Vol. 52 (2021) *Acta Physica Polonica B* No 8

THEORY NEEDS FOR FUTURE e^+e^- COLLIDERS*

Ayres Freitas

Pittsburgh Particle-physics Astro-physics & Cosmology Center (PITT-PACC)
Department of Physics & Astronomy, University of Pittsburgh
Pittsburgh, PA 15260, USA

(Received April 16, 2021; accepted April 24, 2021)

New high-luminosity e^+e^- colliders have been proposed to perform precision measurements of electroweak and Higgs physics to scrutinize the mechanism of electroweak symmetry breaking and search for signs of new physics. The interpretation of the data from such a machine is only possible with the help of accurate theoretical calculations of the Standard Model expectations, including higher-order radiative corrections. This contribution provides an overview of the current knowledge and required improvements for our theoretical understanding of a range of key observables. Furthermore, it gives a short summary of the calculational techniques that can be leveraged to realize these improvements.

DOI:10.5506/APhysPolB.52.929

1. Introduction

Studies of electroweak precision observables (EWPOs) and Higgs precision observables (HPOs) at future e^+e^- colliders require theory inputs on several fronts:

— In order to probe any contributions of physics beyond the Standard Model (BSM physics), experimental data for the EWPOs/HPOs need to be compared to precise predictions of these quantities within the Standard Model (SM). The computation of these predictions necessitates the inclusion of multi-loop corrections in the full SM, *i.e.* with many massive particles in the loops.

— However, the quantities that are commonly referred to as EWPOs or HPOs are technically not observables. Instead, they are obtained by correcting the experimental data to account for detector acceptances and selection cuts, smearing of the center-of-mass energy due

* Presented at XXVII Cracow Epiphany Conference on *Future of Particle Physics*, Cracow, Poland, January 7–10, 2021.

to initial-state radiation, and background contributions. These tasks are typically accomplished with the help of Monte-Carlo generators that simulate QED and QCD radiation and the background processes. For more information, see *e.g.* Ref. [1].

— Moreover, to perform the comparison of EWPOs/HPOs and SM predictions, other electroweak parameters are needed as inputs. Examples for such input parameters are: the top-quark mass m_t, the strong coupling α_s, the electromagnetic coupling α, *etc.* The distinction between EWPOs and input parameters is somewhat arbitrary, but broadly speaking, an "input parameter" is a quantity that (a) can be measured very precisely and (b) whose interpretation is not affected by new physics to first approximation (*e.g.* a particle mass that is extracted from a kinematical feature).

In the following, the impact of theory inputs, in particular higher-order radiative corrections, is discussed both for the input parameters and for genuine EWPOs and HPOs. The size of the theory corrections is compared to the anticipated experimental precision for the proposed ring colliders CC-ee [2] and CEPC [3], but similar conclusions can be drawn for the linear colliders ILC [4, 5] or CLIC [6, 7].

2. Input parameters

For detailed reviews on this topic, see *e.g.* Refs. [8, 9]. The following inputs are important for EWPOs and HPOs:

— m_Z, Γ_Z: The mass and width of the Z boson can be determined from measuring the line-shape of the cross section $\sigma[e^+e^- \to f\bar{f}\,]$ for different center-of-mass energies near the Z resonance. An experimental precision substantially below 1 MeV is expected for CEPC and FCC-ee. The main theory uncertainty stems from the QED initial-state radiation [10].

— m_t: The currently most precise determination of the top-quark mass is obtained at the LHC, with an experimental precision of about 0.3 GeV [11]. It has not been fully resolved what is the proper theoretical definition of this measured value, and additional theory uncertainties may need to be included when translating to one of the commonly used mass definitions, such as the $\overline{\rm MS}$ top mass [12, 13].

Future e^+e^- collider can determine $m_t^{\overline{\rm MS}}$ from a threshold scan at $\sqrt{s} \sim 2m_t$, with a statistical precision of much less than 50 MeV [2]. This method requires precise prediction for the $t\bar{t}$ production cross section near threshold. The currently most precise result includes NNNLO

QCD and NNLO EW corrections in non-relativistic perturbation theory [14, 15]. The estimated uncertainty on m_t from missing higher orders is about 50 MeV. Additional uncertainties arise from the input value for α_s (see below) and from the translation to the $\overline{\text{MS}}$ mass definition. With future theory efforts and a measurement precision for α_s of $\delta\alpha_s \lesssim 0.0002$, the combined theory uncertainty for m_t can be reduced to less than 50 MeV [8]. In addition, these calculations for the total $e^+e^- \to t\bar{t}$ cross section need to be matched to a Monte-Carlo event generator to account for experimental acceptances and selection cuts [16, 17].

— m_b, m_c: The bottom and charm quark masses are important inputs for computing Higgs branching fractions. They can be extracted from quarkonia spectra with the help of lattice QCD calculations, with current uncertainties of $\delta m_b \sim 30$ MeV and $\delta m_c \sim 25$ MeV, respectively [18]. It is estimated that the lattice errors can be reduced to the level of $\delta m_b \sim 13$ MeV and $\delta m_c \sim 7$ MeV [19].

— m_H: The Higgs mass can be measured from the final-state kinematics of $e^+e^- \to HZ$ at $\sqrt{s} \sim 240$ GeV with a precision of 10...20 MeV. Theory uncertainties, *e.g.* from final-state radiation, are subdominant.

— α_s: The strong coupling is determined through a variety of different methods. The current most precise approach is based on lattice QCD calculations. Two recent studies yield

$$\text{Lattice:} \qquad \alpha_s = 0.1185 \pm 0.0008 \quad [20]\,, \tag{1}$$
$$\alpha_s = 0.1172 \pm 0.0011 \quad [21]\,. \tag{2}$$

The achievable precision is limited by systematic lattice errors, which are difficult to evaluate comprehensively.

Alternatively, α_s can be determined from event shape variables for $e^+e^- \to$ jets, from deep inelastic scattering, from τ decays, and from EWPOs, in particular the branching ratio $R_\ell \equiv \Gamma[Z \to \text{had.}]/\Gamma[Z \to \ell^+\ell^-]$ ($\ell = e, \mu, \tau$). The last method has the advantage that it has negligible QCD uncertainties (both perturbative and non-perturbative). A future measurement of $\delta R_\ell \sim 0.001$ [2] would translate to $\delta\alpha_s \sim 0.0001$. However, this translation requires the inclusion of electroweak corrections at the 3-loop and leading 4-loop level to match this precision.

Furthermore, one should keep in mind that R_ℓ and other Z decay quantities are intended to be used to probe possible BSM effects. Such BSM contributions, if they exist, would spoil the determination of α_s.

— $\Delta\alpha$: Due to fermion loop corrections, the electromagnetic fine-structure constant $\alpha(Q^2)$ depends on the momentum transfer Q^2, leading to a shift $\Delta\alpha \equiv 1 - \alpha(0)/\alpha(m_Z^2)$. While the contribution from leptons can be computed reliably in perturbation theory [22, 23], the quark contribution is non-perturbative. It can be determined from data for the cross section for $e^+e^- \to$ hadrons using a dispersion integral, with a current uncertainty of $\delta(\Delta\alpha_{\rm had}) \sim 10^{-4}$ [24–26]. Future data from BES III, VEPP and Belle II may reduce this uncertainty to about 5×10^{-5} [24].

With sufficiently high luminosity, a future e^+e^- collider could directly determine $\alpha(m_Z^2)$ from measurements of the forward–backward asymmetry $A_{\rm FB}^{\mu\mu}$ at two center-of-mass energies $\sqrt{s} = m_Z \pm 3$ GeV. Since these energies are slightly off the Z peak, there are sizeable γ–Z interference contributions (compared to the dominant Z–Z term) and thus they are sensitive to $\alpha(m_Z^2)$ via the photon s-channel contribution. At FCC-ee, an experimental precision of $\delta(\Delta\alpha) \sim 3 \times 10^{-5}$ could be achieved this way [27]. To match the experimental precision, 2-loop and leading 3-loop corrections to the process $e^+e^- \to \mu^+\mu^-$ will need to be taken into account [8].

3. EWPOs: *Z* pole and *W* mass

In the following, we will focus on an important group of EWPOs that include various properties of the Z boson, as well as the W-boson mass.

From measurements of the cross section for $e^+e^- \to f\bar{f}$ near $\sqrt{s} \sim m_Z$, one can extract the following (see also Fig. 1):

— The Z-boson mass, m_Z, and total width, $\Gamma_Z = \sum_f \Gamma_f$ (Γ_f is the partial width for $Z \to f\bar{f}$).

— Branching fractions $\mathrm{BR}_f = \Gamma_f/\Gamma_Z$ ($f = e, \mu, \tau, b, c, \text{had.}$).

— The peak cross section $\sigma_f^0 \approx \left.\dfrac{12\pi\Gamma_e\Gamma_f}{(s - m_Z^2) + m_Z^2\Gamma_Z^2}\right|_{s=m_Z^2} = \dfrac{12\pi}{m_Z^2}\mathrm{BR}_e\mathrm{BR}_f$.

All of these quantities can be written in terms of the partial widths $\Gamma_f \propto (g_{\rm L}^f)^2 + (g_{\rm R}^f)^2$, where $g_{\rm L,R}^f$ are the effective couplings of the Z boson to left- and right-handed fermions, respectively.

To determine $g_{\rm L}^f$ and $g_{\rm R}^f$ independently, one needs to combine these quantities with measurements of different asymmetries:

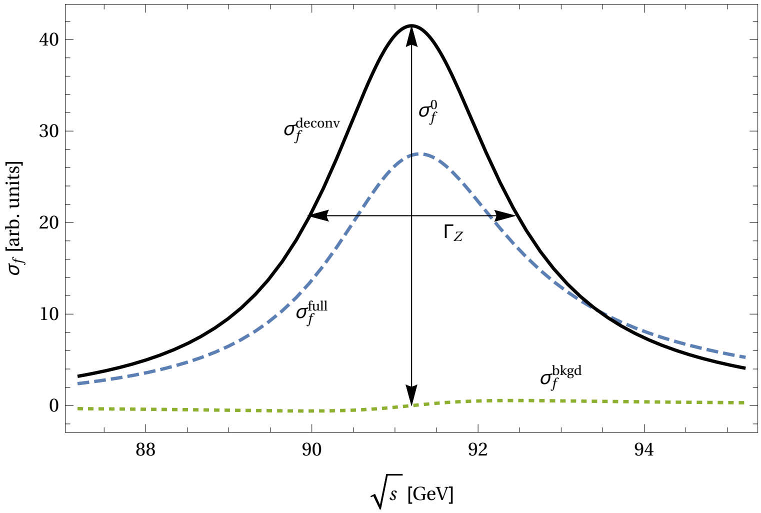

Fig. 1. Sketch of the Z-pole cross section as a function of center-of-mass energy with (dashed blue) and without (solid black) initial-state QED radiation effects. The background contribution to the latter from photon exchange and box diagrams is shown separately (green dotted).

— The forward–backward asymmetry

$$A_{\rm FB}^f = \frac{\sigma_{\rm F} - \sigma_{\rm B}}{\sigma_{\rm F} + \sigma_{\rm B}} = \frac{3}{4}\mathcal{A}_e\mathcal{A}_f\,,$$

where $\sigma_{\rm F} \equiv= \int_0^1 {\rm d}\cos\theta\,\frac{{\rm d}\sigma}{{\rm d}\cos\theta}$, $\sigma_{\rm B} \equiv= \int_{-1}^0 {\rm d}\cos\theta\,\frac{{\rm d}\sigma}{{\rm d}\cos\theta}$.

Here, $\mathcal{A}_f = \frac{2(1-4\sin^2\theta_{\rm eff}^f)}{1+(1+4\sin^2\theta_{\rm eff}^f)^2}$ can be written in terms of the effective weak mixing angle $\sin^2\theta_{\rm eff}^f$, which in turn can be expressed through the effective couplings, *viz.* $\sin^2\theta_{\rm eff}^f = \frac{g_{\rm R}^f}{2|Q_f|\,(g_{\rm R}^f - g_{\rm L}^f)}$.

— With a polarized electron beam, one can obtain a left–right asymmetry, $A_{\rm LR} = \frac{1}{P_e}\frac{\sigma_{\rm L}-\sigma_{\rm R}}{\sigma_{\rm L}+\sigma_{\rm R}} = \frac{1}{P_e}\mathcal{A}_e$, where P_e is the polarization degree.

— For $\tau^+\tau^-$ final states, one can also reconstruct the average tau polarization asymmetry, $\langle\mathcal{P}_\tau\rangle = -A_\tau$.

The above expressions for the cross section, branching ratios and asymmetries are valid only for the leading Z-pole contribution to the process $e^+e^- \to f\bar{f}$.

The full expressions are given by

$$\sigma_f^{\rm full} = \mathcal{R}_{\rm ini}\left(s,s'\right) \otimes \sigma_f^{\rm deconv}\,, \tag{3}$$

$$\sigma_f^{\rm deconv} = \sigma_f^Z + \sigma_f^{\rm bkgd}\,, \tag{4}$$

$$\sigma_f^Z = \frac{R}{\left(s-M_Z^2\right)^2 + M_Z^2 \mathit{\Gamma}_Z'^2} + \sigma^Z_{f,{\rm non-res}}\,, \tag{5}$$

where $R_{\rm ini}(s,s')$ accounts for initial-state radiation through a convolution integral, $\sigma_f^{\rm bkgd}$ includes background contributions from photon exchange, photon–Z interference and box diagrams, and (5) expands the Z-exchange contribution into a leading pole term and a non-resonant remainder.

Note that M_Z and $\mathit{\Gamma}'_Z$ defined as in (5) differ from the commonly reported values for m_Z, $\mathit{\Gamma}_Z$, according to the relations [28]

$$\begin{aligned} M_Z &= m_Z\left(1+\mathit{\Gamma}_Z^2/m_Z^2\right)^{-1/2} \approx m_Z - 34\,{\rm MeV}\,,\\ \mathit{\Gamma}'_Z &= \mathit{\Gamma}_Z\left(1+\mathit{\Gamma}_Z^2/m_Z^2\right)^{-1/2} \approx \mathit{\Gamma}_Z - 0.9\,{\rm MeV}\,. \end{aligned} \tag{6}$$

$R_{\rm ini}(s,s')$ can be computed as an expansion in collinear logarithms, which are known up to sixth logarithmic order [29]. $\sigma_f^{\rm bkgd}$ and $\sigma^Z_{f,{\rm non-res}}$ are currently known at NLO (see *e.g.* Ref. [30]), but these calculations have not been performed in a consistent pole expansion framework, in line with (5). In addition, for FCC-ee/CEPC precision, leading NNLO effects may need to be included.

Another important EWPO is the W-boson mass, m_W, since its direct measurement can be compared to its computed value (within the SM or a BSM theory) from the Fermi constant G_μ. Its currently most precise determination is obtained at hadron colliders (LHC and TeVatron), with a combined precision of $\delta m_W \sim 12$ MeV [11]. Future e^+e^- colliders can make high-precision measurements of m_W (and the width $\mathit{\Gamma}_W$) from a threshold scan at $\sqrt{s} \sim 2m_W$.

To adequately describe the cross section for $e^+e^- \to W^+W^-$ from theory, one must account for the fact that near threshold some radiative corrections are enhanced by factors of β^{-1} and $\ln\beta$, where β is the W-boson velocity, $\beta \sim \sqrt{1-4(m_W^2 - im_W\mathit{\Gamma}_W)/s} \sim \sqrt{\mathit{\Gamma}_W/m_W}$. In addition, non-resonant contributions (with off-shell Ws) are also important.

There are two main approaches to deal with these difficulties: One possibility is to compute the full process $e^+e^- \to f_1\bar{f}_2f_3\bar{f}_4$, including the W decays and all non-resonant diagrams. This process has been computed at full one-loop order [31]. However, for the anticipated precision at FCC-ee/CEPC, two-loop corrections must be included, which would be very challenging for a $2\to 4$ process.

An alternative approach makes use of a non-relativistic effective theory based on the power counting $\alpha \sim \Gamma_W/m_W \sim \beta^2$ [32]. Including NLO corrections and NNLO Coulomb-enhanced corrections in this framework, the theoretical error on the threshold determination of m_W is estimated to be $\delta_{\rm th} m_W \sim 3$ MeV [33]. This could be improved by including full 2-loop calculations of the building blocks $e^+e^- \to W^+W^-$ and $W \to f\bar{f}'$, as well as higher-order initial-state radiation and Coulomb-enhanced effects, resulting in $\delta_{\rm th} m_W \lesssim 0.6$ MeV [34].

To probe the presence of new physics in EWPOs, the values determined from experimental data (as described above) need to be compared to theoretical SM predictions. For a few sample quantities, Table I shows a comparison of the current experimental precision to the projected precision achievable at CEPC and FCC-ee. Also shown are the current and projected theory uncertainties for the SM predictions.

TABLE I

Uncertainties for the determination of a few sample EWPOs from current data [11] and as projected for CEPC [3] and FCC-ee [2] (rows 2–4), as well as for the theoretical prediction of these quantities within the SM [8] (rows 5–8). In addition to the theory error estimate from existing calculational results (row 5), the table shows an estimate for the improvement if about one more order of perturbation theory becomes available (row 6). Also shown are the projected parametric errors due to input quantities needed for the theoretical SM prediction (rows 7, 8). Note that m_W is predicted from the Fermi constant G_μ, which is extracted from the muon decay rate with a very small uncertainty, which has been neglected here.

	m_W [MeV]	Γ_Z [MeV]	R_b [10^{-5}]	$\sin^2\theta^\ell_{\rm eff}$ [10^{-5}]
Current exp.	12	2.3	66	14
CEPC direct	1.0	0.5	4.3	2.3
FCC-ee direct	0.7	0.1	6.0	0.5
Current th. error	4	0.4	10	4.5
Future th. scen.	1	0.15	5	1.5
CEPC param.[1]	2.1	0.15	< 1	2
FCC-ee param.[2]	0.6	0.1	< 1	1

[1] $\delta m_t = 600$ MeV, $\delta\alpha_{\rm s} = 0.0002$, $\delta m_Z = 0.5$ MeV, $\delta(\Delta\alpha) = 5 \times 10^{-5}$.
[2] $\delta m_t = 50$ MeV, $\delta\alpha_{\rm s} = 0.0002$, $\delta m_Z = 0.1$ MeV, $\delta(\Delta\alpha) = 3 \times 10^{-5}$.

The SM predictions are affected by two types of uncertainties: The first source is due to missing higher-order corrections. The current state-of-the-art for the quantities shown in Table I includes full two-loop corrections and y_t-enhanced higher-order corrections, where y_t is the top Yukawa coupling (see Refs. [35–39] and references therein). The estimated error of these results is clearly significantly larger than the target precision of CEPC or FCC-ee. If the theory predictions are improved by computing 3-loop corrections with closed fermion loops and y_t-enhanced 4-loop corrections, the perturbative theory error would become comparable to the CEPC/FCC-ee experimental accuracy [8] ("future th. scen." in Table I).

The second source of error for the SM predictions is due to the input parameters needed for the numerical evaluation of these predictions. The projected precision for these input parameters at CEPC and FCC-ee mainly differ due to the following aspects: (a) The default CEPC plan does not include a run at the $t\bar{t}$ threshold, so that m_t would need to be taken from LHC. In Table I, the LHC uncertainty has been slightly inflated to account for the calibration and conversion of the theoretical top mass definition. (b) Due to its higher luminosity, it is expected that FCC-ee can achieve a much higher precision for m_Z and determine $\Delta\alpha$ directly from $A_{\rm FB}^{\mu\mu}$ (see the previous section).

One can see from the numbers in the table that precise determinations of these inputs parameters are equally important as improvements in the measurement of EWPOs themselves.

4. HPOs: Higgs decay and production

For a more detailed discussion of this topic, see *e.g.* Refs. [8, 19]. The current state-of-the-art of SM prediction for Higgs decay is as follows:

— $h \to bb$: For Higgs decays to quarks, $\mathcal{O}(\alpha_{\rm s}^4)$ QCD corrections [40], NLO electroweak corrections [41, 42], and mixed QCD-electroweak $\mathcal{O}(\alpha\alpha_{\rm s})$ [43] are known, as well as y_t-enhanced two-loop $\mathcal{O}(\alpha^2)$ corrections [44]. The latter are found to be small and other electroweak two-loop corrections may be of similar size. The error from missing higher-order contributions is estimated to be less than 0.4%. By including full two-loop corrections, which is possible with existing calculational methods, it may be further reduced to about 0.2%.

The most important input parameters for the SM prediction are m_b and $\alpha_{\rm s}$. Assuming future uncertainties of $\delta m_b \sim 13$ MeV and $\delta\alpha_{\rm s} \sim 0.0002$, the impact on $\Gamma_{h\to bb}$ is $\sim 0.6\%$.

— $h \to \tau\tau$: Compared to $h \to bb$, higher order corrections are smaller since there are no QCD effects at LO and NLO. With the inclusion of full two-loop corrections, the theory error is expected to be below 0.1%. The parametric error is negligible.

— $h \to WW^*/ZZ^* \to 4f$: While the complete NLO corrections for the $h \to 4f$ process have been computed [45], higher order $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha\alpha_s)$ and $\mathcal{O}(\alpha\alpha_s^2)$ are only known in the large-y_t limit [46–49]. The effect of the latter is small (0.2%). Overall, the theory error from missing complete NNLO corrections is estimated to be about 0.5%. For sufficiently inclusive treatment of the final state, the NNLO QCD effects are expected to be calculable based on existing calculational approaches. This would reduce the theory error to about 0.3%. The most important parametric uncertainty is due to the Higgs mass. An uncertainty of $\delta m_H \sim 10$ MeV would have an impact of $\sim 0.1\%$ on $\Gamma_{h\to 4f}$.

Besides the total decay rate, various distributions of the four-fermion final states are also important for constraining BSM physics. An accurate description of these distributions requires the implementation of the corrections in a Monte-Carlo event generator (see *e.g.* Ref. [45]).

— $h \to gg$: This decay is mediated by a one-loop process at LO. QCD corrections are known up to $\mathcal{O}(\alpha_s^2)$ for the full process [50] and $\mathcal{O}(\alpha_s^3)$ for the leading large-y_t contribution [51, 52]. The two-loop NLO electroweak corrections have also been computed [53, 54]. The theory uncertainty is dominated by missing higher-order QCD corrections. Computing the $\mathcal{O}(\alpha_s^4)$ contribution in the large-y_t limit, which may be within reach with existing calculational methods, would reduce the theory error to about 1%. The SM prediction also strongly depends on the uncertainty of α_s. An uncertainty of $\delta\alpha_s \sim 0.0002$ translates into $\delta\Gamma_{h\to gg} \sim 0.5\%$.

— $h \to \gamma\gamma$: This process is also loop-induced at LO, but does not have any final-state QCD contributions. Therefore, both the perturbative uncertainty and the parametric error are very small ($< 1\%$).

Table II summarizes the above information and compares it to the expected experimental precision at FCC-ee and CEPC for the determination of the effective squared couplings, g^2_{HXX}.

The main production processes for Higgs bosons at e^+e^- colliders with $\sqrt{s} \sim 240$ GeV are $e^+e^- \to hZ$ and $e^+e^- \to h\nu\bar{\nu}$ (WW fusion). The latter contributes at the few-percent level.

TABLE II

Projected precision for the determination of several Higgs squared couplings, g^2_{HXX}, from Higgs decay measurements at CEPC and FCC-ee, compared to estimates of the uncertainties of the SM predictions from current and projected future theoretical calculations. Also shown are the projected parametric errors due to input quantities needed for the theoretical SM prediction.

	$b\bar{b}$	$\tau^+\tau^-$	WW^*/ZZ^*	gg	$\gamma\gamma$
CEPC direct	2.0%	2.4%	2.2%	2.4%	3.2%
FCC-ee direct	0.8%	1.1%	0.4%	1.6%	3.0%
Current th. error	$< 0.4\%$	$< 0.3\%$	0.5%	3%	$< 1\%$
Future th. scen.	0.2%	$< 0.1\%$	0.3%	1%	$< 1\%$
Future param. error	0.6%	—	0.1%	0.5%	—

The one-loop corrections to the dominant production process, $e^+e^- \to hZ$, have been computed a long time ago for an on-shell final-state Z boson [55, 56]. Since the Z width is relatively large, it is desirable to have $\mathcal{O}(\alpha)$ corrections also for the full process $e^+e^- \to hf\bar{f}$, including Z production and decay and off-shell Z contributions. This calculation has been carried out for $f = e$ [57] and $f = \nu$ [58, 59], and could also be straightforwardly done for other final states.

More recently, the $\mathcal{O}(\alpha\alpha_{\rm s})$ two-loop corrections have been computed [60, 61]. The theoretical uncertainty, mainly from missing two-loop electroweak corrections, is estimated to amount of about 1%. If the full two-loop corrections to $e^+e^- \to hZ$ become available, this error should be reduced to less than 0.3%. Parametric uncertainties are very small since the LO cross section involves particles (e, Z, h) whose masses can be precisely measured and that have only electroweak interactions.

Similarly, the NLO corrections to the WW fusion process are known [57]. The calculation of $\mathcal{O}(\alpha\alpha_{\rm s})$ corrections should be possible with the techniques used in Refs. [60, 61]. Other higher orders may contribute at the level of $\mathcal{O}(1\%)$, which is an acceptable theory error for measurements at $\sqrt{s} \sim 240$ GeV. However, for $\sqrt{s} \geq 500$ GeV, where the WW-fusion cross section is significantly larger, additional higher-order corrections will need to be included, in particular terms that are enhanced by electroweak Sudakov logarithms.

5. Calculational techniques

Beyond the one-loop level, there is currently no standard technique that works for any physical process. The complexity of the problem increases with the number of loops, the number of external legs, and the number of independent mass and momentum scales that appear in the loops. Different techniques have been developed and applied to a specific class of problems, but they may not work for other applications.

For corrections that involve few different mass scales (such as QCD corrections), analytical techniques are very effective. They proceed in two steps: First, the full set of multi-loop integrals that appear for a certain physical process are reduced to a small set of master integrals with the help of integration-by-parts identities and other identities [62–64]. There are several public programs that can perform this task [65–68], but for very complicated problems, they need large amounts of computing time and memory. The master integrals can then be evaluated using several methods, with the differential equation method [69–71] being the most commonly used one. The final analytical results can be expressed in terms of generalized harmonic polylogarithms [72, 73], and elliptic polylogarithms [74, 75], but for generic two-loop and higher-order integrals, additional special functions may be needed.

For applications with more independent mass scales, one can perform an expansion of the loop integrals before integration [76, 77]. The series expansion contains simpler integrals (which fewer scales) in its coefficients. This works most effectively if one can take advantage of a large mass ratio, *e.g.* $m_Z^2/m_t^2 \approx 1/4$. It has the advantage that the expansion coefficients can be computed analytically, thus permitting fast numerical evaluation of the final results. However, the computation of the expansion itself may be very complex and resource intensive. In addition, the series may not always converge very well, in particular if multi-variable expansions are required. There are a few public programs that perform certain classes of expansions [78, 79].

On the other hand, numerical integration methods do not have any conceptual limitation in the number of mass scales or external momenta. However, any UV and IR divergencies need to be separated before the numerical integration. There are two generic methods that have been implemented in public programs, sector decomposition [80–82] and Mellin–Barnes representations [83–85]. However, the multi-dimensional numerical integrals often converge slowly and require large amounts of computing time. In addition, cases with internal thresholds and pinched contours can lead to numerical instabilities.

For a particular set of loop topologies, it may be possible to find low-dimensional integral representations that can be evaluated robustly and with modest amounts of computing time. These techniques are typically restricted to a limited class of applications, and they are difficult to fully automatize, but they have advantages in speed and numerical precision and accuracy. A recent example is a technique proposed in Ref. [86] for the evaluation of two-loop double boxes that appear in the NNLO correction to

$e^+e^- \to hZ$. This method uses Feynman parameters and dispersion relations to write one of the two loops in a form such that it effectively factorizes from the other loop. The latter can then be evaluated with the standard one-loop Passarino–Veltman reduction [87, 88]. In the end, one obtains 3-dimensional numerical integrals that can be evaluated to permille precision within minutes on a single CPU core.

6. Summary

Precision studies of electroweak and Higgs physics at future e^+e^- colliders (so-called "Higgs factories") crucially depend on theoretical inputs for higher-order radiative corrections. For Higgs and W-pair production, electroweak NNLO corrections for $2 \to 2$ scattering processes will be needed. Corrections at this order are also desirable for Higgs decays, but should be relatively straightforward to compute. Furthermore, N^{4}LO QCD corrections for Higgs-to-gluon decays are important. These fixed-order calculations also need to be matched to Monte-Carlo tools to simulate experimental acceptances and selection cuts [1]. It is still an open question whether it is also necessary to include NLO corrections for various backgrounds, such as $e^+e^- \to \ell^+\ell^- b\bar{b}$, $e^+e^- \to \nu\bar{\nu}b\bar{b}$, *etc.* The technology for computing such processes at NLO exists, but additional work will be needed to make these results available in suitable tools.

For Z-pole electroweak studies, 3-loop electroweak and mixed electroweak-QCD corrections to the $Zf\bar{f}$ will be needed, as well as leading 4-loop effects that are enhanced by the top Yukawa coupling. Matching these higher-order loop corrections to a QED Monte-Carlo program is an intricate problem, in particular for QED initial–final interference (IFI) contributions [1].

To fully exploit the scope of these precision measurements, the uncertainties of other SM input parameters (α_s, $m_{t,b,c}$, $\alpha(m_Z^2)$, …) also must be reduced. On the theory side, this requires improvements of both perturbative and non-perturbative uncertainties. On the experimental side, the required high-statistics data may be obtained through complementary efforts at the Higgs factories themselves and at other facilities (Belle II, BES III, …).

REFERENCES

[1] S. Jadach, «On the Role of the Precision Monte Carlo Generators in Future Electron Colliders», *Acta Phys. Pol. B* **52**, 947 (2021), this issue.

[2] FCC Collaboration, «FCC-ee: The Lepton Collider: Future Circular Collider Conceptual Design Report Volume 2», *Eur. Phys. J. Spec Top.* **228**, 261 (2019).

[3] CEPC Study Group, «CEPC Conceptual Design Report: Volume 2 — Physics & Detector», arXiv:1811.10545 [hep-ex].

[4] H. Baer *et al.*, «The International Linear Collider Technical Design Report — Volume 2: Physics», arXiv:1306.6352 [hep-ph].

[5] P. Bambade *et al.*, «The International Linear Collider: A Global Project», arXiv:1903.01629 [hep-ex].

[6] L. Linssen *et al.*, «Physics and Detectors at CLIC: CLIC Conceptual Design Report», *CERN Yellow Rep. Monogr.* **3**, 257 (2012), arXiv:1202.5940 [physics.ins-det].

[7] CLICdp, CLIC collaborations, «The Compact Linear Collider (CLIC) — 2018 Summary Report», *CERN Yellow Rep. Monogr.* **2**, 1 (2018), arXiv:1812.06018 [physics.acc-ph].

[8] A. Freitas *et al.*, «Theoretical uncertainties for electroweak and Higgs-boson precision measurements at FCC-ee», arXiv:1906.05379 [hep-ph].

[9] A. Freitas, «TASI 2020 Lectures on Precision Tests of the Standard Model», arXiv:2012.11642 [hep-ph].

[10] S. Jadach, B. Pietrzyk, M. Skrzypek, «On the precision of calculations of initial state radiation in the LEP Z line shape fits», *Phys. Lett. B* **456**, 77 (1999).

[11] Particle Data Group, «Review of Particle Physics», *Prog. Theor. Exp. Phys.* **2020**, 083C01 (2020).

[12] M. Butenschoen *et al.*, «Top Quark Mass Calibration for Monte Carlo Event Generators», *Phys. Rev. Lett.* **117**, 232001 (2016), arXiv:1608.01318 [hep-ph].

[13] S. Ferrario Ravasio, P. Nason, C. Oleari, «All-orders behaviour and renormalons in top-mass observables», *J. High Energy Phys.* **1901**, 203 (2019), arXiv:1810.10931 [hep-ph].

[14] M. Beneke *et al.*, «Next-to-Next-to-Next-to-Leading Order QCD Prediction for the Top Antitop S-Wave Pair Production Cross Section Near Threshold in e^+e^- Annihilation», *Phys. Rev. Lett.* **115**, 192001 (2015), arXiv:1506.06864 [hep-ph].

[15] M. Beneke, A. Maier, T. Rauh, P. Ruiz-Femenía, «Non-resonant and electroweak NNLO correction to the e^+e^- top anti-top threshold», *J. High Energy Phys.* **1802**, 125 (2018), arXiv:1711.10429 [hep-ph].

[16] B. Chokoufé Nejad *et al.*, «NLO QCD predictions for off-shell $t\bar{t}$ and $t\bar{t}H$ production and decay at a linear collider», *J. High Energy Phys.* **1612**, 075 (2016), arXiv:1609.03390 [hep-ph].

[17] J. Reuter *et al.*, «Top Physics in WHIZARD», presented at the International Workshop on Future Linear Colliders (LCWS15), Whistler, Canada, November 2–6, 2015, arXiv:1602.08035 [hep-ph].

[18] LHC Higgs Cross Section Working Group, «Handbook of LHC Higgs Cross Sections: 4. Deciphering the Nature of the Higgs Sector», *CERN Yellow Rep. Monogr.* **2**, 1 (2017), arXiv:1610.07922 [hep-ph].

[19] G.P. Lepage, P.B. Mackenzie, M.E. Peskin, «Expected Precision of Higgs Boson Partial Widths within the Standard Model», arXiv:1404.0319 [hep-ph].

[20] ALPHA Collaboration, «QCD Coupling from a Nonperturbative Determination of the Three-Flavor Λ Parameter», *Phys. Rev. Lett.* **119**, 102001 (2017), arXiv:1706.03821 [hep-lat].

[21] S. Zafeiropoulos *et al.*, «Strong Running Coupling from the Gauge Sector of Domain Wall Lattice QCD with Physical Quark Masses», *Phys. Rev. Lett.* **122**, 162002 (2019), arXiv:1902.08148 [hep-ph].

[22] M. Steinhauser, «Leptonic contribution to the effective electromagnetic coupling constant up to three loops», *Phys. Lett. B* **429**, 158 (1998), arXiv:hep-ph/9803313.

[23] C. Sturm, «Leptonic contributions to the effective electromagnetic coupling at four-loop order in QED», *Nucl. Phys. B* **874**, 698 (2013), arXiv:1305.0581 [hep-ph].

[24] F. Jegerlehner, «$\alpha_{\mathrm{QED,eff}}(s)$ for precision physics at the FCC-ee/ILC», *CERN Yellow Rep. Monogr.* **3**, 9 (2020).

[25] M. Davier, A. Hoecker, B. Malaescu, Z. Zhang, «A new evaluation of the hadronic vacuum polarisation contributions to the muon anomalous magnetic moment and to $\alpha(m_Z^2)$», *Eur. Phys. J. C* **80**, 241 (2020), arXiv:1908.00921 [hep-ph].

[26] A. Keshavarzi, D. Nomura, T. Teubner, «$g-2$ of charged leptons, $\alpha(M_Z^2)$, and the hyperfine splitting of muonium», *Phys. Rev. D* **101**, 014029 (2020), arXiv:1911.00367 [hep-ph].

[27] P. Janot, «Direct measurement of $\alpha_{\mathrm{QED}}(m_Z^2)$ at the FCC-ee», *J. High Energy Phys.* **1602**, 053 (2016); *Erratum ibid.* **11**, 164 (2017), arXiv:1512.05544 [hep-ph].

[28] D. Bardin, A. Leike, T. Riemann, M. Sachwitz, «Energy dependent width effects in e^+e^--annihilation near the Z-boson pole», *Phys. Lett. B* **206**, 539 (1988).

[29] J. Ablinger, J. Blümlein, A. De Freitas, K. Schönwald, «Subleading Logarithmic QED Initial State Corrections to $e^+e^- \to \gamma^*/Z^{0^*}$ to $O(\alpha^6 L^5)$», *Nucl. Phys. B* **955**, 115045 (2020), arXiv:2004.04287 [hep-ph].

[30] D.Y. Bardin *et al.*, «ZFITTER v.6.21: A Semianalytical program for fermion pair production in e^+e^- annihilation», *Comput. Phys. Commun.* **133**, 229 (2001), arXiv:hep-ph/9908433.

[31] A. Denner, S. Dittmaier, M. Roth, L.H. Wieders, «Electroweak corrections to charged-current $e^+e^- \to 4$ fermion processes: Technical details and further results», *Nucl. Phys. B* **724**, 247 (2005); *Erratum ibid.* **854**, 504 (2012), arXiv:hep-ph/0505042.

[32] M. Beneke *et al.*, «Four-fermion production near the W pair production threshold», *Nucl. Phys. B* **792**, 89 (2008), arXiv:0707.0773 [hep-ph].

[33] S. Actis, M. Beneke, P. Falgari, C. Schwinn, «Dominant NNLO corrections to four-fermion production near the W-pair production threshold», *Nucl. Phys. B* **807**, 1 (2009), arXiv:0807.0102 [hep-ph].

[34] C. Schwinn, «Prospects for higher-order corrections to W-pair production near threshold in the EFT approach», *CERN Yellow Rep. Monogr.* **3**, 77 (2020).

[35] M. Awramik, M. Czakon, A. Freitas, G. Weiglein, «Precise prediction for the W boson mass in the standard model», *Phys. Rev. D* **69**, 053006 (2004), arXiv:hep-ph/0311148.

[36] M. Awramik, M. Czakon, A. Freitas, G. Weiglein, «Complete Two-Loop Electroweak Fermionic Corrections to the Effective Leptonic Weak Mixing Angle $\sin^2\theta_{\rm eff}^{\rm lept}$ and Indirect Determination of the Higgs Boson Mass», *Phys. Rev. Lett.* **93**, 201805 (2004), arXiv:hep-ph/0407317.

[37] A. Freitas, «Numerical multi-loop integrals and applications», *Prog. Part. Nucl. Phys.* **90**, 201 (2016), arXiv:1604.00406 [hep-ph].

[38] I. Dubovyk *et al.*, «Complete electroweak two-loop corrections to Z boson production and decay», *Phys. Lett. B* **783**, 86 (2018), arXiv:1804.10236 [hep-ph].

[39] I. Dubovyk *et al.*, «Electroweak pseudo-observables and Z-boson form factors at two-loop accuracy», *J. High Energy Phys.* **1908**, 113 (2019), arXiv:1906.08815 [hep-ph].

[40] P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, «Scalar Correlator at $\mathcal{O}(\alpha_{\rm s}^4)$, Higgs Decay Into Bottom Quarks and Bounds on the Light Quark Masses», *Phys. Rev. Lett.* **96**, 012003 (2006), arXiv:hep-ph/0511063.

[41] B.A. Kniehl, «Radiative corrections for $H \to f\bar{f}(\gamma)$ in the standard model», *Nucl. Phys. B* **376**, 3 (1992).

[42] A. Dabelstein, W. Hollik, «Electroweak corrections to the fermionic decay width of the standard Higgs boson», *Z. Phys. C* **53**, 507 (1992).

[43] L. Mihaila, B. Schmidt, M. Steinhauser, «$\Gamma(H \to b\bar{b})$ to order $\alpha\alpha_{\rm s}$», *Phys. Lett. B* **751**, 442 (2015), arXiv:1509.02294 [hep-ph].

[44] M. Butenschoen, F. Fugel, B.A. Kniehl, «$\mathcal{O}(G_F^2 m_t^4)$ two-loop electroweak correction to Higgs-boson decay to bottom quarks», *Nucl. Phys. B* **772**, 25 (2007), arXiv:hep-ph/0702215.

[45] A. Bredenstein, A. Denner, S. Dittmaier, M.M. Weber, «Precise predictions for the Higgs-boson decay $H \to WW/ZZ \to 4$ leptons», *Phys. Rev. D* **74**, 013004 (2006), arXiv:hep-ph/0604011.

[46] B.A. Kniehl, M. Spira, «Two loop $\mathcal{O}(G_F^2 m_t^4)$ corrections to Higgs production at LEP», *Nucl. Phys. B* **443**, 37 (1995), arXiv:hep-ph/9501392.

[47] B.A. Kniehl, M. Spira, «Low-energy theorems in Higgs physics», *Z. Phys. C* **69**, 77 (1995), arXiv:hep-ph/9505225.

[48] B.A. Kniehl, M. Steinhauser, «Virtual top effects on low mass Higgs interactions at next-to-leading order in QCD», *Phys. Lett. B* **365**, 297 (1996), arXiv:hep-ph/9507382.

[49] B.A. Kniehl, O.L. Veretin, «Low-mass Higgs decays to four leptons at one loop and beyond», *Phys. Rev. D* **86**, 053007 (2012), arXiv:1206.7110 [hep-ph].

[50] M. Schreck, M. Steinhauser, «Higgs decay to gluons at NNLO», *Phys. Lett. B* **655**, 148 (2007), arXiv:0708.0916 [hep-ph].

[51] P.A. Baikov, K.G. Chetyrkin, «Top Quark Mediated Higgs Boson Decay into Hadrons to Order α_s^5», *Phys. Rev. Lett.* **97**, 061803 (2006), arXiv:hep-ph/0604194.

[52] S. Moch, A. Vogt, «On third-order timelike splitting functions and top-mediated Higgs decay into hadrons», *Phys. Lett. B* **659**, 290 (2008), arXiv:0709.3899 [hep-ph].

[53] U. Aglietti, R. Bonciani, G. Degrassi, A. Vicini, «Two-loop light fermion contribution to Higgs production and decays», *Phys. Lett. B* **595**, 432 (2004), arXiv:hep-ph/0404071.

[54] G. Degrassi, F. Maltoni, «Two-loop electroweak corrections to Higgs production at hadron colliders», *Phys. Lett. B* **600**, 255 (2004), arXiv:hep-ph/0407249.

[55] B.A. Kniehl, «Radiative corrections for associated ZH production at future e^+e^- colliders», *Z. Phys. C* **55**, 605 (1992).

[56] A. Denner, J. Kublbeck, R. Mertig, M. Bohm, «Electroweak radiative corrections to $e^+e^- \to HZ$», *Z. Phys. C* **56**, 261 (1992).

[57] F. Boudjema *et al.*, «Electroweak corrections to Higgs production through ZZ fusion at the linear collider», *Phys. Lett. B* **600**, 65 (2004), arXiv:hep-ph/0407065.

[58] G. Belanger *et al.*, «Full one-loop electroweak radiative corrections to single Higgs production in e^+e^-», *Phys. Lett. B* **559**, 252 (2003), arXiv:hep-ph/0212261.

[59] A. Denner, S. Dittmaier, M. Roth, M.M. Weber, «Electroweak radiative corrections to single Higgs boson production in e^+e^- annihilation», *Phys. Lett. B* **560**, 196 (2003), arXiv:hep-ph/0301189.

[60] Y. Gong *et al.*, «Mixed QCD-EW corrections for Higgs boson production at e^+e^- colliders», *Phys. Rev. D* **95**, 093003 (2017), arXiv:1609.03955 [hep-ph].

[61] Q.-F. Sun, F. Feng, Y. Jia, W.-L. Sang, «Mixed electroweak-QCD corrections to $e^+e^- \to HZ$ at Higgs factories», *Phys. Rev. D* **96**, 051301 (2017), arXiv:1609.03995 [hep-ph].

[62] K. Chetyrkin, A. Kataev, F. Tkachov, «Higher order corrections to $\sigma_{\rm tot}(e^+e^- \to \text{hadrons})$ in quantum chromodynamics», *Phys. Lett. B* **85**, 277 (1979).

[63] T. Gehrmann, E. Remiddi, «Differential equations for two-loop four-point functions», *Nucl. Phys. B* **580**, 485 (2000), arXiv:hep-ph/9912329.

[64] S. Laporta, «High precision calculation of multiloop Feynman integrals by difference equations», *Int. J. Mod. Phys. A* **15**, 5087 (2000), arXiv:hep-ph/0102033.

[65] A. von Manteuffel, C. Studerus, «Reduze 2 — Distributed Feynman Integral Reduction», arXiv:1201.4330 [hep-ph].

[66] R.N. Lee, «LiteRed 1.4: a powerful tool for reduction of multiloop integrals», *J. Phys.: Conf. Ser.* **52**, 012059 (2014), arXiv:1310.1145 [hep-ph].

[67] A.V. Smirnov, F.S. Chuharev, «FIRE6: Feynman Integral REduction with Modular Arithmetic», *Comput. Phys. Commun.* **247**, 106877 (2020), arXiv:1901.07808 [hep-ph].

[68] J. Klappert, F. Lange, P. Maierhöfer, J. Usovitsch, «Integral Reduction with Kira 2.0 and Finite Field Methods», arXiv:2008.06494 [hep-ph].

[69] A.V. Kotikov, «Differential equations method: New technique for massive Feynman diagrams calculation», *Phys. Lett. B* **254**, 158 (1991).

[70] E. Remiddi, «Differential Equations for Feynman Graph Amplitudes», *Nuovo Cim. A* **110**, 1435 (1997), arXiv:hep-th/9711188.

[71] J.M. Henn, «Multiloop Integrals in Dimensional Regularization Made Simple», *Phys. Rev. Lett.* **110**, 251601 (2013), arXiv:1304.1806 [hep-ph].

[72] A.B. Goncharov, «Multiple polylogarithms, cyclotomy and modular complexes», *Math. Res. Lett.* **5**, 497 (1998), arXiv:1105.2076 [math.AG].

[73] J. Vollinga, S. Weinzierl, «Numerical evaluation of multiple polylogarithms», *Comput. Phys. Commun.* **167**, 177 (2005), arXiv:hep-ph/0410259.

[74] J. Ablinger *et al.*, «Iterated elliptic and hypergeometric integrals for Feynman diagrams», *J. Math. Phys.* **59**, 062305 (2018), arXiv:1706.01299 [hep-th].

[75] J. Broedel, C. Duhr, F. Dulat, L. Tancredi, «Elliptic polylogarithms and iterated integrals on elliptic curves. Part I: general formalism», *J. High Energy Phys.* **1805**, 093 (2018), arXiv:1712.07089 [hep-th].

[76] M. Beneke, V.A. Smirnov, «Asymptotic expansion of Feynman integrals near threshold», *Nucl. Phys. B* **522**, 321 (1998), arXiv:hep-ph/9711391.

[77] B. Jantzen, «Foundation and generalization of the expansion by regions», *J. High Energy Phys.* **1112**, 076 (2011), arXiv:1111.2589 [hep-ph].

[78] R. Harlander, T. Seidensticker, M. Steinhauser, «Corrections of $O(\alpha\alpha_{\mathrm{s}})$ to the decay of the Z boson into bottom quarks», *Phys. Lett. B* **426**, 125 (1998), arXiv:hep-ph/9712228.

[79] A. Pak, A. Smirnov, «Geometric approach to asymptotic expansion of Feynman integrals», *Eur. Phys. J. C* **71**, 1626 (2011), arXiv:1011.4863 [hep-ph].

[80] S. Borowka *et al.*, «SecDec-3.0: numerical evaluation of multi-scale integrals beyond one loop», *Comput. Phys. Commun.* **196**, 470 (2015), arXiv:1502.06595 [hep-ph].

[81] S. Borowka *et al.*, «pySecDec: a toolbox for the numerical evaluation of multi-scale integrals», *Comput. Phys. Commun.* **222**, 313 (2018), arXiv:1703.09692 [hep-ph].

[82] A.V. Smirnov, «FIESTA4: Optimized Feynman integral calculations with GPU support», *Comput. Phys. Commun.* **204**, 189 (2016), arXiv:1511.03614 [hep-ph].

[83] M. Czakon, «Automatized analytic continuation of Mellin–Barnes integrals», *Comput. Phys. Commun.* **175**, 559 (2006), arXiv:hep-ph/0511200.

[84] A.V. Smirnov, V.A. Smirnov, «On the resolution of singularities of multiple Mellin–Barnes integrals», *Eur. Phys. J. C* **62**, 445 (2009), arXiv:0901.0386 [hep-ph].

[85] I. Dubovyk, J. Gluza,T. Riemann, «Non-planar Feynman diagrams and Mellin–Barnes representations with AMBRE 3.0», *J. Phys.: Conf. Ser.* **608**, 012070 (2015).

[86] Q. Song, A. Freitas, «On the evaluation of two-loop electroweak box diagrams for $e^+e^- \to HZ$ production», *J. High Energy Phys.* **2104**, 179 (2021), arXiv:2101.00308 [hep-ph].

[87] G.J. van Oldenborgh, J.A.M. Vermaseren, «New Algorithms for one-loop integrals», *Z. Phys. C* **46**, 425 (1990).

[88] G.J. van Oldenborgh, «FF: a package to evaluate one-loop Feynman diagrams», *Comput. Phys. Commun.* **66**, 1 (1991).