A CONJECTURAL BOUND ON THE SECOND BETTI NUMBER FOR HYPER-KÄHLER MANIFOLDS

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Abstract. In previous work ([GKLR19]), we noted that the known cases of hyper-Kähler manifolds satisfy a natural condition on the LLV decomposition of the cohomology; informally, the Verbitsky component is the dominant representation in the LLV decomposition. Assuming this condition holds for all hyper-Kähler manifolds, we obtain an upper bound for the second Betti number in terms of the dimension.

1. Introduction

A fundamental open question in the theory of compact hyper-Kähler manifold is the boundedness question: *are there finitely many diffeomorphism types of hyper-Kählers in a given dimension?* In accordance with the Torelli principle, Huybrechts [Huy03, Thm 4.3] proved that there are finitely many diffeomorphism types of hyper-Kähler manifolds once the dimension and the (unnormalized) Beauville–Bogomolov lattice $(H^2(X,\mathbb{Z}), q_X)$ are fixed. Thus, bounding the hyper-Kähler manifolds is equivalent to bounding the second Betti number $b_2 = b_2(X)$, and then the Beauville–Bogomolov form (e.g., the discriminant). In dimension 2, a compact hyper-Kähler manifold is always a K3 surface, thus $b_2 = 22$. In dimension 4, Beauville and Guan [Gua01] gave a sharp bound $b_2 \leq 23$ (in fact, Guan showed $3 \leq b_2 \leq 8$ or $b_2 = 23$). For some further partial results on bounding b_2 see Remark 1.5. The purpose of this note is to give a conjectural bound on $b_2(X)$ for an arbitrary compact hyper-Kähler manifold X of dimension $2n$. Our bound depends on a natural conjectural condition satisfied by the Looijenga–Lunts–Verbitsky (LLV) decomposition of the cohomology $H^*(X)$ for hyper-Kähler manifolds X.

To state our results, let us recall that Verbitsky [Ver95] and Looijenga–Lunts [LL97] noted that the cohomology $H^*(X)$ of a hyper-Kähler manifold admits a natural action by the Lie algebra $\mathfrak{g} = \mathfrak{so}(b_2+2)$, generalizing the usual hard Lefschetz theorem. As a $\mathfrak{g}$-module, the cohomology of a hyper-Kähler manifold X decomposes as

$$H^*(X) = \bigoplus\nolimits_{\mu} V_{\mu}^{\oplus m_{\mu}}, \tag{1.1}$$

where V_μ indicates an irreducible $\mathfrak{g}$-module of highest weight $\mu = (\mu_0, \cdots, \mu_r)$, with $r = \left\lfloor \frac{b_2(X)}{2} \right\rfloor = \operatorname{rk} \mathfrak{g} - 1$. We refer to $\mathfrak{g}$ as the *LLV algebra* of X, and to (1.1) as the *LLV decomposition* of $H^*(X)$ (see [GKLR19] for further discussion). Motivated by the behavior of the LLV decomposition in the known cases of hyper-Kähler manifolds [GKLR19], we have made the following conjecture.

Conjecture ([GKLR19])**.** *Let X be a compact hyper-Kähler manifold of dimension $2n$. Then the weights $\mu = (\mu_0, \cdots, \mu_r)$ occurring in the LLV decomposition* (1.1) *of $H^*(X)$ satisfy*

$$\mu_0 + \cdots + \mu_{r-1} + |\mu_r| \leq n. \tag{1.2}$$

The conjecture holds for all currently known examples of compact hyper-Kähler manifolds (cf. [GKLR19, §1]). Furthermore, the equality in (1.2) holds for *the Verbitsky component*, an irreducible $\mathfrak{g}$-submodule with highest weight $\mu = (n, 0, \cdots, 0)$ that is always present in $H^*(X)$. This shows (1.2) is sharp. Beyond the evidence given by the validity of (1.2) in the known cases, we have some partial arguments of motivic nature (and depending on standard conjectures) showing that at least (1.2) is plausible. This will be discussed elsewhere.

The purpose of this note is to show that conjecture (1.2) implies a general bound on $b_2(X)$.

The two authors were partially supported by NSF grant DMS-1802128.

Main Theorem. *Let X be a compact hyper-Kähler manifold of dimension $2n$. If the condition* (1.2) *holds for X, then*

$$b_2(X) \leq \begin{cases} \frac{21+\sqrt{96n+433}}{2} & \text{if } H^*_{\mathrm{odd}}(X) = 0 \\ 2k+1 & \text{if } H^k(X) \neq 0 \text{ for some odd } k \end{cases}. \tag{1.3}$$

Remark 1.4. A slightly weaker version of (1.3) is

$$b_2(X) \leq \max\left\{\tfrac{21+\sqrt{96n+433}}{2}, 4n-1\right\},$$

which reads explicitly

n	1	2	3	4	5	6	7	≥ 8
$b_2(X) \leq$	22	23	23	24	25	26	27	$4n-1$

.

In low dimensions, our bounds agree with the known results and seem fairly sharp. For instance, we know $\mathrm{K3}^{[n]}$ type hyper-Kähler manifolds have $b_2 = 23$ and OG10 manifolds have $b_2 = 24$. These examples almost reach our maximum bound of b_2 for low dimensions. Similarly, Kum_n type hyper-Kähler manifolds have $H^3(X) \neq 0$ and $b_2 = 7$, showing also that the second inequality in (1.3) is sharp.

Remark 1.5. Sawon [Saw15] and Kurnosov [Kur15] have previously obtained the same bounds for $3 \leq n \leq 5$, and also predicted the general formula (1.3) when $H^*_{\mathrm{odd}}(X) = 0$. However, their results were based on the assumption that an irreducible module V_μ is determined by the shape of its Hodge diamond. In general, the shape of the Hodge diamond of V_μ is controlled only by the first two coefficients μ_0, μ_1 (see [GKLR19, §2.2]).

A few words about the proof of our conjectural bound. First, in [GKLR19, §1], we have already obtained that the condition (1.2) has consequences on the odd cohomology (specifically, if $b_2 \geq 4n$ then there should be no odd cohomology). A slight generalization of the argument in loc. cit. then gives the second inequality in (1.3). The main content of this note is the control of the even cohomology under the assumption (1.2). Essentially, our argument is a representation theoretic refinement of Beauville's argument that $b_2 \leq 23$ for hyper-Kähler fourfolds. Namely, the starting point is Salamon's relation [Sal96], a linear relation satisfied by the Betti numbers of hyper-Kähler manifolds. Inspired by the shape of it, we define a numerical function $s(W)$ for a $\mathfrak{g}$-module W and verify its basic properties, most importantly $s(W_1 \oplus W_2) \leq \max\{s(W_1), s(W_2)\}$. In this setting, Salamon's relation reads $s(H^*(X)) = \frac{n}{3}$. Now the punchline is an explicit formula for $s(V_\mu)$ for irreducible $\mathfrak{g}$-modules V_μ (Theorem 3.5), which is obtained by applying the Weyl character formula. Combining it with (1.2), we conclude

$$\frac{n}{3} = s(H^*(X)) \leq s(V_{(n,0,\cdots,0)}) = \frac{8n(b_2+n)}{(b_2+1)(b_2+2)},$$

which in turn gives the first inequality in (1.3).

2. Cohomology of compact hyper-Kähler manifolds

We briefly review some relevant results on the cohomology of hyper-Kähler manifolds. Let X be a compact hyper-Kähler manifold of dimension $2n$ and $H^*(X) = H^*(X, \mathbb{C})$. Let $\mathfrak{g} \subset \mathfrak{gl}(H^*(X))$ be the Lie algebra generated by all the Lefschetz and dual Lefschetz operators associated to elements in $H^2(X)$ (cf. [Ver95], [LL97]). We call this the Looijenga–Lunts–Verbitsky (LLV) algebra of X. Let

$$(V, q) = (H^2(X), q_X) \oplus U$$

be the Mukai completion of $H^2(X)$ equipped with the Beauville–Bogomolov form, and set $r = \left\lfloor \frac{b_2(X)}{2} \right\rfloor$. Then $\mathfrak{g}$ is isomorphic to the special orthogonal Lie algebra $\mathfrak{so}(V, q) \cong \mathfrak{so}(b_2 + 2, \mathbb{C})$ of rank $r+1$. The cohomology $H^*(X)$ of a hyper-Kähler manifold X admits a $\mathfrak{g}$-module structure, generalizing the hard Lefschetz theorem. We refer to the $\mathfrak{g}$-module irreducible decomposition (1.1) of $H^*(X)$ as the *LLV decomposition* of the cohomology (see [GKLR19, §3] for some examples).

Fix a Cartan and a Borel subalgebra of $\mathfrak{g}$. Representation theory of $\mathfrak{so}(V, q)$ depends on the parity of $\dim V = b_2 + 2$. If $b_2 = 2r$ is even, then we can fix a suitable basis $\varepsilon_0, \cdots, \varepsilon_r$ of the dual Cartan subalgebra

such that the $2r+2$ associated weights of the standard module V are $\pm\varepsilon_0, \cdots, \pm\varepsilon_r$. Similarly, if $b_2 = 2r+1$ is odd, then we can choose ε_i such that V has the $2r+3$ associated weights $0, \pm\varepsilon_0, \cdots, \pm\varepsilon_r$. (Note that the index of the basis starts from 0.) Any dominant integral weight μ can be expressed in this basis as

$$\mu = (\mu_0, \cdots, \mu_r) = \sum_{i=0}^{r} \mu_i \varepsilon_i.$$

Here if $b_2 = 2r$ is even, then μ_i satisfy the condition $\mu_0 \geq \cdots \geq \mu_{r-1} \geq |\mu_r| \geq 0$ and μ_i are either all integers or all half-integers. If $b_2 = 2r+1$ is odd, then $\mu_0 \geq \cdots \geq \mu_r \geq 0$ and μ_i are again either all integers or all half-integers. It will be important whether μ_i are integers or half-integers, so we define:

Definition 2.1. Let $\mu = (\mu_0, \cdots, \mu_r)$ be a dominant integral weight of $\mathfrak{g} = \mathfrak{so}(V, q)$.

(i) If all μ_i are integers, we say μ is *even*. If all μ_i are half-integers (i.e., $\mu_i \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), we say μ is *odd*.
(ii) An irreducible $\mathfrak{g}$-module V_μ of highest weight μ is called even (resp. odd) if μ is even (resp. odd).
(iii) A $\mathfrak{g}$-module W is called even (resp. odd) if all of its irreducible components V_μ are even (resp. odd).

By [GKLR19, Prop 2.35], the even (odd) cohomology $H^*_{\mathrm{even}}(X)$ is always an even (resp. odd) $\mathfrak{g}$-module.

When μ is even and has multiple 0's at the end of its coordinate expression $(\mu_0, \cdots, \mu_r)$, we will simply omit the last 0's. For example, the notation $(m) = (m, 0, \cdots, 0)$ refers to the integral weight $m\varepsilon_0$. In geometric situation for hyper-Kähler manifolds, the subalgebra of $H^*(X)$ generated by $H^2(X)$ becomes an irreducible $\mathfrak{g}$-submodule of $H^*(X)$, which we call *the Verbitsky component* of $H^*(X)$. As a $\mathfrak{g}$-module, Verbitsky component is isomorphic to $V_{(n)}$ and it always occurs with multiplicity 1 in the LLV decomposition.

Let h be the degree operator on $H^*(X)$, the operator acting as multiplication by k on $H^{2n+k}(X)$. For a suitable choice of a Cartan and a Borel subalgebra, we can assume $h = \varepsilon_0^\vee$ (e.g., [GKLR19, (2.28)]). By definition, the degree decomposition of the cohomology

$$H^*(X) = \bigoplus_{k=-2n}^{2n} H^{2n+k}(X)$$

is the h-eigenspace decomposition. In general, an arbitrary $\mathfrak{g}$-module admits the h-eigenspace decomposition

$$W = \bigoplus_{k \in \mathbb{Z}} W_k, \tag{2.2}$$

where W_k denotes the eigenspace of W with eigenvalue k. The eigenvalues k are always integers by the following reason. Let $W(\theta)$ be the weight subspace of W associated to a weight $\theta = \theta_0\varepsilon_0 + \cdots + \theta_r\varepsilon_r$. Then $h = \varepsilon_0^\vee$ acts on $W(\theta)$ by $\langle \varepsilon_0^\vee, \theta \rangle = 2\theta_0$, which is an integer since $\theta_0 \in \frac{1}{2}\mathbb{Z}$ for any weight θ.

Consider the LLV decomposition of the cohomology

$$H^*(X) = \bigoplus\nolimits_\mu V_\mu^{\oplus m_\mu}. \tag{1.1 (restated)}$$

If V_μ is contained in the odd cohomology, then μ is odd by the above discussion. Hence all μ_i are half-integers, and in particular we have $\mu_i \geq \frac{1}{2}$ (possibly except for the last $|\mu_r| \geq \frac{1}{2}$, if b_2 is even). If we specifically assume $H^k(X) \neq 0$ for odd $k < 2n$, then there exists at least one irreducible component V_μ with $(V_\mu)_{k-2n} \neq 0$. This means $h = \varepsilon_0^\vee$ acts on some part of V_μ by $k - 2n$, so V_μ has an associated weight $\theta = \theta_0\varepsilon_0 + \cdots + \theta_r\varepsilon_r$ with $\theta_0 = \frac{k}{2} - n$. This forces $\mu_0 \geq n - \frac{k}{2}$. Summarizing, we have

$$\mu_0 \geq n - \frac{k}{2}, \qquad \mu_1, \cdots, \mu_{r-1}, |\mu_r| \geq \frac{1}{2},$$

which gives the following.

Corollary 2.3. *Let X be a compact hyper-Kähler manifold of dimension $2n$. Assume $H^k(X) \neq 0$ for some odd integer $k < 2n$. Then there exists a weight μ in (1.1) with $\mu_0 + \cdots + \mu_{r-1} + |\mu_r| \geq n - \frac{k}{2} + \frac{r}{2}$.* □

Finally, let us recall Salamon's relation. Let $b_k = b_k(X)$ be the k-th Betti number of X. Salamon [Sal96] proved that the Betti numbers of hyper-Kähler manifolds X satisfy a linear relation:

$$\sum_{k=1}^{2n} (-1)^k (6k^2 - 2n) b_{2n+k} = n b_{2n}.$$

One can manipulate the identity into the following form

$$\sum_{k=-2n}^{2n} (-1)^k k^2 b_{2n+k} = \frac{n}{3} e(X), \tag{2.4}$$

where $e(X) = \sum_{k=-2n}^{2n} (-1)^k b_{2n+k}$ is the topological Euler characteristic of X.

3. Proof of Main Theorem

Inspired by Salamon's relation (2.4), we define a constant $s(W)$ associated to an arbitrary $\mathfrak{g}$-module W.

Definition 3.1. Let W be a $\mathfrak{g}$-module and $W = \bigoplus_k W_k$ its h-eigenspace decomposition in (2.2). Assume $\sum_k (-1)^k \dim W_k \neq 0$ (N.B. This is automatic if W is either even or odd). Then we define a constant $s(W) \in \mathbb{Q}$ associated to W by

$$s(W) = \frac{\sum_{k\in\mathbb{Z}} (-1)^k k^2 \dim W_k}{\sum_{k\in\mathbb{Z}} (-1)^k \dim W_k}.$$

In particular, if $e(X) \neq 0$, Salamon's relation (2.4) reads

$$s(H^*(X)) = \frac{n}{3}. \tag{3.2}$$

The case of odd cohomology will be easily handled by Corollary 2.3. Thus, we can focus on the case of vanishing odd cohomology (in particular, $e(X) \neq 0$). The main content then is to bound the value $s(H^*(X))$ in terms of b_2 and the LLV decomposition (1.1). Once this is done, assuming our conjecture (1.2), Salamon's relation (3.2) leads to the desired inequality (1.3) between b_2 and n. Let us start from some straightforward properties of the constant $s(W)$.

Proposition 3.3. *Let $\{W_i\}_{i\in I}$ be a finite set of $\mathfrak{g}$-modules with well-defined $s(W_i)$.*

(i) *If all W_i are simultaneously even or odd, then* $\min_i\{s(W_i)\} \leq s\left(\bigoplus_i W_i\right) \leq \max_i\{s(W_i)\}$.
(ii) $s\left(\bigotimes_i W_i\right) = \sum_i s(W_i)$.

Proof. It is enough to prove the proposition for two $\mathfrak{g}$-modules W and W'. Assume without loss of generality $s(W) \leq s(W')$, and let us consider the case when W and W' are even (the odd case is similar). In this case, all eigenvalues k of W are even, so we have

$$\sum_k k^2 \dim W_k = s(W) \dim W, \qquad \sum_k k^2 \dim W'_k = s(W') \dim W'.$$

Adding the two equalities and using $s(W) \leq s(W')$ gives us the first item.

For the second item, we compute

$$\begin{aligned}
\sum_k (-1)^k k^2 \dim(W \otimes W')_k &= \sum_k (-1)^k k^2 \left(\sum_{i+j=k} \dim W_i \dim W'_j \right) \\
&= \sum_{i,j} (-1)^{i+j} (i^2 + 2ij + j^2) \dim W_i \dim W'_j \\
&= \left(\sum_i (-1)^i i^2 \dim W_i \right) e(W') + \left(\sum_j (-1)^j j^2 \dim W'_j \right) e(W) \\
&\quad + 2 \left(\sum_i (-1)^i i \dim W_i \right) \left(\sum_j (-1)^j j \dim W'_j \right).
\end{aligned}$$

Here we used the notation $e(W) = \sum_i (-1)^i \dim W_i$ and $e(W') = \sum_j (-1)^j \dim W'_j$ for simplicity. Notice that $\sum_i (-1)^i i \dim W_i = 0$, since by Weyl symmetry we always have $\dim W_i = \dim W_{-i}$. This proves $\sum_k (-1)^k k^2 \dim(W \otimes W')_k = (\sum_i (-1)^i i^2 \dim W_i) e(W') + (\sum_j (-1)^j j^2 \dim W'_j) e(W)$. Dividing both hand sides by $e(W \otimes W') = e(W) e(W')$ gives us the result. □

Remark 3.4. In fact, we can associate to an arbitrary $\mathfrak{g}$-module W the following formal power series

$$S(W) = \sum_k (-1)^k \dim W_k \cdot \exp(kt) \quad \in \quad \mathbb{Q}[[t]].$$

One can easily show

$$S(W \oplus W') = S(W) + S(W'), \quad S(W \otimes W') = S(W) \cdot S(W'),$$

so that S defines a *ring homomorphism* from the representation ring $K(\mathfrak{g})$ of $\mathfrak{g}$

$$S : K(\mathfrak{g}) \to \mathbb{Q}[[t]].$$

By Weyl symmetry, we have $\dim W_k = \dim W_{-k}$, giving that all the odd degree terms of $S(W)$ vanish. Thus, we can write

$$S(W) = s_0 + s_2 t^2 + s_4 t^4 + \cdots \quad \in \quad \mathbb{Q}[[t]], \qquad s_i = \frac{1}{i!} \sum_k (-1)^k k^i \dim W_k.$$

From this perspective, our constant $s(W)$ is the ratio between the first two coefficients

$$s(W) = \frac{2s_2}{s_0}$$

of the formal power series $S(W)$.

A more interesting result is the explicit computation of $s(W)$ for irreducible $\mathfrak{g}$-modules $W = V_\mu$. Recall that the Lie algebra $\mathfrak{g}$ was isomorphic to $\mathfrak{so}(b_2+2, \mathbb{C})$ and $r = \lfloor \frac{b_2}{2} \rfloor$, so that $\mathfrak{g}$ has rank $r+1$ and a dominant integral weight μ can be written as a tuple $(\mu_0, \cdots, \mu_r)$.

Theorem 3.5. *With notations as above, let V_μ be an irreducible $\mathfrak{g}$-module of highest weight μ. If $\mu_r \geq 0$, then*

$$s(V_\mu) = 8 \cdot \frac{\left(\sum_{i=0}^r \mu_i\right) b_2 + \left(\sum_{i=0}^r (\mu_i - i)^2 - i^2\right)}{(b_2+1)(b_2+2)}.$$

If b_2 is even and $\mu_r < 0$, then $s(V_\mu) = s(V_{\mu'})$ where $\mu' = (\mu_0, \cdots, \mu_{r-1}, -\mu_r)$.

We postpone the proof of Theorem 3.5 to the following section. For now let us conclude the proof of our Main Theorem using this result. First, we note the following consequence of Theorem 3.5. Recall from Section 2 that the notation $(m) = (m, 0, \cdots, 0)$ refers to the integral weight $m\varepsilon_0$.

Corollary 3.6. (i) *$s(V_{(m)}) = \frac{8m(b_2+m)}{(b_2+1)(b_2+2)}$ for $m \in \mathbb{Z}_{\geq 0}$.*

(ii) *If μ is even, then $s(V_\mu) \leq s(V_{(m)})$ for $m = \mu_0 + \cdots + \mu_{r-1} + |\mu_r|$.*

(iii) *$s(V_{(m)}) \leq s(V_{(n)})$ for $m \leq n$.*

Proof. The first item is immediate from letting $\mu = m\varepsilon_0$ in Theorem 3.5. The third item follows from it directly. For the second item, using $s(V_\mu) = s(V_{\mu'})$ in Theorem 3.5, we may assume $\mu_r \geq 0$. Let us temporarily define a function $A(\mu)$ of a dominant integral weight μ by

$$A(\mu) = \sum_{i=0}^r (\mu_i - i)^2.$$

Again using Theorem 3.5, one finds that the second item is equivalent to the inequality $A(\mu) \leq A(m\varepsilon_0)$. For it, one first proves an inequality

$$(\mu_i - i)^2 + (\mu_j - j)^2 < (\mu_i + 1 - i)^2 + (\mu_j - 1 - j)^2 \qquad \text{for } \ 0 \leq i < j \leq r, \tag{3.7}$$

which easily follows from $\mu_i \geq \mu_j$. The desired $A(\mu) \leq A(m\varepsilon_0)$ follows from inductively applying the inequality (3.7) to modify the dominant integral weight μ until it reaches $m\varepsilon_0$. □

Proof of Main Theorem. Assume $H^k(X) \neq 0$ for some odd integer k. By Corollary 2.3, there exists at least one component $V_\mu \subset H^*_{\text{odd}}(X)$ with $\mu_0 + \cdots + \mu_{r-1} + |\mu_r| \geq n - \frac{k}{2} + \frac{r}{2}$. Thus, under the condition (1.2), we get $r \leq k$ and hence $b_2 = (2r \text{ or } 2r+1) \leq 2k+1$.

Now assume $H^*_{\mathrm{odd}}(X) = 0$. Among the irreducible components V_μ of the LLV decomposition (1.1), we always have the Verbitsky component, which as a $\mathfrak{g}$-module is isomorphic to $V_{(n)}$. Thus, if we assume the condition (1.2) holds for X, then combining Corollary 3.6 with Proposition 3.3 gives us

$$s(H^*(X)) \leq \max\{s(V_\mu) : \mu \text{ appearing in (1.1)}\} = s(V_{(n)}) = \frac{8n(b_2+n)}{(b_2+1)(b_2+2)}.$$

On the other hand, we have Salamon's relation $s(H^*(X)) = \frac{n}{3}$ in (3.2). We conclude

$$s(H^*(X)) = \frac{n}{3} \leq \frac{8n(b_2+n)}{(b_2+1)(b_2+2)},$$

giving the desired bound on b_2 in Main Theorem. □

4. Computation of $s(W)$ for irreducible $\mathfrak{g}$-modules

In this section, we prove Theorem 3.5 by using standard representation theoretic methods.

Let us first fix the notation. Here, we simply write $b = b_2(X)$. Let (V, q) be a quadratic space of dimension $b+2$ and $\mathfrak{g} = \mathfrak{so}(V, q)$ be the associated simple Lie algebra of type B_{r+1} / D_{r+1}. We fix a Cartan and a Borel subalgebra of $\mathfrak{g}$ so that the positive and simple roots are well defined. We also use the following notation:

- $\mathfrak{W}$ is the Weyl group of $\mathfrak{g}$;
- R_+ is the set of positive roots of $\mathfrak{g}$;
- For $w \in \mathfrak{W}$, $\ell(w)$ is the length of w. That is, $\ell(w)$ is the minimum length of the decomposition of w into a product of simple reflexions $w = s_{\alpha_1} \cdots s_{\alpha_\ell}$ where all α_i are simple roots of $\mathfrak{g}$;
- ρ is the half sum of all the positive roots

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha. \tag{4.1}$$

Throughout, we consider an irreducible representation $W = V_\mu$ of highest weight μ.

Our proof is inspired by the proof of Weyl dimension formula (following [Kir08, §8.5]). The Weyl dimension formula is a closed formula computing $\dim V_\mu$, which can be derived from the Weyl character formula.

Theorem 4.2 (Weyl character formula)**.** *The formal character of the irreducible $\mathfrak{g}$-module V_μ of highest weight μ can be computed from a formal power series expansion of the rational function*

$$\mathrm{ch}(V_\mu) = \frac{\sum_{w \in \mathfrak{W}} (-1)^{\ell(w)} e^{w.(\mu+\rho)}}{\prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Due to its importance to our proof of Theorem 3.5, let us review first the proof of Weyl dimension formula. To start, recall the $\mathfrak{g}$-module V_μ has a weight decomposition $V_\mu = \bigoplus_{\theta \in \Lambda} V_\mu(\theta)$, where Λ is the weight lattice of $\mathfrak{g}$, θ runs through the weights of $\mathfrak{g}$, and $V_\mu(\theta)$ indicates the weight θ subspace of V_μ. The formal character of V_μ is an element in the group algebra $\mathbb{Q}[\Lambda]$ encoding dimensions of the weight subspaces $V_\mu(\theta)$:

$$\mathrm{ch}(V_\mu) = \sum_\theta \dim V_\mu(\theta) \cdot e^\theta \quad \in \mathbb{Q}[\Lambda].$$

Let us introduce a ring homomorphism "projection to ρ-direction"

$$\mathrm{pr}_\rho : \mathbb{Q}[\Lambda] \to \mathbb{Q}[q^{\pm 1}], \qquad e^\theta \mapsto q^{4(\rho,\theta)},$$

where $(,)$ is the Killing form of $\mathfrak{g}$ and ρ is defined in (4.1). Set $f(q) = f_\mu(q) \in \mathbb{Q}[q^{\pm 1}]$ to be the image of the formal character $\mathrm{ch}(V_\mu)$ by the homomorphism pr_ρ:

$$f(q) = \mathrm{pr}_\rho(\mathrm{ch}(V_\mu)) = \sum_\theta \dim V_\mu(\theta)\, q^{4(\rho,\theta)}. \tag{4.3}$$

Since $\dim V_\mu = \sum_\theta \dim V_\mu(\theta)$, the dimension of V_μ can be recovered from $f(q)$ by

$$\dim V_\mu = f(1). \tag{4.4}$$

On the other hand, if we apply pr_ρ to the Weyl character formula above, then using Weyl denominator identity (e.g., [Kir08, Thm 8.39]), the Weyl character formula is translated into

$$f(q) = \prod_{\alpha \in R_+} \frac{q^{2(\mu+\rho,\alpha)} - q^{-2(\mu+\rho,\alpha)}}{q^{2(\rho,\alpha)} - q^{-2(\rho,\alpha)}}. \tag{4.5}$$

The Weyl dimension formula is obtained by computing $f(1) = \lim_{q\to 1} f(q)$ with the aid of (4.5).

Now let us begin the proof of Theorem 3.5. First, notice that for irreducible modules V_μ, we can ignore the sign terms $(-1)^k$ in the definition of $s(V_\mu)$ (i.e., V_μ is either even or odd). Thus, we have

$$s(V_\mu) = \frac{\sum_k k^2 \dim(V_\mu)_k}{\dim V_\mu}.$$

The following lemma expresses $s(V_\mu)$ in terms of $f(q)$, imitating (4.4) above.

Lemma 4.6. *Let $f = f(q)$ be as in (4.3). Then $s(V_\mu) = \frac{6}{b(b+1)(b+2)}(\log f)''(1)$.*

Proof. Consider the derivates of f

$$f'(q) = \sum_\theta 4(\rho,\theta) \dim V_\mu(\theta)\, q^{4(\rho,\theta)-1}, \qquad f''(q) = \sum_\theta 4(\rho,\theta)(4(\rho,\theta)-1) \dim V_\mu(\theta)\, q^{4(\rho,\theta)-2}.$$

The Weyl symmetry gives us $\dim V_\mu(\theta) = \dim V_\mu(-\theta)$. From it, we obtain $f'(1) = 0$ and

$$f''(1) = 16 \sum_\theta (\rho,\theta)^2 \dim V_\mu(\theta).$$

Let us now specialize the discussion to $\mathfrak{g} = \mathfrak{so}(V,q)$ and use the precise value of ρ. For special orthogonal Lie algebras, one can compute all the positive roots explicitly in terms of our preferred basis ε_i and hence obtain the half sum of all the positive roots

$$\rho = \begin{cases} r\varepsilon_0 + (r-1)\varepsilon_1 + \cdots + \varepsilon_{r-1} & \text{when } \ b = 2r \text{ is even} \\ (r+\frac{1}{2})\varepsilon_0 + (r-\frac{1}{2})\varepsilon_1 + \cdots + \frac{1}{2}\varepsilon_r & \text{when } \ b = 2r+1 \text{ is odd} \end{cases}. \tag{4.7}$$

Assume $b = 2r$ is even. Letting $\theta = \sum_{i=0}^r \theta_i \varepsilon_i$, we have $(\rho,\theta) = \sum_{i=0}^r (r-i)\theta_i$. This gives us

$$f''(1) = 16 \sum_\theta \left(\sum_{i=0}^r (r-i)^2 \theta_i^2 + 2 \sum_{0\le i<j\le r} (r-i)(r-j)\theta_i\theta_j \right) \dim V_\mu(\theta).$$

Again by Weyl symmetry, we have $\dim V_\mu(\theta) = \dim V_\mu(w.\theta)$ for any $w \in \mathfrak{W}$. Note that the Weyl group $\mathfrak{W}$ in this case is isomorphic to an order 2 subgroup of $\mathfrak{S}_{r+1} \ltimes (\mathbb{Z}/2)^{\times(r+1)}$, consisting of the elements with even number of $1 \in \mathbb{Z}/2$. The symmetric group part $\mathfrak{S}_{r+1}$ acts on a weight $\theta = (\theta_0, \cdots, \theta_r)$ by permuting coordinates, and $(\mathbb{Z}/2)^{\times(r+1)}$ part acts on it by flipping the signs of θ_i's. With these symmetries in mind, one deduces

(1) $\sum_\theta \theta_i^2 \dim V_\mu(\theta) = \sum_\theta \theta_j^2 \dim V_\mu(\theta)$; and
(2) $\sum_\theta \theta_i\theta_j \dim V_\mu(\theta) = 0$ for $i \neq j$.

This finally leads us to the identity

$$\begin{aligned} f''(1) &= 16 \cdot \left(r^2 + (r-1)^2 + \cdots + 1^2\right) \sum_\theta \theta_0^2 \dim V_\mu(\theta) \\ &= 16 \cdot \frac{r(r+1)(2r+1)}{6} \sum_k \left(\frac{k}{2}\right)^2 \dim(V_\mu)_k = \frac{b(b+1)(b+2)}{6} \sum_k k^2 \dim(V_\mu)_k. \end{aligned}$$

Combining it with $f(1) = \dim V_\mu$ and $f'(1) = 0$, we have $(\log f)''(1) = \frac{f''(1)}{f(1)} = \frac{b(b+1)(b+2)}{6} s(V_\mu)$, as claimed.

Next, assume $b = 2r+1$ is odd. Similar argument gives us the computation

$$f''(1) = \frac{(r+1)(2r+1)(2r+3)}{3} \sum_k k^2 \dim(V_\mu)_k = \frac{b(b+1)(b+2)}{6} \sum_k k^2 \dim(V_\mu)_k.$$

Hence the same result follows, regardless of the parity of b. □

The next step is to use the Weyl character formula (4.5) and compute the value $(\log f)''(1)$.

Lemma 4.8. *Let $f = f(q)$ be as in (4.3). If $\mu_r \geq 0$, then $(\log f)''(1) = \frac{4}{3} b \left[\left(\sum_{i=0}^r \mu_i\right) b + \left(\sum_{i=0}^r \mu_i^2 - 2i\mu_i\right)\right]$.*

Proof. From the q-polynomial version of the Weyl character formula (4.5), we derive

$$\log f(q) = \sum_{\alpha \in R_+} \log\left(\frac{q^{2(\mu+\rho,\alpha)} - q^{-2(\mu+\rho,\alpha)}}{q-1}\right) - \log\left(\frac{q^{2(\rho,\alpha)} - q^{-2(\rho,\alpha)}}{q-1}\right).$$

Here $q-1$ on the denominators are inserted to make each log terms well-defined in the neighborhood of $q = 1$. Notice that $(\log f)''(1)$ is twice the coefficient of the term $(q-1)^2$ in the Taylor series of $\log f$. For a general *positive integer* a, the Taylor series expansion of $\log\left(\frac{q^a - q^{-a}}{q-1}\right)$ at $q = 1$ is

$$\log\left(\frac{q^a - q^{-a}}{q-1}\right) = \log(2a) - \frac{1}{2}(q-1) + \frac{1}{24}(4a^2+5)(q-1)^2 + \cdots,$$

which has a degree 2 coefficient $\frac{1}{24}(4a^2+5)$. Since $2(\mu+\rho,\alpha)$ and $2(\rho,\alpha)$ are both positive integers for any $\alpha \in R_+$ (N.B. Here we used the fact $\mu_r \geq 0$), we conclude

$$\begin{aligned}(\log f)''(1) &= 2\sum_{\alpha\in R_+} \frac{1}{24}\big(16(\mu+\rho,\alpha)^2+5\big) - \frac{1}{24}\big(16(\rho,\alpha)^2+5\big)\\ &= \frac{4}{3}\sum_{\alpha\in R_+} (\mu+\rho,\alpha)^2 - (\rho,\alpha)^2.\end{aligned}$$

Recalling (4.7), let us get into an explicit computation for $\mathfrak{g} = \mathfrak{so}(V,q)$. Assume $b = 2r$ is even. The positive roots are $R_+ = \{\varepsilon_i \pm \varepsilon_j : 0 \leq i < j \leq r\}$. We get

$$\begin{aligned}&\sum_{\alpha\in R_+} (\mu+\rho,\alpha)^2 - (\rho,\alpha)^2\\ &= \sum_{0\leq i<j\leq r} (\mu_i-\mu_j)^2 + 2(\mu_i-\mu_j)(j-i) + (\mu_i+\mu_j)^2 + 2(\mu_i+\mu_j)(2r-i-j)\\ &= \sum_{0\leq i<j\leq r} \big[2(\mu_i^2+\mu_j^2) + 4r(\mu_i+\mu_j) - (i\mu_i + j\mu_j)\big]\\ &= 2r\left[2r\sum_{i=0}^r \mu_i + \sum_{i=0}^r (\mu_i^2 - 2i\mu_i)\right] = b\left[\left(\sum_{i=0}^r \mu_i\right) b + \left(\sum_{i=0}^r \mu_i^2 - 2i\mu_i\right)\right].\end{aligned}$$

This proves the result in this case.

Similarly for $b = 2r+1$ odd, the positive roots are $R_+ = \{\varepsilon_i : 0 \leq i \leq r\} \cup \{\varepsilon_i \pm \varepsilon_j : 0 \leq i < j \leq r\}$, giving:

$$\begin{aligned}&\sum_{\alpha\in R_+} (\mu+\rho,\alpha)^2 - (\rho,\alpha)^2\\ &= \sum_{0\leq i<j\leq r} (\mu_i-\mu_j)^2 + 2(\mu_i-\mu_j)(j-i) + (\mu_i+\mu_j)^2 + 2(\mu_i+\mu_j)(2r+1-i-j)\\ &\quad + \sum_{i=0}^r \mu_i^2 + 2\mu_i(r+\tfrac{1}{2}-i)\\ &= \sum_{0\leq i<j\leq r} \big[2(\mu_i^2+\mu_j^2) + (4r+2)(\mu_i+\mu_j) - (i\mu_i+j\mu_j)\big] + \sum_{i=0}^r \mu_i^2 + (2r+1)\mu_i - 2i\mu_i\\ &= (2r+1)\left[(2r+1)\sum_{i=0}^r \mu_i + \sum_{i=0}^r (\mu_i^2 - 2i\mu_i)\right] = b\left[\left(\sum_{i=0}^r \mu_i\right) b + \left(\sum_{i=0}^r \mu_i^2 - 2i\mu_i\right)\right].\end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 3.5. Combining Lemma 4.6 and 4.8, the theorem follows for the case $\mu_r \geq 0$. Now assume $b_2 = 2r$ is even and $\mu_r < 0$. In this case, ρ does not have the ε_r-coordinate by (4.7). Hence, the Weyl character formula (Theorem 4.2) implies that the weights associated to V_μ and $V_{\mu'}$ are bijective via the action $(\theta_0, \cdots, \theta_{r-1}, \theta_r) \mapsto (\theta_0, \cdots, \theta_{r-1}, -\theta_r)$. By definition, the constant $s(W)$ captures only the h-eigenspaces, i.e., only the ε_0-coordinates of the weights associated to W. This means $s(V_\mu) = s(V_{\mu'})$. □

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