

A relation between chiral central charge and ground state degeneracy in 2+1-dimensional topological orders

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A bosonic topological order on d -dimensional closed space Σ^d may have degenerate ground states. The space Σ^d with different shapes (different metrics) form a moduli space $\mathcal{M}_{\Sigma^d}$. Thus the degenerate ground states on every point in the moduli space $\mathcal{M}_{\Sigma^d}$ form a complex vector bundle over $\mathcal{M}_{\Sigma^d}$. It was suggested that the collection of such vector bundles for d -dimensional closed spaces of all topologies completely characterizes the topological order. Using such a point of view, we propose a direct relation between two seemingly unrelated properties of 2+1-dimensional topological orders: (1) the chiral central charge c that describes the many-body density of states for edge excitations (or more precisely the thermal Hall conductance of the edge), (2) the ground state degeneracy D_g on closed genus- g surface. We show that $cD_g/2 \in \mathbb{Z}$, $g \geq 3$ for bosonic topological orders. We explicitly checked the validity of this relation for over 140 simple topological orders. For fermionic topological orders, let $D_{g,\sigma}^e$ ($D_{g,\sigma}^o$) be the degeneracy with even (odd) number of fermions on genus- g surface with spin structure σ . Then we have $2cD_{g,\sigma}^e \in \mathbb{Z}$ and $2cD_{g,\sigma}^o \in \mathbb{Z}$ for $g \geq 3$.

I. INTRODUCTION

In 1989, through a theoretical study of chiral spin liquid,[1, 2] we realized that there exists a new kind of order – topological order[3–5] – beyond Landau symmetry breaking theory. Topological order cannot be characterized by the local order parameters associated with the symmetry breaking. However, topological order can be characterized/defined by the following macroscopic properties: (a) the topology-dependent ground state degeneracy[3, 4] and (b) the non-Abelian geometric phases of the degenerate ground states as we deform the metrics of the space.[5, 6] Both of the above macroscopic properties are *robust against any local perturbations* that can break any symmetries.[4] This is just like superfluid order which is characterized/defined by zero-viscosity and quantized vorticity that are robust against any local perturbations that preserve the $U(1)$ symmetry.

For topological orders in 2+1-dimensional (2+1D) spacetime, the non-Abelian geometric phases of the degenerate ground states encode the chiral central charge c of the topological order via the gravitational Chern-Simons term in the effective action.[7–9] (The edge of a topological order contains right-movers with central charge c_R , and left-movers with central charge c_L . The chiral central charge $c = c_R - c_L$.[10, 11]) In this paper, we propose a direct relation between the ground state degeneracy D_g on a genus- g space and the chiral central charge c of the edge states, for 2+1D topological orders in bosonic systems:

$$\frac{1}{2}cD_g \in \mathbb{Z} \quad \text{for } g \geq 3. \quad (1)$$

For 2+1D topological orders in fermionic systems, we propose

$$2cD_{g,\sigma}^e \in \mathbb{Z}, \quad 2cD_{g,\sigma}^o \in \mathbb{Z} \quad \text{for } g \geq 3, \quad (2)$$

where $D_{g,\sigma}^e$ ($D_{g,\sigma}^o$) is the degeneracy with even (odd) number of fermions on genus- g surface with spin structure σ . This result can be derived from the characterization of a topological order in terms of a complex vector bundle on the moduli space $\mathcal{M}_{\Sigma^d}$ of a closed space Σ^d , where the fiber is the degenerate ground states on Σ^d . For an introduction on vector bundle, as well as Chern and Pontryagin classes, see Ref. 12. In Appendix, we also discuss how to use the partition function on arbitrary closed spacetime M^D and the resulting complex function on $\mathcal{M}_{M^D}$, the moduli space of M^D , to characterize topological orders.

In this paper, we will use d or 1d, 2d, *etc* to denote the space dimension and D or 1+1D, 2+1D, *etc* to denote the spacetime dimension. We will only consider anomaly-free topological orders, *i.e.* the topological orders that can be realized by local lattice models in the same dimension. In contrast, an anomalous topological order can only be realized as a boundary of the local lattice model in one higher dimension.[13, 14]

II. PROBE AND MEASURE THE TOPOLOGICAL ORDERS – HAMILTONIAN APPROACH

The main issue that we are going to discuss in this paper is how to probe and measure different topological orders using macroscopic properties. Here “probe and measure” means the methods in experiments and/or numerical calculations that allow us to distinguish different topological orders.

A. Complex vector bundle on moduli space of space manifold

In Hamiltonian approach, an anomaly-free topological order is described by a local bosonic Hamiltonian acting on a Hilbert space $\mathcal{V}_{\text{tot}}$ with a tensor product decomposition

$$\mathcal{V}_{\text{tot}} = \bigotimes_i \mathcal{V}_i \quad (3)$$

where $\mathcal{V}_i$ is the finite-dimensional Hilbert space for site i . The Hamiltonian is required to be well defined for arbitrary space Σ^d with arbitrary triangulation. Since anomaly-free topological orders are gapped, we require the Hamiltonian on a closed space Σ^d to be gapped, whose degenerate ground states form a finite dimensional vector space $\mathcal{V}_{\text{grnd}}$, which is a subspace of the total Hilbert space $\mathcal{V}_{\text{tot}}$ of the boson system. Let $\mathcal{M}_{\Sigma^d}$ be the moduli space for closed space Σ^d with different metrics and $\mathcal{M}$ the disjoint union of these moduli spaces. We see that there is a ground-state vector space $\mathcal{V}_{\text{grnd}}$ for every point in $\mathcal{M}_{\Sigma^d}$. Therefore, for each Σ^d , an anomaly-free topological order gives rise to a complex vector bundle $\mathcal{E}_{\Sigma^d}$ with fiber $\mathcal{V}_{\text{grnd}}$ and base space $\mathcal{M}_{\Sigma^d}$. It was proposed in Ref. 5 that

Claim 1: *the complex vector bundle*

$$pt \rightarrow \mathcal{V}_{\text{grnd}} \rightarrow \mathcal{E}_{\Sigma^d} \rightarrow \mathcal{M}_{\Sigma^d} \rightarrow pt \quad (4)$$

of degenerate ground states on $\mathcal{M}_{\Sigma^d}$ (for all Σ^d 's) fully characterize an anomaly-free topological order in Hamiltonian formalism.

To understand the vector bundle $\mathcal{E}_{\Sigma^d}$, let us use $G_{\text{homeo}}(\Sigma^d)$ to denote the orientation preserving homeomorphism group of the space Σ^d . Note that $G_{\text{homeo}}(\Sigma^d)$ only depends on the topology of Σ^d and is the same for every point $\Sigma^d \in \mathcal{M}_{\Sigma^d}$. Let us use $G_{\text{homeo}}^0(\Sigma^d)$ to denote the subgroup of $G_{\text{homeo}}(\Sigma^d)$ which is the connected component of $G_{\text{homeo}}(\Sigma^d)$ that contain identity. The mapping class group is formed by the discrete components of the homeomorphism group:

Definition 1. mapping class group

$$\text{MCG}(\Sigma^d) \equiv G_{\text{homeo}}(\Sigma^d)/G_{\text{homeo}}^0(\Sigma^d) = \pi_0[G_{\text{homeo}}(\Sigma^d)].$$

We note that every homeomorphism $g : \Sigma^d \rightarrow \Sigma^d$ in $\text{MCG}(\Sigma^d)$ defines a mapping torus $\Sigma^d \times_g S^1$ that describes how Σ^d deform around a loop S^1 , and corresponds to an element in $\pi_1(\mathcal{M}_{\Sigma^d})$. Thus $\pi_1(\mathcal{M}_{\Sigma^d}) = \text{MCG}(\Sigma^d)$.

Along a loop g in $\pi_1(\mathcal{M}_{\Sigma^d})$, the fiber bundle gives us a monodromy $U(g)$ which is a unitary matrix acting on the ground state vector space $\mathcal{V}_{\text{grnd}}$. We may view g as an element in the group $\text{MCG}(\Sigma^d)$. So $U(g)$ gives a projective representation of $\text{MCG}(\Sigma^d)$. [7]

To understand why we only get a projective representation, we note that the topological robustness of the ground state degeneracy implies that the unitary matrix $U(\text{id})$ for a contractible loop $g = \text{id}$ in the moduli space

$\mathcal{M}_{\Sigma^d}$ must be a pure over-all phase (which can be path dependent), so that $U(\text{id})$ cannot distinguish (or split) the degenerate ground states. This is because the periodic time evolution along a contractible loop over and over again can be simulated by a local Hamiltonian. If $U(\text{id})$ can distinguish the degenerate ground states, then there is a local Hamiltonian that can distinguish and split the degenerate ground states. This contradicts with the fact that the topological degeneracy of ground states cannot be lifted by any local Hamiltonian. Similarly, $U(g)$ may also depend on paths, but the path-dependent part must be an over-all phase. This leads to the projective representation of $\text{MCG}(\Sigma^d)$. We also like to mention that the trace of $U(g)$ is the volume-independent partition function (see Section A 1 for details) on the corresponding mapping torus:

$$\text{Tr } U(g) = Z^{\text{top}}(\Sigma^d \times_g S^1). \quad (5)$$

As a result, we obtain

$$|Z^{\text{top}}(\Sigma^d \times S^1)| = \text{ground state degeneracy on } \Sigma^d. \quad (6)$$

For spaces with different topologies, we get different projective representations. Those finite dimensional projective representations are the non-Abelian geometric phases of the degenerate ground states introduced in Ref. 5 and 6. Certainly, the non-Abelian geometric phases contain more information than the projective representations. They contain all the information about the vector bundle $\mathcal{E}_{\Sigma^d}$ on $\mathcal{M}_{\Sigma^d}$. Therefore, we believe that such geometric phases for closed d -dimensional spaces Σ^d of all topologies fully characterize the topological order.

We like to remark that for a generic system, its vector bundle $\mathcal{E}_{\Sigma^d}$ is generally not flat. The curvature of the vector bundle can change as we deform the Hamiltonian locally. However, for some topological orders, its vector bundle $\mathcal{E}_{\Sigma^d}$ cannot be made flat no matter how we fine tune the Hamiltonian. In this case, the vector bundle $\mathcal{E}_{\Sigma^d}$ is topologically different from a flat bundle. In this case, the volume-independent partition function on mapping torus $Z^{\text{top}}(\Sigma^d \times_g S^1)$ cannot be topological (*i.e.* the volume-independent partition function cannot be a constant on a connected piece of the moduli space $\mathcal{M}_{\Sigma^d}$). It must depend on the metrics of the space-time $\Sigma^d \times_g S^1$.

It is very strange since the bosonic system has short range correlation and a finite energy gap. In the thermodynamical limit, the space-time becomes flat, and the bosonic system should not be able to sense the geometry of the space-time. The fact that the partition function does depend on the metrics of the space-time means that the entanglement in the ground state can still sense the geometry of the space in the flat limit. We like to link such a geometric sensitivity to the gapless nature of boundary excitations and entanglement spectrum:

Claim 2: *The ground state vector bundle $\mathcal{E}_{\Sigma^d}$ over $\mathcal{M}_{\Sigma^d}$ can be deformed into a flat bundle if and only if the boundary of the corresponding anomaly-free topological order is gappable.*

What is the obstruction that prevent the vector bundle to be flat? First, for a contractible loop $g = \text{id}$, $U(\text{id})$ is a pure $U(1)$ phase. So the non-flat part is only contained in the $U(1)$ phase of the complex vector bundle. We can examine it by considering the determinant line bundle $\mathcal{E}_{\Sigma^d}^{\text{det}}$ of the vector bundle $\mathcal{E}_{\Sigma^d}$, which is a complex line bundle over $\mathcal{M}_{\Sigma^d}$.

To connect the determinant line bundle to the partition function of the system, let us consider a contractible loop S^1 in $\mathcal{M}_{\Sigma^d}$. We have mentioned that the partition function for the spacetime $\Sigma^d \times S^1$ is given by the monodromy $U(\text{id}) = e^{i\theta}$ along the loop (see eqn. (5) and remember that $U(\text{id})$ is pure phase factor). Therefore the partition function of the system on spacetime $\Sigma^d \times S^1$ (see eqn. (13) for an example) is given by

$$Z^{\text{top}}(\Sigma^d \times S^1) = D_{\Sigma^d} e^{i\theta}$$

or $[Z^{\text{top}}(\Sigma^d \times S^1)]^{D_{\Sigma^d}} = (D_{\Sigma^d})^{D_{\Sigma^d}} \text{Det}U(\text{id}), \quad (7)$

where $D_{\Sigma^d} = \text{Dim}U(\text{id})$ is the ground state degeneracy on the closed space Σ^d , and $\text{Det}U(\text{id})$ is the monodromy of the determinant line bundle around the loop S^1 . Now, let us assume that the loop S^1 is the boundary of a 2-dimension submanifold $B \subset \mathcal{M}_{\Sigma^d}$: $S^1 = \partial B$. We can rewrite that above as (for the $U(1)$ phase factor)

$$[Z^{\text{top}}(\Sigma^d \times S^1)]^{D_{\Sigma^d}} \propto \text{Det}U(\text{id}) = e^{i2\pi \int_B C}$$

$$= e^{i2\pi D_{\Sigma^d} \int_{\Sigma^d \times B} P}, \quad (8)$$

where C is the curvature tensor on the moduli space $\mathcal{M}_{\Sigma^d}$ for the determinant line bundle. The next expression $e^{i2\pi D_{\Sigma^d} \int_{\Sigma^d \times B} P}$ is motivated by noticing that $Z^{\text{top}}(\Sigma^d \times S^1)$ is given by a gravitational Chern-Simon term ω

$$Z^{\text{top}}(\Sigma^d \times S^1) \propto e^{i2\pi \int_{\Sigma^d \times S^1} \omega} \quad (9)$$

In Appendix A1, we discussed how to define the volume-independent partition function Z^{top} . A key character of Z^{top} is that it is invariant under the scaling transformation of spacetime $(t, x, y, \dots) \rightarrow (ct, cx, cy, \dots)$. Since the low energy effective action for gapped liquid state is local and does not contain operators like $\frac{\partial x}{\partial y}$, the above isotropic scale invariance implies an invariance of Z^{top} under a more general non-isotropic scaling transformation $(t, x, y, \dots) \rightarrow (c_t t, c_x x, c_y y, \dots)$. The gravitational Chern-Simon term ω is believed to be the only term that is invariant under the non-isotropic scaling transformation, and at the same time depends on the spacetime metrics. There are other topological terms that are invariant under the non-isotropic scaling transformation. But those terms are invariant under any local deformation of the spacetime metrics. Here we ignore such local constant terms, because they do not contribute to $\int_{\Sigma^d \times B} P$ in eqn. (8). In 2+1D, some explicit calculations have been done,[7–9] which confirm the above picture.

We can rewrite eqn. (9) in term of a linear combination of Pontryagin class on $\Sigma^d \times B$

$$Z^{\text{top}}(\Sigma^d \times S^1) \propto e^{i2\pi \int_{\Sigma^d \times S^1} \omega} = e^{i2\pi \int_{\Sigma^d \times B} P} \quad (10)$$

since $\partial(\Sigma^d \times B) = \Sigma^d \times S^1$. Here $\Sigma^d \times B$ is a fiber bundle with the space Σ^d as the fiber and B as the base manifold. Also $P = d\omega$ is a linear combination of Pontryagin class on $\Sigma^d \times B$:

$$P = \sum_{n_1, n_2, \dots} \kappa_{n_1 n_2 \dots} P_{n_1 n_2 \dots}$$

$$P_{n_1 n_2 \dots} = p_{n_1} p_{n_2} \dots, \quad \kappa_{n_1 n_2 \dots} \in \mathbb{Q}, \quad (11)$$

and p_n is the n^{th} Pontryagin class. This leads to the expression eqn. (8). This is a key assumption in this paper.

Now, let us shrink the loop S^1 to a point and B becomes a closed 2-dimensional submanifold in $\mathcal{M}_{\Sigma^d}$. Then, $\int_B C$ becomes the Chern number of the line bundle $\mathcal{E}_{\Sigma^d}^{\text{det}}$ on B , which always is an integer. We obtain

$$D_{\Sigma^d} \int_{\Sigma^d \times B} P = \text{integer}. \quad (12)$$

This expression gives us a constraint between the ground state degeneracy D_{Σ^d} and the gravitational Chern-Simons term in the effective theory. It is the main result of this paper. We remark that the above result is obtained with an assumption that the ground states of topological order can be put the closed space Σ^d without the need to create some topological excitations. Otherwise, the above can still be valid if we set $D_{\Sigma^d} = 0$ when we have to create topological excitations. We also like to point out the first Chern class C (*i.e.* the collection Chern numbers $\int_B C$ for all closed 2-dimensional subspaces of $\mathcal{M}_{\Sigma^d}$) completely classify the line bundle $\mathcal{E}_{\Sigma^d}^{\text{det}}$.

Let us consider an example of 2d theory whose gravitational response contains the gravitational Chern-Simons term:[7–9]

$$Z^{\text{top}}(\Sigma^2 \times S^1) = e^{i \frac{2\pi c}{24} \int_{\Sigma^2 \times S^1} \omega_3}, \quad d\omega_3 = p_1, \quad (13)$$

where c is the chiral central charge of the edge states. For such a theory, eqn. (12) becomes

$$\frac{c}{24} D_g \int_{\Sigma^2 \times B^2} p_1 = \text{integer}, \quad (14)$$

for any surface bundle $\Sigma^2 \times B^2$, where D_g is the ground state degeneracy on Σ^2 , and g is the genus of Σ^2 .

Since $\int_{\Sigma^2 \times B^2} p_1 \neq 0$ for some surface bundle, $\int_{B^2} C \neq 0$ for some B and the vector bundle $\mathcal{E}_{\Sigma^d}$ is not flat if $c \neq 0$. So the appearance of the gravitational Chern-Simons term implies that the vector bundle $\mathcal{E}_{\Sigma^d}$ is not flat.

It was shown that $\int_{\Sigma^2 \times B^2} p_1 = 0 \pmod{12}$ for any orientable surface bundles.[15, 16] If the genus of the fiber Σ^2 is equal or less than 2, then $\int_{\Sigma^2 \times B^2} p_1 = 0$. [15, 17, 18] If the genus of the fiber Σ^2 is equal or greater than 3, then we can always find a base manifold B^2 with a genus equal or less than 111, such that there is a surface bundle

$\Sigma^2 \times B^2$ with $\int_{\Sigma^2 \times B^2} p_1 = \pm 12$.^[18] Therefore,¹

Claim 3: for a 2d bosonic topological orders, the chiral central charge of the edge state is quantized as $cD_g/2 \in \mathbb{Z}$ for $g \geq 3$, where D_g is the ground state degeneracy on genus- g space.

The above result implies that the chiral central charge c is a rational number, which was proven via some other methods.^[19, 20]

Let us give some non-trivial checks for Claim 3.

Application 1. For a bosonic quantum Hall state with one branch of edge mode (*i.e.* $c = 1$), the ground state degeneracy D_g must be even for $g \geq 3$.

Application 2. For a 2d bosonic topological order, we can use i, j, k to label the topological excitations. The fusion rule is given by $i \otimes j = \oplus_k N_{ij}^k k$. The ground state degeneracy D_g is then given by^[21]

$$\begin{aligned} D_g &= \text{Tr} \left(\sum_i N_i N_{\bar{i}} \right)^{g-1} = \sum_i S_{1i}^{-2(g-1)} \\ &= \left(\sum_i d_i^2 \right)^{g-1} \sum_i d_i^{-2(g-1)} \end{aligned} \quad (15)$$

where $\bar{i}$ is the antiparticle of i , the matrix N_i is given by $(N_i)_j^k = N_{ij}^k$, and d_i is the quantum dimension (the largest eigenvalue of N_i). Also S_{ij} is the matrix elements of the S -matrix that characterizes the topological order.^[5, 6] We have $S_{1i} = \frac{d_i}{\sqrt{\sum_i d_i^2}}$

For filling fraction $\nu = 1$ bosonic Pfaffian quantum Hall state, we have $(d_i) = (1, 1, \sqrt{2})$. We find that $D_1 = 3$, $D_2 = 10$, $D_3 = 36$, $D_4 = 136$, $D_5 = 528$, *etc.* Therefore the chiral central charge must be quantized as $c = 0 \bmod 1/2$, which agrees with $c = 3/2$. We also see that $cD_g/2 = \text{integer}$ is not valid for $g = 2$.

For the Fibonacci topological order with $(d_i) = (1, \frac{\sqrt{5}+1}{2})$ and $c = 14/5$, we find that $D_1 = 2$, $D_2 = 5$, $D_3 = 15$, $D_4 = 50$, $D_5 = 175$, *etc.* Indeed, $cD_g/2 = \text{integer}$ for $g \geq 3$. We explicitly checked over 140 simple topological orders listed in Ref. 22, and find that Claim 3 is valid for those bosonic topological orders.

Application 3. The chiral central charge of 2d invertible anomaly-free topological order is quantized as $c = 0 \bmod 2$, since $D_g = 1$. A known 2d invertible anomaly-free topological order is the E_8 state, which has $c = 8$. At the moment, we do not know if the minimal chiral central charge $c = 2$ can be realized by a 2d invertible anomaly-free topological order.

If we have a fermionic system, both Σ^d and $\Sigma^d \times B^2$ should be chosen to be spin manifolds. In this case Σ^d can have a spin structure, denoted as σ , which can be

extended to $\Sigma^d \times B^2$. The ground states on Σ^d can carry even or odd numbers of fermions. We denote the ground state degeneracy with even fermions as $D_{\Sigma^d, \sigma}^e$ and that with odd fermions as $D_{\Sigma^d, \sigma}^o$. We note that the even and odd sectors do not mix due to the conservation of fermion number parity. Therefore, we have two vector bundles on the modular space $\mathcal{M}_{\Sigma^d}$.

In 2-dimensional space ($d = 2$), when $\Sigma^2 \times B^2$ is spin, we have $\int_{\Sigma^2 \times B^2} p_1 = 0 \bmod 48$ for any spin surface bundles.^[15, 17] Assuming that $\int_{\Sigma^2 \times B^2} p_1 = \pm 48$ can be realized for some surface bundle $\Sigma^2 \times B^2$ if the genus of Σ^2 is greater than 2, we find that

Claim 4: For fermionic topological orders, the chiral central charge is quantized as

$$2cD_{g, \sigma}^e \in \mathbb{Z}, \quad 2cD_{g, \sigma}^o \in \mathbb{Z}, \quad g \geq 3, \quad (16)$$

where $D_{g, \sigma}^e$ ($D_{g, \sigma}^o$) is the ground state degeneracy on closed genus- g surface with spin structure σ and even (odd) number of fermions.

For fermionic invertible topological orders, we have $D_{g, \sigma}^e + D_{g, \sigma}^o = 1$ and the chiral central charge is quantized as $c = 0 \bmod 1/2$. The minimal chiral central charge $c = 1/2$ for fermionic invertible topological orders can be realized by $p + ip$ superconductor, which indeed contain no non-trivial topological excitations.

For a 2d fermionic topological order, the quantum dimensions of excitations appear pairs of equal values: $d_{2i} = d_{2i+1}$ ^[23]. Many 2d fermionic topological orders are stacking of a fermionic trivial product state and bosonic topological orders with quantum dimensions d_i^B . In this case, we either have $D_{g, \sigma}^e \neq 0$, $D_{g, \sigma}^o = 0$ or $D_{g, \sigma}^e = 0$, $D_{g, \sigma}^o \neq 0$. The total ground state degeneracy $D_{g, \sigma} = D_{g, \sigma}^e + D_{g, \sigma}^o$ is independent of spin structure. To compute $D_{g, \sigma}$, we note that the quantum dimensions for the resulting fermion topological order are given by $d_{2i} = d_{2i+1} = d_i^B$, and the ground state degeneracies are the same as the corresponding bosonic topological order: $D_{g, \sigma} = D_{g, \sigma}^e + D_{g, \sigma}^o = \left(\sum_i (d_i^B)^2 \right)^{g-1} \sum_i (d_i^B)^{-2(g-1)}$, and we obtain

$$D_{g, \sigma} = D_{g, \sigma}^e + D_{g, \sigma}^o = \left(\frac{1}{2} \sum_i d_i^2 \right)^{g-1} \frac{1}{2} \sum_i d_i^{-2(g-1)}. \quad (17)$$

Amazingly, when we apply the above formula to more general fermionic topological orders obtained in [23], the above expression always give us integers which satisfy $2cD_{g, \sigma} \in \mathbb{Z}$ for $g \geq 3$.

We like to remark that for fermionic topological orders, $D_{\Sigma^d, \sigma}^e$ and $D_{\Sigma^d, \sigma}^o$ may depend on spin structure σ (see Ref. 24 and 25). It is not clear whether $D_{\Sigma^d, \sigma}^e + D_{\Sigma^d, \sigma}^o$ depends on spin structure or not. For the examples examined in Ref. 24 and 25, $D_{\Sigma^d, \sigma}^e + D_{\Sigma^d, \sigma}^o$ does not depend on spin structure.

¹ Claim 3 was first obtained in our long unpublished paper Ref. 14. This paper simplifies and extends the result to get it published.

B. No non-trivial bosonic topological order in 1d space

Next let us consider bosonic 1d topological orders. Since $\text{MCG}(S^1)$ is trivial, $\mathcal{M}_{\Sigma^1}$ is simply connected. Since the Pontryagin classes for circle bundle $S^1 \times B$ all vanishes, the determinant bundle of the vector bundle $\mathcal{E}_{\Sigma^1}^{\text{det}}$ over $\mathcal{M}_{\Sigma^1}$ can always be deformed into a flat one. Thus the vector bundle $\mathcal{E}_{\Sigma^1}$ can be flat. Such a vector bundle is always trivial since $\mathcal{M}_{\Sigma^1}$ is simply connected. Therefore, all bosonic anomaly-free 1d topological orders are trivial.

It appears that the vector bundle $\mathcal{E}_{\Sigma^d}$ on $\mathcal{M}_{\Sigma^d}$ is a high resolution characterization of the anomaly-free topological order. The non-trivial anomaly-free topological order should lead to a non-trivial vector bundle $\mathcal{E}_{\Sigma^d}$. On the other hand, since the structure of the vector bundle can be so rich, it is very likely that not every allowed vector bundle $\mathcal{E}_{\Sigma^d}$ on $\mathcal{M}_{\Sigma^d}$ can be realized by anomaly-free topological orders.

C. How to probe and measure the boundary-gappable topological orders

For a boundary-gappable topological order, the vector bundle on $\mathcal{M}_{\Sigma^d}$ can always be deformed into a flat one. In fact, the boundary-gappable topological orders can be realized by renormalization-group fixed-point Hamiltonians, which are formed by commuting projectors[26], or by renormalization-group fixed-point Lagrangians which are re-triangulation invariant[27, 28] (see next Section). The vector bundles on $\mathcal{M}_{\Sigma^d}$ obtained from those fixed-point systems are always flat, and the partition functions on mapping torus are always topological (which are the state-sum topological invariants[27, 28]). For a flat vector bundle, the unitary matrices $U(g)$ (the monodromies for non-contractible loops) form a representation (instead of a projective representation) of the mapping class group $\text{MCG}(\Sigma^d)$ which fully characterize the flat bundle:[5]

Claim 5: *A boundary-gappable topological order is fully characterized by a collection of representations of the mapping class groups $\text{MCG}(\Sigma^d)$ for various spatial topologies.*

In particular, the representations of $\text{MCG}(\Sigma^d)$ can be computed via the universal wave function overlap[29–31] or tensor network calculations.[32–35]

For 2d boundary-gappable topological orders, the representations of the mapping class group $\text{MCG}(\Sigma_1)$ for genus-1 torus are called the modular data, which already carry a lot of information about the topological orders[6, 22, 36]. However, recently in Ref. 37, Mignard and Schauenburg found some different topological orders that have the same modular data and chiral central charge. Thus, modular data and chiral central charge are not enough to fully characterize topological order. In Ref. 38, it was shown that if we include the rep-

resentations of the mapping class group $\text{MCG}(\Sigma_2)$ for genus-2 surface, then those topological orders can be distinguished. This supports the conjecture proposed in Ref. 5, that the representations of the mapping class groups $\text{MCG}(\Sigma_g)$ for all genus- g surfaces, plus the chiral central charge, can fully characterize 2d topological orders.

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Appendix A: Probe and measure topological orders – path integral approach

In this appendix, we will only consider bosonic systems. Bosonic topological orders can also be realized by path integral on triangulated spacetime M^D . Here, we will discuss how to characterize topological orders using path integral approach and the resulting partition functions.

1. Topological partition function

If the path integral is described by a well defined quantum field theory (such as those that can be regularized by a tensor network path integral) that has no long range correlations, it will describe an anomaly-free topological order. But how to determine which anomaly-free topological order that the path integral produces? How to determine whether two path integrals give rise to the same anomaly-free topological order or not?

One universal way to characterize all the topological orders is via the partition function of the system. In general, a partition function on a closed spacetime M^D may have a form

$$Z(M^D) = e^{-c_D L^D - c_{D-1} L^{D-1} - \dots - c_0 L^0 - c_{-1} L^{-1} - \dots}, \quad (\text{A1})$$

where L is the linear size of M^D . If the ground state does not contain point-like defects, then $c_1 = 0$ since there is no worldline in spacetime. Similarly, if the ground state does not contain string-like, membrane-like *etc* defects, then $c_2 = c_3 = \dots = c_{d-1} = 0$. In this case, we can define a volume-independent partition function via

$$Z^{\text{top}}(M^D) \equiv \lim_{L \rightarrow \infty} \frac{Z(M^D)}{e^{-c_D L^D}} = e^{-c_0 L^0} \quad (\text{A2})$$

When the calculated volume-independent partition function vanishes, it does not mean the partition function

vanishes. It just means that $c_i > 0$, for some $0 < i < D$. This implies that the given space-time topology M^D must contain point-like, string-like, *etc* topological excitations.

We like to remark that it is not yet proven that using the above procedure to define volume-independent partition function $Z^{\text{top}}(M^D)$ always works. The assumption that $Z^{\text{top}}(M^D)$ can be well defined equivalent to the assumption that the partition function of a topological quantum field theory can be defined via a microscopic path integral calculation. However, in Ref. 39, it is shown that only some ratios of $Z^{\text{top}}(M^D)$ are well defined. For example we may choose triangulated manifold which contain four disjoint pieces: $\tilde{M} = M_U \sqcup M_D \sqcup N_U \sqcup N_D$, where the boundaries of M_U , M_D , N_U , and N_D are simplicial complexes all isomorphic to B . Gluing the bound-

aries in different ways gives rise to $M_1 = \left(\begin{smallmatrix} M_U \\ M_D \end{smallmatrix} \right)_B \left(\begin{smallmatrix} N_U \\ N_D \end{smallmatrix} \right)_B$ and

$M_2 = \left(\begin{smallmatrix} N_U \\ M_D \end{smallmatrix} \right)_B \left(\begin{smallmatrix} M_U \\ N_D \end{smallmatrix} \right)_B$. The ratio of $Z^{\text{top}}(M^D)$'s is given by

$$\begin{aligned} & \frac{Z^{\text{top}}\left(\left(\begin{smallmatrix} M_U \\ M_D \end{smallmatrix}\right)_B \left(\begin{smallmatrix} N_U \\ N_D \end{smallmatrix}\right)_B\right)}{Z^{\text{top}}\left(\left(\begin{smallmatrix} N_U \\ M_D \end{smallmatrix}\right)_B \left(\begin{smallmatrix} M_U \\ N_D \end{smallmatrix}\right)_B\right)} \\ &= \frac{Z\left(\left(\begin{smallmatrix} M_U \\ M_D \end{smallmatrix}\right)_B \left(\begin{smallmatrix} N_U \\ N_D \end{smallmatrix}\right)_B\right)}{Z\left(\left(\begin{smallmatrix} N_U \\ M_D \end{smallmatrix}\right)_B \left(\begin{smallmatrix} M_U \\ N_D \end{smallmatrix}\right)_B\right)} \\ &= \frac{Z(M_U \cup M_D)Z(N_U \cup N_D)}{Z(N_U \cup M_D)Z(M_U \cup N_D)} \end{aligned} \quad (\text{A3})$$

since the volume term cancel exactly. Such ratios are topological invariants.

2. The gravitational Chern-Simons term and winding numbers

To use partition function to characterize topological order, we consider volume-independent partition function $Z^{\text{top}}(M^D)$ on closed space-time manifolds M^D . To understand the universal structures in the partition functions, let us use $\mathcal{M}_{M^D}$ to denote the moduli space of the closed space-time M^D with different metrics but the same topology. Then the partition function $Z^{\text{top}}(-)$ can be viewed as a map from $\mathcal{M}_{M^D}$ to $\mathbb{C}$.

Claim 6: *If a path integral describes a short-range correlated systems, then its volume-independent partition function on the moduli space $\mathcal{M}_{M^D}$ of a closed space-time M^D is either always non-zero: $Z^{\text{top}}(M^D) \neq 0$, or always zero: $Z^{\text{top}}(M^D) = 0$.*

Proof. If a volume-independent partition function is zero at some isolated points (or regions) of the moduli space $\mathcal{M}_{M^D}$, then a small local perturbation will make it non-zero. This will represent a diverging response, which

should not occur for short-range correlated systems. Therefore, $Z^{\text{top}}(M^D)$ is either always non-zero or always zero. $\square$

So the non-zero volume-independent partition function $Z^{\text{top}}(\cdot)$ is actually a map $Z^{\text{top}} : \mathcal{M}_{M^D} \rightarrow \mathbb{C} - \{0\} \sim U(1)$. If $\pi_1(\mathcal{M}_{M^D}) \neq 0$, such map may have a non-trivial winding number.

Since $\pi_1(\mathcal{M}_{M^D}) = \text{MCG}(M^D)$, the winding number is a group homomorphism $\text{MCG}(M^D) \rightarrow \mathbb{Z} = \pi_1(U(1))$. So the winding numbers (*i.e.* the group homomorphisms) always form integer classes $\mathbb{Z}$. This leads us to believe that the winding numbers (or the group homomorphism $\text{MCG}(M^D) \rightarrow \mathbb{Z}$) are always realized by the partition function $Z^{\text{top}}(M^D)$ that contains the gravitational Chern-Simons term ω_D

$$Z^{\text{top}}(M^D) \sim e^{i \int_{M^D} \omega_D} \quad (\text{A4})$$

where the gravitation Chern-Simons term ω_D is given by

$$d\omega_D = \sum \kappa_{n_1 n_2 \dots} P_{n_1 n_2 \dots}, \quad (\text{A5})$$

and $P_{n_1 n_2 \dots}$ is a combination of Pontryagin classes which are the only integer characteristic classes of oriented manifolds. Noticing that the winding numbers are given by $\frac{1}{2\pi} \int_{M^D \searrow_f S^1} d\omega_D$, we have

Claim 7:

$$\int_{M^D \searrow_f S^1} \frac{d\omega_D}{2\pi} = \sum \frac{\kappa_{n_1 n_2 \dots}}{2\pi} \int_{M^D \searrow_f S^1} P_{n_1 n_2 \dots} \in \mathbb{Z} \quad (\text{A6})$$

for any mapping torus $M^D \searrow_f S^1$ where $Z^{\text{top}}(M^D)$ is non zero.

Such type of winding numbers and the partition function exist only when the spacetime dimension $D = 4k+3$, since the gravitational Chern-Simons term exists only in $D = 4k+3$ spacetime dimensions. We also note that there is always one combination of Pontryagin classes for each $D = 4k+3$ (corresponding to the signature σ of the manifold), whose value on mapping torus is always zero (see Corollary 1.0.6. in Ref. 40). For such Pontryagin class, the corresponding gravitational Chern-Simons term ω_D^σ can have an unquantized coefficient. For example, in 2+1D, the naive consideration of diffeomorphism invariance appear to require the gravitational Chern-Simons term ω_3 to have a coefficient that is quantized as $\kappa = \frac{2\pi c}{24}$, $c = 0 \bmod 8$. However, the above more careful consideration indicates that c does not have to be quantized as $c = 0 \bmod 8$. This is consistent with the well known fact that there are many 2+1D anomaly-free topological orders with c different from $0 \bmod 8$, despite c must satisfy certain conditions (see Claim 3 and 4).

Clearly, two bosonic systems that give rise to partition functions with different winding numbers must belong to two different phases. So the winding numbers of partition functions are a type of topological invariants that can be used to probe and measure the anomaly-free topological orders.

3. Beyond winding numbers

To have an example of topological orders with non-zero winding numbers, we note that invertible topological order are described by volume-independent partition functions that are pure $U(1)$ phase.

Claim 8: *Since it has only trivial excitations, the volume-independent partition function of an invertible topological order is non-zero for any closed spacetime manifold M^D .*

In particular the $\mathbb{Z}$ -class of invertible topological order (such as the E_8 quantum Hall state in $D = 3$) are described by

$$Z^{\text{top}}(M^D) = e^{i2\pi \int_{M^D} \omega_D}. \quad (\text{A7})$$

We have shown that $\kappa_{n_1 n_2 \dots}$ is quantized (see eqn. (A5)) if $P_{n_1 n_2 \dots}$ is non zero on some mapping torus.

However, for invertible topological orders, $\kappa_{n_1 n_2 \dots}$ can be quantized even if $P_{n_1 n_2 \dots}$ is zero on any mapping torus. To see this, we need to consider more general “loop”, *i.e.* more general “mapping torus”, where the topology of the fiber M^d can change as we go along the loop. In this case, the more general “mapping torus” can be any closed $D+1$ -dimensional manifold M^{D+1} . Thus we require that

Claim 9: *for invertible bosonic topological orders*

$$e^{i2\pi \int_{M^{D+1}} d\omega_D} = e^{i2\pi \sum \kappa_{n_1 n_2 \dots} \int_{M^{D+1}} P_{n_1 n_2 \dots}} = 1 \quad (\text{A8})$$

for any closed M^{D+1} , which leads to a quantization of $\kappa_{n_1 n_2 \dots}$.

We note that even ω_D^σ is required to have a quantize coefficient. For example, for 2+1D invertible topological orders

$$\begin{aligned} Z^{\text{top}}(M^3) &= e^{i\kappa \int_{M^3} \omega_3} = e^{i \frac{2\pi c}{24} \int_{M^3} \omega_3} \\ &= e^{i \frac{2\pi c}{24} \int_{M^4} p_1}, \quad \partial M^4 = M^3, \end{aligned} \quad (\text{A9})$$

where $c \equiv 12\kappa/\pi$ must be quantized as $0 \bmod 8$, since $\int_{N^4} p_1 = 0 \bmod 3$ for closed 4-manifold N^4 . In fact c is the chiral central charge of the edge states and the above partition function describes the stacking of $c/8$ E_8 quantum Hall states.

4. Probe and measure the anomaly-free topological orders

In the last section, we have discussed the quantization of $\kappa_{n_1 n_2 \dots}$ for invertible topological orders. For non-invertible topological orders, we also have gravitational Chern-Simons terms, and have non-zero $\kappa_{n_1 n_2 \dots}$. In this case, $\kappa_{n_1 n_2 \dots}$ are also quantized. However, the quantization condition are weaker, since the volume-independent partition function is non-zero only on some sub-class of closed manifolds. For example, for bosonic topological orders with emergent fermions, the volume-independent

partition function vanishes on spacetime that is not spin. The volume-independent partition function can be non-zero only on spin-manifolds. Only the spin-manifolds can impose the the quantization conditions on $\kappa_{n_1 n_2 \dots}$.

Also for non-invertible topological orders, the volume-independent partition function is not just a phase factor.

Claim 10: *The non-invertible topological orders are characterized by the following topological invariants:*

- (1) Quantized gravitational Chern-Simons terms (*i.e.* quantized $\kappa_{n_1 n_2 \dots}$'s)
- (2) The absolute values of volume-independent partition function $|Z^{\text{top}}(M^D)|$, on spacetime M^D with vanishing Euler and Pontryagin numbers (*i.e.* $|Z^{\text{top}}(M^D)|$ is a constant on moduli space $\mathcal{M}_{M^D}$).

5. How to probe and measure the boundary-gappable topological orders

We know that a boundary-gappable topological order can be described by a topological path integral that is independent of triangulation of space-time and independent of local change of space-time metrics. The topological path integral directly give rise to the volume-independent partition function $Z^{\text{top}}(M^D)$, which is constant on $\mathcal{M}_{M^D}$ locally. Such a topological path integral is a fixed-point of the renormalization group transformation. We propose that

Claim 11: *For boundary-gappable topological orders, we can use the volume-independent partition function $Z^{\text{top}}(M^D)$ of topological path integral to probe and measure them.*

This conjecture has lead to some related researches and is confirmed for simple boundary-gappable topological orders.[29–31] Since the topological path integral is triangulation invariant, we see that $Z^{\text{top}}(M^D)$ is not only independent of volume, it is also independent of shape. It only depends on the topology of M^D . Therefore, the volume-independent partition function $Z^{\text{top}}(M^D)$ is a topological invariant for D -manifold M^D . It might be even true that different boundary-gappable topological orders give different topological invariants for at least some M^D 's. In 2+1D, the topological invariants from boundary-gappable topological orders are the Turaev-Viro invariants for 3-manifolds.[27]

We like to remark that the relation between volume-independent partition functions $Z_{\text{top}}(M^D)$ and boundary-gappable topological orders is not one-to-one. Two volume-independent partition functions $Z_{\text{top}}(M^D)$ differ by a factor $W^{\chi(M^D)} e^{i \sum_{\{n_i\}} \phi_{n_1 n_2 \dots} \int_{M^D} P_{n_1 n_2 \dots}}$ actually describe the same topological order. Here W is a constant. This is because the factor $W^{\chi(M^D)} e^{i \sum_{\{n_i\}} \phi_{n_1 n_2 \dots} \int_{M^D} P_{n_1 n_2 \dots}}$ can be produced by local counter terms, which are deformations within the same phase.

6. Applications

For bosonic 2+1D invertible topological orders, its volume-independent partition function is non-zero in any closed orientable manifold M^3 . From the quantization $\int_M p_1 = 0 \bmod 3$, we find that the chiral central charge $c = 0 \bmod 8$. For 2+1D non-invertible topological orders with emergent fermions, its volume-independent partition functions must be zero on any closed orientable manifold M^3 that is not spin. From the quantization $\int_{M^4} p_1 = 0 \bmod 48$ for any spin manifold M^4 , we see that c only need to satisfy $2cn = 0 \bmod 1$ for a certain set of integers n . Since the partition function may not be non-zero for all spin manifolds, we cannot conclude $2cn = 0 \bmod 1$ for all integers.

The simplest class of 2+1D bosonic topological orders with emergent fermion has quantum dimensions $(d_i) = (1, 1, \sqrt{2})$. Their chiral central charge is given by

$c = \frac{2k+1}{2}$, $k \in \mathbb{Z}$. We believe that the partition functions for those topological orders are non-zero for any 2+1D spin manifolds since those topological orders can be obtained by gauging the fermion-number-parity in $p + ip$ superconductors in 2+1D. So the chiral central charge for those bosonic topological orders must satisfy $2cn = 0 \bmod 1$ for all integers n , i.e. $2c = 0 \bmod 1$. This is consistent with $c = \frac{2k+1}{2}$.

There is a 2+1D topological order whose edge states are described $SU(2)$ level 6 conformal field theory with chiral central charge $c = \frac{9}{4}$. Such a topological order also has an emergent fermion. Since the chiral central charge does not satisfy $2c = 0 \bmod 1$, the volume-independent partition functions for such a topological order cannot be non-zero on all spin manifolds. It is an open problem to understand on which class of manifolds that the volume-independent partition functions are non-zero.

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