F_*R)\,\lambda(R/I_e(I + (\underline{x}^{N}))) \\ &\quad \le \lambda(F_*R/I_{e+1}(I + (\underline{x}^{N}))) + \lambda(T/I_e(I + (\underline{x}^{N}))T). \end{aligned} \tag{4.1}$$

Suppose that $c \in R$ is a nonzero element that annihilates T. Because $(I + (\underline{x}^{N}))^{[p^e]} \subseteq I_e(I + (\underline{x}^{N}))$, there exists a surjective map

$$(R/(c, (I + (\underline{x}^{N}))^{[p^e]}))^{\oplus \mu(T)} \to T/I_e(I + (\underline{x}^{N}))T,$$

and we have that

$$\lambda(T/I_e(I + (\underline{x})T)) \le \mu(T)\,\lambda(R/(c, (I + (\underline{x}^{N}))^{[p^e]})).$$

It is well known that there exists $C \in \mathbb{R}$, depending only on the ring R, such that

$$\lambda(R/(c, (I + (\underline{x}^{N})))^{[p^e]}) \le Cp^{(e\dim(R)-1)}\,\lambda(R/(I + (\underline{x}^{N})));$$

see, for example, [Pol18, Proposition 3.3]. Because R/I is Cohen–Macaulay, we know that

$$\lambda(R/(I + (\underline{x}^{N}))) = \mathrm{e}(\underline{x}^{N}; R/I) = N\,\mathrm{e}(\underline{x}; R/I) = N\,\lambda(R/(I + (\underline{x}))).$$

Dividing inequality (4.1) by $\operatorname{rank}(F_*R)p^{(e+1)d}$, we obtain that

$$\frac{\lambda(R/I_e(I + (\underline{x}^{N})))}{p^{ed}} \le \frac{\lambda(R/I_{e+1}(I + (\underline{x}^{N})))}{p^{(e+1)d}} + \frac{\mu(T)C\,\lambda(R/(I + \underline{x}))N}{p^e}$$

for every $e \in \mathbb{N}$. The constant $\mu(T)C\,\lambda(R/(I + \underline{x}))$ has no dependence on e or N, so we replace C by this constant and utilize [PT18, Lemma 3.5] to obtain that

$$\frac{\lambda(R/I_e(I + (\underline{x}^{N})))}{p^{ed}} \le \mathrm{s}(I + (\underline{x}^{N})) + \frac{2CN}{p^e}.$$

Obtaining inequalities of the form

$$\mathrm{s}(I + (\underline{x}^{N})) \le \frac{\lambda(R/I_e(I + (\underline{x}^{N})))}{p^{ed}} + \frac{CN}{p^e} \tag{4.2}$$

is almost identical to the above. Begin by examining a short exact sequence of the form

$$0 \to R^{\oplus \operatorname{rank}(F_*R)} \xrightarrow{\psi} F_*R \to T' \to 0,$$

where T' is a torsion R-module. By (3) of Lemma 2.3 we have that $\psi(I_e(I + (\underline{x}^{\underline{N}}))^{[p]}) \subseteq I_{e+1}(I + (\underline{x}^{\underline{N}}))F_*R$, and so there are right exact sequences

$$(R/I_e(I + (\underline{x}^{\underline{N}}))^{[p]})^{\oplus \operatorname{rank}(F_*R)} \xrightarrow{\psi} F_*R/I_{e+1}(I + (\underline{x}^{\underline{N}}))F_*R \to T'_e \to 0,$$

where T'_e is the homomorphic image of $T'/I_{e+1}(I + (\underline{x}^{\underline{N}}))T'$. The reader is now encouraged to follow the techniques above and the techniques of [PT18, Theorem 4.3] to obtain inequalities as described in (4.2). $\square$

For the proof of the Theorem C, we recall the following standard result: if $a_{m,n}$ is a bisequence such that

- $\lim_{m,n\to\infty} a_{m,n}$ exists, and
- $\lim_{n\to\infty} a_{m,n}$ exists for all m,

then $\lim_{m,n\to\infty} a_{m,n} = \lim_{m\to\infty} \lim_{n\to\infty} a_{m,n}$.

THEOREM 4.3. *Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p and of Krull dimension d. Suppose that $I \subseteq R$ is an ideal such that R/I is Cohen–Macaulay of dimension h and $\underline{x} = x_1, \ldots, x_h$ is a parameter sequence on R/I. Then*

$$\lim_{n_1,\ldots,n_h\to\infty} \frac{1}{n_1\cdots n_h} \mathrm{s}(I, (x_1^{n_1}, \ldots, x_h^{n_h})) = \sum_P \mathrm{e}(x_1, \ldots, x_h; R/P)\, \mathrm{s}(IR_P),$$

where the sum is taken over all prime ideals $P \supseteq I$ such that $\dim(R/I) = \dim(R/P)$.

Proof. Lemma 4.2 allows us to swap limits and identify

$$\begin{aligned}
&\lim_{n_1,\ldots,n_h\to\infty} \frac{1}{n_1\cdots n_h} \mathrm{s}(I, (x_1^{n_1}, \ldots, x_h^{n_h}))\\
&= \lim_{n_1,\ldots,n_h\to\infty} \lim_{e\to\infty} \frac{1}{n_1\cdots n_h p^{ed}} \lambda(R/I_e(I + (x_1^{n_1}, \ldots, x_h^{n_h}))\\
&= \lim_{e\to\infty} \lim_{n_1,\ldots,n_h\to\infty} \frac{1}{n_1\cdots n_h p^{ed}} \lambda(R/I_e(I + (x_1^{n_1}, \ldots, x_h^{n_h})).
\end{aligned}$$

Furthermore, by Lemma 4.1

$$\begin{aligned}
&\lim_{n_1,\ldots,n_h\to\infty} \frac{\lambda(R/I_e(I + (x_1^{n_1}, \ldots, x_h^{n_h})))}{n_1\cdots n_h}\\
&= \sup_{n_1,\ldots,n_h} \frac{\lambda(R/I_e(I + (x_1^{n_1}, \ldots, x_h^{n_h})))}{n_1\cdots n_h}\\
&= \sup_n \frac{\lambda(R/I_e(I + (x_1^{n}, \ldots, x_h^{n})))}{n^h}.
\end{aligned}$$

We prove the theorem by induction on h. Let us start with the case of $h = 1$.

In this case, let us introduce an auxiliary bisequence that will link the two sides of the formula together.

CLAIM 4.4. *For each pair of natural numbers $n, m \in \mathbb{N}$, let*

$$a_{n,m} = \lambda(R/(I_e(I + (x^{n+m})) + (x^{np^e}))).$$

Then the bisequence $a_{n,m}$ satisfies the following properties:

(1) $a_{n,0} = \lambda(R/I_e(I + (x^n)))$;
(2) $\lim_{m\to\infty} a_{n,m} = \lambda(R/(I_e(I) + (x^{np^e})))$;
(3) $a_{n,m} = a_{n+m,0} - a_{m,0}$.

Proof. The first two properties are immediate from the definition.

For the third formula, we first recall that if J is an ideal and $x \notin J$, then $\lambda(R/(J, x)) = \lambda(R/J) - \lambda(R/J : x)$. Applying this to $J = I_e(I + (x^{n+m}))$ and x^{np^e}, we obtain by (7) of Lemma 2.3 that

$$\begin{aligned} a_{n,m} &= \lambda(R/I_e(I + (x^{n+m}))) - \lambda(R/I_e(I + (x^{n+m})) : x^{np^e}) \\ &= a_{n+m,0} - \lambda(R/I_e((I + (x^{n+m})) : x^n)) \\ &= a_{n+m,0} - a_{m,0}. \end{aligned}$$

□

Recall that for any bisequence, $\sup_n \sup_m a_{n,m} = \sup_{n,m} a_{n,m} = \sup_m \sup_n a_{n,m}$. By definition the sequence $a_{n,m}$ is increasing in m, so by Claim 4.4

$$\sup_n \sup_m \frac{a_{n,m}}{n} = \sup_n \frac{1}{n} \lambda(R/(I_e(I) + (x^{np^e}))) = p^e \operatorname{e}(x, R/I_e(I)),$$

because x is a regular element modulo $I_e(I)$ by Lemma 2.3(9). On the other hand, by Lemma 4.1 $a_{n,0}/n$ is an increasing function in $n \in \mathbb{N}$, so Claim 4.4 also shows that

$$\sup_n \frac{a_{n,m}}{n} = \sup_n \frac{a_{n+m,0} - a_{m,0}}{n} = \lim_{n\to\infty} \frac{a_{n,0}}{n}.$$

Thus

$$p^e \operatorname{e}(x, R/I_e(I)) = \sup_n \sup_m \frac{a_{n,m}}{n} = \sup_m \lim_{n\to\infty} \frac{a_{n,0}}{n} = \lim_{n\to\infty} \frac{\lambda(R/I_e(I + (x^n)))}{n},$$

which proves the theorem in the case $h = 1$ after using the additivity of multiplicity and passing to the limit as $e \to \infty$.

For $h \geq 2$, we may first consider the ideal $I' = I + (x_1^n, \dots, x_{h-1}^n)$ and get that

$$\lim_{m\to\infty} \frac{\lambda(R/I_e(I' + (x_h^m)))}{m} = \sum_Q \operatorname{e}(x_h; R/Q)\, \lambda(R_Q/I_e(I')R_Q),$$

where Q varies through the prime ideals P containing $I_e(I')$ such that $\dim R_Q/PR_Q = \dim R_Q/I_e(I')R_Q$. By induction,

$$\lim_{n\to\infty} \frac{\lambda(R_Q/I_e(I')R_Q)}{n^{h-1}} = \sum_P \operatorname{e}(x_1, \dots, x_{h-1}; R_Q/PR_Q)\, \lambda(R_P/I_e(I)R_P),$$

where P varies through the prime ideals P containing $I_e(I)$ such that $\dim R_Q/PR_Q = \dim R_Q/I_e(I)R_Q$. Thus

$$\lim_{n\to\infty} \frac{\lambda(R/I_e(I + (x_1^n, \dots, x_h^n)))}{n^h}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\lim_{m\to\infty}\frac{\lambda(R/I_e(I'+(x_h^m)))}{mn^{h-1}} \\
&= \sum_Q\sum_P \mathrm{e}(x_h; R/Q)\,\mathrm{e}(x_1,\dots,x_{h-1}; R_Q/PR_Q)\,\lambda(R_P/I_e(I)R_P).
\end{aligned}
$$

The theorem follows by changing the order of summation and using the associativity formula for parameter ideals ([Lec57, Theorem 1]); see the proof of [Smi19, Theorem 4.9]. □

ACKNOWLEDGMENTS. The authors thank Alessandro De Stefani for valuable feedback on a preliminary draft of this paper.

References

[AE06] I. M. Aberbach and F. Enescu, *When does the F-signature exists?* Ann. Fac. Sci. Toulouse Math. (6) 15 (2006), no. 2, 195–201.

[AE08] ———, *Lower bounds for Hilbert–Kunz multiplicities in local rings of fixed dimension,* Michigan Math. J. 57 (2008), 1–16, Special volume in honor of Melvin Hochster.

[AL03] I. M. Aberbach and G. J. Leuschke, *The F-signature and strong F-regularity,* Math. Res. Lett. 10 (2003), no. 1, 51–56.

[AL08] I. M. Aberbach and J. Li, *Asymptotic vanishing conditions which force regularity in local rings of prime characteristic,* Math. Res. Lett. 15 (2008), no. 4, 815–820.

[dSHB17] A. De Stefani, C. Huneke, and L. N. Betancourt, *Frobenius Betti numbers and modules of finite projective dimension,* J. Commut. Algebra 9 (2017), no. 4, 455–490.

[dSPY18] A. De Stefani, T. Polstra, and Y. Yao, *Global Frobenius Betti numbers and F-splitting ratio,* arXiv e-prints (2018), arXiv:1811.11022.

[Fed83] R. Fedder, *F-purity and rational singularity,* Trans. Amer. Math. Soc. 278 (1983), no. 2, 461–480.

[Gla96] D. Glassbrenner, *Strong F-regularity in images of regular rings,* Proc. Amer. Math. Soc. 124 (1996), no. 2, 345–353.

[HL02] C. Huneke and G. J. Leuschke, *Two theorems about maximal Cohen–Macaulay modules,* Math. Ann. 324 (2002), no. 2, 391–404.

[HY02] C. Huneke and Y. Yao, *Unmixed local rings with minimal Hilbert–Kunz multiplicity are regular,* Proc. Amer. Math. Soc. 130 (2002), no. 3, 661–665.

[Kun69] E. Kunz, *Characterizations of regular local rings of characteristic p,* Amer. J. Math. 91 (1969), 772–784.

[Kun76] ———, *On Noetherian rings of characteristic p,* Amer. J. Math. 98 (1976), no. 4, 999–1013.

[Lec57] C. Lech, *On the associativity formula for multiplicities,* Ark. Mat. 3 (1957), 301–314.

[Li08] J. Li, *Characterizations of regular local rings in positive characteristics,* Proc. Amer. Math. Soc. 136 (2008), no. 5, 1553–1558.

[Mon83] P. Monsky, *The Hilbert–Kunz function,* Math. Ann. 263 (1983), no. 1, 43–49.

[Pol18] T. Polstra, *Uniform bounds in F-finite rings and lower semi-continuity of the F-signature,* Trans. Amer. Math. Soc. 370 (2018), no. 5, 3147–3169.

[PT18] T. Polstra and K. Tucker, *F-signature and Hilbert–Kunz multiplicity: a combined approach and comparison,* Algebra Number Theory 12 (2018), no. 1, 61–97.

[Smi19] I. Smirnov, *Equimultiplicity in Hilbert–Kunz theory,* Math. Z 291 (2019), no. 1–2, 245–278.

[SvdB97] K. E. Smith and M. Van den Bergh, *Simplicity of rings of differential operators in prime characteristic,* Proc. Lond. Math. Soc. (3) 75 (1997), no. 1, 32–62.

[Tuc12] K. Tucker, *F-signature exists,* Invent. Math. 190 (2012), no. 3, 743–765.

[WY00] K. Watanabe and K. Yoshida, *Hilbert–Kunz multiplicity and an inequality between multiplicity and colength,* J. Algebra 230 (2000), no. 1, 295–317.

T. Polstra
Department of Mathematics
University of Utah
Salt Lake City, UT 84112
USA

polstra@math.utah.edu

I. Smirnov
Department of Mathematics
Stockholm University
Stockholm, SE-106 91
Sweden

smirnov@math.su.se