On Parseval Frames of Kernel Functions in de Branges Spaces of Entire Vector Valued Functions

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Abstract. We consider the existence and structure properties of Parseval frames of kernel functions in vector valued de Branges spaces. We develop some sufficient conditions for Parseval sequences by identifying the main construction with Naimark dilation of frames. The dilation occurs by embedding the de Branges space of vector valued functions into a dilated de Branges space of vector valued functions. The embedding also maps the kernel functions associated with a frame sequence of the original space into a Riesz basis for the embedding space. We also develop some sufficient conditions for a dilated de Branges space to have the Kramer sampling property.

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1. Introduction

The theory of de Branges spaces of entire functions can be extended with suitable hypotheses to spaces of entire vector valued functions. Spaces of entire vector valued functions were introduced and extensively studied by Louis de Branges and have been developed in view of the model theory for linear transformations in Hilbert spaces [13]. These spaces have played a central role in applications to direct and inverse problems for canonical systems of differential and integral equations and Dirac-Krein systems, see for example [5–7].

The main goal of the present paper is to extend some results on de Branges spaces of scalar valued functions obtained in [1] to de Branges spaces of vector valued functions. We consider the existence and structure properties of Parseval frames of kernel functions in vector valued de Branges spaces. In subsection 1.3 we shall review some definitions and necessary facts from the theory of reproducing kernel Hilbert spaces of vector valued functions. As a special case of such spaces the de Branges spaces of vector valued functions is reviewed in Subsection 1.4. Sections 2 and 3 are devoted to developing new results on the construction of dilated de Branges spaces of vector valued functions and orthogonality of embeddings within the dilation spaces. We develop some necessary conditions for Parseval sequences in vector valued de Branges spaces by identifying the main construction with Naimark dilation of frames via embedding the de Branges space into a dilated de Branges space. The embedding identifies the kernel functions associated with a frame sequence as a summand for a Riesz basis for the dilated space. We also obtain some sufficient conditions for a dilated de Branges space to have the Kramer sampling property in Section 4 as well as results concerning the multiplexing of samples in the dilated space.

1.1. Notation

Some notations are necessary to describe the spaces we will consider here, see [7,12] for additional information. $\mathbb{C}$ will denote the complex plane, $\mathbb{C}^+$ (resp., $\mathbb{C}^-$) the open upper (resp., lower) half plane, $\mathbb{C}^p$ the complex $p \times 1$ vectors. The notation $\mathbb{C}^{p\times q}$ stands for the set of all $p \times q$ matrices with complex entries, the identity matrix that belongs to $\mathbb{C}^{p\times p}$ will be denoted by I_p. A $\mathbb{C}^p$ vector valued function $f(z)$, defined in a region Ω of the complex plane $\mathbb{C}$, is said to be analytic in Ω if the complex valued function $u^*f(z)$ is analytic in the region for every choice of vector $u \in \mathbb{C}^p$. A continuous $\mathbb{C}^{p\times p}$ matrix valued function $F(z)$, defined in Ω, is said to be analytic in the region if $u^*F(z)v$ is analytic in the region for every choice of vectors u and v in $\mathbb{C}^p$. A matrix valued function with entries that are analytic in the full complex plane is said to be entire matrix valued function. $f^*(z)$ is the Hermitian transpose of the matrix valued function $f(z)$, and $f^\#(z) = f^*(\bar{z})$.
$\mathbb{H}_2^{p\times q}$ is the Hardy space of $p\times q$ matrix valued functions with entries in the classical Hardy space $\mathbb{H}_2$ with respect to $\mathbb{C}^+$, with norm

$$\|f\|_2^2 = \sup_{y>0} \int_{-\infty}^{\infty} trace\{f^*(x+iy)f(x+iy)\}dx \ < \infty.$$

$(\mathbb{H}_2^{p\times q})^\perp = \{f : \ f^\# \in \mathbb{H}_2^{q\times p}\}$ (the superscript $\perp$ means that $\mathbb{H}_2^{p\times q}$ and $(\mathbb{H}_2^{p\times q})^\perp$ are orthogonal to each other when regarded as subspaces of $L_2^{p\times q}$). We shall use the symbol $\mathbb{H}_2^p$ for $\mathbb{H}_2^{p\times 1}$, and $(\mathbb{H}_2^q)^\perp$ for $(\mathbb{H}_2^{q\times 1})^\perp$.

$\mathbb{H}_\infty^{p\times q}$ is the Hardy space of holomorphic $p\times q$ matrix valued functions in $\mathbb{C}^+$ with

$$\|f\|_\infty = \sup\{\|f(z)\| : z \in \mathbb{C}^+\} < \infty.$$

The Schur class $\mathcal{S}^{p\times p}$ is the class of $p \times p$ matrix valued functions $s(z)$ that are holomorphic and contractive in $\mathbb{C}^+$, i.e.,

$$I_p - s^*(z)s(z) \succeq 0, \text{ for } z \in \mathbb{C}^+.$$

$\mathcal{S}_{in}^{p\times p}$ is the class of matrix valued functions $f \in \mathcal{S}^{p\times p}$ which are inner, i.e., $I_p - f^*(t)f(t) = 0$ for a.e. point $t \in \mathbb{R}$.
The generalized backward-shift operator R_ω is defined for entire vector valued functions by

$$(R_\omega f)(z) = \begin{cases} \frac{f(z)-f(\omega)}{z-\omega} & \text{if } z \neq \omega \\ f'(\omega) & \text{if } z = \omega \end{cases}$$

for every z, $\omega \in \mathbb{C}$.

1.2. Frame Theory

A sequence $\{f_n\}_{n=1}^{\infty}$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exists constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{H}, \tag{1}$$

The constants A and B are called lower and upper frame bounds, respectively. The frames for which $A = B = 1$ are called *Parseval frames*. A frame which is a basis is called a *Riesz basis*. It is easy to see that a Parseval frame $\{f_n\}_{n=1}^{\infty}$ for a Hilbert space $\mathcal{H}$ is an orthonormal basis if and only if each f_n is a unit vector. If the upper bound in (1) is satisfied, then we say that $\{f_n\}_{n=1}^{\infty}$ is a *Bessel* sequence with Bessel bound B.

Let $\{f_n\}_{n=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$. The analysis operator $\Theta : \mathcal{H} \to \ell^2$, which is bounded because of (1), is defined by

$$\Theta : f \to (\langle f, f_n\rangle);$$

and the synthesis operator $\Theta^* : \ell^2 \to \mathcal{H}$, which is the adjoint operator of Θ, is defined by

$$\Theta^* : (c_n)_{n=1}^{\infty} \to \sum_{n=1}^{\infty} c_n f_n.$$

Additionally, the sum $\sum_{n=1}^{\infty} c_n f_n$ converges in $\mathcal{H}$ for all $(c_n)_{n=1}^{\infty} \in l^2$ (see [14]), and so the synthesis operator is also well defined and bounded.

The operator $S := \Theta^*\Theta : \mathcal{H} \to \mathcal{H}$ is called the frame operator, and we have

$$Sf = \sum_{n=1}^{\infty} \langle f, f_n\rangle f_n \ , \forall f \in \mathcal{H}.$$

The *canonical dual frame* is denoted by $\{\tilde{f}_n\}_{n=1}^{\infty}$, and is defined by $\tilde{f}_n = S^{-1}f_n$. Furthermore, for each $f \in \mathcal{H}$ we have the *frame expansions*

$$f = \sum_{n=1}^{\infty} \langle f, f_n\rangle \tilde{f}_n = \sum_{n=1}^{\infty} \langle f, \tilde{f}_n\rangle f_n, \tag{2}$$

with unconditional convergence of these series.

If $\mathbb{F} = \{f_n\}_{n=1}^{\infty}$ and $\mathbb{G} = \{g_n\}_{n=1}^{\infty}$ are two Bessel sequences in $\mathcal{H}$, define the operator

$$\Theta_{\mathbb{G}}^{*}\Theta_{\mathbb{F}} : \mathcal{H} \to \mathcal{H} : f \to \sum_{n=1}^{\infty} \langle f, f_n \rangle g_n.$$

If $\Theta_{\mathbb{G}}^{*}\Theta_{\mathbb{F}} = 0$ then the two Bessel sequences $\mathbb{F}$ and $\mathbb{G}$ are said to be orthogonal [18]. An extensive study of orthogonal frames can be found in the papers [10, 23]. If $\mathbb{F}$ and $\mathbb{G}$ are both Parseval frames and orthogonal to each other, then for any $f, g \in \mathcal{H}$

$$f = \sum_{n} (\langle f, f_n \rangle + \langle g, g_n \rangle) f_n, \quad \text{and} \quad g = \sum_{n} (\langle f, f_n \rangle + \langle g, g_n \rangle) g_n$$

In other words, both functions can be recovered from the summed coefficients $\langle f, f_n \rangle + \langle g, g_n \rangle$. This procedure is called *multiplexing*, and can be used in multiple access communication systems. In the proof of our main results we also need a concept of *similar frames*: two frames $\mathbb{F} = \{f_n\}_{n=1}^{\infty}$ and $\mathbb{G} = \{g_n\}_{n=1}^{\infty}$ are said to be similar if there is an invertible operator $T : \mathcal{H} \to \mathcal{H}$ such that $Tf_n = g_n$. Two frames $\mathbb{F}$ and $\mathbb{G}$ are similar if and only if $\Theta_{\mathbb{F}}(\mathcal{H}) = \Theta_{\mathbb{G}}(\mathcal{H})$ [11].

Let P be an orthogonal projection from a Hilbert space $\mathcal{K}$ onto a closed subspace $\mathcal{H}$, and $\{f_n\}$ be a sequence in $\mathcal{K}$. Then $\{Pf_n\}$ is called *orthogonal compression* of $\{f_n\}$ under P, and $\{f_n\}$ is called an *orthogonal dilation* of $\{Pf_n\}$. A classical fact on dilation of frames, which can be attributed to Han and Larson [17], says that a Parseval frame in a Hilbert space $\mathcal{H}$ is an image of an orthonormal basis under an orthogonal projection of some larger Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ onto $\mathcal{H}$. This result can be considered as a special case of Naimark's dilation theorem for positive operator valued measures, see [20, 21]. In particular, Han and Larson proved the following result.

Theorem 1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in a Hilbert space $\mathcal{H}$. Then*

(*i*) *$\{f_n\}$ is a Parseval frame for $\mathcal{H}$ if and only if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{e_n\}$ for $\mathcal{K}$ such that if P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ then $f_n = Pe_n$, for all $n \in \mathbb{N}$.*

(*ii*) *$\{f_n\}$ is a frame for $\mathcal{H}$ if and only if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{u_n\}$ for $\mathcal{K}$ such that if P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ then $f_n = Pu_n$, for all $n \in \mathbb{N}$.*

Orthogonality of frames and Naimark dilation of frames are related in the following way (see [8, 17]): If $\{u_n\}$ is a Riesz basis for $\mathcal{K}$ and P is the projection onto $\mathcal{H} \subset \mathcal{K}$, then $\{Pu_n\}$ and $\{(I - P)u_n\}$ are orthogonal frames for $\mathcal{H}$ and $\mathcal{H}^{\perp}$, respectively. Conversely, if $\mathbb{F} = \{f_n\}$ and $\mathbb{G} = \{g_n\}$ are orthogonal frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $\{f_n + g_n\}$ is a frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. Note that the sum of the frames need not be a basis for the direct sum in general–however, it will be provided that

$$\Theta_{\mathbb{F}}(\mathcal{H}_1) \oplus \Theta_{\mathbb{G}}(\mathcal{H}_2) = \ell^2.$$

1.3. Reproducing Kernel Hilbert Spaces of Vector Valued Functions

In this section a number of facts about reproducing kernel Hilbert spaces of vector valued functions that will be used frequently are reviewed briefly; more details and supporting proofs may be found in [4–7]. For related results concerning operator valued reproducing kernel spaces, see e.g. [2,3].

A Hilbert space $\mathcal{H}$ of $p\times 1$ vector valued functions defined on a subset Ω of $\mathbb{C}$ is said to be a reproducing kernel Hilbert space (RKHS) if there exists a $p\times p$ matrix valued function $K_w(z)$ (for $(z,w)\in\Omega\times\Omega$) such that for every choice of $w\in\Omega$, $u\in\mathbb{C}^p$, and $f\in\mathcal{H}$:

1. $K_w(z)u\in\mathcal{H}$, as a vector valued function of z,
2. The reproducing kernel property

$$\langle f(.), K_w(.)u\rangle_{\mathcal{H}} = \langle f(w), u\rangle_{\mathbb{C}} = u^* f(w) \tag{3}$$

The matrix valued function $K_w(z)$ is called a reproducing kernel (RK) of the RKHS $\mathcal{H}$. The existence and uniqueness of a RK is guaranteed by the Riesz representation theorem [15]. The following properties of RKHS are well known and easily checked, see [16] for more details:

1. $\langle K_w(.)u_1, K_v(.)u_2\rangle_{\mathcal{H}} = u_2^* K_w(v)u_1$, for all $w,v\in\mathbb{C}$, $u_1,u_2\in\mathbb{C}^p$, and
$$\|K_w u\|^2_{\mathcal{H}} = u^* K_w(w)u. \tag{4}$$
2. $\|f(w)\|\le\|f\|_{\mathcal{H}}\|K_w(w)\|^{1/2}$, for all $w\in\Omega$ and $f\in\mathcal{H}$.
3. The RK is positive in the sense that
$$\sum_{i,j=1}^{n} u_j^* K_{w_i}(w_j)u_i \ge 0 \tag{5}$$
for every choice of points $w_1,\dots,w_n\in\Omega$ and vectors $u_1,\dots,u_n\in\mathbb{C}^p$ and every positive integer n. Consequently, the set $\{K_w(.)u : w\in\Omega,\, u\in\mathbb{C}^p\}$ is total in $\mathcal{H}$, that is
$$\mathcal{H} = \overline{\text{span}}\{K_w(.)u : w\in\Omega,\, u\in\mathbb{C}^p\}.$$

The following theorem is a matrix version of a theorem of Aronszajn in [4].

Theorem 2. *Let Ω be a subset of $\mathbb{C}$ and let the $p\times p$ matrix valued kernel $K_\omega(z)$ be positive on $\Omega\times\Omega$. Then there is a unique Hilbert space $\mathcal{H}$ of $p\times 1$ vector valued functions $f(z)$ on Ω such that*

$$K_\omega u\in\mathcal{H}, \quad \textit{and} \quad \langle f, K_\omega u\rangle_{\mathcal{H}} = u^* f(\omega)$$

for every $\omega\in\Omega$, $u\in\mathbb{C}^p$ and $f\in\mathcal{H}$.

Example 1. ([7]) The Hardy space $\mathbb{H}_2^p$ is a RKHS of $p\times 1$ vector valued functions that are holomorphic in $\mathbb{C}^+$ with RK

$$K_\omega(z) = \frac{I_p}{-2\pi i(z-\bar\omega)}, \text{ for } z,\omega\in\mathbb{C}^+$$

A RKHS $\mathcal{H}$ of $p \times 1$ vector valued functions is said to have the **Kramer sampling property** if there is a sequence of points $\{w_n\}_{n=1}^{\infty} \subset \Omega$ and a sequence of vectors $\{\xi_n\}_{n=1}^{\infty} \in \mathbb{C}^p$, such that the set $\{K_{w_n}(.)\xi_n\}_{n=1}^{\infty}$ is a complete orthogonal set in $\mathcal{H}$, i.e., every $f \in \mathcal{H}$ can be expressed in the form

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \langle f, K_{w_n}\xi_n\rangle_{\mathcal{H}} \frac{K_{w_n}(z)\xi_n}{\|K_{w_n}\xi_n\|^2} \\ &= \sum_{n=1}^{\infty} \xi_n^* f(w_n) \frac{K_{w_n}(z)\xi_n}{\|K_{w_n}\xi_n\|^2} \end{aligned}$$

In other words, functions of the space $\mathcal{H}$ are uniquely determined and reconstructible from their samples [19].

The notation

$$\mathcal{H}_\omega = \{f \in \mathcal{H} : f(\omega) = 0\}$$

for RKHS's $\mathcal{H}$ of entire vector valued functions will be useful.

1.4. de Branges Spaces of Vector Valued Functions

In this section we shall present a number of facts from the theory of de Branges spaces of vector valued functions that will be needed in the sequel. Most of this information can be found in the papers [5–7, 12].

An entire $p \times 2p$ matrix valued function $\mathfrak{E}(z) = [E_-(z) \quad E_+(z)]$ is called an entire **de Branges matrix** with $p \times p$ blocks $E_\pm(z)$ that are matrix valued entire functions, if

$$\det E_+(z) \not\equiv 0, \text{ in } \mathbb{C}, \quad \text{and} \quad \chi_{\mathfrak{E}} := E_+^{-1}E_- \in \mathcal{S}_{in}^{p\times p}. \tag{6}$$

The determinant of an entire matrix valued function is an entire function. Consequently, if the determinant of the entire matrix valued function $E_+(z)$ does not vanish identically, the given entire matrix valued function has invertible values at all but isolated points in the complex plane. Since $E_\pm(z)$ are entire matrix valued functions, the condition in (6) ensures that (see [16])

$$E_+(z)E_+^{\#}(z) = E_-(z)E_-^{\#}(z), \text{ for all } z \in \mathbb{C}. \tag{7}$$

Definition 1. Given a de Branges matrix $\mathfrak{E}$, the set of entire $\mathbb{C}^p$ vector valued functions $f(z)$ satisfying

$$E_+^{-1}f \in \mathbb{H}_2^p \quad \text{and} \quad E_-^{-1}f \in (\mathbb{H}_2^p)^\perp \tag{8}$$

is a reproducing kernel Hilbert space with reproducing kernel

$$K_w^{\mathfrak{E}}(z) = \begin{cases} \frac{E_+(z)E_+^*(w)-E_-(z)E_-^*(w)}{2\pi i(\bar{w}-z)} & \text{, if } z \neq \bar{w} \\ \frac{E_+'(\bar{w})E_+^*(w)-E_-'(\bar{w})E_-^*(w)}{-2\pi i} & \text{, if } z = \bar{w} \end{cases} \tag{9}$$

with respect to the inner product

$$\langle f, g\rangle_{\mathcal{B}} = \langle E_+^{-1}f, E_+^{-1}g\rangle_{st} = \int_{-\infty}^{\infty} g^*(t)\Delta_{\mathcal{E}}(t)f(t)\, dt, \tag{10}$$

where

$$\Delta_{\mathcal{E}}(t) = \{E_+(t)E_+^*(t)\}^{-1} = \{E_-(t)E_-^*(t)\}^{-1},$$

for all $t \in \mathbb{R}$ at which $\det E_\pm(z) \neq 0$.

The Hilbert space corresponding to the de Branges matrix $\mathfrak{E}$ is called the de Branges space $\mathcal{B}(\mathfrak{E})$; for every $w \in \mathbb{C}$, every $u \in \mathbb{C}^p$, and every $f \in \mathcal{B}(\mathfrak{E})$

1. $K_w^{\mathfrak{E}} u \in \mathcal{B}(\mathfrak{E})$ and
2. $\langle f, K_w^{\mathfrak{E}} u\rangle_{\mathcal{B}(\mathfrak{E})} = u^* f(w)$

Remark 1. If $E(z)$ is a scalar valued entire function which has no real zeros and $|E(z)| > |E(\bar{z})|$ for all $z \in \mathbb{C}^+$, then $\mathcal{B}(\mathfrak{E})$ with $\mathfrak{E} = [E^\#(z) \quad E(z)]$ is just the usual de Branges space corresponding to the de Branges function $E(z)$.

Example 2. ([16]) If $E_+^t(z) = e^{-izt} I_p$ and $E_-^t(z) = e^{izt} I_p$ for $t > 0$, then it is easy to see that $\mathfrak{E}_t(z) = [E_-^t(z) \quad E_+^t(z)]$ is an entire de Branges matrix, and the space $\mathcal{B}(\mathfrak{E}_t)$ is a vector Paley-Wiener space with RK

$$K_w^{\mathfrak{E}_t}(z) = \frac{\sin(z - \bar{w})t}{\pi(z - \bar{w})} I_p.$$

There is a connection between de Branges spaces $\mathcal{B}(\mathfrak{E})$ of entire vector valued functions that are invariant under the action of the generalized backward-shift operator R_ω and the Kramer sampling property, the following is found in [16, Theorem 9.4].

Theorem 3. *Let $\mathcal{H}$ be the de Branges space $\mathcal{B}(\mathfrak{E})$ based on the de Branges matrix $\mathfrak{E}$ with RK $K_\omega(z)$. If*

(1) *$R_\omega \mathcal{H}_\omega \subseteq \mathcal{H}$ for every point $\omega \in \mathbb{C}$, and*
(2) *$K_\omega(\omega) \succ 0$ for at least one point $\omega \in \mathbb{C}$,*

then $\mathcal{B}(\mathfrak{E})$ has the Kramer sampling property.

A sufficient condition for the space $\mathcal{H}_\omega$ to be invariant under the operator R_ω is given by the next lemma [16, Lemma 6.4].

Lemma 1. *Let $\mathcal{H}$ be the de Branges space $\mathcal{B}(\mathfrak{E})$ based on the de Branges matrix $\mathfrak{E}$, then:*

(1) *$R_\omega \mathcal{H}_\omega \subseteq \mathcal{H}$ for every point $\omega \in \overline{\mathbb{C}^+}$ at which $E_+(\omega)$ is invertible.*
(2) *$R_\omega \mathcal{H}_\omega \subseteq \mathcal{H}$ for every point $\omega \in \overline{\mathbb{C}^-}$ at which $E_-(\omega)$ is invertible.*

2. The de Branges Space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$

In this section a number of results on constructing the dilated de Branges space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ will be obtained. The space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ is a simultaneous dilation of two de Branges spaces $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$. We will consider the class of $p \times p$ entire matrix valued functions $F(z)$ such that $\det(F(z)) \not\equiv 0$ in $\mathbb{C}$, and $F^{-1}F^\# \in \mathcal{S}_{in}^{p\times p}$. We will denote this class by $\mathbb{N}_{inv}(\mathbb{C}^{p\times p})$. If $F \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$, the conditions in (6) and (7) imply that

$$F(z)F^\#(z) = F^\#(z)F(z), \text{ for all } z \in \mathbb{C}.$$

Hence, the $p \times 2p$ matrix

$$\mathfrak{F} := [F^{\#}(z) \quad F(z)]$$

is a de Branges matrix, with corresponding de Branges space $\mathcal{B}(\mathfrak{F})$.

Example 3. For $n \in \mathbb{N}$, define the family of $2n \times 2n$ entire matrix-valued functions

$$F(z) = \begin{bmatrix} e^{f_1(z)} I_n & 0 \\ 0 & e^{f_2(z)} I_n \end{bmatrix},$$

where $f_1(z) = g_1(z) + \alpha_1 + \beta_1 iz$, $f_2(z) = g_2(z) + \alpha_2 + \beta_2 iz$, for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, and entire functions g_1, g_2 which are real on the real line. Then it is readily checked that the matrix valued functions $UF(z)U^*$ belongs to the class $\mathbb{N}_{inv}(\mathbb{C}^{2n\times 2n})$ for any $2n \times 2n$ constant unitary matrix U.

Definition 2. Given de Branges matrices $\mathfrak{F} := [F^{\#}(z) \quad F(z)]$, $\mathfrak{E} = [E_-(z) \quad E_+(z)]$, where $F(z) \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$, we define

$$\mathfrak{F} \diamond \mathfrak{E} := [F^{\#}(z)E_-(z) \quad F(z)E_+(z)].$$

Our main results will utilize the following additional commutation assumption:

$$F^{\#}E_- = E_-F^{\#} \quad \text{and} \quad FE_+ = E_+F. \tag{11}$$

Under this additional assumption on the matrix valued functions F and $E_\pm$ we prove that the space $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ is a RKHS whose kernel can be expressed in terms of the kernels for $\mathcal{B}(\mathfrak{F})$ and $\mathcal{B}(\mathfrak{E})$. Throughout the rest of this paper, unless otherwise specified, we will assume that the de Branges matrices $\mathfrak{F} = [F^{\#}(z) \quad F(z)]$ and $\mathfrak{E} = [E_-(z) \quad E_+(z)]$ with $F(z) \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$. We begin with a lemma.

Lemma 2. *Assume $\mathfrak{F}$ and $\mathfrak{E}$ satisfy the hypotheses of Definition* 2 *and Equation* (11). *Then the following hold:*

(*i*) $FE_- = E_-F$;
(*ii*) $F^{\sharp}E_+ = E_+F^{\sharp}$;
(*iii*) $FE_+^{-1} = E_+^{-1}F$;
(*iv*) $F^{-1}E_- = E_-F^{-1}$;
(*v*) $E_-^{-1}(F^{\sharp})^{-1}F = (F^{\sharp})^{-1}FE_-^{-1}$.

Proof. By virtue of $F(z) \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$, we have that $F^*F = FF^*$ on the real axis. Item (*i*) holds by Fuglede's Theorem: F^* is normal on the real axis and $F^*E_- = E_-F^*$ holds on the real axis by Equation (11). An analogous argument shows that (*ii*) holds. Items (*iii*), respectively (*iv*), hold because of Equation (11), respectively (*i*), and a standard Neumann series argument. Item (*v*) holds by Equation (11) and (*iv*). □

Theorem 4. *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition* 2 *and Equation* (11). *Then*

(*i*) *$\mathfrak{F} \diamond \mathfrak{E}$ is a de Branges matrix, and* (*ii*) *the corresponding de Branges space is* $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$*, with RK*

$$K_w^{\mathfrak{F}\diamond\mathfrak{E}}(z) = F(z)K_w^{\mathfrak{E}}(z)F^*(w) + E_-(z)K_w^{\mathfrak{F}}(z)E_-^*(w). \tag{12}$$

Proof. Since $\det(E_\pm(z)) \not\equiv 0$, $\det(F(z)) \not\equiv 0$, and $\det(F^\#(z)) \not\equiv 0$ in $\mathbb{C}$, then

$$\det(F^\#(z)E_-(z)) \not\equiv 0 \text{ and } \det(F(z)E_+(z)) \not\equiv 0 \text{ in } \mathbb{C}.$$

To show that the function $\chi_{\mathfrak{F}\diamond\mathfrak{E}} := (FE_+)^{-1}(F^\# E_-) \in \mathcal{S}_{in}^{p\times p}$, we use the fact that both functions $\chi_{\mathfrak{E}} := E_+^{-1}E_-$ and $\chi_{\mathfrak{F}} := F^{-1}F^\#$ belongs to the class $\mathcal{S}_{in}^{p\times p}$. By Lemma 2 (*ii*), we have $F^\# E_+^{-1} = E_+^{-1}F^\#$. Thus, again using Lemma 2,

$$\chi_{\mathfrak{F}\diamond\mathfrak{E}} = (FE_+)^{-1}(F^\# E_-) = (E_+F)^{-1}(F^\# E_-) = F^{-1}E_+^{-1}F^\# E_- = F^{-1}F^\# E_+^{-1}E_- = \chi_{\mathfrak{F}}\chi_{\mathfrak{E}}.$$

This proves that $\mathfrak{F} \diamond \mathfrak{E}$ is a de Branges matrix.

The RK of the space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ is

$$\begin{aligned} K_w^{\mathfrak{F}\diamond\mathfrak{E}}(z) &= \frac{F(z)E_+(z)(F(w)E_+(w))^* - F^\#(z)E_-(z)(F^\#(w)E_-(w))^*}{2\pi i(\bar{w}-z)} \\ &= \frac{F(z)E_+(z)E_+^*(w)F^*(w) - F^\#(z)E_-(z)E_-^*(w)(F^\#(w))^*}{2\pi i(\bar{w}-z)} \\ &= \frac{F(z)E_+(z)E_+^*(w)F^*(w) - F(z)E_-(z)E_-^*(w)F^*(w)}{2\pi i(\bar{w}-z)} \\ &\quad + \frac{F(z)E_-(z)E_-^*(w)F^*(w) - F^\#(z)E_-(z)E_-^*(w)(F^\#(w))^*}{2\pi i(\bar{w}-z)} \\ &= F(z)K_w^{\mathfrak{E}}(z)F^*(w) + E_-(z)K_w^{\mathfrak{F}}(z)E_-^*(w) \end{aligned}$$

since $FE_- = E_-F$ and $F^\# E_- = E_-F^\#$ by Lemma 2. □

Example 4. Consider the matrix valued function $F(z)$ given in Example 3 and the matrix valued functions $E_+(z)$, $E_-(z)$ given in Example 2, then

$$\mathfrak{F} = [F^\# \quad F], \quad \mathfrak{E} = [E_- \quad E_+]$$

satisfies the conditions of Definition 2.

3. Orthogonality in $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$

Now we prove that the spaces $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$ can be embedded into the larger space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$.

Proposition 1. *Let* $\mathfrak{F}$ *and* $\mathfrak{E}$ *be two de Branges matrices that satisfy Definition* 2 *and Equation* (11)*. The operator* $\mathcal{I} : \mathcal{B}(\mathfrak{E}) \to \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$*, defined by* $\mathcal{I}(f) = Ff$*, is a linear isometry.*

Proof. We first prove that $\mathcal{I}$ is well defined, i.e., for every $f \in \mathcal{B}(\mathfrak{E})$, $Ff \in \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, that is

$$(FE_+)^{-1}Ff \in \mathbb{H}_2^p, \quad \text{and} \quad (F^\# E_-)^{-1}Ff \in (\mathbb{H}_2^p)^\perp,$$

Let $f \in \mathcal{B}(\mathfrak{E})$, then by Definition 1

$$E_+^{-1} f \in \mathbb{H}_2^p, \quad \text{and} \quad E_-^{-1} f \in (\mathbb{H}_2^p)^\perp, \tag{13}$$

hence, $(FE_+)^{-1} F f = E_+^{-1} f \in \mathbb{H}_2^p$. On the other hand, $(F^{\#} E_-)^{-1} F f = E_-^{-1} (F^{\#})^{-1} F f = (F^{\#})^{-1} F E_-^{-1} f$ belongs to $(\mathbb{H}_2^p)^\perp$, since $E_-^{-1} f \in (\mathbb{H}_2^p)^\perp$ and $(F^{\#})^{-1} F$ is the inverse of a matrix valued inner function.

Let $f_1, f_2 \in \mathcal{B}(\mathfrak{E})$, then

$$\begin{aligned}
\langle \mathcal{I}(f_1), \mathcal{I}(f_2) \rangle_{\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})} &= \int_{-\infty}^{\infty} (F(t) f_2(t))^* \Delta_{\mathfrak{F} \diamond \mathfrak{E}}(t) (F(t) f_1(t))\, dt \\
&= \int_{-\infty}^{\infty} f_2^*(t) F^*(t) (FE_+ (FE_+)^*)^{-1}(t) F(t) f_1(t)\, dt \\
&= \int_{-\infty}^{\infty} f_2^*(t) F^*(t)\, (F^*(t))^{-1} (E_+^*(t))^{-1} E_+^{-1}(t) F^{-1}(t) F(t) f_1(t)\, dt \\
&= \int_{-\infty}^{\infty} f_2^*(t) (E_+^*(t))^{-1} E_+^{-1}(t) f_1(t)\, dt \\
&= \int_{-\infty}^{\infty} f_2^*(t) \Delta_{\mathfrak{E}}(t) f_1(t)\, dt = \langle f_1, f_2 \rangle_{\mathcal{B}(\mathfrak{E})}.
\end{aligned}$$

□

A similar argument as in the proof of Proposition 1 can be used to proof the next proposition.

Proposition 2. *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition* 2 *and Equation* (11)*. The operator $\mathcal{J} : \mathcal{B}(\mathfrak{F}) \to \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, defined by $\mathcal{J}(g(z)) = E_-(z) g(z)$ is a linear isometry.*

Theorem 5. *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition* 2 *and Equation* (11)*. The images of the operators $\mathcal{I}$ and $\mathcal{J}$ are orthogonal in $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$.*

Proof. Let $f \in \mathcal{B}(\mathfrak{E})$ and $g \in \mathcal{B}(\mathfrak{F})$, then

$$\left\langle (FE_+)^{-1} F f, (FE_+)^{-1} E_- g \right\rangle = \left\langle (E_+)^{-1} f, (E_+)^{-1} E_- F^{-1} g \right\rangle = 0,$$

because $f \in \mathcal{B}(\mathfrak{E})$ if and only if $E_+^{-1} f \in \mathbb{H}_2^p \ominus (E_+)^{-1} E_- \mathbb{H}_2^p$. □

Remark 2. Given $\omega \in \mathbb{C}$ and $u \in \mathbb{C}^p$ the vector valued function $K_w^{\mathfrak{F} \diamond \mathfrak{E}}(z) u \in \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ as a function of z. Likewise, $K_w^{\mathfrak{E}}(z) F^*(w) u \in \mathcal{B}(\mathfrak{E})$ and $K_w^{\mathfrak{F}}(z) E_-^*(w) u \in \mathcal{B}(\mathfrak{F})$. It follows from (12) that for any $w \in \mathbb{C}$ and $u \in \mathbb{C}^p$

$$\begin{aligned}
K_\omega^{\mathfrak{F} \diamond \mathfrak{E}}(z) u &= F(z) \left(K_\omega^{\mathfrak{E}}(z) F^*(\omega) u \right) + E_-(z) \left(K_w^{\mathfrak{F}}(z) E_-^*(\omega) u \right) \\
&= \mathcal{I}\left(K_w^{\mathfrak{E}}(z) F^*(w) u \right) + \mathcal{J}\left(K_w^{\mathfrak{F}}(z) E_-^*(w) u \right)
\end{aligned} \tag{14}$$

Consequently, since the set $\{K_w^{\mathfrak{F} \diamond \mathfrak{E}}(z) u : \ w \in \mathbb{C},\ u \in \mathbb{C}^p\}$ spans the space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, the set

$$\mathcal{I}\left(\{K_w^{\mathfrak{E}}(z) F^*(w) u : w \in \mathbb{C},\ u \in \mathbb{C}^p\} \right) \cup \mathcal{J}\left(\{K_w^{\mathfrak{F}}(z) E_-^*(w) u : \ w \in \mathbb{C},\ u \in \mathbb{C}^p\} \right)$$

spans $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ whenever $\det(F^*(\omega))\not\equiv 0$ and $\det(E_-^*(\omega))\not\equiv 0$. Indeed, for any finite set of points $\omega_1,\dots,\omega_n\in\mathbb{C}$ and vectors $u_1,\dots,u_n\in\mathbb{C}^p$, then by (12) we have

$$K_{\omega_k}^{\mathfrak{F}\diamond\mathfrak{E}}(z)u_k = F(z)K_{\omega_k}^{\mathfrak{E}}(z)F^*(\omega_k)u_k + E_-(z)K_{\omega_k}^{\mathfrak{F}}(z)E_-^*(\omega_k)u_k.$$

Setting $\xi_k = F^*(\omega_k)u_k$ and $\eta_k = E_-^*(\omega_k)u_k$ we get

$$\sum_{k=1}^n K_{\omega_k}^{\mathfrak{F}\diamond\mathfrak{E}}(z)u_k = F(z)\left(\sum_{k=1}^n K_{\omega_k}^{\mathfrak{E}}(z)\xi_k\right) + E_-(z)\left(\sum_{k=1}^n K_{\omega_k}^{\mathfrak{F}}(z)\eta_k\right).$$

On the other hand, for any $\omega\in\mathbb{C}$ and $u\in\mathbb{C}^p$, by Equation (4) we have

$$\begin{aligned}
\|K_\omega^{\mathfrak{F}\diamond\mathfrak{E}}u\|^2_{\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})} &= u^*K_\omega^{\mathfrak{F}\diamond\mathfrak{E}}(\omega)u\\
&= u^*F(\omega)K_\omega^{\mathfrak{E}}(\omega)F^*(\omega)u \,+\, u^*E_-(\omega)K_\omega^{\mathfrak{F}}(\omega)E_-^*(\omega)u\\
&= \|K_\omega^{\mathfrak{E}}F^*(\omega)u\|^2_{\mathcal{B}(\mathfrak{E})} \,+\, \|K_\omega^{\mathfrak{F}}E_-^*(\omega)u\|^2_{\mathcal{B}(\mathfrak{F})}.
\end{aligned}$$

Let $P_{\mathfrak{E}}$ be the orthogonal projection of $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ onto the image of $\mathcal{I}$, and $P_{\mathfrak{F}}$ be the orthogonal projection of $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ onto the image of $\mathcal{J}$. We have

$$P_{\mathfrak{E}}(h) = Ff_1 \quad \text{and} \quad P_{\mathfrak{F}}(h) = E_-f_2,$$

for some $f_1\in\mathcal{B}(\mathfrak{E})$ and $f_2\in\mathcal{B}(\mathfrak{F})$. The next Theorem shows that the space $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ admits an orthogonal direct sum decomposition using the spaces $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$. For this, we define

$$\begin{aligned}
F\mathcal{B}(\mathfrak{E}) &= \{Ff:\, f\in\mathcal{B}(\mathfrak{E})\}\\
E_-\mathcal{B}(\mathfrak{F}) &= \{E_-f:\, f\in\mathcal{B}(\mathfrak{F})\}
\end{aligned}$$

Theorem 6. *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition* 2 *and Equation* (11)*. Then*

$$\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E}) = F\mathcal{B}(\mathfrak{E})\oplus E_-\mathcal{B}(\mathfrak{F})$$

i.e., for any $h\in\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$, there exist a unique $f_1\in\mathcal{B}(\mathfrak{E})$ and $f_2\in\mathcal{B}(\mathfrak{F})$ such that $h = Ff_1 + E_-f_2$, and

$$\|h\|^2_{\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})} = \|f_1\|^2_{\mathcal{B}(\mathfrak{E})} + \|f_2\|^2_{\mathcal{B}(\mathfrak{F})}.$$

Proof. It is easily checked that $K_\omega^{(1)}(z) := F(z)K_\omega^{\mathfrak{E}}(z)F^*(\omega)$ is a reproducing kernel with corresponding RKHS $\mathcal{B}_1 = F\mathcal{B}(\mathfrak{E})$, and $K_\omega^{(2)}(z) := E_-(z)K_\omega^{\mathfrak{F}}(z)E_-^*(\omega)$ is a reproducing kernel with corresponding RKHS $\mathcal{B}_2 = E_-\mathcal{B}(\mathfrak{F})$. Furthermore, Theorem 5 implies that $\mathcal{B}_1\cap\mathcal{B}_2 = \{0\}$.

Since $K_\omega^{(1)}(z) + K_\omega^{(2)}(z)$ is a RK, and $K_\omega^{\mathfrak{F}\diamond\mathfrak{E}}(z) = K_\omega^{(1)}(z) + K_\omega^{(2)}(z)$, this implies that

$$\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E}) = \mathcal{B}_1\oplus\mathcal{B}_2 = F\mathcal{B}(\mathfrak{E})\oplus E_-\mathcal{B}(\mathfrak{F}).$$

It follows that the orthogonal complement of $\mathcal{B}_1$ in $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ is the space $\mathcal{B}_2$. The claim now follows from orthogonality and the isometry properties of $\mathcal{I}$ and $\mathcal{J}$. $\square$

Theorem 7. *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition 2 and Equation (11). If $\{\omega_n\} \subset \mathbb{C}$ and $\{u_n\} \subset \mathbb{C}^p$ are such that $\left\{ \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}} \right\}$ is a complete orthonormal set for $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ then*

1. $\left\{ \frac{K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}} \right\}$ *is a Parseval frame for $\mathcal{B}(\mathfrak{E})$, and for every $f \in \mathcal{B}(\mathfrak{E})$*
$$f(z) = \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{K_{\omega_n}^{\mathfrak{E}}(z)F^*(\omega_n)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}. \tag{15}$$
2. $\left\{ \frac{K_{\omega_n}^{\mathfrak{F}}(.)E_-^*(\omega_n)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \right\}$ *is a Parseval frame for $\mathcal{B}(\mathfrak{F})$, and for every $g \in \mathcal{B}(\mathfrak{F})$*
$$g(z) = \sum_n u_n^* E_-(\omega_n) g(\omega_n) \frac{K_{\omega_n}^{\mathfrak{F}}(z)E_-^*(\omega_n)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}. \tag{16}$$

Proof. By Equation (14) we have
$$\frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} = \frac{\mathcal{I}\left(K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n\right)}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} + \frac{\mathcal{J}(K_{\omega_n}^{\mathfrak{F}}(.)E_-^*(\omega_n)u_n)}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}},$$
hence,
$$P_{\mathfrak{E}}\left(\frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \right) = \frac{\mathcal{I}\left(K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n\right)}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}}.$$

Since $\left\{ \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \right\}$ is an orthonormal set for $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ and $\mathcal{I}$ is an isometric from $\mathcal{B}(\mathfrak{E})$ onto $\mathcal{I}(\mathcal{B}(\mathfrak{E}))$ then
$$\frac{\mathcal{I}\left(K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n\right)}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \tag{17}$$
is a Parseval frame for $\mathcal{I}(\mathcal{B}(\mathfrak{E}))$. Applying $\mathcal{I}^*$ to (17) we obtain the first claim. Consequently, given any $f \in \mathcal{B}(\mathfrak{E})$ we have
$$\begin{aligned} f(z) &= \sum_n \left\langle f, \frac{K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \right\rangle_{\mathcal{B}(\mathfrak{E})} \frac{K_{\omega_n}^{\mathfrak{E}}(z)F^*(\omega_n)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \\ &= \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{K_{\omega_n}^{\mathfrak{E}}(z)F^*(\omega_n)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}. \end{aligned}$$
Using an analogous argument we obtain the second claim. □

Now we show that the Parseval frames for $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$ given in Theorem 7 are orthogonal.

Theorem 8. *Assume the hypothesis of Theorem 7, then*

1. *For every* $f \in \mathcal{B}(\mathfrak{E})$,

$$\sum_n u_n^* F(\omega_n) f(\omega_n) \frac{K_{\omega_n}^{\mathfrak{F}}(.) E_-^*(\omega_n) u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.) u_n} = 0. \tag{18}$$

2. *For every* $g \in \mathcal{B}(\mathfrak{F})$,

$$\sum_n u_n^* E_-^*(\omega_n) g(\omega_n) \frac{K_{\omega_n}^{\mathfrak{E}}(.) F^*(\omega_n) u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.) u_n} = 0. \tag{19}$$

Proof. Let $f \in \mathcal{B}(\mathfrak{E})$. Since $Ff \in \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ and $\left\{ \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.) u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.) u_n}} \right\}$ is a complete orthonormal set for $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ then

$$\begin{aligned}
\mathcal{I}(f)(z) &= F(z) f(z) \\
&= \sum_n \langle Ff, \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.) u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n}} \rangle \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(z) u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n}} \\
&= \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(z) u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n} \\
&= \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{F(z) K_{\omega_n}^{\mathfrak{E}}(z) F^*(\omega_n) u_n + E_-(z) K_{\omega_n}^{\mathfrak{F}}(z) E_-^*(\omega_n) u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n} \\
&= \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{\mathcal{I}\left(K_{\omega_n}^{\mathfrak{E}}(z) F^*(\omega_n) u_n\right) + \mathcal{J}\left(K_{\omega_n}^{\mathfrak{F}}(z) E_-^*(\omega_n) u_n\right)}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n}
\end{aligned}$$

Applying $\mathcal{J}^*$ to the last line above, and using the fact that $\mathcal{J}^*(Ff) = 0$ we obtain Equation (18). Similar argument applying $\mathcal{I}^*$ to $E_- g$ yields Equation (19). □

4. Sampling in the Space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$

The next theorem shows that if a de Branges matrix $\mathfrak{G} = [G_-(z) \quad G_+(z)]$ can be factored as

$$G_-(z) = F^{\#}(z) E_-(z), \quad \text{and} \quad G_+(z) = F(z) E_+(z),$$

with $F(z) \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$ and Equation (11) holds, then the space $\mathcal{B}(\mathfrak{G})$ will have the Kramer sampling property whenever the de Branges space $\mathcal{B}(\mathfrak{E})$ satisfies the conditions of Theorem 3. The sampling problem can be considered dual to the interpolation problem [22]; results concerning interpolation in vector valued reproducing kernel spaces can be found in [9].

Theorem 9. *Let* $\mathfrak{F}$ *and* $\mathfrak{E}$ *be two de Branges matrices that satisfy Definition* 2 *and Equation* (11). *Suppose further that* $\det E_+(\cdot)$ *is nonvanishing in* $\overline{\mathbb{C}^+}$ *and* $\det E_-(\cdot)$ *is nonvanishing in* $\overline{\mathbb{C}^-}$. *If* $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0$ *for some point* $\alpha \in \mathbb{C}$, *then the space* $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ *will have the Kramer sampling property.*

Proof. Using Theorem 3 it is enough to show that $R_\omega \mathcal{B}_\omega(\mathfrak{F} \diamond \mathfrak{E}) \subseteq \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ for every point $\omega \in \mathbb{C}$, and $K_\alpha^{\mathfrak{F}\diamond\mathfrak{E}}(\alpha) \succ 0$ for the given $\alpha \in \mathbb{C}$.

First, let $\omega \in \overline{\mathbb{C}^+}$ then $F(\omega)E_+(\omega)$ is invertible because $\det E_+(\omega) \neq 0$ by the hypothesis. Hence $R_\omega \mathcal{B}_\omega(\mathfrak{F} \diamond \mathfrak{E}) \subseteq \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ for every point $\omega \in \overline{\mathbb{C}^+}$ by Lemma 1. Similarly, $F(\omega)E_-(\omega)$ is invertible because $\det E_-(\omega) \neq 0$ by the hypothesis, hence $R_\omega \mathcal{B}_\omega(\mathfrak{F} \diamond \mathfrak{E}) \subseteq \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ for every point $\omega \in \overline{\mathbb{C}^-}$.

Let $\alpha \in \mathbb{C}$ be such that $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0$. Then $u^* K_\alpha^{\mathfrak{E}}(\alpha) u > 0$ for every nonzero vector $u \in \mathbb{C}^p$. Hence, by Equation (12) and using the fact that $F^*(\alpha)u \in \mathbb{C}^p$, $E_-^*(\alpha)u \in \mathbb{C}^p$, $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0$, and $K_\alpha^{\mathfrak{F}}(\alpha) \succeq 0$, by (5) we get

$$u^* K_\alpha^{\mathfrak{F}\diamond\mathfrak{E}}(\alpha) u = u^* F(\alpha) K_\alpha^{\mathfrak{E}}(\alpha) F^*(\alpha) u + u^* E_-(\alpha) K_\alpha^{\mathfrak{F}}(\alpha) E_-^*(\alpha) u > 0$$

i.e., $K_\alpha^{\mathfrak{F}\diamond\mathfrak{E}}(\alpha) \succ 0$ for the given $\alpha \in \mathbb{C}$. This completes the proof of the theorem. □

Example 5. Consider the de Branges space $\mathcal{B}(\mathfrak{G})$ with

$$\mathfrak{G} = [G_-(z) \quad G_+(z)]$$

and

$$G_-(z) = F^{\#}(z) E_-(z), \quad G_+(z) = F(z) E_+(z)$$

where $F(z)$ and $E_\pm(z)$ as in Example 4. Then it is evident that the space $\mathcal{B}(\mathfrak{G})$ have the Kramer sampling property by Theorem 9.

4.1. Multiplexing the Sampled Vector Valued Functions

Multiplexing refers to the transmission of several signals simultaneously over a single communications channel. Generically, multiplexing occurs when two (or more) signals x and y are encoded into X and Y in such a way that x and y can each be recovered from $X + Y$. The signals we consider here are elements of a de Branges space and the encoding involves the sampling of the signal. Specifically, if $f \in \mathcal{B}(\mathfrak{E})$ and $g \in \mathcal{B}(\mathfrak{F})$, we encode both f and g into the *multiplexed samples*:

$$\{u_n^* F(\omega_n) f(\omega_n) + u_n^* E_-^*(\omega_n) g(\omega_n)\}_n \tag{20}$$

which are transmitted in some fashion. The goal then is to recover f and g from these mixed samples.

Corollary 1. *Assume the hypotheses of Theorem* 7*,* $f \in \mathcal{B}(\mathfrak{E})$ *and* $g \in \mathcal{B}(\mathfrak{F})$*. Given the samples* $\{f(\omega_n)\}$ *and* $\{g(\omega_n)\}$*,* f *and* g *can be reconstructed from the multiplexed samples in* (20) *as follows:*

$$f(z) = \sum_n \left(u_n^* F(\omega_n) f(\omega_n) + u_n^* E_-^*(\omega_n) g(\omega_n)\right) \frac{K_{\omega_n}^{\mathfrak{E}}(z) F^*(\omega_n) u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n} \tag{21}$$

$$g(z) = \sum_n \left(u_n^* F(\omega_n) f(\omega_n) + u_n^* E_-^*(\omega_n) g(\omega_n)\right) \frac{K_{\omega_n}^{\mathfrak{F}}(z) E_-^*(\omega_n) u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n}. \tag{22}$$

Proof. Equations (21) and (22) follow immediately from Equations (15), (16), (18), and (19). □

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Conflict of interest

The authors declare that they have no conflict of interest.

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