Controllability of Sobolev-Type Linear Ensemble Systems

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***Abstract*— Systems composed of large ensembles of isolated or interacted dynamic units are prevalent in nature and engineered infrastructures. Linear ensemble systems are inarguably the simplest class of ensemble systems and have attracted intensive attention to control theorists and practionars in the past years. Comprehensive understanding of dynamic properties of such systems yet remains far-fetched and requires considerable knowledge and techniques beyond the reach of modern control theory. In this paper, we explore the classes of linear ensemble systems with system matrices that are not globally diagonalizable. In particular, we focus on analyzing their controllability properties under a Sobolev space setting and develop conditions under which uniform controllability of such ensemble systems is equivalent to that of their diagonalizable counterparts. This development significantly facilitates controllability analysis for linear ensemble systems through examining diagonalized linear systems.**

I. INTRODUCTION

Population systems arising from numerous practical applications exhibit variations in the system parameters that characterize the dynamics. Typical examples include the dispersion of Larmor frequencies of a sample of nuclear spins immersed in a static magnetic field in nuclear magnetic resonance spectroscopy and imaging [1], [2], [3], and the variation of circadian rhythms generated by suprachiasmatic nuclei in the mammal brain in chronobiology [4]. The control of such ensemble systems is challenged by the inherited inhomogeneous dynamics and the underactuated nature where typically control can only be implemented at the population level [5], [6].

In the past decade, considerable efforts have been made towards understanding fundamental properties of ensemble systems, especially controllability and observability of linear ensemble systems [7], [8], [9], [10], [11], [12], [13]. A recent development based on the notion and technique of separating points led to a necessary and sufficient condition of uniform ensemble controllability for quite general linear ensemble systems, and, more importantly, revealed the connection between classical and ensemble controllability [14]. However, ensemble systems of Sobolev-type, i.e., those with differentiable system and control matrices with respect to the system parameters, may fall outside the scope of this technique. Motivated by this observation, in this paper, we focus on analyzing controllability of these exceptional systems. In particular, we formulate ensemble control problems of such systems in a Sobolev space setting and derive explicit algebraic conditions that guarantee the equivalence between uniform ensemble controllability of these systems and that of their diagonalizable counterparts. This work then facilitates controllability analysis for general linear ensemble systems through their diagonalizable counterparts.

The paper is organized as follows. In the next section, we will briefly review the concept of ensemble controllability and the separating point techniques for linear ensemble systems with globally diagonalizable system matrices. In Section III, we will introduce the formulation of linear ensemble control problems in a Sobolev space setting, and derive the conditions leading to the equivalence of controllabiltiy properties between linear ensemble systems and their diagonalizable counterpart.

This work was supported in part by the National Science Foundation under the awards CMMI-1933976 and ECCS-1810202.

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II. ENSEMBLE SYSTEMS AND ENSEMBLE CONTROL

In this section, we introduce the notion of ensemble systems and ensemble controllability, and review the recent work on characterizing controllability of linear ensemble systems based on the techniques of separating points, which is most related to our development [14].

A. Ensemble controllability

An *ensemble system* is a parameterized family of dynamical systems defined on a common manifold M of the form

$$\frac{d}{dt}x(t,\beta) = f(t,\beta,x(t,\beta),u(t)), \tag{1}$$

where β is the parameter taking values on $\Omega \subseteq \mathbb{R}^d$, $x(t,\beta) \in M$ holds for all $t \geq 0$ and $\beta \in \Omega$, $f(t,\beta,\cdot,u(t))$ is a vector field on M, and $u(t) \in \mathbb{R}^m$ is an external input. Note that the ensemble system in (1) is indeed a control system defined on $\mathscr{F}(\Omega,M)$, the space of M-valued functions defined on Ω. An *ensemble control* task then focuses on the design of a β-independent control input $u:[0,T] \to \mathbb{R}^m$ to steer the ensemble system in (1) from an initial state $x(0,\cdot) \in \mathscr{F}(\Omega,\beta)$ to the desired final state $x_F \in \mathscr{F}(\Omega,\beta)$ in a finite time T. In the case that the parameter space Ω is an infinite set, i.e., the ensemble system in (1) contains infinitely many individual systems, the state space $\mathscr{F}(\Omega,M)$ is generally an infinite-dimensional manifold, which significantly challenges the study ensemble control problems. For example, exact control is not always feasible for infinite-dimensional systems [15], and hence we introduce the notion of ensemble controllability to characterize the ability of the control input to steer an ensemble system between the states of interest in the approximate sense.

Definition 1 (Ensemble controllability): An ensemble system in the form of (1) is said to be *ensemble controllable* on $\mathscr{F}(\Omega,M)$ if for any $\varepsilon>0$ and starting with any initial condition $x_0\in\mathscr{F}(\Omega,M)$, there is a piecewise constant control input $u:[0,T]\to\mathbb{R}^m$ steering the system into an ε-neighborhood of the desired final state $x_F\in\mathscr{F}(\Omega,M)$ at a finite time $T>0$, i.e., $d\big(x(T,\cdot),x_F(\cdot)\big)<\varepsilon$, where $d:\mathscr{F}(\Omega,M)\times\mathscr{F}(\Omega,M)\to\mathbb{R}$ is a metric on $\mathscr{F}(\Omega,M)$.

Note that ensemble controllability of the system in (1) indeed depends on the metric d, meaning, the system may express different ensemble controllability for different choices of d on $\mathscr{F}(\Omega,M)$, as shown in the later discussion.

In this paper, we particularly focus on time-invariant linear ensemble systems in the form of

$$\frac{d}{dt}x(t,\beta)=A(\beta)x(t,\beta)+B(\beta)u(t), \tag{2}$$

where β is the system parameter taking values on a compact subspace K of $\mathbb{R}$, the state $x(t,\cdot)$ is in the space $C(K,\mathbb{R}^n)$ of continuous $\mathbb{R}^n$-valued functions defined on K, and the system matrix $A\in C(K,\mathbb{R}^{n\times n})$ and control matrix $B\in C(K,\mathbb{R}^{n\times m})$ are continuous $\mathbb{R}^{n\times n}$- and $\mathbb{R}^{n\times m}$-valued functions defined on K, respectively. In this case, the state-space $C(K,\mathbb{R}^n)$ is a Banach space under the uniform norm, that is, $\|g\|_\infty=\sup_{\beta\in K}\|g(\beta)\|$ for any $g\in C(K,\mathbb{R}^n)$, where $\|\cdot\|:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a norm on $\mathbb{R}^n$, and the uniform norm induces the uniform metric $d(g_1,g_2)=\|g_1-g_2\|_\infty$ for any $g_1,g_2\in C(K,\mathbb{R}^n)$. Ensemble controllability of the system in (2) under this metric is referred to as *uniform ensemble cotnrollability*.

B. Uniform control of diagonalizable linear ensembles

To acquire a deep understanding of the relationship between ensemble controllability and well-studied classical controllability , in our recent work, we developed the separating point technique so that uniform ensemble controllability of time-invariant linear ensemble systems can be equivalently examined by classical controllability of individual systems in the reparameterized ensembles [14]. Broadly speaking, this also provide a tool to analyze infinite-dimensional systems by using finite-dimensional techniques, in which the notion of ensemble controllability Criterion Matrix plays a crucial role.

Lemma 1: Consider a scalar multi-input linear ensemble system indexed by the parameter β varying on a compact set $K\subset\mathbb{R}$, given by

$$\frac{d}{dt}x(t,\beta)=a(\beta)x(t,\beta)+\sum_{i=1}^{m}b_i(\beta)u_i(t), \tag{3}$$

where $x(t,\cdot)\in C(K,\mathbb{R})$, $a\in C(K,\mathbb{R})$, $b_i\in C(K,\mathbb{R})$, and $u_i:[0,T]\to\mathbb{R}$ are piecewise constant for $i=1,\dots,m$, and let $a^{-1}(\eta)=\{\beta_\eta^1,\dots,\beta_\eta^{\kappa(\eta)}\}$ be the preimage of $\eta\in a(K)$ with $\kappa(\eta)=|a^{-1}(\eta)|$ denoting its cardinality. Then, this system is uniformly ensemble controllable on $C(K,\mathbb{R})$ if and only if the *Ensemble Controllability Criterion Matrix* $D(\eta)\in\mathbb{R}^{\kappa(\eta)\times m}$, defined by

$$D(\eta)=\begin{bmatrix} b_1(\beta_\eta^1) & \cdots & b_m(\beta_\eta^1)\\ \vdots & \ddots & \vdots\\ b_1(\beta_\eta^{\kappa(\eta)}) & \cdots & b_m(\beta_\eta^{\kappa(\eta)})\end{bmatrix}, \tag{4}$$

has full rank, i.e., $\mathrm{rank}(D(\eta))=\kappa(\eta)\le m$, for all $\eta\in a(K)$.

Proof: See [14]. ∎

Lemma 1 sheds light on the idea of separating points by using a 1-dimensional linear ensemble system and, more importantly, provides an effective tool, that is, the Ensemble Controllability Criterion Matrix D, to examine whether the points in K with the same image under the drift a are separated, i.e., the separation of the injective branches of a, by multiple control inputs, which is the essence to guarantee uniform ensemble controllability of the system. However, for multi-dimensional linear ensemble systems, it is also required to take the points in the shared spectra of the system matrices into consideration, and the main idea can be well illuminated by globally diagonalizable linear ensembles.

Proposition 1: Suppose that the system matrix $A\in C(K,\mathbb{R}^{n\times n})$ of the time-invariant linear ensemble system in (2) is diagonalizable with eigenvalue functions $\lambda_1,\dots,\lambda_n\in C(K,\mathbb{R})$. Let

$$\frac{d}{dt}\begin{bmatrix} y_1(t,\beta)\\ \vdots\\ y_n(t,\beta)\end{bmatrix}=\begin{bmatrix}\lambda_1(\beta)y_1(t,\beta)\\ \vdots\\ \lambda_n(\beta)y_n(t,\beta)\end{bmatrix}+\begin{bmatrix}\tilde b_1(\beta)\\ \vdots\\ \tilde b_n(\beta)\end{bmatrix}u(t) \tag{5}$$

with $y_i(t,\cdot)\in C(K,\mathbb{R})$ and $\tilde b_i\in C(K,\mathbb{R}^{1\times m})$ for all $i=1,\dots,n$ be the corresponding diagonalized system, transformed by the eigenvalue decomposition. Then, the system in (2), as well as the diagonalized system in (5), is uniformly ensemble controllable on $C(K,\mathbb{R}^n)$ if and only if the system obtained by parameterizing the system in (5) by $\eta_1=\lambda_1(\beta),\dots,\eta_n=\lambda_n(\beta)$, given by

$$\frac{d}{dt}\begin{bmatrix} z_1(t,\eta_1)\\ \vdots\\ z_n(t,\eta_n)\end{bmatrix}=\begin{bmatrix}\eta_1 I_{\kappa_1(\eta_1)}z_1(t,\eta_1)\\ \vdots\\ \eta_n I_{\kappa_n(\eta_n)}z_n(t,\eta_n)\end{bmatrix}+\begin{bmatrix}D_1(\eta_1)\\ \vdots\\ D_n(\eta_n)\end{bmatrix}u(t),$$

is controllable on $\mathbb{R}^{N(\eta)}$ for each n-tuple $\eta=(\eta_1,\dots,\eta_n)\in K_1\times\cdots\times K_n$ with $K_i=\lambda_i(K)$, $i=1,\dots,n$, where $N(\eta)=\sum_{i=1}^n\kappa_i(\eta_i)$, $\kappa_i(\eta_i)=|\lambda_i^{-1}(\eta_i)|$ is the cardinality of the preimage of η_i under λ_i, $I_{\kappa_i(\eta_i)}$ is the $\kappa_i(\eta_i)\times\kappa_i(\eta_i)$ identity matrix, and $D_i(\eta_i)\in\mathbb{R}^{\kappa_i(\eta_i)\times m}$ is the Ensemble Controllability Criterion Matrix associated with the scalar system $\frac{d}{dt}y_i(t,\beta)=\lambda_i(\beta)y_i(t,\beta)+\tilde b_i(\beta)u(t)$.

Proof: See [14]. ∎

III. Controllability of Sobolev-Type Linear Ensembles

In our previous work [14], we have also shown that a time-invariant linear ensemble system in the form of (2) is uniformly ensemble controllable if and only if its diagonalizable counterpart is uniformly ensemble controllable, under the condition that all of the system matrix, control matrix and state of the system are defined on the space of continuous functions equipped with the uniform norm [14]. This surprising result characterizes a fundamental difference between ensemble and classical linear systems: classical controllability of a linear system is definitely implied by that of its diagonalizable counterpart, but the converse is generally not true. For example, the system

$$\frac{d}{dt}x(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

is controllable on $\mathbb{R}^2$, but its *diagonalizable counterpart*,

$$\frac{d}{dt}x(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

is not controllable on $\mathbb{R}^2$.

Unfortunately, if some smoothness conditions are imposed on the control and system matrices of the system in (2), e.g., differentiability up to certain orders, together with the nondiagonalizability of the system matrix, then it may lay outside the scope of the separating point technique so that the equivalence between the system and its diagonalizable counterpart in terms of uniform ensemble controllability on the space of continuous function may fail. The following example gives a glance at this situation.

Example 1: Consider the linear ensemble system evolving on $C([1,2],\mathbb{R}^2)$, given by

$$\begin{aligned}\frac{d}{dt}x(t,\beta) &= J(\beta)x(t,\beta) + B(\beta)u(t) \\ &= \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & \beta \end{bmatrix} u(t) \end{aligned} \tag{6}$$

Without loss of generality, we assume that the initial condition satisfies $x(0,\beta)=0$ for all β, then the solution of the system is

$$\begin{aligned} x(t,\beta) &= \int_0^t e^{(t-s)J(\beta)} B(\beta)u(s)ds \\ &= \int_0^t \begin{bmatrix} (t-s)e^{(t-s)\beta} & (1+\beta(t-s))e^{(t-s)\beta} \\ e^{(t-s)\beta} & \beta e^{(t-s)\beta} \end{bmatrix} u(s)ds. \end{aligned}$$

It is straight forward to check that $x(t,\beta) = \begin{bmatrix} x_1(t,\beta) & x_2(t,\beta) \end{bmatrix}'$, with "$'$" denoting the matrix transpose, satisfies $x_1(t,\beta) = \frac{d}{d\beta}x_2(t,\beta)$ for any choice of the control input u and hence cannot be an arbitrary element in $C([1,2],\mathbb{R}^2)$, i.e., the system in (6) is not uniformly ensemble controllable on $C([1,2],\mathbb{R}^2)$.

However, the diagonalizable counterpart of this system, given by

$$\frac{d}{dt}x(t,\beta) = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} x(t,\beta) + \begin{bmatrix} 0 & 1 \\ 1 & \beta \end{bmatrix} u(t), \tag{7}$$

is uniformly ensemble controllable on $C([1,2],\mathbb{R}^2)$. To see this, we parameterize the system in (7) by the eigenvalue functions $\eta_1 = \lambda_1(\beta) = \beta$ and $\eta_2 = \lambda_2(\beta) = \beta$ of its system matrix, which gives

$$\frac{d}{dt}z(t,\eta) = \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix} z(t,\eta) + \begin{bmatrix} 0 & 1 \\ 1 & \eta_2 \end{bmatrix} u(t). \tag{8}$$

For each $\eta = (\eta_1,\eta_2) \in [1,2]\times[1,2]$, controllability of the system in (8) on $\mathbb{R}^2$ is implied by the full rank of its controllability matrix

$$W(\eta) = \begin{bmatrix} 0 & 1 & 0 & \eta_1 \\ 1 & \eta_2 & \eta_2 & \eta_2^2 \end{bmatrix}.$$

Then, by Proposition 1, the system in (7) is uniformly ensemble controllable on $C([1,2],\mathbb{R}^2)$.

A careful deliberation on the above example, especially the analytical properties of $B(\beta)$, discloses the cause of the nonequivalence between the systems in (6) and (7): the first row of $B(\beta)$ is the derivative of its second row with respect to β, and this is exactly the property inherited by the solution $x(t,\beta)$ of the system in (6), which destroys uniform ensemble controllability of the system. This observation can be directly extended to the case that the eigenvalue of the system matrix is an arbitrary injective function of β.

Lemma 2: Given a linear ensemble system defined on $C(K,\mathbb{R}^2)$ of the form

$$\frac{d}{dt}x(t,\beta) = J(\beta)x(t,\beta) + B(\beta)u(t), \tag{9}$$

where $\beta \in K$ with $K \subset \mathbb{R}$ compact, $u(t) \in \mathbb{R}^m$, and

$$J(\beta) = \begin{bmatrix} \lambda(\beta) & 1 \\ 0 & \lambda(\beta) \end{bmatrix}$$

is the Jordan block with an injective eigenvalue function $\lambda \in C(K,\mathbb{R})$. Let $b_i \in C(K,\mathbb{R}^{1\times m})$ denote the i^{th} row of B, $i=1,2$, then the system in (9) is not uniformly ensemble controllable on $C(K,\mathbb{R}^2)$ if $b_2 \circ \lambda^{-1}$ is differentiable and satisfies $\frac{d}{d\eta} b_2 \circ \lambda^{-1}(\eta) = b_1 \circ \lambda^{-1}(\eta)$ for any $\eta \in \lambda(K)$.

Proof: The injectivity of λ allows the parametrization of the ensemble in (9) by $\eta \in \lambda(K)$ as

$$\frac{d}{dt}y(t,\eta) = J\circ\lambda^{-1}(\eta)y(t,\eta) + B\circ\lambda^{-1}(\eta)u(t), \tag{10}$$

and $\lambda(K) \subset \mathbb{R}$ is compact since λ is continuous and K is compact. Then, the system in (9) is uniformly ensemble controllable on $C(K,\mathbb{R}^2)$ if and only if the system in (10) is uniformly ensemble controllable on $C(\lambda(K),\mathbb{R}^2)$. Because the reachable set of the system in (10) is the closure (under the topology of uniform convergence) of the space $\mathscr{L} = \text{span}\{(J^kB_i)\circ\lambda^{-1} \in C(\lambda(K),\mathbb{R}^2) : k \in \mathbb{N},\ i=1,2\}$ with B_i the ith column of B[10], denoted by $\overline{\mathscr{L}}$ and any element $p \in \mathscr{L}$ follows the form

$$\begin{aligned} p(\eta) = \begin{bmatrix} p_1(\eta) \\ p_2(\eta) \end{bmatrix} = \sum_{i=1}^m \Bigg\{ a_{0i} \begin{bmatrix} b_{1i}(\lambda^{-1}(\eta)) \\ b_{2i}(\lambda^{-1}(\eta)) \end{bmatrix} \\ + \sum_{k_i=1}^{N_i} a_{k_i i} \begin{bmatrix} \eta^{k_i} b_{1i}(l^{-1}(\eta)) + k_i \eta^{k_i-1} b_{2i}(\lambda^{-1}(\eta)) \\ \eta^{k_i} b_{2i}(\lambda^{-1}(\eta)) \end{bmatrix} \Bigg\}, \end{aligned}$$

where $N_i \in \mathbb{N}$, $a_{k_i i} \in \mathbb{R}$, and b_{ji} denote the $(j,i)^{\text{th}}$ entry of B for all $i=1,\ldots,m$ and $j=1,2$, the condition $b_1(\lambda^{-1}(\eta)) = \frac{d}{d\eta} b_2(\lambda^{-1}(\eta))$ implies $p_1(\eta) = \frac{d}{d\eta}p_2(\eta)$. Therefore, we conclude $\overline{\mathscr{L}} \neq C(\lambda(K),\mathbb{R}^2)$, which then leads to uncontrollability of the system in (10) on $C(\lambda(K),\mathbb{R}^2)$ and then so is the original system in (9) on $C(K,\mathbb{R}^2)$. ■

In the sequel, we will focus on imposing conditions on these exceptional systems for guaranteeing the equivalence between them and their diagonalizable counterparts in terms of uniform ensemble controllability.

A. Sobolev-type linear ensemble systems

Lemma 2 gives a strong hint that the failure of the controllability equivalence between a linear ensemble system and its diagonalizable counterpart arises from the differentiability of the control matrix. However, it is common that the derivatives of a uniformly convergent sequence of differentiable functions are not uniformly convergent [16]. This fact may lead to the situation that, taking the element $p = [\ p_1\ p_2\ ]' \in \mathscr{L}$ in the proof of Lemma 2 as an example, $\lim_{N_i\to\infty} p_1 \notin C(K,\mathbb{R}^2)$ for some i. To resolve this technical issue, it is inevitable to restrict our attention to linear ensemble systems defined on the space of differentiable functions up to some order. In particular, to obtain the appropriate order, we investigate more about the analytic properties of Jordan blocks. Given

$$J = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix} \in \mathbb{R}^{n\times n}$$

and an analytic function $f:\mathbb{R}\to\mathbb{R}$, it can be shown by using the Taylor's series of f that $f(J)$ is equal to

$$\begin{bmatrix} f(\lambda) & \frac{d}{d\lambda}f(\lambda) & \frac{1}{2!}\frac{d^2}{d\lambda^2}f(\lambda) & \cdots & \frac{1}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}f(\lambda) \\ & f(\lambda) & \frac{d}{d\lambda}f(\lambda) & \cdots & \frac{1}{(n-2)!}\frac{d^{n-2}}{d\lambda^{n-2}}f(\lambda) \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \frac{d}{d\lambda}f(\lambda) \\ & & & & f(\lambda) \end{bmatrix}$$

which only involves the derivative of f with respect to λ up to the $(n-1)^{\text{th}}$ order [17]. Consequently, for an n-dimensional linear ensemble system in the form of (2), it is enough to require that each entry of the system matrix A, together with its eigenvalue functions, and control matrix B, is $(n-1)$ times continuously differentiable with respect to the system parameter β, and then so is the state $x(t,\cdot)$.

Notionally, we denote the space of $\mathbb{R}^n$-valued functions defined on K possessing continuous derivatives up to order k, including one-side derivatives on the boundary of K, by $C^k(K,\mathbb{R}^n)$, and equip it with the norm

$$\|f\|_{k,\infty} = \sum_{i=0}^{k}\sup_{\beta\in K}\big\|\frac{d^i}{d\beta^i}f(\beta)\big\|$$

for any $f\in C^k(K,\mathbb{R}^m)$. Note that this norm is exactly the norm on the Sobolev space $W^{k,\infty}(K,\mathbb{R}^n)$. Although, under the norm topology, $C^k(K,\mathbb{R}^n)$ is not dense in $W^{k,\infty}(K,\mathbb{R}^n)$. it is complete and hence a Banach space, which coincides with the Hölder space of k-times differentiable functions with the Hölder exponent 0 [18].

We then focus on time-invariant linear ensemble systems evolving on the space $C^{n-1}(K,\mathbb{R}^n)$, and still refer to ensemble controllability defined through $\|\cdot\|_{n-1,\infty}$ as uniform ensemble controllability. Moreover, Proposition 1 can be directly extended to diagonalizable linear ensemble systems defined on $C^{n-1}(K,\mathbb{R}^n)$.

B. Controllability of Sobolev-type ensemble systems

After the technical preparation in Section III-A, the focus of this section is the derivation of conditions under which a linear ensemble system in the form of (2) is uniformly ensemble controllable on $C^{n-1}(K,\mathbb{R}^n)$ if and only if its diagonalizable counterpart is uniformly ensemble controllable on $C^{n-1}(K,\mathbb{R}^n)$.

The main idea can be well illuminated by using the 2-dimensional linear ensemble system in the form of (9), i.e.,

$$\frac{d}{dt}x(t,\beta) = J(\beta)x(t,\beta) + B(\beta)u(t),$$

where $J\in C^1(K,\mathbb{R}^2)$ is a Jordan block with the injective eigenvalue function $\lambda\in C^1(K,\mathbb{R})$ and $B\in C^1(K,\mathbb{R}^{2\times m})$ is the control matrix. Its diagonalizable counterpart is given by

$$\frac{d}{dt}x(t,\beta) = \lambda(\beta)Ix(t,\beta) + B(\beta)u(t), \tag{11}$$

where I denotes the 2×2 identity matrix. Moreover, by Proposition 1, the system in (11) can be uniformly ensemble controllable on $C^1(K,\mathbb{R}^2)$ driven by two control inputs. Therefore, in the context of establishing the controllability equivalence between the systems in (9) and (11), it suffices to assume $B\in C^1(K,\mathbb{R}^{2\times 2})$. Let $B_i\in C^1(K,\mathbb{R}^2)$ denotes the i^{th} column of B for $i=1,2$, because the reachable sets of the systems in (9) and (11) are the closure of

$$\mathscr{L} = \text{span}\{J^kB_i \in C^1(K,\mathbb{R}^2) : k\in\mathbb{N},\ i=1,2\}$$

and

$$\mathscr{K} = \text{span}\{\lambda^kB_i \in C^1(K,\mathbb{R}^2) : k\in\mathbb{N},\ i=1,2\}$$

under the Sobolev norm $\|\cdot\|_{1,\infty}$, respectively, it amounts to impose conditions such that for any $f\in C^1(K,\mathbb{R}^2)$, $f\in\mathscr{L}$ if and only if $f\in\mathscr{K}$. Following the same computation as in the proof of Lemma 2, $f\in\mathscr{L}$ if and only if there exist polynomials p_1 and p_2 over $\mathbb{R}$ such that

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} (p_1\circ\lambda)b_{11} + \frac{dp_1\circ\lambda}{d\lambda}b_{21} + (p_2\circ\lambda)b_{12} + \frac{dp_2\circ\lambda}{d\lambda}b_{22} \\ (p_1\circ\lambda)b_{21} + (p_2\circ\lambda)b_{22} \end{bmatrix},$$

or equivalently, the system of functional equations

$$\begin{bmatrix} f_1 - \frac{d}{d\lambda}f_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} b_{11} - \frac{d}{d\lambda}b_{21} & b_{12} - \frac{d}{d\lambda}b_{22} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} p_1\circ\lambda \\ p_2\circ\lambda \end{bmatrix} \tag{12}$$

has a set of polynomial solutions p_1 and p_2, where b_{ij} is the $(i,j)^{\text{th}}$ entry of B. The necessary and sufficient condition for the existence of solutions of the system in (12) is

$$\begin{aligned} 0 &\neq \det\begin{bmatrix} b_{11} - \frac{d}{d\lambda}b_{21} & b_{12} - \frac{d}{d\lambda}b_{22} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \det\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} - \det\begin{bmatrix} \frac{d}{d\lambda}b_{21} & \frac{d}{d\lambda}b_{22} \\ b_{21} & b_{22} \end{bmatrix} \end{aligned} \tag{13}$$

for all $\beta \in K$. On the other hand, $f \in \mathscr{K}$ is equivalent to

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} (q_1 \circ \lambda) b_{11} + (q_2 \circ \lambda) b_{12} \\ (q_1 \circ \lambda) b_{21} + (q_2 \circ \lambda) b_{22} \end{bmatrix}$$

for some polynomials q_1 and q_2 over $\mathbb{R}$, i.e., the solvability of the system of functional equations

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} q_1 \circ \lambda \\ q_2 \circ \lambda \end{bmatrix} \tag{14}$$

whose necessary and sufficient condition is given by

$$\det \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \neq 0 \tag{15}$$

for all $\beta \in K$. Comparing the inequalities in (13) and (15) immediately reveals a sufficient condition under which the systems of functional equations in (12) and (14) are simultaneously solvable: for all $\beta \in K$,

$$\det \begin{bmatrix} \frac{d}{d\lambda} b_{21} & \frac{d}{d\lambda} b_{22} \\ b_{21} & b_{22} \end{bmatrix} = 0. \tag{16}$$

Proposition 2: Consider the time-invariant linear ensemble system defined on $C^1(K, \mathbb{R}^2)$ as in (9), i.e.,

$$\frac{d}{dt} x(t, \beta) = J(\beta) x(t, \beta) + B(\beta) u(t),$$

where β takes values on a compact set $K \subset \mathbb{R}$, $J \in C^1(K, \mathbb{R}^{2\times 2})$ is the Jordan canonical with an injective eigenvalue function $\lambda \in C^1(K, \mathbb{R})$. If the control matrix $B \in C^1(K, \mathbb{R}^{2\times 2})$ satisfies one of the following conditions,

1) $\det \begin{bmatrix} \frac{d}{d\lambda} b_{21} & \frac{d}{d\lambda} b_{22} \\ b_{21} & b_{22} \end{bmatrix} = 0$ for all $\beta \in K$,
2) b_{21}/b_{22} is a constant function in β,

then the system is uniformly ensemble controllable on $C^1(K, \mathbb{R}^2)$ if and only if its diagonalizable counterpart in (11), i.e.,

$$\frac{d}{dt} x(t, \beta) = \lambda(\beta) I x(t, \beta) + B(\beta) u(t),$$

is uniformly ensemble controllable on $C^1(K, \mathbb{R}^2)$, where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix and $b_{ij} \in C^1(K, \mathbb{R})$ denotes the $(i, j)^{\text{th}}$ entry of B for all $i, j = 1, 2$.

Proof: It remains to derive Condition 1) from Condition 2). To this end, we explicitly calculate the determinant in Condition 1) as follows

$$\det \begin{bmatrix} \frac{d}{d\lambda} b_{21} & \frac{d}{d\lambda} b_{22} \\ b_{21} & b_{22} \end{bmatrix} = b_{22} \frac{d}{d\lambda} b_{21} - b_{21} \frac{d}{d\lambda} b_{22}$$
$$= (b_{21}^2 + b_{22}^2) \frac{d}{d\lambda} \arctan \frac{b_{21}}{b_{22}}.$$

If b_{21}/b_{22} is a constant function, then so is $(b_{21} \circ \lambda^{-1})/(b_{22} \circ \lambda^{-1})$, which yields $\frac{d}{d\lambda} \arctan(b_{21}/b_{22}) = 0$ leading to Condition 1). ■

Remark 1:

1) If b_{22} vanishes at some points in K, Condition 2) should be interpreted as $\lim_{\beta \to \beta_0} b_{21}(\beta)/b_{22}(\beta)$ is a constant function in $\beta_0 \in K$.
2) Conditions 1) and 2) in Porposition 2 are not equivalent: as indicated by the calculation in the proof, Condition 1) holds if and only if b_{21}/b_{22} is a constant function or $b_{21}^2 + b_{22}^2 = 0$ is the zero function. The latter situation then implies $b_{21} = 0$ and $b_{22} = 0$ are both the zero function, and consequently so is $B = 0$. However, this case is meaningless in the sense that there is no control input applied to the system.

The establishment of the condition guaranteeing the uniform ensemble controllability equivalence between 2-dimensional linear ensembles and their diagonalizable counterparts can be directly generalized to the n-dimensional case.

Theorem 1: Consider the linear ensemble system define on $C^{n-1}(K, \mathbb{R}^n)$, given by

$$\frac{d}{dt} x(t, \beta) = J(\beta) x(t, \beta) + B(\beta) u(t), \tag{17}$$

where β varies on a compact set $K \subset \mathbb{R}$, $J \in C^{n-1}(K, \mathbb{R}^{n\times n})$ is in the Jordan block with an injective eigenvalue function $\lambda \in C^{n-1}(K, \mathbb{R})$. If the control matrix $B \in C^{n-1}(K, \mathbb{R}^{n\times n})$ satisfies one of the following conditions,

1) $\det \begin{bmatrix} \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} \frac{d^{i-1} b_{i1}}{d\lambda^{i-1}} & \cdots & \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} \frac{d^{i-1} b_{in}}{d\lambda^{i-1}} \\ \vdots & & \vdots \\ \sum_{i=j}^n \frac{(-1)^{i-j}}{(i-j)!} \frac{d^{i-j} b_{i1}}{d\lambda^{i-j}} & \cdots & \sum_{i=1}^n \frac{(-1)^{i-j}}{(i-j)!} \frac{d^{i-j} b_{in}}{d\lambda^{i-j}} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = 0$ for all $\beta \in K$,
2) $b_{i,j-1}/b_{ij}$ is a constant function for each $i, j = 2, \ldots, n$,

then the system in (17) is uniformly ensemble controllable on $C^{n-1}(K, \mathbb{R}^n)$ if and only if its diagonalizable counterpart

$$\frac{d}{dt} x(t, \beta) = \lambda(\beta) I x(t, \beta) + B(\beta) u(t), \tag{18}$$

is uniformly ensemble controllable on $C^{n-1}(K, \mathbb{R}^n)$, where $b_{ij} \in C^{n-1}(K, \mathbb{R})$ denotes the $(i, j)^{\text{th}}$ entry of B, and $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

Proof: The proof follows from direct computations and the induction on n by using Proposition 2. ■

Note that Condition 2) in Theorem 1 can also be equivalently represented as b_{i1}/b_{ij} is constant for all $i, j = 2, \ldots, n$. This further implies that the functions $b_{i1}, \ldots, b_{in}$ are not linearly independent over $C^1(K, \mathbb{R})$. However, in this case, by Proposition 1, both of the systems in (18) and (17) cannot be uniformly ensemble controllable on $C^{n-1}(K, \mathbb{R})$ if λ is not injective.

In addition, for the system in (17), if the control matrix $B \in \mathbb{R}^{n\times n}$ is constant, then Conditions 1) and 2) in Theorem 1 holds trivially. Specifically, in this case, all the terms involving derivatives in Condition 1) are vanishing. As a result, the controllability equivalence indeed holds for such types of systems defined on $C(K, \mathbb{R})$.

Corollary 1: Consider the linear ensemble system as in (2), i.e.,

$$\frac{d}{dt} x(t, \beta) = A(\beta) x(t, \beta) + B(\beta) u(t),$$

where the system matrix

$$A(\beta)=\begin{bmatrix} \lambda_1(\beta) & & & & & \\ & \ddots & & & & \\ & & \lambda_k(\beta) & & & \\ & & & J_{k+1}(\beta) & & \\ & & & & \ddots & \\ & & & & & J_l(\beta) \end{bmatrix}$$

is in the Jordan canonical form with eigenvalue functions $\lambda_1,\ldots,\lambda_l \in C(K,\mathbb{R})$, $J_s \in C(K,\mathbb{R}^{n_s\times n_s})$ is the Jordan block with the eigenvalue function λ_s for each $s=k+1,\ldots,l$, and the corresponding partition of the rows of the control matrix $B\in C(K,\mathbb{R}^{n\times m})$ has the form

$$B(\beta)=\begin{bmatrix} b_1(\beta) \\ \vdots \\ b_k(\beta) \\ B_{k+1} \\ \vdots \\ B_l \end{bmatrix}$$

with $b_i \in C(K,\mathbb{R}^{1\times m})$ for $i=1,\ldots,k$ and $B_s \in \mathbb{R}^{n_s\times m}$ for $s=k+1,\ldots,l$. This system is uniformly ensemble controllable on $C(K,\mathbb{R}^n)$ if and only if its diagonalizable counterpart

$$\frac{d}{dt}x(t,\beta)=\Lambda(\beta)x(t,\beta)+B(\beta)u(t)$$

is uniformly ensemble controllable on $C(K,\mathbb{R}^n)$, where

$$\Lambda(\beta)=\begin{bmatrix} \lambda_1(\beta) & & & & & \\ & \ddots & & & & \\ & & \lambda_k(\beta) & & & \\ & & & \lambda_{k+1}(\beta)I_{k+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_l(\beta)I_l \end{bmatrix}$$

and $I_s \in \mathbb{R}^{n_s\times n_s}$ denotes the identity matrix for each $s=k+1,\ldots,l$.

Proof: The proof follows from applying Theorem 1 to the subsystems $\frac{d}{dt}x_s(t,\beta)=J_s(\beta)x(t,\beta)+B_su(t)$ defined on $C(K,\mathbb{R}^{n_s})$ for all $s=k+1,\ldots,l$. ∎

IV. CONCLUSIONS AND FUTURE WORKS

In this paper, we reveal the relation of controllability properties between linear ensemble systems with diagonalizable and nondiagonalizable system matrices. In particular, we extend the notion of uniform ensemble controllability to linear ensemble systems defined on the Hölder space, and derive explicit algebraic conditions under which such an ensemble is uniformly ensemble controllable if and only if its diagonalizable counterpart is uniformly ensemble controllable. This result not only generalizes the uniform ensemble controllability conditions developed in our previous work to broader classes of systems, in particular, those with system matrices neither globally diagonalizable nor globally similar to Jordan canonical forms, but also sheds light on a fundamental difference between linear ensemble systems and classical linear systems. Based on this line of research, in the future, we will focus on the extension of linear ensemble control problems defined on the Sobolev space, which will allow the utilization of well-established theoretical and computational techniques, e.g., gradient flow and pseudospectral methods, developed in the domain of Sobolev space to push forward the boundaries of ensemble control theory.

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